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CONVEXIFICATION ESTIMATES FOR MINKOWSKI AVERAGES IN INFINITE DIMENSIONS

ZVI ARTSTEIN

Abstract. The Minkowski averages of subsets of a finite dimensional vector space posses a convexification property. Estimates for the convexification phenomenon have been derived, employing the Shapley-Folkman lemma. We examine the infinite dimensional case. We show that the convexification property may not hold in infinite dimensions. We identify conditions that guarantee the convexification property, and provide estimates for the Hausdorff distance between the average and its convex hull.

1. Introduction

Convexity and convexification are major tools in many areas of mathematical analysis and applications. Among the key tools in the area is the convexification effect of the Minkowski sums of sets in linear spaces. Estimates for the convergence in the Hausdorff distance in finite dimensional spaces, of the Minkowski averages, have been offered, based on the Shapley-Folkman Lemma. The estimate for uniformly bounded sets in the N-dimensional Euclidean space is, roughly, big oh of $\frac{\sqrt{N}}{k}$, where k the number of the sets that are averaged. For general N-dimensional spaces, the estimate is big oh of $\frac{N}{k}$. In the next section we collect the necessary definitions. In Section 3 we revisit the finite dimensional case. In particular, we point out that these estimates are tight.

The appearance of N in the finite dimensional estimates may hint that in infinite dimensional spaces one may not derive universal convexification estimates. We show that, indeed, the convexification of the Minkowski averages of bounded sets may not occur in general spaces. It is known to hold in a Hilbert space. We validate it in Section 4 (and display an open problem). Within a compact framework, however, convexification always holds, and convexification estimates can be derived based on the Kolmogorov widths of the compact set. This is done in Section 5.

The present paper examines convexification estimates with respect to the Hausdorff distance. Other measures of non-convexity are also of interest. The elaborate survey by Fradelizi, Madiman, Marsiglietti and Zvavitch [5], offers an overview, and novel results, for the convexification effect of Minkowski averages with respect to the Hausdorff distance and other measures of non-convexity. The survey [5] also contains many references concerning applications of the convexification effect, including

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applications to mathematical economics, optimization, numerical integration, combinatorics, computer science, and more.

2. The setting

Convexification estimates are established in the literature for, typically, compact sets. With only a bit of effort it can be shown that most of the results are valid for general sets, even unbounded. Since sets that may not be closed will arise in the infinite dimensional considerations below, we revisit the entire theory for general sets (and point in parenthesis the effort that can be saved if the sets in question are closed or compact).

Let X be a normed space with norm $\|\cdot\|$. The distance $d(x, A)$ between a point $x \in X$ and a subset A of X is defined by $\inf\{\|x - a\| : a \in A\}$. The Hausdorff distance between two susets A_1 and A_2 of X is denoted $haus(A_1, A_2)$, and defined as $\max(\sup\{d(a_1, A_2) : a_1 \in A_1\}, \sup\{d(a_2, A_1) : a_2 \in A_2\})$. The Hausdorff distance does not distinguish between sets that share the same closure. It satisfies the triangle inequality.

The convex hull of a subset A of X, denoted coA, is the smallest closed convex set in X that contains A. In a finite dimensional space, say N-dimensional, Carathéodory Theorem implies that the set of points that are convex combinations of $N + 1$ points in A, is convex (notice that for unbounded sets this set may not be closed even if A is). The closure of this collection is the convex hull of the set. Since we later work in infinite dimensional spaces, we prefer to define the convex hull as a closed set.

The Hausdorff distance between A and coA is a natural measure of non-convexity of A. We use the indefinite article “a” since the literature offers, examines and compares, alternative measures of non-convexity, see [5]. In this paper we concentrate on the Hausdorff distance.

A useful parameter for subsets of a normed space is the *inner radius* of the set, as follows. Let A be a set in the space. The inner radius $ir(A)$ of A is the smallest non-negative number r, such that for every $a \in coA$ and every $\varepsilon > 0$, the point a is in the convex hull of the intersection of A with the ball of radius $r + \varepsilon$ centered in a. (In finite dimensions, if A is closed there is no need to employ the ε, namely, $\varepsilon = 0$ will do.)

Remark 2.1. Clearly, $haus(A, coA)$ is less than or equal to $ir(A)$ which, in turn, is less than or equal to the diameter of A. The latter two may coincide. For instance, when $A = \{0, 1\}$ in the real line, then $coA = [0, 1]$ and $ir(A) = 1$. The inner radius $ir(A)$ may be very large, even equal to ∞, while $haus(A, coA)$ being small. For instance, take A in the two-dimensional plane (ξ, η), to be the set determined by $\eta \geq he^{-|\xi|}$. Then $ir(A) = \infty$ while $haus(A, coA) < h$. A similar phenomenon, namely, inner radius being orders of magnitude bigger than the Hausdorff distance, may occur also for compact sets.

Let $A_1, \dots, A_k$ be sets in X. The Minkowski sum $A_1 + \cdots + A_k$ of the k sets is the set $\{a_1 + \cdots + a_k : a_i \in A_i, i = 1, \dots, k\}$. Let $\rho > 0$ be a real number, then $\rho A = \{\rho a : a \in A\}$. The Minkowski average of the k sets $A_1, \dots, A_k$ is the set $\frac{1}{k}(A_1 + \cdots + A_k)$. The convex hull operation on subsets of X is not linear, namely,

$co(A_1 + A_2)$ may not be equal to $coA_1 + coA_2$, since the latter may not be closed (e.g., in the (ξ, η) plane, take $A_1 = \{(\xi, \eta) : \xi > 0, \eta \geq \frac{1}{\xi}\}$ and $A_2 = \{(\xi, \eta) : \xi < 0, \eta \geq \frac{-1}{\xi}\}$); its closure, however, is equal to the former.

3. The finite dimensional case

The convexification estimates alluded to in the introduction, are related to the distance (Hausdorff distance in our case) between the Minkowski average and its convex hull. Since the finite dimensional case will play a prime role in infinite dimensions, we revisit it. We start by stating a key observation in the theory, namely, the Shapley-Folkman Lemma.

Lemma 3.1. *Let $A_1, \ldots, A_k$ be subsets of an N-dimensional Banach space X. Let the vector a be a convex combination of points in $A_1 + \cdots + A_k$. Then a can be written as $a_1 + \cdots + a_k$ where for at least $k - N$ indices i the points a_i are in A_i, and for the rest of the indices (at most N) the point a_i is a convex combination of points in A_i.*

Proof. Proofs in case the sets A_i are compact can be found, e.g., in Starr [10, Lemma 2 in Appendix 2], Arrow and Hahn [1], Artstein [2, Theorem 5.1], Zhou [12], Fradelizi et al. [5, Lemma 2.3]. The general case can be reduced to the compact one by considering the finite (at most $N + 1$ elements in each) sets in A_i whose Minkowski sum gives rise to the convex combination that yields a. □

In what follows, throughout the paper, we assume that $k \geq N$.

The following result is well known and was documented in many publications (it is verified in [5, Corollary 7.8] for spaces with asymmetric norms). The proof is simple, and it is displayed here for further reference.

Theorem 3.2. *Let $A_1, \ldots, A_k$ be sets in the N-dimensional Banach space X. Let $h = \max\{haus(A_i, coA_i) : i = 1, \ldots, k\}$. Then the Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull is less than or equal to $h\frac{N}{k}$.*

Proof. The estimate follows easily from the Shapley-Folkman Lemma 3.1. Indeed, let $a = a_1 + \cdots + a_k$ be in the associated Minkowski sum where for at most N indices the term a_i is not in A_i. For every $\varepsilon > 0$, these terms can be replaced by b_i with $b_i \in A_i$, with $\|a_i - b_i\| \leq h + \varepsilon$. Since ε is arbitrarily small the result follows. (Had all the sets been closed, invoking ε would not be needed.) □

As Example 4.1 below shows, the estimate $h\frac{N}{k}$ is tight for general normed spaces of finite dimensions. Further information on the norm may yield alternative estimates. Here is an estimate (also known as the Shapley-Folkman-Starr Theorem) for Euclidean spaces. Example 4.3 below shows that the estimate is tight for Euclidean spaces. Recall that the Euclidean norm of a vector $(\xi_1, \ldots, \xi_N)$ is $(\sum_{i=1}^{N} \xi_i^2)^{\frac{1}{2}}$.

Theorem 3.3. *Let $A_1, \ldots, A_k$ be sets in the N-dimensional Euclidean space E^N. Let $r = \max\{ir(A_i) : i = 1, \ldots, k\}$. Then the Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull, is less than of equal to $r\frac{\sqrt{N}}{k}$.*

Proof. A proof in case the sets are compact can be found in Starr [11]. It is easy to make the necessary changes to include general sets. (In the next section we provide a proof in the infinite dimensional case, that covers the present case.) □

Remark 3.4. At times the parameters h or r that appear in the preceding results are not available. If only a bound on the diameters of the sets is available, say D, then h and r can be replaced, respectively, by $\frac{D}{2}$ and D. If it is only known that the sets in question are included in a ball of radius β, then h and r can be replaced, respectively, by β and 2β. Many publications choose to state the theorems while referring to these parameters. Notice, however, that our statements apply also for unbounded sets.

Remark 3.5. At first look, the estimate in the Euclidean case seems better than the general one, since it employs $\sqrt{N}$ rather than N. However, the coefficient r in the Euclidean estimate is larger, and may be much larger, than the Hausdorff distance h used in the general case. See Remark 2.1. Hence, in some cases the general estimate may provide an estimate better than the one specific to the Euclidean norm.

4. The infinite dimensional case

We examine here the convexification phenomenon of the Minkowski averages of general sets in an infinite dimensional space. We start with some examples.

Example 4.1. Consider the space l_1 of summable sequences $x = (\xi_1, \xi_2, \dots)$ of real numbers with the l_1-norm, namely $\|x\| = \sum_{i=1}^{\infty} |\xi_i|$. Consider the sequence of sets in this space given by $A_i = \{0, e_i\}$, where 0 is the origin, and e_i is the unit vector with the entry 1 in the $i - th$ coordinate and 0 otherwise. It is a bounded set, and $haus(A_i, coA_i) = \frac{1}{2}$ for all i. The Minkowski average of $A_1, \dots, A_k$ is the set of sequences with entries being either 0 or $\frac{1}{k}$ in the first k coordinates, and 0 otherwise. The convex hull of the Minkowski average contains the vector whose first k entries are equal to $\frac{1}{2k}$, and 0 otherwise. Clearly, the Hausdorff distance between the Minkowski average and its convex hull is $\frac{1}{2}$ for all k, namely, there is no convexification effect as $k \to \infty$. (Taking a finite number of summations shows that the estimate in Theorem 3.2 is tight.)

A similar lack of convexification may occur even when the sequence of sets is constant.

Example 4.2. Consider again the space l_1 as in the previous example. Let $A = \{0, e_1, e_2, \dots\}$. The convex hull of the set consists of all the summable sequences $(\xi_1, \xi_2, \dots)$ with $0 \le \xi_i \le 1$, and sum (i.e. norm) less than or equal to 1. Then $haus(A, coA) = 1$. Indeed, the closest point in A to the vector in the convex hull whose first m entries are equal to $\frac{1}{m}$, is the origin, and the distance between the two is 1, and this is the largest possible such distance. Any vector in the Minkowski average of k sets, all equal to A, has coordinates either equal to 0 or equal to $\frac{m_i}{k}$, with m_i integers with sum less than or equal to k. The vector with $2k$ coordinates being equal to $\frac{1}{2k}$ is in the convex hull. The closest vector to it in the Minkowski average, is the zero vector, and the distance between the two is 1. In particular, the Hausdorff distance between the Minkowski average and its convex hull does not converge to zero as $k \to \infty$.

Notice that the set A in the previous example is not compact, neither do the sets in Example 4.1 belong to one compact set. In the next section we show that the Minkowski averages of sets within one compact set, exhibit the convexifiation phenomenon. In a Hilbert space the situation is known to be different, as shown in the following two examples.

Example 4.3. Consider the sequence of sets as in Example 4.1, this time, however, interpreted as square summable sequences with the l_2-norm. The Minkowski averages are the same as in Example 4.1. The Hausdorff distance between the Minkowski average of $A_1, \dots, A_k$ and its convex hull is given by the estimate for the finite k-dimensional space, namely $\frac{\sqrt{k}}{k}$ (notice that the inner radius of A_i is 1). There is a convexification effect as $k \to \infty$, but the rate of convergence in not of order big oh of $\frac{1}{k}$ as in the finite dimensional case, but, rather, it is big oh of $\frac{1}{\sqrt{k}}$. (Taking a finite number of summations shows that the estimate in Theorem 3.3 is tight.)

The same rate of convergence may also occur in a Hilbert space when the sequence A_i is constant, as follows.

Example 4.4. Consider the set A as in Example 4.2, this time interpreted as a bounded set (not compact though) in the Hilbert space l_2. The linear structure determines the Minkowski averages, as described in Example 4.2. Since the coordinates of the vectors involved are less than 1, the convex hull (here it is the closure of all the Minkowski averages) in l_2 is identical to the convex hull in l_1. In particular, the Hausdorff distance between the convex hull of A and the Minkowski average of k replicas of A, is attained at the vector in the convex hull that has $2k$ coordinates with value $\frac{1}{2k}$. The distance from the nearest element in the Minkowski average is then $\frac{1}{\sqrt{2k}}$. The convexification holds, with convergence rate of big oh of $\frac{1}{\sqrt{k}}$.

We state now the general results, to which the former examples correspond. The first one relates to a general Banach space. It does not guarantee convexification, as Examples 4.1 and 4.2 demonstrate.

Theorem 4.5. *Let $A_1, \dots, A_k$ be sets in a normed space X. Let h be the maximum of $haus(A_i, coA_i)$ for $i = 1, \dots, k$. Then the Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull is less than of equal to h.*

Proof. For $a = a_1 + \cdots + a_k$ where $a_i \in coA_i$, and a given $\varepsilon > 0$, we can find $b_i \in A_i$ such that $\|a_i - b_i\| \le h + \varepsilon$. Then $\frac{1}{k}(b_1 + \cdots + b_k)$ is in the Minkowski average of the sets, with distance from a being less than or equal to $h + \varepsilon$. Since ε is arbitrarily small the result follows. □

The next result establishes the convexification of the Minkowski averages in a Hilbert space, analogous to Theorem 3.3. The convexification property in a Hilbert space was established in Cassels [4] for an equivalent measure of nonconvexity. Using arguments similar to [4], Puri and Ralescu [9] established the convexification property for compact sets in spaces of p-type with $p > 1$ (a Banach space X is of p-type if there exists a constant $K > 0$ such that for every sequence $f_1, f_2, \dots, f_n$, of independent X-valued random variables with mean zero the inequality $E\|\sum_{j=1}^{n} f_i\|^p \le K \sum_{j=1}^{n} E\|f_i\|^p$ holds, where E denotes expectation, see [9]). Our approach considers general sets (as demonstrated by Examples 4.3 and 4.4). The convexification issue for unbounded sets and for sets in normed spaces was also examined by Khan [6] and [7], using non-standard analysis arguments. The proof here is an adaptation of the arguments in Starr [11] to the infinite dimensional case.

Theorem 4.6. *Let $A_1, \dots, A_k$ be sets in a Hilbert space H, with inner product $< \cdot, \cdot >$. Let $r = \max\{ir(A_i) : i = 1, \dots, k\}$. Then the Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull, is less than or equal to $r\frac{\sqrt{k}}{k}$.*

Proof. We shall prove an equivalent claim, namely, that the Hausdorff distance between $A_1 + \cdots + A_k$ and its convex hull is less than or equal to $r\sqrt{k}$. We proceed by induction on k. For $k = 1$ the result is obvious. Suppose it holds for k and consider $a \in (coA_1 + \cdots + coA_{k+1})$ (the latter is convex, hence it is dense in its convex hull, i.e, its closure). Then $a = a_1 + \cdots + a_{k+1}$ with $a_i \in coA_i$. The induction hypothesis implies that for every $\varepsilon > 0$ there exist $b_i \in A_i$ for $i = 1, \dots, k$, such that $\|(a_1 + \cdots + a_k) - (b_1 + \cdots + b_k)\| \le r\sqrt{k} + k\varepsilon$. The definition of the inner radius implies that for every $\varepsilon > 0$ the point a_{k+1} is in the convex hull of the intersection of A_{k+1} and the ball of radius $r + \varepsilon$ centered in a_{k+1}. Denote this set by T. Then for every vector v in the space, $\sup\{< v, c - a_{k+1} >: c \in T\}$ is greater than or equal to 0 (otherwise a_{k+1} can be strictly separated from the convex hull of T). For $v = b_1 + \cdots + b_k$ choose $b_{k+1} \in T$ such that $< v, b_{k+1} - a_{k+1} >$ is greater or equal to $-\varepsilon$. Then

$$\begin{aligned} & \|(b_1 + \cdots + b_k + b_{k+1}) - (a_1 + \cdots + a_k + a_{k+1})\|^2 \\ (4.1) \quad & \le \|(b_1 + \cdots + b_k) - (a_1 + \cdots + a_k)\|^2 + \|b_{k+1} - a_{k+1}\|^2 - 2 < b_{k+1}, a_{k+1} > \\ & \le (r\sqrt{k} + k\varepsilon)^2 + (r + \varepsilon)^2 + 2\varepsilon. \end{aligned}$$

Since ε is arbitrarily small, the result follows. □

The previous results lead to some interesting problems. We may say that a Banach space has the *convexification property* if the Hausdorff distance between the Minkowski average of a sequence $A_1, \dots, A_k$, all contained in the unit ball, and its convex hull, converges to 0 as $k \to \infty$. We may say that the Banach space has the *weak convexification property* if for any bounded set A in the space, the Minkowski average of k replicas of A, converges, in the Hausdorff distance, to the convex hull of A. The previous result establishes that a Hilbert space has the convexification property (likewise for spaces of p-type with $p > 1$, as follows from [9]). It is not clear to me if every space that has the weak convexification property has also the convexification property. It seems that uniformly convex spaces have the convexification property, but I am not aware of a proof. Just strict convexity, namely, no intervals on the boundary of the unit ball, does not guarantee even the weak convexification property, as the following example shows.

Example 4.7. Consider the space X of sequences $(x_1, x_2, \dots)$, where x_k belongs to the finite dimensional space R_p^k, namely, the k-dimensional space with the l_p-norm, $\|(\xi_1, \dots, \xi_k)\|_p = (\sum_{i=1}^k |\xi_i|^p)^{\frac{1}{p}}$. Furthermore, let $p = p(k)$ depend on k and be equal to $1 + \frac{1}{k}$. Let the norm in X of a sequence be the l_2-norm of the sequence $\|x_k\|_{p(k)}$. The square summable sequences form a Banach space which is strictly convex (but not uniformly convex). For a given k define $A_{k,1}, \dots, A_{k,k}$ in the space by $A_{k,j} = \{0, e_{k,j}\}$, where $e_{k,j}$ is the vector whose entry in the k-th coordinate of the sequence is the unit vector in $R_{p(k)}^k$ with 1 in the j-th coordinate. The Minkowski average of $A_{k,1}, \dots, A_{k,k}$ employs only the k-th coordinate of the Banach space, and within this coordinate the linear structure is as examined in Example 4.1 above. In particular, the vector $(\frac{1}{2k}, \dots, \frac{1}{2k})$ is in the convex hull of the average, but not in the average itself. Its distance from the average is $\frac{1}{2}k^{\frac{-1}{k+1}}$. The latter is also the Hausdorff distance between the Minkowski average of the k sets and its convex hull. For large k this distance tends to $\frac{1}{2}$, namely, the space does not have the convexification property.

The subset of X which consists of the union of all the sets $A_{k,j}$ for all k and all $j = 1, \dots, k$, validates that the space does not posses also the weak convexification property.

5. Compact sets in infinite dimensions

The lack of convexification, demonstrated in the first two examples of the previous section, does not occur when all the sets participating in the Minkowski averages belong to one compact set. The existence of convexification within a compact set can be deduced from Artstein and Hansen [3, Lemma]. The existence and convexification estimates can be derived employing the Kolmogorov widths of the set. The resulting estimates of convexification may also improve the rates established for Hilbert spaces. We start with the relevant definition.

Let X be a Banach space and let C be a set in X. For an N-dimensional linear subspace V of X we denote $\kappa_N(C, V) = \sup\{d(x, V) : x \in C\}$. The Kolomogorov N-width of C in X is the infimum of $\kappa_N(C, V)$ over all N-dimensional subspaces V of X. We denote the Kolomogorov N-width of C by $\kappa_N(C)$.

The notion of N-width was introduced by Kolmogorov in 1936, and has become a cornerstone in many areas of applied mathematics and approximation theory. See Pinkus [8]. Extensive research, including looking for estimates and for rates of convergence as $N \to \infty$, is still going on, as a simple literature search would show.

An underlying observation is documented here for completeness, as follows.

Lemma 5.1. *Let X be a Banach space and let C be a compact subset of X. Then $\kappa_N(C)$ converge to* 0 *as $N \to \infty$.*

Proof. The claim follows from the existence, for every $\varepsilon > 0$, of a finite cover of C with balls of radius ε. $\square$

A converse of the previous observation also holds, namely, if C is bounded and closed in X and $\kappa_N(C) \to 0$ as $N \to \infty$, then C is compact. This also follows from the possibility to find finite ε-covers. There may exist, however, unbounded, say closed, sets where $\kappa_N(C) \to 0$ as $N \to \infty$. The intersection of such a set with a closed bounded set in X, is compact.

In view of the previous analysis we provide two results, one for a general Banach space and one for a Hilbert space.

Theorem 5.2. *Let C be a subset of a Banach space X such that $\kappa_N(C) \to 0$ as $N \to \infty$ (for instance, C being compact). Given $h \geq 0$, for each k define*

$$e(k) = \min\{(h + 2\kappa_N(C))\frac{N}{k} + 2\kappa_N(C) : N = 1, 2, \ldots, k\}. \tag{5.1}$$

Then $e(k) \to 0$ as $k \to \infty$. Let $A_1, \ldots, A_k$ be subsets of C, and let $h = \max\{haus(A_i, coA_i) : i = 1, \ldots, k\}$. The Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull, is less than of equal to $e(k)$, in particular it tends to 0 as $k \to \infty$.

Proof. Since $\kappa_N(C)$ decreases to 0 as $k \to \infty$, we can first choose N that makes $\kappa_N(C)$ small, then choose k that makes $h\frac{N}{k}$ small. This shows that $e(k) \to 0$ as $k \to \infty$.

Now, given $\varepsilon > 0$ and a natural number N, let V_N be an N-dimensional linear subspace of X such that for every $x \in C$ the distance $d(x, V_N) \leq \kappa_N(C) + \varepsilon$. Such a linear subspace exists by the definition of the N-width. For each $a \in C$ let $b(a) \in V_N$ be such that $\|a - b(a)\| = d(a, V_N)$ (such a nearest vector exists since V_N is finite dimensional). Let $B_i = \{b(a) : a \in A_i\}$. Clearly, $haus(A_i, B_i) \leq \kappa_N(C) + \varepsilon$ for each i. A simple triangle inequality of distances in X implies then that $haus(B_i, coB_i) \leq h + 2(\kappa_N(C) + \varepsilon)$.

Let A and B be the Minkowski averages of $A_1, \ldots, A_k$ and of $B_1, \ldots, B_k$ respectively. Then, clearly,

$$haus(A, B) \leq \kappa_N(C) + \varepsilon. \tag{5.2}$$

The convex hull operation does not increase the Hausdorff distance, hence

$$haus(coA, coB) \leq \kappa_N(C) + \varepsilon. \tag{5.3}$$

Finally, the finite dimensional estimate exhibited in Theorem 3.2 implies that

$$haus(B, coB) \leq (h + 2(\kappa_N(C) + \varepsilon))\frac{N}{k}. \tag{5.4}$$

Applying now the triangle inequality of the Hausdorff distance, along with the former inequalities, reveals that

$$\begin{aligned} haus(A, coA) \leq &haus(A, B) + haus(B, coB) + haus(coB, coA) \\ &\leq 2(\kappa_N(C) + \varepsilon) + (h + 2(\kappa_N(C) + \varepsilon))\frac{N}{k}. \end{aligned} \tag{5.5}$$

Since ε is arbitrarily small the desires inequality $haus(A, coA) \leq e(k)$ follows from (5.1). This completes the proof. □

Similar arguments can be applied to the Hilbert space case, as follows.

Theorem 5.3. *Let C be a subset of a Hilbert space H such that $\kappa_N(C) \to 0$ as $N \to \infty$ (e.g., C being compact). Given $r \geq 0$, for each k define*

$$e_2(k) = \min\{r\frac{\sqrt{N}}{k} + 2\kappa_N(C) : N = 1, 2, \ldots, k\}. \tag{5.6}$$

Then $e_2(k) \to 0$ as $k \to \infty$. Let $A_1, \ldots, A_k$ be subsets of C, and let $r = \max\{ir(A_i) : i = 1, \ldots, k\}$. The Hausdorff distance between the Minkowski average $\frac{1}{k}(A_1 + \cdots + A_k)$ of the sets and its convex hull, is less than or equal to $e_2(k)$, in particular it tends to 0 as $k \to \infty$.

Proof. The convergence of $e_2(k) \to 0$ as $k \to \infty$, is justified as in the proof of the previous result.

Given $\varepsilon > 0$ and a natural number N, let V_N be an N-dimensional linear subspace of X such that for every $x \in C$ the distance $d(x, V_N) \leq \kappa_N(C) + \varepsilon$. For each $a \in C$ let $b(a)$ be the projection of a on V_N. Let $B_i = \{b(a) : a \in A_i\}$. Clearly, $haus(A_i, B_i) \leq \kappa_N(C) + \varepsilon$ for each i. Since the projection is a non-expansive linear operation it follows that $haus(B_i, coB_i) \leq h$.

Let A and B be the Minkowski averages of $A_1, \ldots, A_k$ and of $B_1, \ldots, B_k$ respectively. Then, clearly the inequalities (5.2) and (5.3) hold.

The linearity of the projection also implies that $ir(B_i) \leq ir(A_i)$, hence, by the finite dimensional estimate exhibited in Theorem 3.3, we get

$$haus(B, coB) \leq r\frac{\sqrt{N}}{k}. \tag{5.7}$$

Applying now the triangle inequality of the Hausdorff distance, along with the former inequalities, reveals that

$$\begin{aligned} haus(A, coA) &\leq haus(A, B) + haus(B, coB) + haus(coB, coA) \\ &\leq 2(\kappa_N(C) + \varepsilon) + r\frac{\sqrt{N}}{k}. \end{aligned} \tag{5.8}$$

Since ε is arbitrarily small, the desired inequality, $haus(A, coA) \leq e_2(k)$, follows from (5.6). This completes the proof. □

We wish to point out that the arguments in Remark 3.5 apply also to the infinite dimensional case, namely, $e(k)$ of (5.1) may be much smaller than $e_2(k)$ of (5.2).

We conclude with an example.

Example 5.4. Consider the space l_1, of summable sequences $x = (\xi_1, \xi_2, \ldots)$ of real numbers, with the l_1 norm. Let $f(i)$ be a fixed sequence in the space, with positive coordinates, say, $f(i) = \beta 2^{-i}$ for some constant β. Then $\|f\| = \beta$. Let $C = \{x : |\xi_i| \leq f(i)\}$. Then C is a compact subset of l_1. Given N, consider the finite dimensional subspace of l_1 determined by $\xi_k = 0$ for $k > N$. It is clear then that the N-width of C is less than or equal to $\beta 2^{-N}$. Any point in C is included in the ball of radius β. Hence, $haus(A, coA) \leq \beta$ for any subset A of C. Let now $A_i, i = 1, 2, \ldots,$ be subsets of C. Recalling (5.1), given k, we get $e(k) \leq \beta((1 + 2^{1-N})\frac{N}{k} + 2^{1-N})$, for any $N \leq k$. Choosing, for instance, $N = \log_2(k)$ (or rather, the nearby integer) we get $e(k) \leq \beta(1 + \frac{2}{k})\frac{\log_2(k)+2}{k}$ as an estimate for the convexification. Recall that in l_1 the convexification for a non-compact set is not guaranteed.

When the same example is examined in l_2, it is easy to see that the l_2-norm of $f(\cdot)$ is $\beta\sqrt{\frac{2}{3}}$. Likewise, $\kappa_N(C) \leq \beta 2^{-N}\sqrt{\frac{2}{3}}$. Clearly, the inner radius $ir(A)$ of any subset of C is less than or equal to $2\beta\sqrt{\frac{2}{3}}$. We get then from (5.6) and the choice $N = \log_2(k)$, the estimate $e_2(k) \leq 2\beta\sqrt{\frac{2}{3}}\frac{\sqrt{\log_2(k)}+1}{k}$. It is a much better estimate than the one deduced from the general case.

References

[1] K. J. Arrow and F. H. Hahn, *General Competitive Analysis*, Holden-Day, San Francisco, 1971.

[2] Z. Artstein, *Discrete and continuous bang-bang and facial spaces, or: Look for the extreme points*, SIAM Review **22** (1980), 172–185.

[3] Z. Artstein and G.C. Hansen, *Convexification in limit laws of random sets in Banach spaces*, Annals Probability **13** (1985), 307–309.

[4] J. W. S. Cassels, *Measures of non-convexity of sets and the Shapley-Folkman-Starr theorem*, Math. Proc. Cambridge Philos. Soc. **78** (1975), 433–436.

[5] M. Fradelizi, M. Madiman, A. Marsiglietti and A. Zvavitch, *The convexification effect of Minkowski summation*, EMS Surv. Math. Sci. **5** (2018), 1–164.

[6] M. A.Khan, *Approximately convex average sums of unbounded sets*, Proc. Amer. Math. Soc. **43** (1974), 181–185.

[7] M. A. Khan, *Approximately convex average sums of sets in normed spaces*, Appl. Math. Comput. **9** (1981), 27–34.

[8] A. Pinkus, *n-Widths in Approximation Theory*, Springer-Verlag, Berlin, 1985.

[9] M. L. Puri and D. A. Ralescu, *Limit theorems for random compact sets in Banach space*, Math. Proc. Cambridge Philos. Soc.**97** (1985), 151–158.

[10] R. M. Starr, *Quasi-Equilibria in markets with non-convex preferences*, Econometrica **37** (1969), 25–38.

[11] R. M. Starr, *Approximation of points of the convex hull of a sum of sets by points of the sum: An elementary approach*, J. Economic Theory **25** (1981), 314–317.

[12] L. Zhou, *A simple proof of the Shapley-Folkman theorem*, Econ. Theory **3** (1993), 371–372.

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Z. Artstein
Department of Mathematics, The Weizmann Institute of Science, Rehovot 76100, Israel
E-mail address: zvi.artstein@weizmann.ac.il