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ANALYSIS OF OPTIMAL INVESTMENT IN CONTINUUM OF CAPITAL VINTAGES

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Abstract. The considered nonlinear optimal control problem describes distributed investments into new and old capital under limited substitutability among vintages of different ages. The production output is defined by the CES production function with continuum of capital vintage inputs, learning, and technological change. Both balanced growth and transition dynamics are explored. The obtained outcomes lead to new interesting insights about the structure of optimal investment into old vintages.

1. Introduction

Vintage capital models in economic growth theory describe the optimal investment in age-dependent capital and replacement of obsolete capital under technological change. The first vintage capital models were suggested in the 1960's [17] and have been intensively studied since that time. Mathematically, they are presented by difference, partial differential, or integral equations or their combination. This paper considers a vintage capital model in integral setting with additional features that make the model more appealing to practice though bring more challenges to its investigation.

A key novel feature of our model is that it considers simultaneous investments into old and new vintages using the CES production function [16] with continuum of capital vintage inputs. We assume that firms can invest into old and new capital and that the vintages of various ages can substitute each other to a certain degree. Then, the evolution of heterogeneous capital stock is governed by the first-order linear partial differential equation, well known in the theory of controlled biological populations as the Lotka-McKendrick model [1]. Such PDE-based models have become common in mathematical economics, starting with [18].

Most VCMs [4–6, 9, 10, 12, 13, 15] assume that all investments flow into the latest vintage. Only few vintage models [8, 11, 14, 16] consider a limited substitutability among vintages, which leads to coexistence of several vintages and investment into older vintages. Chari and Hopenhayn [8] analyze a VCM with limited substitutability among two (old and new) vintages and establish its bell-shaped investment age-profile. Jovanovic and Yatsenko [16] introduce the novel CES production

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function whose inputs are different capital vintages and study properties of a balanced growth in their model. They obtain various investment age-profiles along balanced growth for different levels of vintages substitutability in the case of linear utility. Hritonenko, Yatsenko, and Boranbayev [14] formally prove that, when the elasticity of substitution among vintages increases indefinitely, bell-shaped investment profiles in the model of [16] converge to a Dirac-type spiked investment into one vintage with the largest productivity. Both papers [14] and [16] are restricted to the analysis of balanced growth dynamics. The transition dynamics in such a class of models is notoriously challenging to analyze. The only paper related to the transition dynamics in our model is the paper of Hritonenko, Kato, and Yatsenko [11], that studies the structure of solutions in the optimization problem with a linear utility and shows that initially bang-bang optimal investment eventually converges to a steady-state trajectory that represents a balanced economic growth. The present paper is the first which treats at the same time: (i) nonlinear preferences; (ii) nonlinear production function; (iii) investment in old vintages, and (iv) characterization of stationary solutions (balanced growth paths) and transitional dynamics.

The optimal trajectory in most vintage capital models consists of two parts: balanced growth path and transition dynamics [5, 6, 9–13]. The related mathematical analysis includes finding a solution to the corresponding optimal control problem without its initial conditions (a balanced growth path), and the most efficient way to get to the balanced growth path from the given initial conditions (a transition dynamics). Both balanced and transition dynamics are analyzed in this paper. Economic interpretation of all theoretical outcomes is provided, and practical recommendations are outlined. They lead to better understanding of the investment pattern into old and new vintages of capital.

The rest of the paper is as follows. In Section 2, the optimal control problem, its applied relevance to economics and place in the related research are presented. Necessary optimal conditions in form of maximum principle are proven in Section 3. Section 4 describes a unique balanced growth path at concave utility and compares the results with a linear utility case. Section 5 addresses the transition dynamics in the nonlinear utility model, which appear to be different from the previously obtained results for the model with linear utility. Section 6 concludes and discusses the obtained outcomes.

2. Optimal control problem and its applied relevance

Let us consider the following optimal control problem (OCP):

$$\max_{u,x} I = \max \int_0^\infty e^{-rt} \frac{c^{1-\eta}(t)}{1-\eta} dt, \; x \geq 0, u \geq 0, \tag{2.1}$$

subject to

$$y(t) = \left(\int_{-\infty}^t A(t-v) \Big(z(v) k^\beta (v,t) \Big) dv \right)^{\alpha/\beta}, \tag{2.2}$$

$$k(v,t) = e^{-\delta(t-v)}x(v) + \int_v^t e^{-\delta(t-s)}u(v,s)ds \texttt{ for } 0 < v < t, \tag{2.3}$$

$$k(v,t) = e^{-\delta(t-v)}k_0(v) + \int_0^t e^{-\delta(t-s)}u(v,s)ds \texttt{ for } -\infty < v \leq 0, \tag{2.4}$$

$$y(t) = c(t) + x(t) + \int_{-\infty}^t u(v,t)dv. \tag{2.5}$$

with respect to the unknown functions $u(v,t) \in L^\infty(-\infty,\infty] \times [0,\infty), x(t) \in L^\infty[0,\infty), y(t) \in L^\infty[0,\infty), c(t) \in L^\infty[0,\infty)$, and $k(v,t) \in L^\infty(-\infty,\infty) \times [0,\infty)$ The constants r, α, β, δ, η and functions $A(t-v)$ and $z(t) \in L^\infty[0,\infty), k^o(v), v \in (-\infty, 0]$, are given. The initial condition $k(v,0) = k^o(v), v \in (-\infty, t]$, for capital is incorporated in (2.4).

In economic growth theory, the OCP (2.1)-(2.5) is known as a *vintage capital model* (VCM) with distributed investments into new and old capital and limited substitutability among vintages of different ages. A production output is described by the CES production function with continuum of capital vintage inputs and concave utility [16], and the model functions and parameters are interpreted as follows:

$c(t)$ is a consumption,
$x(t)$ is the investment in new capital,
$u(v,t)$ is the investment in the various vintages of old capital,
$y(t)$ is a product output,
$z(v)$ is a unit efficiency of the capital of vintage v,
$k(v,t)$ is the amount of capital at date t embodying technology of vintage v,
$A(t\text{-}v)$ is the age-dependent "learning curve" for the capital of vintage $v, v \in (-\infty, t]$, $t \in [0,\infty)$,
$k^o(v)$, $v \in (-\infty,0]$, is a distribution of capital over past vintages $v<0$ at date zero,
r is the discount rate,
η is a parameter of the isoelastic utility function, $\eta > 0$,
δ is a deterioration rate,
α, $0 < \alpha < 1$, describes diminishing returns to scale,
β, $0 < \beta < 1$, reflects substitutability of vintages, which are perfectly substitutable at $\beta = 1$. The elasticity of substitution $\sigma = 1/(1\text{-}\beta)$ is a common economic measure of how easy it is to switch between production inputs.

The decision variables in (2.1)-(2.5) are the investment $x(t)$ in new capital and the investment $u(v,t)$, in vintages of old capital, $v \in (-\infty,t]$, $t \in [0,\infty)$. The unknown capital amount $k(v,t)$, product output $y(t)$, and consumption $c(t)$ are determined from (2.3)-(2.4) and (2.2) respectively.

The linear version of the OCP (2.1)-(2.5) with $\eta = 0$ was first suggested and studied in [16]. In this paper we consider both balanced and transition dynamics at nonlinear utility, $0 < \eta < 1$. Our goal is to better understand investment patterns into capital vintages in the modern technologically advanced world.

3. Optimality conditions

Following standard optimization techniques, let us derive the first-order extremum condition for any (including corner) solution $u(v,t)$:

Lemma 3.1 (necessary conditions for an extremum)**.** *If $(x(t),\ u(v,t))$, $0\le t<\infty$, $-\infty<v\le t$, is a solution of the problem (2.1)-(2.5), then*

$$I'(t,t)\le 0 \ \textit{at}\ x(t)=0, I'(t,t)=0 \ \textit{at}\ x(t)>0, 0\le t<\infty,$$

$$I'(v,t)\le 0 \quad \textit{at}\ u(v,t)=0, I'(v,t)=0 \quad \textit{at}\ u(v,t)>0, 0\le t<\infty -\infty<v<t,$$

where

$$\begin{aligned}(3.1)\quad & I'(v,t)\\ &=\alpha z^{\beta}(v)\int_t^{\infty} e^{-(r+\delta)s+\delta t}y^{(\alpha-\beta)/\alpha}(s)A(s-v)k^{\beta-1}(v,s)c^{-\eta}(s)ds-e^{-rt}c^{-\eta}(t),\end{aligned}$$

$$0<v<t,\ \ 0<t<\infty.$$

Proof. The increment of the Lagrangean of the OCP(2.1)-(2.5) is

$$\begin{aligned}\delta L=&\int\limits_0^{\infty} e^{-rt}\frac{(c(t)+\delta c(t))^{1-\eta}}{1-\eta}dt\\ &+\int\limits_0^{\infty} e^{-rt}\lambda(t)[(x(t)+\delta x(t))+(c(t)+\delta c(t))\\ &+\int\limits_{-\infty}^{t}(u(t,v)+\delta u(t,v))dv\\ &-(\int\limits_{-\infty}^{t} A(t-v)((z(v)(k(v,t)+\delta k(v,t)))^{\beta}dv)^{\alpha/\beta}]dt\\ &-\int\limits_0^{\infty} e^{-rt}\frac{c^{1-\eta}(t)}{1-\eta}dt-\int\limits_0^{\infty} e^{-rt}\lambda(t)[x(t)+c(t)\\ &+\int\limits_{\infty}^{t} u(t,v)dv-(\int\limits_{-\infty}^{A} A(t-v)(z(v)k(v,t))^{\beta}dv)^{\alpha/\beta}]dt.\end{aligned}$$

Applying Taylor series expansion and using (2.3) as

$$\delta k(v,t)=e^{-\delta(t-v)}\delta x(v)+\int\limits_v^{t} e^{-\delta(t-s)}\delta u(v,s)ds,$$

we get

$$\delta L = \int_0^\infty [e^{-rt}c^{-\eta}(t) + e^{-rt}\lambda(t)]\delta c(t)dt$$

$$+ \int_0^\infty [e^{-rt}\lambda(t)\delta x(t) - e^{-rt}\lambda(t)\alpha y^{\frac{\alpha-\beta}{\alpha}}(t)$$

$$\int_{-\infty}^t (A(t-v)z^\beta(v)k^{\beta-1}(v,t)e^{-\delta(t-v)}\delta x(v))dv]dt$$

$$+ \int_0^\infty [e^{-rt}\lambda(t) \int_{-\infty}^t \delta u(v,t)dv - e^{-rt}\lambda(t)\alpha y^{\frac{\alpha-\beta}{\alpha}}(t)$$

$$\int_{-\infty}^t (A(t-v)z^\beta(v)k^{\beta-1}(v,t) \int_v^t e^{-\delta(t-s)}\delta u(v,s)ds)dv]dt,$$

that after interchanging limits of integration, collecting coefficients of $\delta x(t), \delta u(v,t)$, $\delta c(t)$, and setting the coefficient of $\delta c(t)$ equal 0, leads to (3.1). The Lemma is proven. □

The function (3.1) characterizes optimal investments into various capital vintages. Formally, (3.1) is the *Fréchet derivative* of the objective function (2.1) of the OCP (2.1)-(2.5). Economically, it describes the future rental value of vintage v at time t. In particular, if $I'(v,t)$ is less than zero, then by Lemma 3.1, the optimal $u(v,t)= 0$, which means no investment at date t into capital of vintage v should be provided, i.e., no old vintages should be bought.

4. Balanced growth analysis

The balanced growth in economics is an interior steady-state solution to the OCP without initial conditions. If there is an exponential balanced growth, then all functions of the OCP should grow with the same rate. The existence of a balanced growth is often taken as an indicator of the quality of an economic model. If a balanced growth exists, then a transition dynamic path can be analyzed. The transition dynamics shows the most efficient way to reach the balanced growth trajectory from the given initial conditions. A combination of a transition dynamic path and a balanced growth path delivers a complete solution to the OCP with initial conditions.

To find a balanced growth, denoted by ($\tilde{y}(t)$, $\tilde{c}(t)$, $\tilde{k}(v,t)$, $\tilde{x}(t)$, $\tilde{u}(v,t)$), let us set (3.1) equal zero for $\infty<v<t$, $0<t<\infty$ and differentiate it with respect to t to get the first-order extremum condition

$$\alpha z^\beta(v)y^{(\alpha-\beta)/\alpha}(t)A(t-v)k^{\beta-1}(v,t) = r + \delta + \eta c'(t)/c(t). \tag{4.1}$$

for an interior solution $u(v,t) > 0$, $x(t) > 0$.

As common in the economic growth theory, we assume the exponential technological change:

$$z(t) = \bar{z}e^{\gamma t}, \gamma > 0, \tag{4.2}$$

where the technological rate γ and the constant z are given, and analyze the possibility of a balanced growth of $\tilde{y}(t)$, $\tilde{k}(v,t)$, $\tilde{u}(v,t)$, -∞<v<t, 0<t<∞. We start with assuming the existence of a balanced capital $\tilde{k}(v,t)$ in the form

$$\tilde{k}(v,t) = e^{g_t t}\chi(t-v), g_t > 0, \tag{4.3}$$

where the factor $\chi(t-v)$ depends on the capital age a=t-v only. In looking for the balanced growth $\tilde{y}(t)$, $\tilde{k}(v,t)$, $\tilde{u}(v,t)$ we disregard [1] the given distribution $k^0(v)$ capital over past vintages $v < 0$ at t=0 and assume that (4.1) holds for all -∞<v<t. The substitution of (4.2) and (4.3) into (4.1) leads to the equality

$$\alpha\bar{z}^{\beta}e^{\gamma\beta v}y^{(\alpha-\beta)/\alpha}(t)A(t-v)\chi(t-v)^{\beta-1}e^{(\beta-1)g_t t} = r + \delta + \eta c'(t)/c(t),$$

that should hold for all -∞<v<t, 0<t<∞. In the variables a=t-v and v, the last equality is written as

$$\alpha\bar{z}^{\beta}e^{\gamma\beta v}y^{(\alpha-\beta)/\alpha}(v+a)A(a)\chi^{\beta-1}(a) = (r+\delta+\eta c'(v+a)/c(v+a))e^{(1-\beta)g_t(v+a)}, \tag{4.4}$$
$$-\infty < v < \infty, 0 < a < \infty.$$

Now, let us take a look at $y(t)$. Substituting (4.1) and (4.2) into the output equation (2.2), we obtain

$$]y(t) = \bar{z}^{\alpha}e^{\alpha g_t t}\left(\int_{-\infty}^{t} e^{\gamma\beta v}A(t-v)\chi^{\beta}(t-v)dv\right)^{\alpha/\beta}, \tag{4.5}$$

or, replacing the integration with respect to the vintage installation time v with the integration with respect to the vintage age a=t-v,

$$y(t) = \bar{z}^{\alpha}e^{\alpha(g_t+\gamma)t}\left(\int_{0}^{\infty} e^{-\gamma\beta a}A(a)\chi^{\beta}(a)da\right)^{\alpha/\beta}. \tag{4.6}$$

The substitution (4.6) into (4.4) delivers the following equation

$$\alpha\bar{z}^{\alpha}e^{[\alpha\gamma-g_t(1-\alpha)]v}\left(\int_{0}^{\infty} e^{-\gamma\beta s}A(s)\chi^{\beta}(s)ds\right)^{\frac{\alpha-\beta}{\beta}} \tag{4.7}$$
$$= (r+\delta++\eta c'(v+a)/c(v+a))e^{[-\gamma(\alpha-\beta)+g_t(1-\alpha)]a}A^{-1}(a)\chi^{1-\beta}(a).$$

Along the balanced growth path, the consumption $c(t)$ should grow with the rate g_t, that is,

$$c(t)'/c(t) = const = g_t. \tag{4.8}$$

In order to hold (4.7) at any age 0<a<∞ and vintage -∞<v<∞, and any unknown $\chi(a)$, the equality

$$g_t = \frac{\gamma\alpha}{1-\alpha}, \tag{4.9}$$

[1]The given capital distribution at -∞<v<0 impacts the transition dynamics of the problem (2.1)-(2.5), which is considered in detail in next sections.

should be valid for the dynamics in v and

$$\alpha\bar{z}^{\alpha}\left(\int_0^{\infty} e^{-\gamma\beta s}A(s)\chi^{\beta}(s)ds\right)^{\frac{\alpha-\beta}{\beta}} = (r+\delta+\eta g_t)e^{\gamma\beta a}A^{-1}(a)\chi^{1-\beta}(a). \tag{4.10}$$

for the dynamics in a. The rate g_t coincides with the growth rate in the linear utility case in [16].

The integral equation (4.10) has a unique solution

$$\chi(a) = \bar{k}e^{-\frac{\gamma\beta}{1-\beta}a}A^{\frac{1}{1-\beta}}(a), \tag{4.11}$$

with the constant

$$\bar{k} = \left(\frac{\alpha\bar{z}^{\alpha}}{r+\delta+\eta\frac{\gamma\alpha}{1-\alpha}}\right)^{\frac{1}{1-\alpha}}\left(\int_0^{\infty} e^{-\frac{\gamma\beta}{1-\beta}s}A^{\frac{1}{1-\beta}}(s)ds\right)^{\frac{\alpha-\beta}{\beta(1-\alpha)}}, \tag{4.12}$$

that can be verified by direct substitution of (4.11) and (4.12) into (4.10) and routine algebraic transformations. Uniqueness of a solution follows from a deeper analysis of the integral equation (4.10).

By (4.6) and (4.12), the optimal balanced output grows as

$$y(t) = \varsigma e^{g_t t}, \tag{4.13}$$

where

$$\varsigma = (k^{\beta-1}\bar{z}^{\beta}\frac{\alpha}{r+\delta+\eta g_t})^{-\frac{\alpha}{\alpha-\beta}}. \tag{4.14}$$

Knowing the optimal balanced $\tilde{k}(v,t)$ by (4.3), (4.8), (4.11), (4.12), we can find from (2.3) that the optimal investment into new capital at v=t is

$$\tilde{x}(t) = \tilde{k}(t,t) = \chi_0 e^{g_t t} = \bar{k}A^{\frac{1}{1-\beta}}(0)e^{\frac{\gamma\alpha}{1-\alpha}t}. \tag{4.15}$$

and obtain the following Volterra integral equation of the first kind

$$e^{-\delta(t-v)}\int_v^t e^{\delta(s-v)}\tilde{u}(v,s)ds = e^{g_t t}[\chi(t-v) - \bar{k}e^{-(g_t+\delta)(t-v)}A^{\frac{1}{1-\beta}}(0)].$$

for the optimal investment into old capital $\tilde{u}(v,t)$ for $0 < v < t$, $0 < t < \infty$. The equation (2.4) for the optimal investment into past vintages $v < 0$ is different. However, the impact of such vintages deteriorates with time, so we focus on the last equation in the long-term analysis of this section. To obtain its balanced solution, we assume that $\tilde{u}(v,t)$ grows exponentially in t:

$$\tilde{u}(v,t) = e^{g_u t}\omega(t-v), g_u > 0, \tag{4.16}$$

where $\omega(t-v)$depends on the capital age a=t-v only. Substituting (4.11) and (4.16) into (4.15), we obtain that g_u=g_t (that is, the same exponential growth) and

$$\int_0^a e^{\left(\delta+\frac{\gamma\alpha}{1-\alpha}\right)s}\omega(s)ds = \bar{k}\left[e^{\left(\delta+\frac{\gamma\alpha}{1-\alpha}-\frac{\gamma\beta}{1-\beta}\right)a}A^{\frac{1}{1-\beta}}(a) - A^{\frac{1}{1-\beta}}(0)\right], \quad a > 0.$$

The differentiation of this equality gives the explicit formula for the optimal age-dependent investment profile

$$\omega(a) = \bar{k}e^{-\frac{\gamma\beta}{1-\beta}a}\left[\left(\delta+\frac{\gamma\alpha}{1-\alpha}-\frac{\gamma\beta}{1-\beta}\right)A^{\frac{1}{1-\beta}}(a) + \frac{d}{da}\left(A^{\frac{1}{1-\beta}}(a)\right)\right]. \tag{4.17}$$

in terms of the given model parameters. Finally

$$\tilde{c}(t) = \left(\varsigma - \bar{k} A^{\frac{1}{1-\beta}}(0) - \int_0^\infty \omega(a) da \right) e^{q_t t}, \tag{4.18}$$

where $\overline{k}$, $\omega(a)$, and ς are calculated by (4.12), (4.14), (4.17).

We can summarize the outcomes obtained above as

Theorem 4.1 (the balanced growth)**.** *If the technological change occurs at a constant rate $\gamma>0$, then the optimal control problem (2.1)-(2.5) has a unique balanced growth solution ($\tilde{y}(t)$, $\tilde{c}(t)$, $\tilde{k}(v,t)$, $\tilde{x}(t)$, $\tilde{u}(v,t)$) that grows at the exponential rate $g_t = \frac{\gamma\alpha}{1-\alpha}$ and is described by (4.13), (4.18), (4.3), (4.15) and (4.16).*

Henceforth, we will restrict ourselves to the case with no learning:

$$A \equiv 1, \beta < \beta_{cr} = \frac{\gamma\alpha + \delta(1-\alpha)}{\gamma + \delta(1-\alpha)}. \tag{4.19}$$

The inequality in (4.19) guarantees a positive optimal investment in the long run for each vintage $\infty < v <$ t [16]. Then, by (4.2), (4.12), (4.13), (4.15), (4.16) and (4.18), the balanced growth is

$$y(t) = \bar{k}^\alpha \bar{z}^\alpha \left(\frac{\gamma\beta}{1-\beta}\right)^{\alpha/\beta} e^{\frac{\gamma\alpha}{1-\alpha}t}, \tag{4.20}$$

$$\tilde{k}(v,t) = \bar{k} e^{-\frac{\gamma\alpha}{1-\beta}(t-v)} e^{\frac{\gamma\alpha}{1-\alpha}t},\ \bar{k}^{1-\alpha} = \frac{\alpha}{r+\delta+\eta g} \left(\frac{\gamma\beta}{1-\beta}\right)^{(\alpha-\beta)/\beta}, \tag{4.21}$$

$$\tilde{u}(v,t) = \bar{k} \left(\delta + g - \frac{\gamma\beta}{1-\beta} \right) e^{-\frac{\gamma\beta}{1-\beta}(t-v)} e^{\frac{\gamma\alpha}{1-\alpha}t}, \tag{4.22}$$

$$\tilde{c}(t) = (y(t) - x(t) - \int_{-\infty}^{\infty} u(v,t) dv) e^{\frac{\gamma\alpha}{1-\alpha}t}. \tag{4.23}$$

The long-term investment into vintage declines monotonically as a function of its age. Its dynamics is shown in Figures 1a and 1b.

At $\beta < \beta_{cr}$, the age-dependent investment profile (4.22) is positive for $0 \le a < \infty$ and decreases exponentially with the rate $g_a = \frac{\gamma\beta}{1-\beta}$ from the initial $\tilde{u}(0) = \bar{k}(\delta + c(t) - c(a))$ to zero at $a \to \infty$. The optimal $\tilde{u}(v,t)$ tends to zero when $\beta \to \beta_{cr}$ and $\tilde{u}(v,t) \equiv 0$ at $\beta > \beta_{cr}$.

The obtained properties of balanced growth lead to the following recommendations:

(a) At $\alpha > \beta$, the firm buys *more* capital of all vintages. The actual amount of a specific vintage v increases in time t (see Figure 1a).

(b) If $\alpha < \beta < \beta_{cr}$, then the actual investment into a fixed vintage v decreases in time t because of (4.23). So, the firm buys *less* capital of older vintages in the future. The amount $k(v,t)$ of a specific fixed vintage v decreases in time t because the investment into the vintage v does not compensate deterioration (see Figure 1b).

(c) If $\beta_{cr} \leq \ \beta \leq \ 1$, then the firm should not buy old capital at all: $u(v,t)$=0. The amount of a specific vintage v decreases in time as $e^{-\delta t}$ because of deterioration.

Remark 4.2. Case $\alpha = 1$ in (2.1)-(2.5) describes the AK production function, which is known to produce endogenous growth in the classic economic growth models with homogeneous capital [3]. Earlier remarkable analysis of an AK model with vintages was done by R. Boucekkine et al. [7], who assumed no technological change, finite exogenous lifetime of vintages, no investment in old vintages allowed, and $\beta = 1$. Boucekkine et al showed that key properties of the AK model change dramatically in vintage setting.

A preliminary analysis of the case $\alpha = 1$ in our model (2.1)-(2.5) demonstrates interesting qualitive picture that lies somewhere between the AK models of [3] and [7]. Specifically, by (4.2), (4.11), and (4.12), in presence of technological change ($\gamma > 0$), there is no balance growth and the investments and capital grow indefinitely in the model (2.1)-(2.5). In the absence of technological change ($\gamma = 0$), there is no balance growth in our model and the investment and capital grow indefinitely at $t \longrightarrow \infty$ if $z > r + \delta$, while the balance growth path is zero and the investments and capital tend to zero at $z < r + \delta$ (where z is the constant productivity). The latter inequalities resemble well known conditions in [3] for having positive or negative growth rate in the AK model. As in [7], convergence is not instantaneous and may be monotonic or non-monotonic depending on the initial distribution $k^0(v)$ of vintages in (2.4). More diverging nature of the optimal growth occurs because of non-diminishing returns and availability of unlimited resource. More detailed comparative study of the model (2.1)-(2.5) and the one of [7] may bring new insights into the AK model features and represents a prospective venue for future research.

5. Transition dynamics

The short-term (transition) dynamics in the model (2.1)-(2.5) is caused by old vintages that have been installed on the prehistory (-∞,0]. The purpose of the transition dynamics is to change the original non-optimal capital distribution $k^0(v)$ by vintages to the optimal $\tilde{k}(v,t)$ for each past vintage v<0. The transition pattern depends on whether the old capital vintages were well-funded or underfunded on the prehistory -∞<v<0.

In the case of the "ideal" initial distribution of capital

$$k^0(v) = \tilde{k}(v,0) \text{ for all vintages } -\infty < v < 0,$$

there is no transition dynamics and the solution is given by the balanced growth path (4.20)-(4.23) for t>0.

If the vintages are initially overfunded: $k^0(v) > k(v,0)$, then the optimal policy is to wait until the capital decrease to the level $\hat{k}(v,t)$ because of deterioration.

If the vintages are initially underfunded: $k(v)^0 < k(v,0)$, then the optimal policy is to provide the investment up to the level $\hat{k}(v,0)$ from the initial given level $k^0(v)$.

In the last two cases, the dynamic analysis requires solving the model equations when the capital dynamics follow the equation (2.4) or

$$k(v,t) = e^{-\delta t}k^0(v) + \int_0^t e^{-\delta(t-s)}u(v,s)ds \ for - \infty < v < 0. \tag{5.1}$$

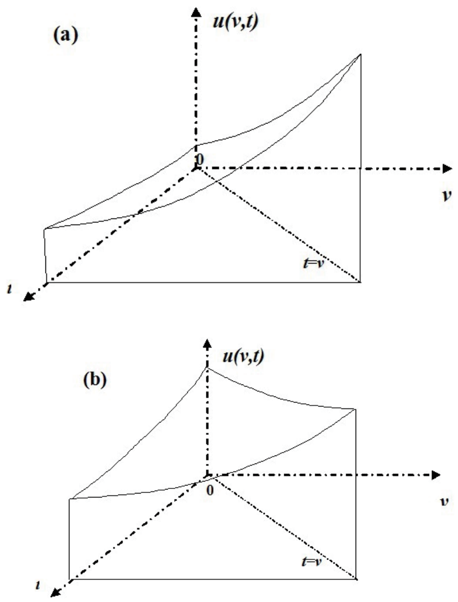

FIGURE 1. The dependence of long-term optimal investment on the vintage v and current time t. Figure (a) corresponds to the case $\alpha>\beta$ while Figure (b) illustrates the case $\alpha<\beta$

The corresponding output $y(t)$ is not equal to $\tilde{y}(t)$ if $k^0(v) \neq \tilde{k}(v,0)$ for, at least, some past vintages.

Transition dynamics in the linear utility case was systematically analyzed in [11], where it is proven that at $\eta = 0$ and $\gamma > 0$, the problem (2.1)-(2.5) has a unique solution such that transition dynamics ends at a finite time $\bar{t}_{max}$ and the optimal $u(v,t) = \tilde{u}(v,t)$, $k(v,t) = \tilde{k}(v,t)$, $y(t) = \tilde{y}(t)$ for $-\infty < v < t$, $\bar{t}_{max} \leq t < \infty$.

5.1. **Transition Dynamics in Concave Utility Case.** The transition dynamics in the model with nonlinear utility is much richer though brings more challenges to its analysis. In particular, the existence and uniqueness of solutions cannot be proven in a general case. However, some relevant structural (myopic) properties of the optimal investments can be established. In the case of nonlinear utility (2.1), by the optimality condition (4.1), the optimal capital

$$k(v,t) = \left[\alpha e^{\gamma\beta v} y^{(\alpha-\beta)/\alpha}(t)/(r+\delta+\eta c'(t)/c(t))\right]^{\frac{1}{1-\beta}}. \tag{5.2}$$

depends on both, the output $y(t)$ and the consumption $c(t)$. Let us rewrite (5.2) as

$$k(v,t) = e^{\frac{\gamma\beta}{1-\beta}v} f(t), \tag{5.3}$$

where

$$f(t) = \left[\alpha y^{(\alpha-\beta)/\alpha}(t)/(r+\delta+\eta c'(t)/c(t))\right]^{\frac{1}{1-\beta}}. \tag{5.4}$$

Then, by (2.3), the investment into new vintages is

$$x(t) = e^{\frac{\gamma\beta}{1-\beta}t} f(t). \tag{5.5}$$

and the optimal investment into the vintages $0 < v < t$ is

$$u(v,t) = e^{\frac{\gamma\beta}{1-\beta}v} \left[\delta f(t) + f'(t)\right] \mathit{for}\ 0 < t < \infty. \tag{5.6}$$

Cases of well-funded or underfunded vintages are considered below.

5.2. **The case of overfunded past vintages.** The optimal policy for overfunded vintages is not to invest into old vintages, that is, keep $u(v,t)$=0, and wait until the capital decreases to an optimal level because of deterioration.

Case A. For simplicity, let us assume that all past vintages are overfunded:

$$k^0(v) > \tilde{k}(v,0) \text{ for each vintage } -\infty < v < 0.$$

In this case, the optimal investment $u(v,t)$= 0 for all vintages v<0, while the capital $k(v,t)$ gradually decreases to the optimal level (5.5) following (1.4). The transition period ends at a time $\bar{t}_{max}$ such that $k(v,t)$ =$\tilde{k}(v,t)$ for all vintages. To find the instant $\bar{t}_{max}$, we need to indentify the "most overfunded" vintage from the condition

$$\breve{v} = \arg\max_{-\infty<v\leq 0} e^{-\frac{\gamma\beta}{1-\beta}v} k^0(v).$$

Then, the length $\bar{t}_{max}$ of transition dynamics can be found from the condition

$$k^0(\breve{v})e^{-\frac{\gamma\beta}{1-\beta}\breve{v}} = \tilde{f}(\bar{t}_{\max}), \tag{5.7}$$

where $\tilde{f}(t) = \bar{k}e^{\left(\frac{\gamma\alpha}{1-\alpha}-\frac{\gamma\beta}{1-\beta}\right)t}$.

The condition (5.7) holds because the dynamics after the transition period coincides with the known long-term balanced growth $\tilde{u}(v,t)$, $\tilde{k}(v,t)$, $\tilde{y}(t)$.

5.3. **The case of underfunded past vintages.** The transition dynamics of "underfunded" vintages is significantly different from the linear utility case. We cannot invest instantaneously into all underfunded vintages because the consumption y should remain positive. So, the transition investment $u(v,t)$ should be finite and determined from a certain balance between the positivity of consumption and the necessity of investing fast in underfunded vintages. The formulas for such interior transitory investment appear to be more complicated compared to the "overfunded" case of Section 5.2.

Case B. Let us assume that all past vintages are underfunded:

$$k^0(v) < \tilde{k}(v,0) \tag{5.8}$$

for each vintage $-\infty < v < 0$.

Then, the structure of the optimality conditions (Lemma 3.1) dictates the following investment choice at the beginning of transition period.

Lemma 5.1. *The optimal strategy at t=0 is to invest only into the "most unbalanced" vintage (or vintages) $\hat{v}$ determined from the condition*

$$\hat{v} = \arg\min_{-\infty<v\leq 0} e^{-\frac{\gamma\beta}{1-\beta}v} k^0(v). \tag{5.9}$$

For the vintages $\hat{v}$, the optimal capital amount is given by (5.3) and the optimal investment is given by (5.4) for $0<t<\infty$.

Proof. Substituting the desired optimal capital distribution (5.3) into (2.3), we obtain the equation

$$e^{-\delta t}k^0(v) + \int_0^t e^{-\delta(t-s)}u(v,s)ds = e^{\frac{\gamma\beta}{1-\beta}v}f(t), -\infty < v < 0, \tag{5.10}$$

for possible interior investment $u(v{,}t)$, where the endogenous function $f(t)$ depends on $y(t)$, $c(t)$, $c'(t)$ by (5.4). In a general case, this equation cannot be satisfied at t=0 for all $-\infty<v<0$ simultaneously. Since $I'(v,t) \le 0$ by Lemma 3.1, the only possible choice is to satisfy (5.10) for $\hat{v}$, then $I'(v,t) = 0$ for $v = \hat{v}$ and $I'(v,t) < 0$ for $v \neq \hat{v}$. We can choose $y(t)$, $c(t)$, $c'(t)$ such that

$$f(0) = k^0(\widehat{v}). \tag{5.11}$$

Indeed, the initial value $y(0) = \left(\int_{-\infty}^0 e^{\gamma\beta v}k^{0\beta}(v)dv\right)^{\alpha/\beta}$ is determined from (2) at t=0. Let the number of vintages $\hat{v}$, be countable, at most. Then, by (2.5), $c(0) = y(0) - x(0) = y(0) - f(0) = y(0) - k^0(\widehat{v})$ is also given. On the other side, $f(0) = \left[\alpha y^{(\alpha-\beta)/\alpha}(0)/(r+\delta+\eta c'(0)/c(0))\right]^{\frac{1}{1-\beta}}$ by (5.4). So, the value $c'(0)$ found from $r+\delta+\eta c'(0)/c(0) = \alpha y^{(\alpha-\beta)/\alpha}(0)f^{\beta-1}(0)$ satisfies (5.11).

Finally, at condition (5.11), the solution of (5.10) at $v = \hat{v}$ exists and is given by (5.6).

The lemma is proven. □

So, the initial optimal investment at $t=0$ is positive only for the "most unbalanced" vintages $\hat{v}$. The transition dynamics at $t>0$ is characterized by the following conjecture.

Proposition 5.2 (transition of underfunded vintages)**.** *At the condition (5.8), for every fixed $t>0$ there exists a set $\hat{V} \subset (-\infty, 0]$, mes$\hat{V}>0$, of past vintages such that investments $u(v,t)>0$ are determined by (5.6) and $u(v{,}t)=0$ for $v \in (-\infty,0] - \hat{V}$. The size (mes$\hat{V}$) of the set $\hat{V}$ increases in t. When $\hat{V}$ becomes equal to $(-\infty, 0]$, the transition ends and the further solution coincides with the balanced growth path.*

The intuition under Proposition 1 is that the initial investing into the most underfunded vintages increases and equalizes their capital. So, when time increases, a larger set $\hat{V}$ of "equally underfunded" vintages appears to invest in. The specific structure of the set $\hat{V}$ heavily depends on the shape of the initial capital distribution $k^0(v)$. For example, if all past vintages are equally underfunded: $k^0(v) = a\tilde{k}(v,0)$, $0<a<1$, then $\hat{V}=(-\infty,0]$ starting t=0.

The set $\hat{V}$ at small $t>0$ is illustrated in Figure 2 in the case when the function $\exp(-\frac{\gamma\beta}{1-\beta}v)k^0(v)$ possesses a unique absolute minimum over $(-\infty, 0]$ at point $\hat{v}$. In this case, the set $\hat{V}$ is simply a certain range $[\underline{v}(t), \bar{v}(t)]$. Still, the exact formulas for $\underline{v}(t)$ and $\bar{v}(t)$ involve the inverse of $k^0(v)$ and should be found from nonlinear functional equations. At $t \to 0$, both $\underline{v}(t)$ and $\bar{v}(t)$ approach the "most unbalanced" vintage $\hat{v}$. The function $\bar{v}(t)$ increases and $\underline{v}(t)$ decreases when time grows. Other structures of the set $\hat{V}$ are quite possible depending on the shape of $k^0(v)$.

The desired properties of the set $\hat{V}$ are:
(1) the uniqueness of $\hat{V}$ for each t and
(2) the convergence of the set $\hat{V}$ to the interval $(-\infty, 0]$ as time t increases.

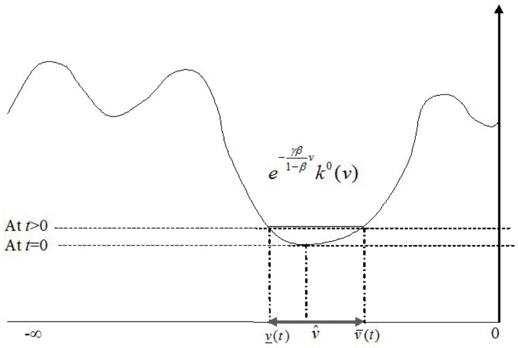

FIGURE 2. The structure of the set $\hat{V}$=[$\underline{v}(t)$, $\bar{v}(t)$] around the most unbalanced vintage $\hat{v}$.

The convergence of the transition dynamics to the balanced growth path during a finite time can be relatively easily proven in Case A of all overfunded past vintages. To prove the convergence in a general case, we can construct a similar problem with constrained investments $0 \leq u(v,t) \leq \bar{u}(t)$. For a certain given level $\bar{u}(t)$>0, the optimal transitory investment will be $u(v,t)$ =$\bar{u}(t)$ and the convergence in a finite time can be justified. The problem (2.1)-(2.5) will have more flexibility, so its transition dynamics should end even sooner.

The uniqueness issue is extremely challenging to consider. The set $\hat{V}$ is unique when the optimal $y(t)$ and $c(t)$ are unique. Because of complex nonlinear dependence, it is difficult to prove the uniqueness of $y(t)$ and $c(t)$ even in simple special cases when all vintages $k^0(v)$ are equally overfunded or underfunded, or $k^0(v)$ is close to $\tilde{k}(v,t)$, and similar. To highlight arising challenges, we consider the following example.

Example 5.1. Let us assume that all past vintages are equally underfunded:

$$k^0(v) = a\tilde{k}(v,0) = a\bar{k}e^{\frac{\gamma\beta}{1-\beta}v}, 0 < a < 1, \text{ for all } -\infty < v < 0.$$

Then, $y(0) = a\tilde{y}^{\alpha}(0) < \tilde{y}(0)$ by (2.2), $\hat{V} = (-\infty, 0]$ by (5.9), and$f(0) = a\tilde{f}(0) < \tilde{f}$ by (5.11). By proposition 5.2, the optimal strategy is to invest $u(v,t)$>0 uniformly into all vintages v starting t=0. So, the investment structure by vintages is simple in this case. Nevertheless, the problem of determining the optimal balance between consumption $c(t)$ and the total investment appears to be tricky. Indeed, by

$$c(t) = y(t) - x(t) - \int_{-\infty}^{t} \hat{u}(v,t)dv$$

or

$$c(t) = f^{\alpha}(t)e^{\frac{\gamma\alpha}{1-\beta}t}\left(\frac{1-\beta}{\gamma\beta}\right)^{\alpha/\beta} - f(t)e^{\frac{\gamma\beta}{1-\beta}t} - \frac{1-\beta}{\gamma\beta}\left(\delta f(t) + f'(t)\right)e^{\frac{\gamma\beta}{1-\beta}t}. \tag{5.12}$$

On the other hand, from (5.4) we have

$$r+\delta+\eta c'(t)/c(t)=\alpha^{-1}y^{(\alpha-\beta)/\alpha}(t)f^{\beta-1}(t). \tag{5.13}$$

So, we obtain the system (5.12),(5.13) of two first-order nonlinear differential equations with respect to $c(t)$ and $f(t)$ for $t\geq 0$. The value $f(0)=a\tilde{f}(0)$ is given, but the initial value $c(0)$ is not fixed. Instead, the relative rate $c'(0)/c(0)$ is fixed by (5.13) and is larger than the long-term rate $\tilde{c}(0)'/\tilde{c}(0)$. So, the initial problem for ODE system (5.12), (5.13) is non-standard and the uniqueness of its solution is not obvious even in this simple case.

5.4. **General Case.** In a general case, assuming that the unique optimal $y(t)$ and $c(t)$ exist, we can summarize the above investment dynamics in the following formal statement about the structure of optimal investments in the case on nonlinear utility.

Proposition 5.3 (on the structure of optimal investments)**.** *Let a unique solution to the OCP (2.1)-(2.5) exist. Then:*

(1) *For vintages $0<v<\infty$, the optimal investment is*

$$x(t)=e^{\frac{\gamma\beta}{1-\beta}t}f(t) \text{ at } t=v, \tag{5.14}$$

$$\hat{u}(v,t)=e^{\frac{\gamma\beta}{1-\beta}v}\left[\delta f(t)+f'(t)\right] \text{ at } t>v, \tag{5.15}$$

and the corresponding optimal capital is

$$k(v,t)=e^{\frac{\gamma\beta}{1-\beta}v}f(t) \text{ for } 0<t<\infty, \tag{5.16}$$

where

$$f(t)=\left[\alpha y^{(\alpha-\beta)/\alpha}(t)/(r+\delta+\eta c'(t)/c(t))\right]^{\frac{1}{1-\beta}}.$$

(2) *For the past vintages $-\infty<v\leq 0$, the transition dynamics starts with the vintage (or vintages) $\hat{v}=\arg\min_{-\infty<v\leq 0} e^{-\frac{\gamma\beta}{1-\beta}v}k^0(v)$ (with the smallest initial "efficient capital"). Once started, the optimal investment into a specific vintage never stops.*

If all past vintages are overfunded: $k^0(\hat{v})>k(\hat{v},0)$, then the investments into all past vintages are delayed until a certain finite time $\underline{t}>0$.

If $k^0(\hat{v})\leq k(\hat{v},0)$, then the optimal investment in vintages $\hat{v}$ starts at $t=0$, while the investment into vintages $v\neq\hat{v}$ is delayed by a certain time $\bar{t}(v)>0$:

$$u(v,t)=\begin{cases} 0 \text{ for } 0<t\leq\bar{t}(v) \\ \hat{u}(v,t) \text{ for } \bar{t}(v)\leq t<\infty \end{cases}.$$

The corresponding optimal capital for these vintages is

$$k(v,t)=\begin{cases} e^{-\delta t}k^0(\mathrm{v}) \text{ for } 0<t<\bar{t}(v) \\ \hat{k}(v,t) \text{ for } \bar{t}(v)\leq t<\infty \end{cases}.$$

After the transition dynamics ends, $f(t)=\bar{k}e^{\left(\frac{\gamma\alpha}{1-\alpha}-\frac{\gamma\beta}{1-\beta}\right)t}$ and the optimal trajectory (5.14)-(5.16) coincides with the long-term balanced growth $\tilde{u}(v,t)$ $\tilde{k}(v,t)$, $\tilde{x}(t)$ given by (4.20)-(4.22).

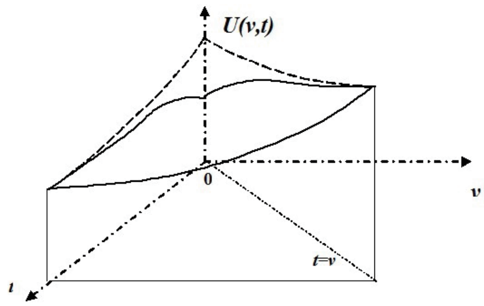

FIGURE 3. The short-term optimal investment into vintage $\hat{v}$ at time t. The dashed lines depict the exponential long-term balance growth from Figure 1a.

So, the dynamics of the optimal investment into past vintages is different from the balanced growth regime until $t = t_{max}$. A robust feature of the optimal investment is the age-dependent vintage profile $e^{\frac{\gamma\beta}{1-\beta}v}$, which is the same during the transition and long-term dynamics (see Figure 3).

6. Discussion

The considered optimal control problem can be interpreted as a vintage capital model with distributed investments into new and old capital and limited substitutability among vintages of different ages. A production output is described by the CES production function with continuum of vintage inputs and concave utility. The model possesses an "ideal" initial distribution of capital on the prehistory $(-\infty, 0]$ such that there is no transition dynamics at all and the optimal controls coincide with the balanced growth trajectory of Section 4.

In the general case, the optimal dynamics for the old vintages (installed before the planning horizon) is richer and includes a transition period. During this period, the optimal investment in each old vintage depends whether the vintage was initially overfunded or underfunded.

For initially underfunded vintages, the optimal investment is finite and balanced with a positive consumption because of the convexity of the problem. Namely, the investment at time zero starts with the most underfunded old vintage (s) $\hat{v}$. Eventually, investing into the most underfunded vintages increases and equalizes their capital. So, when the time increases, the set of "equally underfunded" vintages investments grows. As shown in Figure 2, the structure of investment into and around the most unbalanced vintage $\hat{v}$ looks like filling an uneven surface with water. This analogue can be useful for making ad hoc investment decisions. The described structure of the investment to underfunded vintages is quite interesting and represents a novel effect in the vintage capital theory.

For initially overfunded vintages, the structure of the optimal investment into old vintages is simpler: the optimal policy is to wait until the vintage capital decreases to the optimal level because of deterioration.

Finally, the obtained transition dynamics in model (2.1)-(2.5) does not possess investment echoes or industry shakeouts. To have dynamic patterns with investment echoes, the vintage capital model (2.1)-(2.5) should include a certain economic balance restriction, similar to the given industry-wide consumer demand in [11] or the given labor in [5,6] or various resource restrictions at a firm's level in [10,12,13].

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