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EQUIVALENCE BETWEEN NIKLIBORC'S THEOREM AND FROBENIUS' THEOREM

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Abstract. This paper discusses the relationship between two existence theorems for local solutions of partial/total differential equations, namely Nikliborc's Theorem and Frobenius' Theorem. In this paper, we show that if Nikliborc's Theorem holds, then Frobenius' Theorem also holds, and vice versa. This result also holds for extensions of these theorems. The relationship between Nikliborc's Theorem and Frobenius' Theorem is similar to that between the inverse function theorem and the implicit function theorem, and if one can be proved, then the other can be derived immediately.

1. Introduction

This paper treats the relationship between the partial differential equation:

$$DE(q) = f(q, E(q)),\ E(p) = m,$$

and the total differential equation:

$$Du(x) = \lambda(x)g(x),$$

where the operator D denotes the Fréchet derivative operator: that is, for a function h, $Dh(x)$ is the Fréchet derivative of h at x. Both differential equations have applications in microeconomic theory. This paper reveals that existence theorems regarding a local solution to the above equations are mutually equivalent.

First, we should explain the application of these equations to economics. In classical consumer theory, the behavior of a consumer is described by the utility maximization hypothesis. That is, the consumer is assumed to choose his/her consumption plan x for maximizing his/her 'utility'. The word 'utility' means some real-valued function $u : \Omega \to \mathbb{R}$ that represents the preference of this consumer, where Ω represents the set of all possible consumption plans. Hence, the consumer's choice problem can be represented as follows:

$$\begin{aligned} \max\ & u(x), \\ \text{subject to. } & x \in \Omega, \\ & p \cdot x \le m. \end{aligned}$$

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Usually, we assume that the 'consumption set' Ω is the **nonnegative orthant** $\{x \in \mathbb{R}^n | x_i \ge 0 \text{ for all } i\}$. The positive vector p denotes the price system, and $m > 0$ represents the income. If u satisfies several assumptions, then the solution of the above maximization problem is unique, and is denoted by $f^u(p, m)$. The function f^u is called the **demand function**.

In the usual theory, the consumer is assumed to determine their consumption behavior by solving this problem, and thus u is given and f^u is derived. However, there is another view for this theory, in which a candidate f of the demand function is given, and u is derived from f. The reason for taking such a position is as follows. The utility function is **unobservable**, because it is hidden in the consumer's mind. In contrast, the demand function is **observable**, because it corresponds to the consumer's actual purchase behavior. Therefore, if we wish to obtain the utility function, then we must use the information of the demand function and derive the utility function by some calculation. The research area for obtaining such a calculation method is called **integrability theory**.

In fact, the above argument is a little rough, because the demand function includes **infinitely many information** about the purchase behavior, whereas we can only obtain finite purchase data. Therefore, the statement that "the demand function is observable" is an exaggeration. However, although the demand function is actually unobservable, we can say that integrability theory has a meaning in modern econometric theory. That is, the estimation of a utility function is quite difficult, because it is hidden in consumer's mind. On the other hands, the estimation for a demand function is, at least, easier than that for a utility function, because it corresponds to actual purchase behavior. Although the demand function is actually unobservable, we can obtain the 'estimated' demand function for purchase data; thus, by using integrability theory, we can immediately obtain the 'estimated' utility function.

There are two approaches in integrability theory, the **direct approach** and the **indirect approach**.[1] The direct approach treats the following partial differential equation:

$$DE(q) = f(q, E(q)),\ E(p) = m.$$

If $f = f^u$ for some appropriate function u, then the function $E^x(q) = \inf\{q \cdot y | u(y) \ge u(x)\}$ solves the above equation (Shephard's lemma). Therefore, solving this equation means calculating E^x without the use of any information about u, and using the information of E^x allows us to re-construct u.[2] In contrast, the indirect approach treats the following total differential equation:

$$Du(x) = \lambda(x) g(x),$$

where the function $g(x)$ is assumed to satisfy

$$f(g(x), g(x) \cdot x) = x.$$

That is, $g(x)$ is the price system under which x is chosen. The function g is called the **inverse demand function**. If u is smooth and increasing, then by Lagrange's

[1] These names were assigned by Hurwicz and Uzawa (1971).

[2] For the actual calculation procedure of u in the direct approach, see Hurwicz and Uzawa (1971) or Hosoya (2017).

multiplier rule, there exists a positive function λ such that $Du(x) = \lambda(x)g(x)$ for every x. Therefore, for a given g, we obtain the information of u by solving the above equation.[3]

We clarify the relationship between the above partial and total differential equations. Both have famous results regarding local existence theorems for their solutions. The local existence theorem for the solution of the above partial differential equation is called **Nikliborc's Theorem**, and that of the above total differential equation is called **Frobenius' Theorem**. In this paper, we reveal that if Nikliborc's Theorem is correct, then Frobenius' Theorem is also correct, and vice versa (Theorems 1 and 2). That is, the relationship between Nikliborc's Theorem and Frobenius' Theorem is similar to that between the inverse function theorem and the implicit function theorem, and once one of them is proved, the other can be proved immediately.

This result has an important meaning. That is, if one succeeds in extending one of these theorems, then the other can also be extended. To clarify this, we present extensions of these theorems and show that if the extended version of Nikliborc's Theorem is correct, then the extended form of Frobenius' Theorem is also correct, and vice versa (Theorems 3 and 4).

Section 2 introduces some basic knowledges regarding these equations. In section 3, we present the main results. Several comments on Nikliborc's Theorem and Frobenius' Theorem are given in section 4. Section 5 treats the extensions of our main results.

2. Preliminaries

2.1. **Nikliborc's Theorem.** Consider the following partial differential equation (PDE)

$$DE(q) = f(q, E(q)), \tag{2.1}$$

where $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is continuous. A function $E : V \to \mathbb{R}$ is a **solution** of the above PDE if and only if 1) V is an open set, 2) E is C^1, and 3) $DE(q) = f(q, E(q))$ for all $q \in V$.[4] Note that, if f is C^1, then any solution E is automatically C^2. If $E : V \to \mathbb{R}$ is a solution of (2.1) and $p \in V$, then E is called a **local solution** around p.

If $P \subset \mathbb{R}^{n+1}$ is open and $f : P \to \mathbb{R}^n$ is differentiable, define

$$s_{ij}(p, m) = \frac{\partial f_i}{\partial p_j}(p, m) + \frac{\partial f_i}{\partial m}(p, m) f_j(p, m). \tag{2.2}$$

The function f is said to be **integrable** if and only if $s_{ij}(p, m) = s_{ji}(p, m)$ for every $(p, m) \in P$.

The following classical result was derived by Nikliborc (1929).

[3]For the actual calculation procedure of u in the indirect approach, see Debreu (1972) or Hosoya (2013). Note that, almost all classical results in integrability theory focus on the indirect approach. For example, see Antonelli (1886), Pareto (1906), Samuelson (1950), and Katzner (1970).

[4]Throughout the paper, we assume that $n \geq 2$. If $n = 1$, then (2.1) is just a standard form of the ordinary differential equation. However, because $n \geq 2$, this is a PDE.

Nikliborc's Theorem. *Suppose that $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is C^1. Then, f is integrable if and only if for every $(p, m) \in P$, there exists a local solution $E : V \to \mathbb{R}$ of (2.1) around p such that $E(p) = m$. Moreover, if f is C^k for $k \ge 1$, then any solution E of (2.1) must be C^{k+1}.*

2.2. **Frobenius' Theorem.** Consider the following total differential equation (TDE)

$$Du(x) = \lambda(x)g(x), \tag{2.3}$$

where $g : U \to \mathbb{R}^n \setminus \{0\}$, $U \subset \mathbb{R}^n$ is open, and g is C^1. A pair (u, λ) of functions from V to $\mathbb{R}$ is a **solution** of the above TDE if and only if 1) V is an open set, 2) u is C^1 and λ is positive and continuous, and 3) $Du(x) = \lambda(x)g(x)$ for every $x \in V$. If (u, λ) is a solution defined on V and $x^* \in V$, then this pair is called a **local solution** of (2.3) around x^*.

The function g is said to satisfy **Jacobi's integrability condition** if, for all $i, j, k \in \{1, \dots, n\}$ with $i \ne j \ne k \ne i$,[5]

$$g_i\left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j}\right) + g_j\left(\frac{\partial g_k}{\partial x_i} - \frac{\partial g_i}{\partial x_k}\right) + g_k\left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i}\right) = 0 \tag{2.4}$$

for all $x \in U$.

Then, the following result is known.

Frobenius' Theorem. *Suppose that $g : U \to \mathbb{R}^n \setminus \{0\}$, $U \subset \mathbb{R}^n$ is open, and g is C^1. Then, g satisfies Jacobi's integrability condition if and only if, for every $x^* \in U$, there exists a local solution (u, λ) around x^* such that for every $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an $n - 1$ dimensional C^2 manifold. If g is C^k for $k \ge 2$, then a local solution (u, λ) can be chosen as follows: 1) u is C^k, 2) λ is C^{k-1}, and 3) for every $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an $n - 1$ dimensional C^{k+1} manifold.*

2.3. **A Note on Jacobi's Integrability Condition.** Suppose that g satisfies (2.4) for all $i, j \in \{1, \dots, n-1\}$ and $k = n$ with $i \ne j$, and $g_n(x) \ne 0$. Then, g satisfies (2.4) for all $i, j, k \in \{1, \dots, n\}$. To show this, choose any $i, j, k \in \{1, \dots, n\}$ with $i \ne j \ne k \ne i$. If $i = n$, $j = n$ or $k = n$, then our assumption implies that (2.4) holds. Therefore, we assume that $i, j, k \in \{1, \dots, n-1\}$. By our assumption,

$$\begin{aligned}
g_i\left(\frac{\partial g_j}{\partial x_n} - \frac{\partial g_n}{\partial x_j}\right) + g_j\left(\frac{\partial g_n}{\partial x_i} - \frac{\partial g_i}{\partial x_n}\right) + g_n\left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i}\right) &= 0,\\
g_j\left(\frac{\partial g_k}{\partial x_n} - \frac{\partial g_n}{\partial x_k}\right) + g_k\left(\frac{\partial g_n}{\partial x_j} - \frac{\partial g_j}{\partial x_n}\right) + g_n\left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j}\right) &= 0,\\
g_k\left(\frac{\partial g_i}{\partial x_n} - \frac{\partial g_n}{\partial x_i}\right) + g_i\left(\frac{\partial g_n}{\partial x_k} - \frac{\partial g_k}{\partial x_n}\right) + g_n\left(\frac{\partial g_k}{\partial x_i} - \frac{\partial g_i}{\partial x_k}\right) &= 0.
\end{aligned}$$

Multiplying the first equation by g_k, the second equation by g_i, the third equation by g_j, and summing up all three equations, we obtain

$$g_n\left[g_i\left(\frac{\partial g_j}{\partial x_k} - \frac{\partial g_k}{\partial x_j}\right) + g_j\left(\frac{\partial g_k}{\partial x_i} - \frac{\partial g_i}{\partial x_k}\right) + g_k\left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i}\right)\right] = 0,$$

[5]In this paper, we frequently abbreviate the variables of functions to prevent the formulas becoming too long.

which implies that (2.4) holds.

2.4. **A Note on Nikliborc's Theorem.** First, suppose that $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is C^1. Let $(p, m) \in P$, and $p \in V$, where V is an open and convex set. If there exists a solution $E : V \to \mathbb{R}$ of (2.1) that satisfies $E(p) = m$, then such a solution is unique.[6]

To verify this, suppose that there is another solution $F : V \to \mathbb{R}$ of (2.1) that satisfies $F(p) = m$. Choose any $q \in V$. Define $c_1(t) = E((1-t)p + tq)$ and $c_2(t) = F((1-t)p + tq)$. Then, c_i is the solution of the following ODE:

$$\dot{c}(t) = f((1-t)p + tq, c(t)) \cdot (q - p),\ c(0) = m.$$

Because of the Picard-Lindelöf uniqueness theorem of solutions to ODEs, we have that $c_1(t) \equiv c_2(t)$. In particular,

$$F(q) = c_2(1) = c_1(t) = E(q).$$

Because $q \in V$ is arbitrary, we have that $E \equiv F$, and thus the uniqueness result holds.

Second, suppose again that $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is C^1. Moreover, suppose that f is integrable, and that **Nikliborc's Theorem is correct**. Fix $(p, m) \in P$. Let $E : V \to \mathbb{R}$ be a local solution around p that satisfies $E(p) = m$. Moreover, suppose that W is a bounded, open, convex neighborhood of p and that V includes the closure of W. Then, there exists $\varepsilon > 0$ such that if $|h| \le \varepsilon$, then there exists a local solution $E_h : W \to \mathbb{R}$ around p that satisfies $E(p) = m + h$, and $E_h(q)$ is continuous and increasing in h.

To verify this, consider the following parametrized ODE:

$$\dot{c}(t; q, w) = f((1-t)p + tq, c(t; q, w)) \cdot (q - p),\ c(0; q, w) = w. \tag{2.5}$$

Let $\bar{W}$ denote the closure of W. If $q \in \bar{W}$, then $c(t; q, m) = E((1-t)p + tq)$, and thus the solution function $c(t; q, w)$ is defined on $[0, 1] \times \bar{W} \times \{m\}$. Because the domain of $c(t; q, w)$ is open, there exists $\varepsilon > 0$ such that this domain includes $[0, 1] \times \bar{W} \times [m - \varepsilon, m + \varepsilon]$. Choose $h \in [-\varepsilon, \varepsilon]$ and define $E_h(q) = c(1; q, m + h)$ for all $q \in W$. We will show that E_h is a solution of (2.1) with $E_h(p) = m + h$, and $E_h(q)$ is increasing in h.

Because $c(t; p, m+h) \equiv m+h$ for all $t \in \mathbb{R}$, we have that $E_h(p) = c(1; p, m+h) = m + h$.

Next, choose any $q \in W$, and define $p(t) = (1-t)p + tq$. Because we assume that Nikliborc's Theorem is correct and f is integrable, for every $t \in [0, 1]$, there exists a local solution $E^t : V_t \to \mathbb{R}$ of (2.1) around $p(t)$ that satisfies $E^t(p(t)) = c(t; q, m+h)$. Without loss of generality, we can assume that V_t is an open ball centered at $p(t)$. Suppose that for $t_1, t_2 \in [0, 1]$ with $t_1 < t_2$, there exists $r \in V_{t_1} \cap V_{t_2}$. This implies that there exists $t_3 \in [t_1, t_2]$ such that $p(t_3) \in V_{t_1} \cap V_{t_2}$. Let $c_i(t) = E^{t_i}(p(t))$ for $i \in \{1, 2\}$. Then, c_i satisfies the following ODE:

$$\dot{c}(t) = f(p(t), c(t)) \cdot (q - p),\ c(t_i) = c(t_i; q, m + h),$$

[6] In this subsection, we frequently use some basic knowledges of ordinary differential equations (ODEs), which is shown in the next subsection. For readers unfamiliar with ODEs, we recommend reading the next subsection first, and then returning to this subsection.

and thus, by the Picard-Lindelöf uniqueness theorem and (2.5), we have that $c_i(t) = c(t; q, m+h)$ if both are defined. In particular,

$$E^{t_1}(p(t_3)) = c_1(t_3) = c(t_3; q, m+h) = c_2(t_3) = E^{t_2}(p(t_3)).$$

This implies that both E^{t_1}, E^{t_2} are solutions of (1) on an open and convex set $V_{t_1} \cap V_{t_2}$ that satisfy $E^{t_i}(p(t_3)) = c(t_3; q, m+h)$. Thus, by the above uniqueness result, we have that $E^{t_1}(r) = E^{t_2}(r)$ for all $r \in V_{t_1} \cap V_{t_2}$.

Therefore, if we define

$$F(r) = E^t(r)$$

for $r \in V \equiv \cup_{t\in[0,1]} V_t$, then F is a well-defined solution of (2.1) that satisfies $F(p) = m + h$. Because V is open, there exists an open neighborhood $U \subset \mathbb{R}^n$ of q such that $(1-t)p + tr \in V$ for all $t \in [0,1]$ and $r \in U$. Then, using the Picard-Lindelöf uniqueness theorem again, we have

$$c(t; r, m+h) = F((1-t)p + tr),$$

and thus,

$$E_h(r) = F(r),$$

which implies that E_h is C^1 around q and

$$DE_h(q) = DF(q) = f(q, F(q)) = f(q, E_h(q)),$$

as desired.

The continuity of $E_h(q)$ in h follows from the continuity of $c(t; q, w)$. Finally, suppose that $E_h(q)$ is not increasing in h. Then, there exists $h_1, h_2 \in [-\varepsilon, \varepsilon]$ such that $h_1 < h_2$ and $E_{h_1}(q) \geq E_{h_2}(q)$. Note that $c(0; q, m+h_1) = m + h_1 < m + h_2 = c(0; q, m+h_2)$. Because of the definition and the intermediate value theorem, there exists $t \in [0,1]$ such that $c(t; q, m+h_1) = c(t; q, m+h_2)$. By the Picard-Lindelöf uniqueness theorem, we have that $m + h_1 = c(0; q, m+h_1) = c(0; q, m+h_2) = m + h_2 > m + h_1$, which is a contradiction. This completes the proof of our claims.

Note that the above arguments do not use the differentiability of f. If f is not necessarily C^1 but locally Lipschitz, then all above arguments are still correct when **Nikliborc's Theorem II is correct** (this theorem is explained in section 5).

2.5. **Properties of the Solution to an ODE.** This subsection is intended to aid readers who are unfamiliar with the theory of ODEs.

Consider the following equation:

$$\dot{x}(t) = h(t, x(t)),\ x(t^*) = x^*, \tag{2.6}$$

where $h : X \to \mathbb{R}^n$ and $X \subset \mathbb{R} \times \mathbb{R}^n$. We assume that $(t^*, x^*) \in X$. The notation $\dot{x}$ denotes the derivative of the function x with respect to t. We call this an ordinary differential equation.

We call a convex subset $I \subset \mathbb{R}$ an **interval** if it contains at least two points. A function $x : I \to \mathbb{R}^n$ is called a **solution** of (2.6) if and only if 1) I is an interval and $t^* \in I$, 2) $x(t^*) = x^*$, and 3) for every $t \in I$, $(t, x(t)) \in X$ and $\dot{x}(t) = h(t, x(t))$. A solution $x : I \to \mathbb{R}$ of (2.6) is **nonextendable** if and only if for any other solution $y : J \to \mathbb{R}$, if $I \subset J$ and $x(t) = y(t)$ for all $t \in I$, then $J = I$. Then, the following results hold.

Fact 1. If X is open and h is continuous, then there is at least one solution $x : I \to \mathbb{R}$, where I is an open interval.[7]

Fact 2. If X is open and h is continuous in (t,x) and locally Lipschitz in x,[8] then for any two solutions $x : I \to \mathbb{R}^n$ and $y : J \to \mathbb{R}^n$, $x(t) = y(t)$ for every $t \in I \cap J$. In particular, the nonextendable solution is uniquely determined and its domain is an open interval.[9]

Next, consider the following parametrized equation:

$$\dot{x}(t) = h(t, x(t), y),\ x(t^*) = z, \tag{2.7}$$

where $h : X \to \mathbb{R}^n$ and $X \subset \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m$. We assume that X is open and h is continuos in (t,x,y) and locally Lipschitz in x. Then, for every y, z with $(t^*, z, y) \in X$, there exists a unique nonextendable solution $x^{y,z} : I \to \mathbb{R}$. Let us write $x(t; y, z) = x^{y,z}(t)$, and call the function x the **solution function** of (2.7). Then, the following results hold.

Fact 3. The domain of $x(t; y, z)$ is open, and x is continuous.

Fact 4. If h is locally Lipschitz, then $x(t; y, z)$ is also locally Lipschitz.

Fact 5. If h is C^k, then $x(t; y, z)$ is also C^k.

Facts 1-3 and 5 are famous results, and thus we omit their proofs. To prove these results, see textbooks on ODEs: we recommend Pontryagin (1962), Hartman (1997), or Smale and Hirsch (1974). Only Fact 4 is not standard, and thus we present a proof in the appendix.

3. Main result

Our main results are as follows.

Theorem 3.1. *Frobenius' Theorem implies Nikliborc's Theorem.*

Theorem 3.2. *Nikliborc's Theorem implies Frobenius' Theorem.*

Proof of Theorem 3.1. Suppose that Frobenius' Theorem holds. Let $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ be open, and f be C^1.

First, suppose that for every $(p, m) \in P$, there exists a local solution $E : V \to \mathbb{R}$ around p that satisfies $E(p) = m$. Then, E is C^2, and thus the Hessian matrix

[7] This fact is known as Peano's existence theorem.

[8] The function h is locally Lipschitz in x if and only if for every compact set $C \subset X$, there exists $L > 0$ such that if $(t, x_1), (t, x_2) \in C$, then $\|h(t, x_1) - h(t, x_2)\| \le L\|x_1 - x_2\|$.

[9] This fact is known as the Picard-Lindelöf uniqueness theorem.

$D^2E(p)$ is symmetric. Meanwhile,

$$\begin{aligned}\frac{\partial^2 E}{\partial q_j \partial q_i}(q) &= \frac{\partial}{\partial q_j} f_i(q, E(q))\\ &= \frac{\partial f_i}{\partial p_j}(q, E(q)) + \frac{\partial f_i}{\partial m}(q, E(q))\frac{\partial E}{\partial q_j}(q)\\ &= \frac{\partial f_i}{\partial p_j}(q, E(q)) + \frac{\partial f_i}{\partial m}(q, E(q)) f_j(q, E(q))\\ &= s_{ij}(q, E(q))\end{aligned}$$

by (2.2). Therefore, we have that

$$s_{ij}(p,m) = s_{ij}(p, E(p)) = \frac{\partial^2 E}{\partial q_j \partial q_i}(p) = \frac{\partial^2 E}{\partial q_i \partial q_j}(p) = s_{ji}(p, E(p)) = s_{ji}(p,m).$$

Because (p,m) is arbitrary, we have that f is integrable.

Conversely, suppose that f is integrable. Define

$$g(p,m) = (f(p,m), -1).$$

Then, $g : P \to \mathbb{R}^{n+1}$, P is open, and g is C^1. Choose any $i, j \in \{1, \ldots, n\}$ with $i \neq j$. Then,

$$\begin{aligned}&g_i\left(\frac{\partial g_j}{\partial m} - \frac{\partial g_{n+1}}{\partial p_j}\right) + g_j\left(\frac{\partial g_{n+1}}{\partial p_i} - \frac{\partial g_i}{\partial m}\right) + g_{n+1}\left(\frac{\partial g_i}{\partial p_j} - \frac{\partial g_j}{\partial p_i}\right)\\ &= f_i\frac{\partial f_j}{\partial m} - f_j\frac{\partial f_i}{\partial m} - \frac{\partial f_i}{\partial p_j} + \frac{\partial f_j}{\partial p_i}\\ &= s_{ji} - s_{ij} = 0.\end{aligned}$$

By our arguments in subsection 2.3, we have that g satisfies Jacobi's integrability condition. Therefore, by Frobenius' Theorem, for every $(p,m) \in P$, there exists a local solution (u, λ) of (2.3) around (p,m). Because

$$\frac{\partial u}{\partial m}(p,m) = \lambda(p,m) g_{n+1}(p,m) = -\lambda(p,m) \neq 0,$$

we can use the implicit function theorem, and thus, there exists an open neighborhood V of p and a C^1 function E such that $E(p) = m$ and

$$u(q, E(q)) \equiv u(p,m)$$

for all $q \in V$. Differentiating both sides with respect to q_i, we have

$$\begin{aligned}\frac{\partial E}{\partial q_i}(q) &= -\frac{\frac{\partial u}{\partial p_i}(q, E(q))}{\frac{\partial u}{\partial m}(q, E(q))}\\ &= -\frac{g_i(q, E(q))}{g_{n+1}(q, E(q))}\\ &= f_i(q, E(q)).\end{aligned}$$

Therefore, E is a solution of (2.1) that satisfies $E(p) = m$.

Finally, suppose that for $k \geq 1$, if f is C^k, then the solution E of (2.1) must be C^{k+1}. Suppose that f is C^{k+1}. Then, for any solution E of (2.1),

$$DE(q) = f(q, E(q)),$$

and the right-hand side is C^{k+1}. This implies that E is C^{k+2}, and thus by mathematical induction, we have that all claims of Nikliborc's Theorem are correct. This completes the proof of Theorem 3.1. □

Proof of Theorem 3.2. Suppose that Nikliborc's Theorem holds. Let $g : U \to \mathbb{R}^n \setminus \{0\}$, $U \subset \mathbb{R}^n$ be open, and g be C^1. Throughout this proof, we use the following notation: if $x = (x_1, \ldots, x_n)$, then $\tilde{x} = (x_1, \ldots, x_{n-1})$.

First, suppose that for any $x^* \in U$, there exists a local solution (u, λ) of (2.3) around x^*. Fix an $x^* \in U$ and choose such a solution (u, λ). Because $g(x^*) \neq 0$, we can assume without loss of generality that $g_n(x^*) \neq 0$. Then, there exists an open neighborhood V of x^* such that $g_n(x) \neq 0$ for every $x \in V$. Because $\frac{\partial u}{\partial x_n}(x^*) = \lambda(x^*) g_n(x^*) \neq 0$, the implicit function theorem means that there exists an open neighborhood W of $\tilde{x}^*$ and a C^1 function $E : W \to \mathbb{R}$ such that $E(\tilde{x}^*) = x_n^*$ and $u(\tilde{x}, E(\tilde{x})) = u(x^*)$ for every $\tilde{x} \in W$, where W is an open neighborhood of $\tilde{x}^*$ such that $(\tilde{x}, E(\tilde{x})) \in V$ for every $\tilde{x} \in W$. Define $f_i(x) = -\frac{g_i(x)}{g_n(x)}$ for every $x \in V$. Then,

$$\begin{aligned}\frac{\partial E}{\partial x_i}(\tilde{x}) &= -\frac{\frac{\partial u}{\partial x_i}(\tilde{x}, E(\tilde{x}))}{\frac{\partial u}{\partial x_n}(\tilde{x}, E(\tilde{x}))} \\ &= -\frac{g_i(\tilde{x}, E(\tilde{x}))}{g_n(\tilde{x}, E(\tilde{x}))} \\ &= f_i(\tilde{x}, E(\tilde{x})).\end{aligned}$$

Therefore, E is a local solution of (2.1) around x^* with the above f_i. Thus, E is C^2 at x^*, and thus by Young's theorem, we have that $s_{ij}(x^*) = s_{ji}(x^*)$. To calculate these values, we have that

$$\begin{aligned}s_{ij} &= \frac{\partial f_i}{\partial x_j} + \frac{\partial f_i}{\partial x_n} f_j \\ &= \frac{g_i \frac{\partial g_n}{\partial x_j} - \frac{\partial g_i}{\partial x_j} g_n + \frac{\partial g_i}{\partial x_n} g_j - \frac{\partial g_n}{\partial x_n} \frac{g_i g_j}{g_n}}{g_n^2}, \\ s_{ji} &= \frac{\partial f_j}{\partial x_i} + \frac{\partial f_j}{\partial x_n} f_i \\ &= \frac{g_j \frac{\partial g_n}{\partial x_i} - \frac{\partial g_j}{\partial x_i} g_n + \frac{\partial g_j}{\partial x_n} g_i - \frac{\partial g_n}{\partial x_n} \frac{g_i g_j}{g_n}}{g_n^2},\end{aligned}$$

and thus,

$$\begin{aligned}0 &= s_{ij} - s_{ji} \\ &= -\frac{1}{g_n^2}\left[g_i \left(\frac{\partial g_j}{\partial x_n} - \frac{\partial g_n}{\partial x_j} \right) + g_j \left(\frac{\partial g_n}{\partial x_i} - \frac{\partial g_i}{\partial x_n} \right) + g_n \left(\frac{\partial g_i}{\partial x_j} - \frac{\partial g_j}{\partial x_i} \right) \right].\end{aligned}$$

By our arguments in subsection 2.3, we have that g satisfies Jacobi's integrability condition.

Conversely, suppose that g satisfies Jacobi's integrability condition. Fix an $x^* \in U$, and assume without loss of generality that $g_n(x^*) > 0$. Let $W = \{x \in U | g_n(x) > 0\}$, and define $f_i(x) = -\frac{g_i(x)}{g_n(x)}$. Consider the following PDE:

$$DE(\tilde{x}) = f(\tilde{x}, E(\tilde{x})). \tag{3.1}$$

By repeating the above calculation, we have that $s_{ij} = s_{ji}$ on W, and thus by Nikliborc's Theorem and our arguments in subsection 2.4, there exists $\varepsilon > 0$ such that if $|h| \le \varepsilon$, then there exists a solution $E_h : \prod_{i=1}^{n-1}]x_i^* - \varepsilon, x_i^* + \varepsilon[\to \mathbb{R}$ that satisfies $E_h(\tilde{x}^*) = x_n^* + h$. Moreover, E_h is continuous and increasing in h.

Choose a sufficiently small $\delta > 0$ such that if $\tilde{x} \in \prod_{i=1}^{n-1}[x_i^* - \delta, x_i^* + \delta]$, then $E_{-\varepsilon}(\tilde{x}) < x_n^* - \delta$ and $E_\varepsilon(\tilde{x}) > x_n^* + \delta$. Define $V = \prod_{i=1}^{n}]x_i^* - \delta, x_i^* + \delta[$. Then, by the intermediate value theorem, for every $x \in V$, there uniquely exists $h \in]-\varepsilon, \varepsilon[$ such that $x_n = E_h(\tilde{x})$. Define $u(x)$ as such an h. Consider the following ODE:

$$\dot{c}(t; \tilde{x}, w) = f((1-t)\tilde{x}^* + t\tilde{x}, c(t; \tilde{x}, w)) \cdot (\tilde{x} - \tilde{x}^*),\ c(0; \tilde{x}, w) = w.$$

Note that this equation is equivalent to (2.5). By our arguments in subsection 2.4, we have that $E_h(\tilde{x}) = c(1; \tilde{x}, x_n^* + h)$, and thus we have

$$u(x) = h \Leftrightarrow c(1; \tilde{x}, x_n^* + h) = x_n.$$

By definition, $c(1; \tilde{x}^*, x_n^* + h) = x_n^* + h$, and thus, if $\varepsilon > 0$ and $\delta > 0$ are sufficiently small, then $\frac{\partial}{\partial h} c(1; \tilde{x}, x_n^* + h) > 0$ for all $\tilde{x} \in \prod_{i=1}^{n-1}]x_i^* - \delta, x_i^* + \delta[$ and $h \in]-\varepsilon, \varepsilon[$. Therefore, by the implicit function theorem, we have that u is differentiable and

$$\frac{\partial u}{\partial x_n}(x) = \frac{1}{\frac{\partial c}{\partial w}(1; \tilde{x}, x_n^* + u(x))} > 0.$$

Choose any $w \in \mathbb{R}$. If $u^{-1}(w)$ is nonempty, then

$$u^{-1}(w) = \{(\tilde{x}, E_w(\tilde{x})) | |x_i - x_i^*| < \delta \text{ for all } i \in \{1, \ldots, n-1\}\} \cap V, \tag{3.2}$$

and the right-hand side is an $n-1$ dimensional C^2 manifold. Next, choose any $x \in V$ and suppose that $u(x) = h$. Then,

$$\frac{\partial E_h}{\partial x_i}(\tilde{x}) = -\frac{\frac{\partial u}{\partial x_i}(x)}{\frac{\partial u}{\partial x_n}(x)}$$

by a direct calculation, and

$$\frac{\partial E_h}{\partial x_i}(\tilde{x}) = f_i(x) = -\frac{g_i(x)}{g_n(x)}$$

by definition. Therefore, we have either $g_i(x) = \frac{\partial u}{\partial x_i}(x) = 0$ or

$$\frac{\frac{\partial u}{\partial x_i}(x)}{g_i(x)} = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)}.$$

Thus, if we define $\lambda(x) = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)}$, then it is continuous and positive, and

$$Du(x) = \lambda(x) g(x),$$

as desired.

Finally, if g is C^k, then f is also C^k, and thus E_h is C^{k+1} and $c(t;\tilde{x},w)$ is C^k, and by the implicit function theorem, u is C^k. Therefore, λ is C^{k-1}. Moreover, if $u^{-1}(w)$ is nonempty, then by (3.2), we have that it is an $n-1$ dimensional C^{k+1} manifold. This completes the proof of Theorem 3.2. □

4. Comments on the two theorems

4.1. **Comments on Nikliborc's Theorem.** If f is independent of m, then the integrability condition coincides with

$$\frac{\partial f_i}{\partial p_j}(p) = \frac{\partial f_j}{\partial p_i}(p),$$

and in this case, the existence of E such that $DE(p) = f(p)$ is a famous result derived by Poincaré. There are two alternative proofs of this result. One uses Stokes' theorem, and the other analyzes the De Rham cohomology. Both are well known. Because of the similarity between this result and Nikliborc's Theorem, we doubt that the latter is a classical result, and Nikliborc himself may not be the founder of this theorem. We can at least say that, to the best of our knowledge, Nikliborc's paper contains one of the classical results concerning the PDE in (2.1).

In Nikliborc's original result, f is not C^1, but is differentiable and locally Lipschitz. Theorems 3.1 and 3.2 suggest that an extension of Frobenius' Theorem may hold. That is, if g is differentiable and locally Lipschitz, then we may be able to show the following result: there exists a solution (u,λ) of (2.3) if and only if g satisfies Jacobi's integrability condition (where λ is not necessarily continuous, and u is not necessarily C^1).

To show this, we are confronted with at least two difficulties. First, to prove the differentiability of the solution function $c(t;q,w)$ of (2.5), we usually assume that f is C^1. If f is not C^1, then Hadamard's lemma cannot be directly used, and thus the present proof of the differentiability of this function is broken (see ch.4 of Pontryagin (1962)). Second, we use the implicit function theorem to ensure the differentiability of u. If f is not continuously differentiable, however, $c(t;\tilde{x},w)$ is probably not C^1, and thus the usual implicit function theorem is broken. Hence, whether the above extension of Frobenius' Theorem holds or not is still an open problem.

However, in the next section, we introduce a more extended form of Frobenius' Theorem, and show that this is equivalent to an extended form of Nikliborc's Theorem.

4.2. **Comments on Frobenius' Theorem.** The original version of Frobenius' Theorem is written in the language of differential forms. Suppose that ω is a 1-form on some manifold X that does not vanish. Then, the original Frobenius' Theorem states that the following two claims are equivalent: 1) for every $x^* \in X$, there exists an open neighborhood $U \subset X$ of x^* and a function $u : X \to \mathbb{R}$ such that du does not vanish and is proportional to ω on U; 2) for every $x^* \in X$, there exists an open neighborhood $U \subset X$ of x^* and a 1-form θ defined on U such that $d\omega = \omega \wedge \theta$. The relationship between our version of Frobenius' Theorem and the original form of Frobenius' Theorem is explained by Hosoya (2012).

There exists an extension of Frobenius' Theorem. Suppose that $\omega_1,\dots,\omega_k$ are 1-forms on some n dimensional manifold X, and the intersection of the kernels of ω_i is $n-k$ dimensional at every point. Then, the following statements are equivalent: 1) for every $x^*\in X$, there exists an open neighborhood $U\subset X$ of x^* and $u_1,\dots,u_k:U\to\mathbb{R}$ such that the intersection of the kernels of $du_1,\dots,du_k$ is the same as that of $\omega_1,\dots,\omega_k$ for each point of U; 2) for every $x^*\in X$, there exists an open neighborhood U and a family of 1-forms $(\theta_{ij})_{i,j=1}^k$ defined on U such that $d\omega_i=\sum_{j=1}^k\omega_j\wedge\theta_{ij}$ for every $i\in\{1,\dots,k\}$. For this extension, see, for example, Auslander and MacKenzie (2009), Hicks (1965), Kosinski (1993), Matsushima (1972), and Sternberg (1999). Our consideration suggests that this probably corresponds to some variety of Nikliborc's Theorem on the following differential equation:

$$DE_i(p)=f_i(p,E_1(p),\dots,E_k(p)) \text{ for all } i\in\{1,\dots,k\}, E(p)=m.$$

Finally, we note a fact. If g is C^k and (u,λ) is a solution of (2.3), then $u^{-1}(w)$ is a C^{k+1} manifold. One might think that u itself can be C^{k+1}. However, this is incorrect. The following example was obtained by Debreu (1976). Let

$$g_1(x)=\begin{cases}\frac{x_2^2}{\sqrt{1+x_2^4}} & \text{if } x_2\ge 0,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$g_2(x)=\begin{cases}\frac{1}{\sqrt{1+x_2^4}} & \text{if } x_2\ge 0,\\ 1 & \text{otherwise.}\end{cases}$$

Then, g is C^1, but not C^2. If $x^*=(0,0)$, we can obtain a local solution

$$u(x)=\begin{cases}\frac{x_2}{1-x_1x_2} & \text{if } x_2\ge 0,\\ x_2 & \text{otherwise}\end{cases}$$

and

$$\lambda(x)=\begin{cases}\frac{\sqrt{1+x_2^4}}{(1-x_1x_2)^2} & \text{if } x_2\ge 0,\\ 1 & \text{otherwise,}\end{cases}$$

of the TDE in (2.3). However, no C^2 local solution v of (2.3) around $(0,0)$ exists for the following reason. Suppose that (v,μ) is a solution of (2.3) and v is C^2. Consider the following differential equation:

$$c(t;d)=-\frac{g_1(t,c(t;d))}{g_2(t,c(t;d))},\ c(0;d)=d.$$

Then, there exists $\varepsilon>0$ such that the solution function $c(t;d)$ is defined on $[-\varepsilon,\varepsilon]^2$. Without loss of generality, we can assume that both u,v are defined on $[-\varepsilon,\varepsilon]^2$. Choose $\delta>0$ sufficiently small that if $t\in[-\delta,\delta]$, then $c(t;-\varepsilon)<-\delta$ and $c(t;\varepsilon)>\delta$. Define $W=[-\delta,\delta]^2$. By the intermediate value theorem, we have that for every $x\in W$, there exists $d\in[-\varepsilon,\varepsilon]$ such that $c(x_1;d)=x_2$. By the chain rule, we have that $u(x)=u(0,d)=d$ and $v(x)=v(0,d)$. Define $\varphi(d)=v(0,d)$. Then, φ is C^2 and $\varphi'(d)=\mu(0,d)g_2(0,d)>0$. By the above result, we have that for every $x\in W$,

$$v(x)=\varphi(u(x)),$$

and thus,

$$\frac{\partial v}{\partial x_2}(x) = \varphi'(u(x))\frac{\partial u}{\partial x_2}(x),$$

and the right-hand side is not differentiable if $x_1 \neq 0$ and $x_2 = 0$, which is a contradiction. Therefore, such a (v, μ) does not exist.

5. Equivalence result for extensions

We extend the original version of Nikliborc's result. First, we introduce an important theorem.

Rademacher's Theorem. *Suppose that $f : U \to \mathbb{R}$, $U \subset \mathbb{R}^n$ is open, and f is locally Lipschitz. Then, f is Fréchet differentiable at almost every $x \in U$.*

The proof of this theorem is given in Heinonen (2004). Because of this theorem, if $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is locally Lipschitz, then $s_{ij}(p, m)$ can be defined for almost all $(p, m) \in P$. Thus, we can extend the notion of integrability: f is **integrable** if and only if $s_{ij}(p, m) = s_{ji}(p, m)$ for almost all $(p, m) \in P$.

Our extended form of Nikliborc's Theorem is as follows.

Nikliborc's Theorem II. *Suppose that $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ is open, and f is locally Lipschitz. Then, f is integrable if and only if for every $(p, m) \in P$, there exists a local solution $E : V \to \mathbb{R}$ of PDE (2.1) around p such that $E(p) = m$.*

We do not present the proof of this theorem here, because this is not the main theme of the current paper. In the near future, the proof of this theorem will hopefully be published. This theorem has one important application in the theory of consumer behavior, and thus we are preparing a paper that examines this application. This future paper will contain a proof of this theorem.

Note that if f is also differentiable, then the proof of this theorem was given by Nikliborc (1929), though this paper is written in French. For an English version, see EXISTENCE THEOREM I of Hurwicz and Uzawa (1971). Although the proof of EXISTENCE THEOREM I is missing some details,[10] we can fill the gaps by using a local Lipschitz constant.

The theme of this paper is the relationship between PDE (2.1) and TDE (2.3). Therefore, we must present the counterpart of this theorem. Suppose that $g : U \to \mathbb{R}^n$, $U \subset \mathbb{R}^n$ is open, and g is locally Lipschitz. A pair (u, λ) of real-valued functions defined on some open set V is a **solution** of TDE (2.3) if and only if 1) u is locally Lipschitz, 2) λ is positive, and 3) $Du(x) = \lambda(x)g(x)$ for almost every $x \in V$. If $x^* \in V$, then (u, λ) is called a **local solution** of (2.3) around x^*.

Meanwhile, g is said to satisfy **extended Jacobi's integrability condition** if (2.4) holds for almost all $x^* \in U$.

Frobenius' Theorem II. *Suppose that $g : U \to \mathbb{R}^n \setminus \{0\}$, $U \subset \mathbb{R}^n$ is open, and g is differentiable and locally Lipschitz. Then, g satisfies extended Jacobi's integrability*

[10]They do not assume that f is locally Lipschitz. However, the differentiability of the function $z^{(k)}$ in their proof cannot be proved when f is not locally Lipschitz.

condition if and only if, for every $x^* \in U$, there exists a local solution (u, λ) of TDE (2.3) around x^* such that for every $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an $n-1$ dimensional C^1 manifold.

Our next theorems are as follows.

Theorem 5.1. *Frobenius' Theorem II implies Nikliborc's Theorem II.*

Theorem 5.2. *Nikliborc's Theorem II implies Frobenius' Theorem II.*

Proof of Theorem 5.1. Suppose that Frobenius' Theorem II holds. Let $f : P \to \mathbb{R}^n$, $P \subset \mathbb{R}^{n+1}$ be open, and f be locally Lipschitz.

First, suppose that for every $(p, m) \in P$, there exists a local solution E of (2.1) around p such that $E(p) = m$. Because f is locally Lipschitz, by Rademacher's Theorem, it is differentiable almost everywhere. Choose any $(p, m) \in P$ such that f is differentiable, and let E be a local solution of (2.1) around p such that $E(p) = m$. Then, E is twice differentiable at p. Now, the following result is needed.

Extended Young's Theorem. *Suppose that $W \subset \mathbb{R}^2$ is open, and $g : W \to \mathbb{R}$ is differentiable around $(x^*, y^*) \in W$ and both $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ are differentiable at (x^*, y^*). Then, $\frac{\partial^2 g}{\partial y \partial x}(x^*, y^*) = \frac{\partial^2 g}{\partial x \partial y}(x^*, y^*)$.*

Proof of Extended Young's Theorem. Easy, and thus omitted. □

Using this theorem, we have that

$$s_{ij}(p, m) = \frac{\partial^2 E}{\partial q_j \partial q_i}(p) = \frac{\partial^2 E}{\partial q_i \partial q_j}(p) = s_{ji}(p, m),$$

and thus f is integrable.

Conversely, suppose that f is integrable. Define

$$g(p, m) = (f(p, m), -1).$$

Then, for the same reason as in the proof of Theorem 3.1, we have that g satisfies extended Jacobi's integrability condition. Fix $(p, m) \in P$. Then, by Frobenius' Theorem II, there exists a local solution (u, λ) of (2.3) around (p, m), and for any $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an n dimensional C^1 manifold. Let U be the domain of u and λ.

Recall the ODE (2.5):

$$\dot{c}(t; q, h) = f((1-t)p + tq, c(t; q, h)) \cdot (q - p),\ c(0; q, h) = m + h.$$

Choose a sufficiently small $\varepsilon > 0$ such that if $|q_i - p_i| \le \varepsilon$ for every $i \in \{1, \ldots, n\}$ and $|h| \le \varepsilon$, then $c(t; q, h)$ exists and $((1-t)p + tq, c(t; q, h)) \in U$ for all $t \in [0, 1]$. Define $V = \prod_{i=1}^n]p_i - \varepsilon, p_i + \varepsilon[$ and choose any $q \in V$ and $h \in]-\varepsilon, \varepsilon[$. We will show that $u((1-t)p + tq, c(t; q, h)) = u(p, m + h)$ for every $t \in [0, 1]$.

Choose any $q \in V$. If $q = p$, then $c(t; q, m + h) \equiv m + h$, and thus our claim is correct. Hence, we assume that $q \ne p$. We need a lemma.

Lemma 5.3. *Suppose that $W \subset U$ and the Lebesgue measure of $U \setminus W$ is zero. Moreover, suppose that $q \in V$ and $q_{i^*} \neq p_{i^*}$ for $i^* \in \{1, \dots, n\}$. For every $(t, \tilde{r}, h) \subset \mathbb{R}^{n+1}$ such that $t \in [0, 1]$, $r \in V$ for*

$$r_i = \begin{cases} \tilde{r}_i & \text{if } i < i^*, \\ q_i & \text{if } i = i^*, \\ \tilde{r}_{i-1} & \text{if } i > i^*, \end{cases}$$

and $h \in]-\varepsilon, \varepsilon[$, define

$$\xi(t, \tilde{r}, h) = ((1 - t)p + tr, c(t; r, h)). \tag{5.1}$$

Then, the Lebesgue measure of $\xi^{-1}(U \setminus W)$ is also zero.

Proof of Lemma 5.3. Without loss of generality, we assume that $i^* = n$. Throughout the proof of Lemma 5.3, we use the following notation. If $r \in \mathbb{R}^n$, then $\tilde{r} = (r_1, \dots, r_{n-1}) \in \mathbb{R}^{n-1}$. Conversely, if $\tilde{r} \in \mathbb{R}^{n-1}$, then $r = (r_1, \dots, r_{n-1}, q_n)$.

Let $\tilde{V} = \{\tilde{r} | r \in V\}$. We first show that ξ is one-to-one on the set $]0, 1] \times \tilde{V} \times]-\varepsilon, \varepsilon[$. Suppose that $t_1 \neq 0 \neq t_2$ and $\xi(t_1, \tilde{r}_1, h_1) = \xi(t_2, \tilde{r}_2, h_2) = (v, c)$. Because $v_n = (1 - t_1)p_n + t_1 q_n = (1 - t_2)p_n + t_2 q_n$ and $p_n \neq q_n$, we have $t_1 = t_2$. Because $v_i = (1 - t_1)p_i + t_1 r_{1i} = (1 - t_1)p_i + t_1 r_{2i}$ and $t_1 \neq 0$, we have that $r_{1i} = r_{2i}$, and thus $\tilde{r}_1 = \tilde{r}_2$. Therefore, it suffices to show that $c(t; r, h)$ is increasing in h. Suppose that $h_1 < h_2$ and $c(t; r, h_1) \geq c(t; r, h_2)$. Because $c(0; r, h_1) = m + h_1 < m + h_2 = c(0; r, h_2)$, by the intermediate value theorem, there exists $s \in [0, t]$ such that $c(s; r, h_1) = c(s; r, h_2)$. Then, by the Picard-Lindelöf uniqueness theorem, we have $m + h_1 = c(0; r, h_1) = c(0; r, h_2) = m + h_2$, which is a contradiction.

Next, define

$$W^\ell = \xi([\ell^{-1}, 1[\times \tilde{V} \times] - \varepsilon, \varepsilon[).$$

We show that ξ^{-1} is Lipschitz on W^ℓ. Define

$$t(v) = \frac{v_n - p_n}{q_n - p_n},$$

$$\tilde{r}(v) = \frac{1}{t(v)}[(t(v) - 1)\tilde{p} + \tilde{v}].$$

Suppose that $(v_1, c_1), (v_2, c_2) \in W^\ell$ and $(v_j, c_j) = \xi(t_j, \tilde{r}_j, h_j)$. Then, we have $t_j = t(v_j)$ and $\tilde{r}_j = \tilde{r}(v_j)$. Clearly, the functions $t(v)$ and $\tilde{r}(v)$ are Lipschitz on W^ℓ. Next, consider the following ODE:

$$\dot{d}(s) = f((1 - (s + t - t_2))p + (s + t - t_2)r(v), d(s)) \cdot (r(v) - p), \ d(t_2) = c.$$

Let $d(s; t, v, c)$ be the solution of the above ODE. Define $\hat{V}$ as the closure of $\tilde{V}$. If $(v, c) = \xi(t, \tilde{r}, h)$ for some $(t, \tilde{r}, h) \in [\ell^{-1}, 1] \times \hat{V} \times [-\varepsilon, \varepsilon]$, then $d(s; t, v, c) = c(s + t - t_2; r, h)$. Moreover, the set

$$\{(t, v, c) | t \in [\ell^{-1}, 1], \ (v, c) = \xi(t, \tilde{r}, h) \text{ for some } (\tilde{r}, h) \in \hat{V} \times [-\varepsilon, \varepsilon]\}$$

is compact, and thus, we have that $(t, v, c) \mapsto d(t_2 - t; t, v, c)$ is Lipschitz on this set. Therefore,

$$\begin{aligned} |h_1 - h_2| &= |d(t_2 - t_1; t_1, v_1, c_1) - d(t_2 - t_2; t_2, v_2, c_2)| \\ &\le L[|t_1 - t_2| + \|(v_1, c_1) - (v_2, c_2)\|] \\ &= L[|t(v_1) - t(v_2)| + \|(v_1, c_1) - (v_2, c_2)\|] \\ &\le L(M+1)\|(v_1, c_1) - (v_2, c_2)\|, \end{aligned}$$

where $L, M > 0$ are some constants, and hence our claim is correct.

Now, recall that the Lebesgue measure of $U \setminus W$ is zero. Because ξ^{-1} is Lipschitz on W^ℓ, we have that the Lebesgue measure of

$$\xi^{-1}(W^\ell \cap (U \setminus W))$$

is zero. Therefore, the Lebesgue measure of

$$\cup_\ell \xi^{-1}(W^\ell \cap (U \setminus W))$$

is also zero. Clearly, the Lebesgue measure of

$$\xi^{-1}(U \setminus W) \setminus \left(\cup_\ell \xi^{-1}(W^\ell \cap (U \setminus W))\right)$$

is zero, because this set is included in $\{(t, \tilde{r}, h) | t \in \{0, 1\}\}$. This completes the proof of Lemma 5.3. □

By Rademacher's Theorem and Lemma 5.3, there exists a sequence (q^k, h^k) such that $q^k \to q$, $h^k \to h$ as $k \to \infty$, and for all k and almost all $t \in [0, 1]$, u is differentiable and $Du = \lambda g$ at $((1-t)p + tq^k, c(t; q^k, h^k))$. Fix $t \in [0, 1]$. If $t = 0$, then clearly $u((1-t)p + tq, c(t; q, h)) = u(p, m + h)$. If $t > 0$, then for almost all $s \in [0, t]$,

$$\begin{aligned} \frac{d}{ds} u((1-s)p + tq^k, c(s; q^k, h^k)) &= \sum_{i=1}^n (q_i^k - p_i) \frac{\partial u}{\partial q_i} + \frac{\partial u}{\partial w} \dot{c} \\ &= \lambda \left[\sum_{i=1}^n (q_i^k - p_i) g_i - \sum_{i=1}^n (q_i^k - p_i) f_i \right] \\ &= 0, \end{aligned}$$

and thus $u((1-t)p + tq^k, c(t; q^k, h^k)) = u(p, m + h^k)$. Taking $k \to \infty$, we have that $u((1-t)p + tq, c(t; q, h)) = u(p, m + h)$, as desired.

Define $E_h(q) = c(1; q, h)$ for every $q \in V$ and $h \in]-\varepsilon, \varepsilon[$. Suppose that u is differentiable and $Du = \lambda g$ at $(q, E_h(q))$. We show that E_h is C^1 around q and $DE_h(q) = f(q, E_h(q))$. To solve this, define $X_h = u^{-1}(u(p, m + h))$. Then, $(p(t), c(t; q, h)) \in X_h$. Define the function $\chi : X_h \to \mathbb{R}^n$ as

$$\chi(r_1, \ldots, r_n, w) = (r_1, \ldots, r_n).$$

Then, $E_h(r)$ is the $n+1$-th coordinate of $\chi^{-1}(r)$. Thus, by the inverse function theorem, to prove that E_h is C^1 around q, it suffices to show that $d\chi_{(q, E_h(q))}$ is a surjective function from the tangent space $T_{(q, E_h(q))}(X_h)$ into $\mathbb{R}^n$.[11] Note that,

[11] In this part, we need some basic knowledges about differential manifolds. See sections 1.1-1.4 of Guillemin and Pollack (1976).

because χ is the restriction of a linear mapping $K(r_1,\dots,r_n,w)=(r_1,\dots,r_n)$ into X_h, $d\chi_{(q,E_h(q))}$ is the restriction of the same linear mapping into $T_{(q,E_h(q))}(X_h)$. Choose any $x\in\mathbb{R}^n$, and define $c=f(q,E_h(q))\cdot x$. Then, $(x,c)\cdot g(q,E_h(q))=0$. This implies that $Du(q,E_h(q))(x,c)=0$, and thus $(x,c)\in T_{(q,E_h(q))}(X_h)$ and $d\chi_{(q,E_h(q))}(x,c)=x$ as desired. Moreover,

$$\frac{\partial E_h}{\partial q_i}(q)=-\frac{\frac{\partial u}{\partial q_i}(q,E_h(q))}{\frac{\partial u}{\partial w}(q,E_h(q))}=f_i(q,E_h(q)).$$

Therefore, our claim is correct.

Now, choose any $q\in V$ with $q_n\neq p_n$ and $i\in\{1,\dots,n-1\}$. Then, by Lemma 5.3, there exists $\delta>0$ and a sequence (q^k,h^k) such that $q^k\to q$, $h^k\to 0$ as $k\to\infty$, and for every k and almost every $s\in]-\delta,\delta[$, u is differentiable and $Du=\lambda g$ at $(q^k+se_i,E_{h^k}(q^k+se_i))$, where e_i is the i-th unit vector.[12] Then, if $0<|s|<\delta$,

$$\frac{E_{h^k}(q^k+se_i)-E_{h^k}(q^k)}{s}=\frac{1}{s}\int_0^s f_i(q^k+\tau e_i,E_{h^k}(q^k+\tau e_i))d\tau.$$

By the dominated convergence theorem, we have that

$$\frac{E_0(q+se_i)-E_0(q)}{s}=\frac{1}{s}\int_0^s f_i(q+\tau e_i,E_0(q+\tau e_i))d\tau,$$

and thus,

$$\frac{\partial E_0}{\partial q_i}(q)=f_i(q,E_0(q)).$$

Next, choose any $q\in V$ and $i\in\{1,\dots,n-1\}$. Let $e=(1,1,\dots,1)\in\mathbb{R}^n$, and let $q^k=q+k^{-1}e$. For sufficiently large k, $q^k_n\neq p_n$, and thus there exists $\delta>0$ such that if k is sufficiently large and $0<|s|<\delta$, then

$$\frac{E_0(q^k+se_i)-E_0(q^k)}{s}=\frac{1}{s}\int_0^s f_i(q^k+\tau e_i,E_0(q^k+\tau e_i))d\tau.$$

Therefore, again by the dominated convergence theorem,

$$\frac{E_0(q+se_i)-E_0(q)}{s}=\frac{1}{s}\int_0^s f_i(q+\tau e_i,E_0(q+\tau e_i))d\tau,$$

which implies that

$$\frac{\partial E_0}{\partial q_i}(q)=f_i(q,E_0(q))$$

for every $q\in V$ and $i\in\{1,\dots,n-1\}$. Changing n to 1 and repeating the above arguments, we have that

$$DE_0(q)=f(q,E_0(q))$$

for all $q\in V$. Clearly, $E_0(p)=m$, and thus we obtain a local solution $E_0:V\to\mathbb{R}$ of (2.1). This completes the proof. $\square$

[12]Use Fubini's theorem. Note that $c(t;r,h)=c(1;(1-t)p+tr,h)=E_h((1-t)p+tr)$ by the Picard-Lindelöf uniqueness theorem.

Proof of Theorem **??**. Suppose that Nikliborc's Theorem II holds. Let $g : U \to \mathbb{R}^n \setminus \{0\}$, $U \subset \mathbb{R}^n$ be open, and g be locally Lipschitz. Throughout this proof, we use the following notation: if $x = (x_1, \ldots, x_n)$, then $\tilde{x} = (x_1, \ldots, x_{n-1})$.

First, suppose that for every $x^* \in U$, there exists a local solution (u, λ) of TDE (2.3) around x^* such that for every $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an $n-1$ dimensional C^1 manifold. Choose any $x^* \in U$ such that g is differentiable at x^*, and choose a local solution (u, λ) of TDE (2.3) around x^* such that for every $w \in \mathbb{R}$, $u^{-1}(w)$ is either the empty set or an $n-1$ dimensional C^1 manifold. Let V be the domain of u and λ. Without loss of generality, we assume that $g_n(x^*) \neq 0$. Define $f_i(x) = -\frac{g_i(x)}{g_n(x)}$ for every $x \in V$ with $g_n(x) \neq 0$, and consider the following ODE:

$$\dot{c}(t; \tilde{x}, h) = f((1-t)\tilde{x}^* + t\tilde{x}, c(t; \tilde{x}, h)) \cdot (\tilde{x} - \tilde{x}^*),\ c(0; \tilde{x}, h) = x_n^* + h.$$

Because $t \mapsto c(t; \tilde{x}^*, 0) \equiv x_n^*$ is defined on $\mathbb{R}$, we have that there exists an open neighborhood $W \subset V$ of $\tilde{x}^*$ and an open interval I including 0 such that $c(t; \tilde{x}, h)$ is defined on $[0, 1] \times W \times I$. Define $E_h(\tilde{x}) = c(1; \tilde{x}, h)$. Then, as in the proof of Theorem 5.1, we can show that $u(\tilde{x}, E_h(\tilde{x})) = u(\tilde{x}^*, x_n^* + h)$ and $DE_0(\tilde{x}) = f(\tilde{x}, E_0(\tilde{x}))$ for every $\tilde{x} \in W$. Therefore, by extended Young's theorem, we have that

$$s_{ij}(x^*) = \frac{\partial^2 E_0}{\partial x_j \partial x_i}(\tilde{x}^*) = \frac{\partial^2 E_0}{\partial x_i \partial x_j}(\tilde{x}^*) = s_{ji}(x^*),$$

and thus, by the same arguments as in the proof of Theorem 3.2, we have that g satisfies (2.4) at x^*. Hence, g satisfies extended Jacobi's integrability condition.

Conversely, suppose that g satisfies the extended Jacobi integrability condition. Choose any $x^* \in U$. Without loss of generality, we assume that $g_n(x^*) > 0$. For $i \in \{1, \ldots, n-1\}$, define $f_i(x) = -\frac{g_i(x)}{g_n(x)}$. Then, $f(x)$ is defined and Lipschitz on some open neighborhood W of x^*. Without loss of generality, we assume that $g_n(x) > 0$ on W.

If g is differentiable and (2.4) holds at $x \in W$, then by the same arguments as in the proof of Theorem 3.2, we have that $s_{ij}(x) = s_{ji}(x)$. This implies that f is integrable. Consider the following PDE (3.1):

$$DE(\tilde{x}) = f(\tilde{x}, E(\tilde{x})). \tag{5.2}$$

By Nikliborc's Theorem II and our arguments in subsection 2.4, there exists $\varepsilon > 0$ such that if $|h| \le \varepsilon$, then there exists a solution $E_h : \prod_{i=1}^{n-1}]x_i^* - \varepsilon, x_i^* + \varepsilon[\to \mathbb{R}$ of (5.2) that satisfies $E_h(\tilde{x}^*) = x_n^* + h$. Moreover, E_h is continuous and increasing in h.

Choose a sufficiently small $\delta > 0$ such that if $\tilde{x} \in \prod_{i=1}^{n-1}[x_i^* - \delta, x_i^* + \delta]$, then $E_{-\varepsilon}(\tilde{x}) < x_n^* - \delta$ and $E_\varepsilon(\tilde{x}) > x_n^* + \delta$. Define $V = \prod_{i=1}^{n}]x_i^* - \delta, x_i^* + \delta[$ and $\bar{V}$ as the closure of V. Then, by the intermediate value theorem, for every $x \in \bar{V}$, there uniquely exists $h \in]-\varepsilon, \varepsilon[$ such that $x_n = E_h(\tilde{x})$. Define $u(x)$ as such an h. Consider the following ODE:

$$\dot{c}(t; \tilde{x}, w) = f((1-t)\tilde{x}^* + t\tilde{x}, c(t; \tilde{x}, w)) \cdot (\tilde{x} - \tilde{x}^*),\ c(0; \tilde{x}, w) = w.$$

Note that this equation is equivalent to (2.5), and because f is locally Lipschitz, $c(t; \tilde{x}, w)$ is also locally Lipschitz. By our arguments in subsection 2.4, we have that

$E_h(\tilde{x}) = c(1;\tilde{x},x_n^*+h)$, and thus

$$u(x) = h \Leftrightarrow c(1;\tilde{x},x_n^*+h) = x_n.$$

Define $\tilde{V} = \prod_{i=1}^{n-1}[x_i^*-\delta, x_i^*+\delta]$. For every $(\tilde{x},w) \in \tilde{V}\times]x_n^*-\varepsilon, x_n^*+\varepsilon[$, if $|h|$ is sufficiently small, then

$$\begin{aligned}c(1;\tilde{x},w+h) - c(1;\tilde{x},w) = h + \int_0^1 &[f((1-t)\tilde{x}^*+t\tilde{x}, c(t;\tilde{x},w+h)) \\ &- f((1-t)\tilde{x}^*+t\tilde{x}, c(t;\tilde{x},w))]\cdot(\tilde{x}-\tilde{x}^*)dt,\end{aligned}$$

and

$$|[f((1-t)\tilde{x}^*+t\tilde{x}, c(t;\tilde{x},w+h)) - f((1-t)\tilde{x}^*+t\tilde{x}, c(t;\tilde{x},w))]\cdot(\tilde{x}-\tilde{x}^*)| \le L\|\tilde{x}-\tilde{x}^*\||h|$$

for every $t\in[0,1]$, where $L>0$ is some constant independent of $t,\tilde{x},w,h$. Therefore, if $\|\tilde{x}-\tilde{x}^*\| < (2L)^{-1}$, then $|c(1;\tilde{x},w+h)-c(1;\tilde{x},w)| \ge 2^{-1}|h|$. This implies that if $\delta>0$ is sufficiently small, then $\frac{\partial c}{\partial w}(1;\tilde{x},w) \ge 2^{-1}$ whenever the left-hand side is defined. Because $x_n \mapsto u(\tilde{x},x_n)$ is the inverse function of $h \mapsto c(1;\tilde{x},x_n^*+h)$, we have that if $\delta>0$ is sufficiently small, then u is Lipschitz in x_n on $\bar{V}$. Hence, we hereafter assume that $\delta>0$ is sufficiently small and u is Lipschitz in x_n on $\bar{V}$.

Next, choose any $x\in V$ and suppose that $u(x)=w$. Then, $E_w(\tilde{x}) = x_n$, and the graph of $E_w(\tilde{y})$ coincides with the set $u^{-1}(w)$. Thus, the latter set is an $n-1$ dimensional C^1 manifold. Moreover, because $DE_w(\tilde{x}) = f(x)$, there exists $\delta'>0$ such that if $|h|<\delta'$, then $(x_1,\ldots,x_i+h,\ldots,x_n)\in\bar{V}$, and $y_n(h) = E_w(x_1,\ldots,x_i+h,\ldots,x_{n-1}) \in [x_n-(|f_i(x)|+1)|h|, x_n+(|f_i(x)|+1)|h|]$. Define $y(h) = (x_1,\ldots,x_i+h,\ldots,x_{n-1},y_n(h))$. Then, we have that $u(y(h)) = u(x) = w$. Because u is Lipschitz in x_n on $\bar{V}$, there exists $L'>0$ independent of x,h such that

$$\begin{aligned}|u(x_1,\ldots,x_i+h,\ldots,x_n)-u(x)| &= |u(x_1,\ldots,x_i+h,\ldots,x_n)-u(y(h))| \\ &\le L'(\max_{y\in\bar{V}}\|f(y)\|+1)|h|,\end{aligned}$$

which implies that u is locally Lipschitz on V. By Rademacher's theorem, u is differentiable at almost every point $x\in V$. If u is differentiable at $x\in V$, then for $h=u(x)$, $E_h(\tilde{x}) = x_n$ and the graph of E_h coincides with $u^{-1}(h)$. Thus, by the chain rule,

$$\frac{\partial u}{\partial x_i}(x) + \frac{\partial u}{\partial x_n}(x)f_i(x) = 0,$$

which implies that either $\frac{\partial u}{\partial x_i}(x) = g_i(x) = 0$ or

$$\frac{\frac{\partial u}{\partial x_i}(x)}{g_i(x)} = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)}$$

for every $i\in\{1,\ldots,n-1\}$. Therefore, if we define

$$\lambda(x) = \frac{\frac{\partial u}{\partial x_n}(x)}{g_n(x)},$$

then

$$Du(x) = \lambda(x)g(x).$$

It suffices to show that $\lambda(x)$ is positive for almost all $x\in V$. Now, $x_n\mapsto u(\tilde{x},x_n)$ is the inverse function of $w\to c(1;\tilde{x},x_n^*+w)$, and the latter function is increasing.

Therefore, we have that u is increasing in x_n. Because u is differentiable at almost all points in V, for almost every $\tilde{x}$ such that $|x_i - x_i^*| < \delta$ for every $i \in \{1, \dots, n-1\}$, u is differentiable at $(\tilde{x}, x_n)$ for almost all $x_n \in]x_n^* - \delta, x_n^* + \delta[$. Suppose that $\tilde{x}$ satisfies such a requirement. Let $\xi(x_n) = u(\tilde{x}, x_n)$. Then, ξ is Lipschitz on $[x_n^* - \delta, x_n^* + \delta]$ and $\xi^{-1} : w \mapsto c(1; \tilde{x}, x_n^* + w)$ is Lipschitz on $[\xi(x_n^* - \delta), \xi(x_n^* + \delta)]$. Therefore, for almost all $x_n \in]x_n^* - \delta, x_n^* + \delta[$, ξ is differentiable at x_n and ξ^{-1} is differentiable at $\xi(x_n)$. For such an x_n, $\xi'(x_n)(\xi^{-1})'(\xi(x_n)) = 1$, and thus $\frac{\partial u}{\partial x_n}(\tilde{x}, x_n) = \xi'(x_n) > 0$. Because $g_n(\tilde{x}, x_n) > 0$, we have that $\lambda(x)$ is positive for almost all $x \in V$. We can define $\lambda(x) = 1$ if the original $\lambda(x) = 0$ or $\lambda(x)$ is undefined, and the new $\lambda(x)$ is always positive on V. Thus, (u, λ) is a solution of TDE (2.3). This completes the proof. □

Notes on Theorems 5.1 and 5.2. In PDE (2.1), if f is differentiable, then the solution E must be twice differentiable. In this connection, we obtain a conjecture. In TDE (2.3), if g is differentiable and locally Lipschitz, then for a solution (u, λ), u should be differentiable. To prove this, however, we are confronted with a huge problem: we cannot prove the differentiability of the solution function $c(t; \tilde{x}, w)$ under only the differentiability and locally Lipschitz condition of g. This breaks our proof of Theorem 3.2, and thus this conjecture remains an open problem.

Meanwhile, suppose that $f : P \to \mathbb{R}^n$ is locally Lipschitz on an open set $P \subset \mathbb{R}^{n+1}$ and integrable. The proof of Theorem 5.1 implies that for every $(p, m) \in P$ such that f is differentiable at (p, m), $s_{ij}(p, m) = s_{ji}(p, m)$. In this connection, suppose that $g : U \to \mathbb{R}^n$ is locally Lipschitz on an open set $U \subset \mathbb{R}^n$ and satisfies extended Jacobi's integrability condition. The proof of Theorem 5.2 implies that for every $x^* \in U$ such that g is differentiable at x^*, (2.4) must hold at x^*. Note that these are not obvious even when f, g are differentiable, because the derivative may be not continuous.

Now, suppose that $U \subset \mathbb{R}^n$ is open and $g : U \to \mathbb{R}^n$ is locally Lipschitz and satisfies extended Jacobi's integrability condition. Choose any $x^* \in U$ and suppose that (u, λ) is a local solution of TDE (2.3) around x^* such that $X = u^{-1}(u(x^*))$ is an $n-1$ dimensional C^1 manifold. Then, there is a conjecture: $g(x^*)$ is a normal vector of $T_{x^*}(X)$. Actually, this conjecture can be verified. To show this, without loss of generality, suppose that $g_n(x^*) \neq 0$, and define $f_i(x) = -\frac{g_i(x)}{g_n(x)}$ (if it can be defined). Let E_0 be the function defined in the proof of Theorem 5.2. Then, E_0 is defined on some open and convex neighborhood of $\tilde{x}^*$, $u(\tilde{x}, E_0(\tilde{x})) \equiv u(x^*)$, and E_0 solves the PDE (5.2) with $E_0(\tilde{x}^*) = x_n^*$. Therefore, $\varphi : \tilde{x} \mapsto (\tilde{x}, E_0(\tilde{x}))$ is a local parametrization of the manifold X around x^*, and thus $T_{x^*}(X)$ coincides with the range of $D\varphi(\tilde{x^*})$. Choose any $\tilde{v} \in \mathbb{R}^{n-1}$ and define $\tilde{x}(t) = \tilde{x}^* + t\tilde{v}$. Then,

$$\begin{aligned} D\varphi(x^*)\tilde{v} &= \left.\frac{d}{dt}\varphi(\tilde{x}(t))\right|_{t=0} \\ &= (\tilde{v}, DE_0(\tilde{x}^*)\tilde{v}) \\ &= (\tilde{v}, f(x^*) \cdot \tilde{v}), \end{aligned}$$

and thus, for every $v \in T_{x^*}(X)$,

$$g(x^*) \cdot v = g_n(x^*)[-f(x^*) \cdot \tilde{v} + v_n] = 0,$$

as desired.

However, this proof is heavily dependent on the proofs of Theorem 5.1 and 5.2, and thus this fact cannot be used to prove Theorems 5.1 and 5.2. This makes the proofs of these theorems difficult.

This result is valuable, because it implies the uniqueness of the leaf containing x^*.

Finally, Lemma 5.3 itself is important. We can verify Nikliborc's Theorem II using Lemma 5.3 and the usual proof of Nikliborc's Theorem.

Appendix A. Proofs of fact 4

First, we introduce a famous result.

Gronwall's inequality. Suppose that $u : [t_0, t_1] \to \mathbb{R}$ is continuous and satisfies the following inequality for every $t \in [t_0, t_1]$:

$$u(t) \le \int_{t_0}^t (\alpha u(\tau) + \beta) d\tau,$$

where $\alpha > 0$ and $\beta \ge 0$. Then,

$$u(t) \le \frac{\beta}{\alpha}(e^{\alpha(t-t_0)} - 1).$$

Proof of Gronwall's inequality. See Lemma 3 of Hosoya (2018). □

Suppose that $h(t, x, y)$ is locally Lipschitz. Recall equation (2.7):

$$\dot{x}(t) = h(t, x(t), y),\ x(t^*) = z.$$

Actually, we can omit the value z. Fix $x^* \in \mathbb{R}^n$ and define $k(t, x, y, z)$ as

$$k(t, x, y, z) = h(t, x(t) + (z - x^*), y),$$

and consider the following differential equation:

$$\dot{x}(t) = k(t, x(t), y, z),\ x(t) = x^*.$$

Clearly, $x : I \to \mathbb{R}$ is a solution of the above equation if and only if $x(t)+(z-x^*)$ is a solution of (2.7). Therefore, we omit z and simply consider the following differential equation:

$$\dot{x}(t) = h(t, x(t), y),\ x(t^*) = x^*. \tag{A.1}$$

Let $x(t; y)$ be the solution function of (A.1). By Fact 3, the domain U of $x(t; y)$ is open and $x(t; y)$ is continuous. Choose any $(\bar{t}, \bar{y}) \in U$. Let I be the domain of $t \mapsto x(t; \bar{y})$, and choose any $t_0, t_1 \in I$ such that $t_0 < t_1$ and $t^*, \bar{t} \in]t_0, t_1[$. Because $x(t; y)$ is continuous, there exists a compact neighborhood V of $\bar{y}$ such that 1) if $y \in V$, then the domain of $t \mapsto x(t; y)$ includes $[t_0, t_1]$, and 2) there exist $L > 0$

and $M > 0$ such that if we define $W = \{x(t;y)|t \in [t_0,t_1],\ y \in V\}$, then for every $(t,x,y),(t',x',y') \in [t_0,t_1] \times W \times V$,

$$\|h(t,x,y)\| \le M,$$
$$\|h(t,x,y) - h(t',x',y')\| \le L[|t-t'| + \|x-x'\| + \|y-y'\|].$$

If $t^* \le t \le t_1$ and $y_1, y_2 \in V$, then

$$\begin{aligned}\|x(t;y_1) - x(t;y_2)\| &= \left\|\int_{t^*}^t (h(\tau,x(\tau;y_1),y_1) - h(\tau,x(\tau;y_2),y_2))d\tau\right\| \\ &\le \int_{t^*}^t \|h(\tau,x(\tau;y_1),y_1) - h(\tau,x(\tau;y_2),y_2)\|d\tau \\ &\le \int_{t^*}^t [L\|x(\tau;y_1) - x(\tau;y_2)\| + L\|y_1-y_2\|]d\tau,\end{aligned}$$

and by Gronwall's inequality, we have

$$\|x(t;y_1) - x(t;y_2)\| \le \|y_1-y_2\|(e^{L(t-t^*)}-1) \le \|y_1-y_2\|(e^{L(t_1-t_0)}-1).$$

By the symmetrical arguments, we can show that

$$\|x(t;y_1) - x(t;y_2)\| \le \|y_1-y_2\|(e^{L(t_1-t_0)}-1)$$

for all $t \in [t_0,t^*]$ and $y_1,y_2 \in V$. Therefore, if $(t,y),(t',y') \in [t_0,t_1] \times V$, then

$$\|x(t;y) - x(t';y')\| \le M|t-t'| + (e^{L(t_1-t_0)}-1)\|y-y'\|,$$

and thus the solution function $x(t;y)$ is Lipschitz on $[t_0,t_1] \times V$. Hence, $x(t;y)$ is locally Lipschitz. This completes the proof of Fact 4. □

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