



EXTRAPOLATION: STORIES AND PROBLEMS

SERGEY ASTASHKIN AND MARIO MILMAN

ABSTRACT. We discuss some aspects of Extrapolation Theory. The presentation includes many examples and open problems.

1. INTRODUCTION

Apparently the first result concerning extrapolation of operator inequalities was formulated by Yano [73], although special cases had been considered earlier in papers by Marcinkiewicz and Titchmarsh (cf. [74] for proofs and precise references). Yano's result concerns with operators acting on L^p spaces defined on finite measure spaces¹, where p belongs to a fixed open interval (p_0, p_1) , and the rate of blow up of the norm inequalities, as p approaches the end points of the interval (p_0, p_1) , is prescribed. Yano's theorem was stated before the foundations of the general theory of interpolation were developed in the sixties and seventies. Moreover, during this so called "golden era of interpolation" most researchers were busy..interpolating, and Yano's theorem remained an isolated result for a long time. As a matter of fact, despite its obvious elementary nature, and the many concrete questions that it left open, further developments had to await for the general theory of Extrapolation that was only initiated in the 1980's (cf. [42], [40], [41], [55]). The abstract theory of extrapolation not only substantially extends and improves Yano's original result but it provides a general framework, and a powerful machinery, to extract information from inequalities that decay in a specified way in scales of Banach spaces.

On the occasion of this special issue of Pure and Applied Functional Analysis devoted to Extrapolation Theory² it seemed to us that it would be important to include a paper collecting in some organized fashion open problems, in order to promote more activity in this area. Indeed, it is probably fair to say that the vitality of a field at any given time can be gauged in terms of the quality, and the number, of its open problems.

Let us now say a few words about our intended audience, and how it has influenced the choices, as well as the presentation, of topics we shall discuss. We generally expect that our reader will have some familiarity with the rudiments of interpolation

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¹Although it has a straightforward extension to infinite measure spaces (cf. (2.4) below, for example).

²We refer to the Preface of this Special Issue for a brief discussion of Extrapolation and its connection with Interpolation.

theory as developed, for example, in the first few chapters of [15] or [16], to which we refer for background information and notation. Moreover, since there is a close connection between Extrapolation and Interpolation, we decided to organize the exposition exploiting the familiarity of the reader with Interpolation. In short, we have tried to frame our selection of problems by means of comparing familiar results of interpolation theory with their (possible) counterparts in extrapolation, often providing informal background explanations and, as much as possible, including explicit examples and calculations. These discussions correspond to what we refer to as *stories* in the title of our paper, which we have supplemented with Appendices that contain supporting material in order to facilitate the reading. We hope that these choices will make it easier for newcomers to profitably read the paper, while at the same time, it is also our expectation that experts could also find something of interest in the material as well.

In our effort to streamline the presentation we were led to introduce new concepts and notation that hopefully will clarify the connections and facilitate the formulation of new problems. In particular, there are two natural ways to look at extrapolation methods: either as mechanisms that reverse the process of interpolation or, alternatively, as interpolation processes that involve scales of interpolation spaces, rather than pairs of spaces. The latter is the point of view that we have largely adopted for our presentation in this paper. The unifying concepts that we introduce here are those of “ K and J –functionals for a scale of spaces” (cf. Section 4, Definition 4.2 and Definition 4.5).

The penalty that we have paid is that we produced a narrower set of problems and far more effort was spent on the explanations and motivations than we had originally envisioned. To somewhat mitigate these concerns we now provide a super brief set of references to some of the topics that we have not considered in this paper but have recently received extensive treatments in the literature, and refer the reader to these works for further references. In particular, we mention a number of recent papers devoted to abstract extrapolation methods (cf. [8], [7]); the connections between extrapolation and the so called limiting interpolation spaces was recently explored in detail in [11], [9]. Applications of extrapolation to embeddings of function spaces and other topics in Harmonic Analysis and Approximation are well known, for a recent account see [28]. Some aspects of the theory of extrapolation, as it applies to non-commutative L^p spaces, has been treated in a paper appearing in this issue (cf. [53]), where more references in this direction can be found. We also refer the reader³ to the other articles in this issue for further information on extrapolation, other potential sets of problems, and inspiration.

Finally, producing a list of problems creates, well, problems of its own. For example, one would probably need to exclude some topics and thus some important problems could end up not being mentioned, one could also create, unintentionally, the impression that some questions are more important than others. It is also in the nature of developing such lists that some of the problems will turn out to be easier to solve than others...Indeed, it will be immediately clear that our list covers

³Further sources include the dedicated web page

<https://sites.google.com/site/mariomilman/home/extrapolation-links>
that contains a listing of many relevant publications on Extrapolation Theory.

only a small sample of possible topics and that many important issues are not even mentioned. For these, and other possible shortcomings, we must apologize in advance and stress that, in particular, any omissions (or inclusions) do not represent necessarily a value judgement on our part⁴ but simply reflect our own taste and limitations. Moreover, although we have systematically documented all the topics discussed therein, it was not our intention to present a comprehensive bibliography. To supplement our references we refer the reader to the bibliographies of the papers that we do reference, as well as the collection of papers included in this volume, and, once again, apologize in advance⁵ if your favorite papers are not included in our somewhat limited bibliography.

We should issue one more warning concerning the presentation. In order to try to make the paper more user friendly we tried⁶ to avoid getting bogged down with another long formal paper and decided to adopt the somewhat more informal style of a conference or lecture presentation. This has resulted in a paper that is not linearly ordered, contains some repetitions, and may require the reader to jump around topics⁷, and even though we provide some road maps⁸, the reader is invited to rearrange the order and do her/his own jumping decisions.

2. IMPROVING YANO'S THEOREM

In order to illustrate the general methods of extrapolation we start by showing how the modern ideas led to a substantial improvement of Yano's theorem. Since the classical result deals with L^p spaces on finite measure spaces⁹, **in this section** we shall work with spaces of functions defined on fixed finite measure space that, without loss of generality, we shall take to be the interval $[0, 1]$ with Lebesgue measure. In particular, we shall let $L^p := L^p[0, 1]$, etc.

Theorem 2.1 (Yano [73], [74, Ch. 12, Theorem 4.41]). *Let $\alpha > 0$. Then,*

(i) *Suppose that T is a linear operator with values in the set of measurable functions on $[0, 1]$, such that T is bounded on L^p , for all $p \in (1, p_0)$, $p_0 > 1$, and $\|T\|_{L^p \rightarrow L^p} \leq \frac{c}{(p-1)^\alpha}$, as $p \rightarrow 1$, with a constant $c > 0$ independent of p . Then, T*

⁴Solve at your own peril!

⁵To add one more story, it is perhaps appropriate to quote Barry Simon here. In his talk "Tales of our forefathers" <http://www.math.caltech.edu/simon/biblio.html> he quotes H. Steinhaus (via a story by Steinhaus' student Mark Kac): "*The acceptance of your work by the mathematical public goes through three phases: First, they say it's wrong. Then, they say it's trivial. Finally, they say I did it first.*"

⁶Tried is the operative word here.

⁷Here we were inspired by Cortázar's novel *Hopscotch* [21], which can be described as an open-ended novel, or antinovel; where the reader is invited to rearrange the different parts of the novel according to a plan prescribed by the author. For the Cortázar fans we add that some version of "extrapolación" plays also a rôle in the novel.

⁸It is possible to read the text without jumping around and in such case read no further. Otherwise we suggest: Section 2 \rightarrow 3 \rightarrow 4 \rightarrow 6 \rightarrow 8....

⁹The main effect of this assumption for us is that in this case we work with an *ordered* pair (i.e. $L^\infty \subset L^1$) and therefore a special reiteration formula is available that makes some calculations somewhat easier.

can be extended to be a bounded operator

$$(2.1) \quad T : L(\operatorname{Log} L)^\alpha \rightarrow L^1,$$

where the Zygmund space $L(\operatorname{Log} L)^\alpha := L(\operatorname{Log} L)^\alpha[0, 1]$ consists of all functions f that are measurable on $[0, 1]$ and, moreover, such that

$$\|f\|_{L(\operatorname{Log} L)^\alpha} := \int_0^1 f^*(t) \log^\alpha(e/t) dt < \infty$$

(f^* is the decreasing rearrangement of the function $|f|$, see e.g. [49, Ch. II]).

(ii) Suppose that T is a bounded linear operator on L^p , for all $p > 1$, and such that for some constant $c > 0$, $\|T\|_{L^p \rightarrow L^p} \leq cp^\alpha$, as $p \rightarrow \infty$. Then, T is a bounded operator,

$$(2.2) \quad T : L^\infty \rightarrow e^{L^{1/\alpha}},$$

where the Zygmund space $e^{L^{1/\alpha}} := e^{L^{1/\alpha}}[0, 1]$ consists of all measurable functions $f(t)$ on $[0, 1]$ such that

$$\|f\|_{e^{L^{1/\alpha}}} := \sup_{0 < t \leq 1} (f^*(t) \log^{-\alpha}(e/t)) < \infty.$$

Yano's result is a simple consequence of the modern theory of extrapolation. Indeed, to see (i) we apply the \sum extrapolation functor¹⁰ of [43] to get

$$(2.3) \quad T : \sum_{p>1} \frac{L^p}{(p-1)^\alpha} \rightarrow \sum_{p>1} L^p,$$

and (2.1) follows from the known calculations (cf. [43] and the discussions below)

$$\sum_{p>1} \left(\frac{L^p}{(p-1)^\alpha} \right) = L(\operatorname{Log} L)^\alpha, \quad \sum_{p>1} L^p = L^1.$$

Likewise, if (ii) holds, then applying the Δ -functor yields

$$T : \Delta_{p>1}(L^p) \rightarrow \Delta_{p>1}(p^{-\alpha} L^p).$$

Consequently, (2.2) follows since (cf. [43])

$$\Delta_{p>1}(L^p) = L^\infty; \quad \Delta_{p>1}(p^{-\alpha} L^p) = e^{L^{1/\alpha}}.$$

Example 2.2. The following generalization of Theorem 2.1 holds (cf. [43]). Let $\vec{A} = (A_0, A_1)$, $\vec{B} = (B_0, B_1)$ be Banach pairs, and let $\alpha > 0$. Then,

(i) Suppose that T is a bounded linear operator¹¹, $T : (A_0, A_1)_{\theta, 1; J}^{\blacktriangleleft} \rightarrow (B_0, B_1)_{\theta, \infty; K}^{\blacktriangleleft}$, for all $\theta \in (0, 1)$, and such that there exists a constant $c > 0$, so that $\|T\|_{(A_0, A_1)_{\theta, 1; J}^{\blacktriangleleft} \rightarrow (B_0, B_1)_{\theta, \infty; K}^{\blacktriangleleft}} \leq c\theta^{-\alpha}$. Then, T can be extended to be a bounded operator

$$T : \sum_{\theta} \frac{1}{\theta^\alpha} \vec{A}_{\theta, 1; J}^{\blacktriangleleft} \rightarrow \sum_{\theta} \vec{B}_{\theta, \infty; K}^{\blacktriangleleft}.$$

¹⁰We refer to Appendix 13.2 for a discussion on the \sum and Δ methods of extrapolation.

¹¹We refer to Appendix 13.1 for notation and background on the real method of interpolation.

Moreover,

$$\sum_{\theta} \frac{1}{\theta^{\alpha}} \vec{A}_{\theta,q;J}^{\blacktriangleleft} = \{f : \|f\|_{\sum_{\theta} \frac{1}{\theta} \vec{A}_{\theta,q;J}^{\blacktriangleleft}} = \int_0^1 K(s, f; \vec{A}) (\log \frac{1}{s})^{\alpha-1} \frac{ds}{s} < \infty\},$$

$$\sum_{\theta} \vec{B}_{\theta,\infty;K}^{\blacktriangleleft} = B_0 + B_1.$$

(ii) Suppose that T is a bounded linear operator, $T : (A_0, A_1)_{\theta,q(\theta);K}^{\blacktriangleleft} \rightarrow (B_0, B_1)_{\theta,\infty;K}^{\blacktriangleleft}$, for all $\theta \in (0, 1)$, and such that there exists a constant $c > 0$, so that $\|T\|_{(A_0, A_1)_{\theta,1;K}^{\blacktriangleleft} \rightarrow (B_0, B_1)_{\theta,\infty;K}^{\blacktriangleleft}} \leq \frac{c}{(1-\theta)^{\alpha}}$. Then, T is a bounded operator

$$T : \Delta((A_0, A_1)_{\theta,q(\theta);K}^{\blacktriangleleft}) \rightarrow \Delta((1-\theta)^{\alpha} \vec{B}_{\theta,\infty;K}^{\blacktriangleleft}).$$

Moreover,

$$\Delta((1-\theta)^{\alpha} \vec{B}_{\theta,\infty;K}^{\blacktriangleleft}) = \{f : \|f\|_{\Delta((1-\theta)^{\alpha} \vec{B}_{\theta,\infty;K}^{\blacktriangleleft})} = \sup_{0 < t < 1} (1 + \log \frac{1}{t})^{-\alpha} \frac{K(t, f; \vec{B})}{t} < \infty\}$$

$$A_0 \cap A_1 \subset \Delta((A_0, A_1)_{\theta,q(\theta);K}^{\blacktriangleleft}).$$

Remark 2.3. In particular, the previous Example shows that the conclusions of Yano's Theorem hold under weaker assumptions: We can replace “strong type (p, p) ” by “weak type (p, p) ” (cf. Section 3 below for full details).

Remark 2.4. It follows from the previous example that Yano's theorem holds for L^p -spaces based on infinite measure, if we give a proper interpretation to (2.1) and (2.2). For example, suppose that the underlying measure space is $(0, \infty)$ with Lebesgue measure. Then (2.1) should now read

$$T : L(\text{Log} L)^{\alpha}(0, \infty) + L^{\infty}(0, \infty) \rightarrow L^1(0, \infty) + L^{\infty}(0, \infty).$$

The proof is the same: we apply the \sum -functor but now we need to recall that (cf. [43], see also [11])

$$(2.4) \quad \sum_{p>1} \left(\frac{p^{\alpha}}{(p-1)^{\alpha}} L^p(0, \infty) \right) = L(\text{Log} L)^{\alpha}(0, \infty) + L^{\infty}(0, \infty),$$

$$(2.5) \quad \sum_{p>1} L^p(0, \infty) = L^1(0, \infty) + L^{\infty}(0, \infty).$$

Example 2.5. Consider the Sobolev pair $\vec{A} = (W_{L^1}^k(\mathbb{R}^n), W_{L^{\infty}}^k(\mathbb{R}^n))$, where $k \in \mathbb{N}$, and for a function space $X(\mathbb{R}^n)$, the corresponding Sobolev space is defined using the norm

$$\|f\|_{W_X^k} = \sum_{|j| \leq k} \|D^j f\|_X.$$

Then a classical computation due to DeVore and Scherer yields (cf. [15]),

$$(2.6) \quad K(t, f; \vec{A}) \approx \sum_{|j| \leq k} K(t, D^j f, L^1, L^{\infty}).$$

It follows that $\vec{A}_{1/p', p; K}^\blacktriangleleft = W_{L^p}^k(\mathbb{R}^n)$, and therefore,

$$\sum_{p>1} \frac{1}{(1/p')^\alpha} \vec{A}_{1/p', p; K}^\blacktriangleleft = W_{L(\text{Log } L)^\alpha}^k(\mathbb{R}^n) + W_{L^\infty}^k(\mathbb{R}^n).$$

Let Ω be a Lipschitz domain of finite measure, then the pair $\vec{A} = (W_{L^1}^k(\Omega), \dot{W}_{L^\infty}^k(\Omega))$ is ordered, and the corresponding version of (2.6) holds (cf. [15], [20]), yielding (cf. [43])

$$\sum_{p>1} \frac{1}{(1/p')^\alpha} \vec{A}_{1/p', p; K}^\blacktriangleleft = W_{L(\text{Log } L)^\alpha}^k(\Omega).$$

Example 2.6. We refer to [15, Chapter 5, Section 6] for details and notation. Let H be the Hilbert transform and let N be the nontangential maximal function (cf. [15, (6.2), page 363]). Let $H(L^1)$ be the Hardy space for $L^1(\mathbb{R})$, $H(L^1) = \{f : f \in L^1(\mathbb{R}) \text{ and } Hf \in L^1(\mathbb{R})\}$, with

$$\|f\|_{\text{Re } H^1(\mathbb{R})} = \|f\|_{L^1(\mathbb{R})} + \|Hf\|_{L^1(\mathbb{R})}.$$

Let \vec{A} be the pair $(H(L^1)(\mathbb{R}), L^\infty(\mathbb{R}))$. Then

$$K(t, f; \vec{A}) \approx \int_0^t N(f)^*(s) ds.$$

Therefore, for $\alpha > 0$,

$$\begin{aligned} \|f\|_{\sum_{p>1} \frac{1}{(1/p')^\alpha} \vec{A}_{1/p', p; K}^\blacktriangleleft} &\approx \int_0^1 N(f)^{**}(s) (\log \frac{1}{s})^{\alpha-1} ds \\ &\approx \|N(f)\|_{L(\text{Log } L)^\alpha + L^\infty}. \end{aligned}$$

Example 2.7. Many other computations of extrapolation spaces and the corresponding extrapolation theorems can be read off from the known classical computations of K -functionals (cf. [15], [60], [64], [72] and the references therein). For example, if we consider $\vec{A} = (L^p(\mathbb{R}^n), \dot{W}_{L^p}^k(\mathbb{R}^n))$, $1 \leq p \leq \infty$, then (cf. [15, page 341], and also [47])

$$\begin{aligned} K(t^k, f; \vec{A}) &\approx \omega_p^k(t, f) := \sup_{|h| \leq t} \|\Delta_h^k f\|_{L^p} \\ &\approx \left\{ t^{-n} \int_{|h| \leq t} \|\Delta_h^k f\|_p^p dh \right\}^{1/p}, \quad p \in [1, \infty] \end{aligned}$$

where Δ_h^k denotes the k -th difference operator defined recursively by

$$\Delta_h f(x) = \Delta_h^1 f(x) = f(x+h) - f(x), \quad \Delta_h^k = \Delta_h^1 \Delta_h^{k-1}.$$

Therefore, for $0 < s < 1$, $k \in \mathbb{N}$, the Besov space $\mathcal{B}_{p,q}^{sk} := (L^p(\mathbb{R}^n), \dot{W}_{L^p}^k(\mathbb{R}^n))_{s,q;K}^\blacktriangleleft$ is defined by the condition

$$(2.7) \quad \|f\|_{\mathcal{B}_{p,q}^{sk}} \approx [(1-s)sq]^{1/q} k \left(\int_0^\infty t^{-skq} \left\{ t^{-n} \int_{|h| \leq t} \|\Delta_h^k f\|_p^p dh \right\}^{q/p} \frac{dt}{t} \right)^{1/q} < \infty,$$

with the usual conventions if $q = \infty$.

It follows that for $\alpha > 0$,

$$\|f\|_{\sum_s \frac{\mathcal{B}_{p,q}^{s,k}}{s^\alpha}} \approx \int_0^1 \omega_p^k(t, f) \left(\log \frac{1}{t}\right)^{\alpha-1} \frac{dt}{t}.$$

In particular, for $\alpha = 1$, one obtains the Dini type of spaces $\mathring{B}^{0,1,1}$ that were examined and applied to study the mixing properties of vector fields by Bianchini [17, see (1.16) and (4.6)]. In fact, these spaces are also useful in Harmonic Analysis (cf. [30], [34] and the references therein).

Example 2.8. Suppose that \mathcal{H} is a separable complex Hilbert space. Recall that the Schatten-von Neumann class \mathfrak{S}^p consists of all compact operators $T : \mathcal{H} \rightarrow \mathcal{H}$ such that

$$\|T\|_{\mathfrak{S}^p} := \left(\sum_{j=1}^{\infty} s_j(T)^p \right)^{\frac{1}{p}} < \infty,$$

where $\{s_j(T)\}_{j=1}^{\infty}$ is the non-increasing sequence of s -numbers of T determined by the Schmidt expansion (cf. [33]). The Schatten-von Neumann classes belong to the larger family of symmetrically normed ideals. In particular, the so-called Matsaev operator ideals and their duals, have been singled out and studied as suitable end point ideals for the scale of Schatten-von Neumann classes \mathfrak{S}^p .

Let $\alpha > 0$. The Matsaev ideal \mathcal{M}^α is the ideal of compact operators in a Hilbert space \mathcal{H} , endowed with the norm

$$\|T\|_{\mathcal{M}^\alpha} := \sum_{j=1}^n \frac{\log^{\alpha-1}(ej) s_j(T)}{j}.$$

Similarly, the dual ideal to \mathcal{M}^α , $\mathcal{M}^{\alpha,*}$, consists of all compact operators T such that

$$\|T\|_{\mathcal{M}^{\alpha,*}} := \sup_{n \in \mathbb{N}} \frac{\sum_{j=1}^n s_j(T)}{\log^\alpha(en)} < \infty.$$

By the well-known equivalence (see e.g. [48])

$$K(t, T; \mathfrak{S}^1, \mathfrak{S}^\infty) \approx \sum_{j=1}^{[t]} s_j(T), \quad (\text{where } [t] \text{ is the integer part of } t),$$

and straightforward direct calculations, we have

$$(\mathfrak{S}^\infty, \mathfrak{S}^1)_{1/p, p; K}^\blacktriangleleft = (\mathfrak{S}^1, \mathfrak{S}^\infty)_{1/p', p; K}^\blacktriangleleft = \mathfrak{S}^p.$$

Hence, as above, for each $p_0 > 1$

$$\mathcal{M}^\alpha = \sum_{p > p_0} p (\mathfrak{S}^\infty, \mathfrak{S}^1)_{1/p, p; K}^\blacktriangleleft = \sum_{p > p_0} p \mathfrak{S}^p$$

and

$$\mathcal{M}^{\alpha,*} = \Delta_{1 < p < p_0} ((p-1)^\alpha (\mathfrak{S}^\infty, \mathfrak{S}^1)_{1/p, p; K}^\blacktriangleleft) = \Delta_{1 < p < p_0} ((p-1)^\alpha \mathfrak{S}^p),$$

(see [55, 5.2]). These relations give a version of Yano's theorem in the non-commutative setting, where the ideals \mathcal{M}^α and $\mathcal{M}^{\alpha,*}$ play the role of the $L(\text{Log} L)^\alpha$ and $e^{L^{1/\alpha}}$ spaces (cf. [55, Theorem 42]).

The key to prove these results is that we can reduce the computation of the Σ - and Δ -functors as follows. Let M and N be tempered¹² weights (cf. [43]), then

$$(2.8) \quad \sum_{\theta} M(\theta) \vec{A}_{\theta, q(\theta); K}^{\blacktriangleleft} = \sum_{\theta} M(\theta) \vec{A}_{\theta, 1; J}^{\blacktriangleleft}, \quad \Delta_{\theta}(N(\theta) \vec{A}_{\theta, q(\theta); K}^{\blacktriangleleft}) = \Delta_{\theta}(N(\theta) \vec{A}_{\theta, \infty; K}^{\blacktriangleleft}).$$

With this reduction at hand we can compute the Σ - and Δ -functors via their Fubini type properties described in the next two theorems. Before we go to the statement and proof of these results we recall the concept of characteristic function of an interpolation functor [39], which we shall use freely in what follows.

Definition 2.9. Let F be an interpolation functor¹³, then the characteristic function ρ of F satisfies (cf. [39], [43, (2.7), page 11])

$$F(\mathbb{C}, \frac{1}{t}\mathbb{C}) = \frac{1}{\rho(t)}\mathbb{C}, \quad t > 0.$$

Theorem 2.10 (Jawerth-Milman [43, Theorem 3.1 (i), page 20]). *Let \vec{A} be a Banach pair, and let $\{\rho_{\theta}\}_{\theta \in (0,1)}$ be a family of quasi-concave functions. Suppose that $\rho(t) = \sup_{\theta \in (0,1)} \rho_{\theta}(t)$ is finite at one point (and hence at all points). Then*

$$\sum_{\theta} \left(\vec{A}_{\rho_{\theta}, 1; J} \right) = \vec{A}_{\rho, 1; J}, \quad \text{with } \|f\|_{\sum_{\theta} (\vec{A}_{\rho_{\theta}, 1; J})} = \|f\|_{\vec{A}_{\rho, 1; J}},$$

where for a quasi-concave function τ , we let $\vec{A}_{\tau, 1; J}$ be the space of elements that can be represented by

$$f = \int_0^{\infty} u(s) \frac{ds}{s},$$

where $u : (0, \infty) \rightarrow \Delta(\vec{A})$ is strongly measurable and such that

$$\int_0^{\infty} \frac{J(s, u(s); \vec{A})}{\tau(s)} \frac{ds}{s} < \infty,$$

with

$$\|f\|_{\vec{A}_{\tau, 1; J}} = \inf_{x = \int_0^{\infty} u(s) \frac{ds}{s}} \left\{ \int_0^{\infty} \frac{J(s, u(s); \vec{A})}{\tau(s)} \frac{ds}{s} \right\}.$$

Proof. We repeat with full details¹⁴ the elementary argument given in [43] since it illustrates why the $(1, J)$ functor “commutes” with Σ (Fubini).

By definition $\rho(t) \geq \rho_{\theta}$, for all $\theta \in (0, 1)$, therefore, it is easy to see that $\|f\|_{\vec{A}_{\rho, 1; J}} \leq \|f\|_{\vec{A}_{\rho_{\theta}, 1; J}}$. Indeed, if $f \in \vec{A}_{\rho_{\theta}, 1; J}$, then given $\varepsilon > 0$, we can select a representation $f = \int_0^{\infty} u(s) \frac{ds}{s}$ such that,

$$\int_0^{\infty} \frac{J(s, u(s); \vec{A})}{\rho_{\theta}(s)} \frac{ds}{s} \leq (1 + \varepsilon) \|f\|_{\vec{A}_{\rho_{\theta}, 1; J}}.$$

¹²We say that M is tempered if $M(2\theta) \approx M(\theta)$, when θ is close to zero, and $M(1 - 2(1 - \theta)) \approx M(\theta)$, when θ is close to 1.

¹³cf. Appendix 13.2.

¹⁴In their younger days, before changing their ways, Jawerth-Milman had adopted the terse writing style of [16].

Therefore,

$$\|f\|_{\vec{A}_{\rho,1;J}} \leq \int_0^\infty \frac{J(s, u(s); \vec{A})}{\rho(s)} \frac{ds}{s} \leq \int_0^\infty \frac{J(s, u(s); \vec{A})}{\rho_\theta(s)} \frac{ds}{s} \leq (1 + \varepsilon) \|f\|_{\vec{A}_{\rho_\theta,1;J}}.$$

Now, letting $\varepsilon \rightarrow 0$, we find that for all $\theta \in (0, 1)$,

$$\|f\|_{\vec{A}_{\rho,1;J}} \leq \|f\|_{\vec{A}_{\rho_\theta,1;J}}.$$

The previous inequality can be now extended to all of $\sum_\theta (\vec{A}_{\rho_\theta,1;J})$. Indeed, let $f \in \sum_\theta (\vec{A}_{\rho_\theta,1;J})$. Select a decomposition $f = \sum_\theta f_\theta$ such that¹⁵ $\|f\|_{\sum_\theta (\vec{A}_{\rho_\theta,1;J})} \approx \sum_\theta \|f_\theta\|_{\vec{A}_{\rho_\theta,1;J}}$. Then,

$$\begin{aligned} \|f\|_{\vec{A}_{\rho,1;J}} &= \left\| \sum_\theta f_\theta \right\|_{\vec{A}_{\rho,1;J}} \\ &\leq \sum_\theta \|f_\theta\|_{\vec{A}_{\rho,1;J}} \\ &\leq \sum_\theta \|f_\theta\|_{\vec{A}_{\rho_\theta,1;J}} \\ &\leq \|f\|_{\sum_\theta (\vec{A}_{\rho_\theta,1;J})}. \end{aligned}$$

To show the converse inequality let us first observe that for each $f \in \Delta(\vec{A})$, $\theta \in (0, 1)$, $t > 0$,

$$\begin{aligned} \|f\|_{\sum_\theta (\vec{A}_{\rho_\theta,1;J})} &\leq \|f\|_{\vec{A}_{\rho_\theta,1;J}} \\ &\leq \frac{J(t, f; \vec{A})}{\rho_\theta(t)} \end{aligned}$$

(see [16, Theorem 3.11.2 (4), page 64]). Therefore,

$$\begin{aligned} \|f\|_{\sum_\theta (\vec{A}_{\rho_\theta,1;J})} &\leq \inf_\theta \left\{ \frac{1}{\rho_\theta(t)} \right\} J(t, f; \vec{A}) \\ &\leq \frac{1}{\rho(t)} J(t, f; \vec{A}). \end{aligned}$$

Let $f \in \vec{A}_{\rho,1;J}$, $\varepsilon > 0$, and select a decomposition $f = \int_0^\infty u(s) \frac{ds}{s}$ such that

$$\int_0^\infty \frac{J(s, u(s); \vec{A})}{\rho(s)} \frac{ds}{s} \leq (1 + \varepsilon) \|f\|_{\vec{A}_{\rho,1;J}}.$$

¹⁵Informally, we need to use an epsilon argument, etc.

We have

$$\begin{aligned}
\|f\|_{\sum_{\theta}(\vec{A}_{\rho_{\theta},1;J})} &\leq \left\| \int_0^{\infty} u(s) \frac{ds}{s} \right\|_{\sum_{\theta}(\vec{A}_{\rho_{\theta},1;J})} \\
&\leq \int_0^{\infty} \|u(s)\|_{\sum_{\theta}(\vec{A}_{\rho_{\theta},1;J})} \frac{ds}{s} \\
&\leq \int_0^{\infty} \frac{1}{\rho(s)} J(s, u(s); \vec{A}) \frac{ds}{s} \\
&\leq (1 + \varepsilon) \|f\|_{\vec{A}_{\rho,1;J}}.
\end{aligned}$$

Now we can safely let $\varepsilon \rightarrow 0$ to conclude the proof. \square

Likewise, and even easier “Fubini argument” but this time using the L^{∞} -norm (informally: “sup commutes with sup”) shows that (cf. [43, Theorem 3.1 (ii), page 20-21])

Theorem 2.11. *Suppose that $\rho^*(t) = \inf_{\alpha \in (0,1)} \{\rho_{\alpha}(t)\}$ is non-zero at a point, then*

$$\Delta_{\alpha \in (0,1)}(\vec{A}_{\rho_{\alpha},\infty;K}) = \vec{A}_{\rho^*,\infty;K},$$

where for a quasi-concave function τ ,

$$\vec{A}_{\tau,\infty;K} = \{f : \|x\|_{\vec{A}_{\tau,\infty;K}} := \sup_{s>0} \left\{ \frac{K(s, f; \vec{A})}{\tau(s)} \right\} < \infty\}.$$

Proof. Since for all indices $\alpha \in (0, 1)$, and for all $t > 0$, $\rho^*(t) \leq \rho_{\alpha}(t)$, we readily see that

$$\|f\|_{\Delta(\vec{A}_{\rho_{\alpha},\infty;K})} = \sup_{\alpha \in (0,1)} \sup_{s>0} \frac{K(s, f; \vec{A})}{\rho_{\alpha}(s)} \leq \sup_{s>0} \frac{K(s, f; \vec{A})}{\rho^*(s)} = \|f\|_{\vec{A}_{\rho^*,\infty;K}}.$$

Now, for all $t > 0$, we have

$$\begin{aligned}
K(t, f; \vec{A}) &\leq \rho_{\alpha}(t) \|f\|_{\vec{A}_{\rho_{\alpha},\infty;K}} \\
&\leq \rho_{\alpha}(t) \sup_{\alpha \in (0,1)} \|f\|_{\vec{A}_{\rho_{\alpha},\infty;K}} \\
&= \rho_{\alpha}(t) \|f\|_{\Delta(\vec{A}_{\rho_{\alpha},\infty;K})}.
\end{aligned}$$

Thus, for all indices α , and for all $t > 0$,

$$\frac{K(t, f; \vec{A})}{\rho^*(t)} \leq \frac{\rho_{\alpha}(t)}{\rho^*(t)} \|f\|_{\Delta(\vec{A}_{\rho_{\alpha},\infty;K})}.$$

Taking the infimum over all indices α , we get

$$\begin{aligned}
\frac{K(t, f; \vec{A})}{\rho^*(t)} &\leq \|f\|_{\Delta(\vec{A}_{\rho_{\alpha},\infty;K})} \inf_{0 < \alpha < 1} \left\{ \frac{\rho_{\alpha}(t)}{\rho^*(t)} \right\} \\
&= \|f\|_{\Delta(\vec{A}_{\rho^*,\infty;K})},
\end{aligned}$$

and the result follows taking the supremum over all $t > 0$. \square

Problem 2.12. (The multiplier problem I) Give a complete characterization of the weights $M(\theta)$ for which the formulae (2.8) holds. More generally, let $\{\rho_\theta\}$ be a family of concave functions and consider interpolation functors $\{F_{\rho_\theta}\}$ with characteristic functions ρ_θ . We ask to characterize the weights M and N such that, for all Banach pairs \vec{A} , we have

$$(2.9) \quad \sum_{\theta} M(\theta) \vec{A}_{\rho_\theta, 1; J}^{\blacktriangleleft} = \sum_{\theta} M(\theta) F_{\rho_\theta}(\vec{A}), \Delta(N(\theta) \vec{A}_{\rho_\theta, \infty; K}^{\blacktriangleleft}) = \Delta(N(\theta) F_{\rho_\theta}(\vec{A})).$$

The same question for more general extrapolation functors (cf. Section 13.2), e.g. the Σ_p -methods (cf. [43, (2.6), page 10] and [46]), the Δ_p -methods (cf. [46]), the extrapolation methods of Astashkin-Lykov (cf. [7], [6], etc.).

Remark 2.13. In connection to Problem 2.12 we should like to mention some cases where progress has been made. Consider the pair $(L^1(0, 1), L^\infty(0, 1))$, then it is shown (cf. [8, Theorem 3.5]) that

$$(2.10) \quad \Delta_{1 < p < \infty}(\omega(p)(L^1, L^\infty)_{1/p', \infty; K}^{\blacktriangleleft}) = \Delta_{1 < p < \infty}(\omega(p)(L^1, L^\infty)_{1/p', p; K}^{\blacktriangleleft}),$$

holds for weights of the form $\omega(p) = \psi(e^{-p})$, where ψ is an increasing positive function on $[0, 1]$ such that for some $C > 0$, $\psi(et) \leq C\psi(t)$, $0 < t \leq 1/e$. It is worth to note that, in general, these weights fail to be tempered. Indeed, it is easy to see that the function $\omega(1/(1 - \theta))$ is tempered at 0 if and only if $\omega(2p) \approx \omega(p)$, when p is sufficiently large. Therefore, this is equivalent to the condition $\psi(t) \leq C\psi(t^2)$ for $0 < t \leq 1$. Moreover, formula (2.10) holds even for weights decreasing at a much faster rate at infinity, e.g. weights of the form $\omega(p) = \psi(e^{-e^p})$. Once again we refer to [8]. Likewise, one can ask for a characterization of the weights $\omega(p)$ such that we have

$$(2.11) \quad \sum_{1 < p < \infty} (\omega(p)(L^1, L^\infty)_{1/p', 1; J}^{\blacktriangleleft}) = \sum_{1 < p < \infty} (\omega(p)(L^1, L^\infty)_{1/p', p; J}^{\blacktriangleleft}).$$

It is known (cf. [52, Theorem 3]) that formula (2.11) holds for weights ω of the form $\omega(p) = \psi(p/(p - 1))$, where $\psi : [1, \infty) \rightarrow [1, \infty)$ is an increasing positive function such that for some $C > 0$, $\psi(x + e^{-x}) \leq C\psi(x)$, $x \geq 1$. Note that in this example $\omega(p)$ is tempered at 1 if and only if there exists $C > 0$, such that $\psi(2x) \leq C\psi(x)$ for sufficiently large x . Therefore, (2.11) can be valid for a rather wide class of non-tempered weights.

Problem 2.14. Formulate suitable versions of Theorems 2.10 and 2.11 for other extrapolation functors (cf. the previous Problem).

Problem 2.15. (Open Ended) There is a natural duality associated to Theorems 2.10 and 2.11 (cf. [43]) but more generally the rôle of duality in extrapolation theory has not been studied systematically.

3. YANO'S THEOREM FOR WEAK TYPE OPERATORS

The general method to prove Yano's theorem indicated in Section 2 shows, as a bonus, that we can extrapolate replacing strong type by weak type. Indeed, this is a direct consequence of (2.8). We develop this point in detail. Once again the

underlying measure space will be $(0, 1)$ with Lebesgue measure, but we now assume that T satisfies

$$(3.1) \quad T : L(p, 1) \rightarrow L(p, \infty), \text{ with } \|T\|_{L(p,1) \rightarrow L(p,\infty)} \leq \frac{1}{(p-1)^\alpha}, \text{ for } 1 < p < p_0.$$

Before we go on we remark that we need to be fastidious about how we define the norms of the Lorentz spaces. We shall let¹⁶

$$(3.2) \quad L(p, 1) = (L^1, L^\infty)_{1/p', 1; K}^\blacktriangleleft = \{f : \|f\|_{L(p,1)} = \frac{1}{pp'} \int_0^\infty f^{**}(s) s^{1/p} \frac{ds}{s} < \infty\},$$

$$(3.3) \quad L(p, \infty) = \{f : \|f\|_{L(p,\infty)} = \sup_{t>0} \{f^{**}(s) s^{1/p}\}\} = (L^1, L^\infty)_{1/p', \infty; K}^\blacktriangleleft.$$

The reason we use (3.2) to define the $L(p, 1)$ -spaces is that we can then apply directly the strong form of the fundamental Lemma (cf. [25] and Section 13.1.1 below) to obtain¹⁷ that, with constants of equivalence independent of p ,

$$(3.4) \quad L(p, 1) = (L^1, L^\infty)_{1/p', 1; K}^\blacktriangleleft = (L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft.$$

At this point we see that

$$T : \sum_{1 < p < p_0} \frac{(L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft}{(p-1)^\alpha} \rightarrow \sum_{1 < p < p_0} (L^1, L^\infty)_{1/p', \infty; K}^\blacktriangleleft = \sum_{1 < p < p_0} L(p, \infty).$$

Now using Theorem 2.10 and the reiteration formula (cf. [43] and the recent extensive discussion in [11])

$$\sum_{1 < p < p_0} \frac{(L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft}{(p-1)^\alpha} = \sum_{1 < p < \infty} \frac{(L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft}{(p-1)^\alpha},$$

we have

$$\sum_{1 < p < p_0} \frac{(L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft}{(p-1)^\alpha} = L(\text{Log} L)^\alpha, \quad \sum_{1 < p < p_0} L(p, \infty) = L^1.$$

We can deal in a similar fashion with the second part of Yano's theorem.

¹⁶It is easy to give $L(p, 1)$ a more familiar norm, as we now indicate. For this purpose we may restrict ourselves, without loss of generality, to functions f that are integrable with compact support. Then, integrating by parts we find

$$\begin{aligned} \frac{1}{pp'} \int_0^\infty f^{**}(s) s^{1/p} \frac{ds}{s} &= \frac{1}{p} \int_0^\infty f^*(s) s^{1/p} \frac{ds}{s} \\ &= \int_0^\infty f^*(s) ds^{1/p}. \end{aligned}$$

¹⁷A more general result is discussed in detail in Example 13.2 below.

3.1. Limiting Spaces. To describe extrapolation spaces like the $L(\text{Log}L)^\alpha(0,1)$ spaces it is useful to introduce variants of the usual interpolation constructions. Here we consider the simplest such constructions, the $\langle \vec{X} \rangle_{w,q;K}$ spaces (cf. Appendix 13.1 below, moreover, we refer to [11] for a comprehensive discussion of limiting spaces that can be described as extrapolation spaces). Let $w : (0,1) \rightarrow (0,\infty)$ and define

$$\langle \vec{X} \rangle_{w,q;K} = \{f \in X_0 + X_1 : \|x\|_{\langle \vec{X} \rangle_{w,q;K}} := \left\{ \int_0^1 (w(s)K(s,f;\vec{X}))^q \frac{ds}{s} \right\}^{1/q} < \infty\}.$$

For example, if $w_\theta(s) = s^{-\theta}$, $0 \leq \theta \leq 1$, (cf. (13.4))

$$\langle \vec{X} \rangle_{w_\theta,q;K} := \langle \vec{X} \rangle_{\theta,q;K} = \{f \in X_0 + X_1 : \|x\|_{\langle \vec{X} \rangle_{\theta,q;K}} := \Phi_{\theta,q}(\chi_{(0,1)}K(s,f;\vec{X})) < \infty\},$$

Then, for every $p_0 > 1$ and $\alpha > 0$,

$$\begin{aligned} \sum_{1 < p < p_0} \frac{p^\alpha (L^1, L^\infty)_{1/p',1;J}^\blacktriangleleft}{(p-1)^\alpha} &= \{f : \int_0^1 f^{**}(s) \left(\log \frac{1}{s}\right)^{\alpha-1} ds < \infty\} \\ (3.5) \qquad \qquad \qquad &= L(\text{Log}L)^\alpha. \end{aligned}$$

Thus,

$$\sum_{1 < p < p_0} \frac{p^\alpha (L^1, L^\infty)_{1/p',1;J}^\blacktriangleleft}{(p-1)^\alpha} = \langle L^1, L^\infty \rangle_{(\log \frac{1}{s})^{\alpha-1}, 1;K},$$

and in the special case $\alpha = 1$,

$$\sum_{1 < p < p_0} \frac{p (L^1, L^\infty)_{1/p',1;J}^\blacktriangleleft}{p-1} = \langle L^1, L^\infty \rangle_{w_0,1;K} := \langle L^1, L^\infty \rangle_{0,1;K}.$$

Problem 3.1. The characterization of $L(\text{Log}L)^\alpha$ as an extrapolation space for the \sum -method given by (3.5) leads to the following reiteration formulae, which for simplicity we formulate for $\alpha = 1$:

$$\langle L^1, L^\infty \rangle_{0,1;K} = \langle L^1, L^p \rangle_{0,1;K} = \langle L^1, \langle L^1, L^\infty \rangle_{1/p',1;K} \rangle_{0,1;K}.$$

More generally, the formulae can be stated for ordered Banach pairs. We ask, for conditions on κ, μ , quasi-concave functions for the validity of

$$\langle L^1, L^\infty \rangle_{\mu,1;K} = \langle L^1, \langle L^1, L^\infty \rangle_{\kappa,1;K} \rangle_{\mu,1;K}.$$

4. JAWERTH-MILMAN MEET CALDERÓN: K AND J FUNCTIONALS FOR SCALES OF SPACES AND THEIR RÔLE IN EXTRAPOLATION

The results of the previous sections point to a connection between Extrapolation and Calderón's theory of weak type interpolation. This will be the topic of our discussion in this section. We have reorganized the results of [43] by means of introducing the concept of K -functional for scales of interpolation spaces. This approach to the results of [43] shows more clearly the connection with Calderón's theory, leads to a cleaner presentation and makes it easier to formulate Problems.

Let us recall that Calderón [19] developed methods to characterize weak type interpolation inequalities via rearrangement inequalities and he used it with great success to characterize the corresponding interpolation spaces (cf. [15], [18]). It is somewhat less well known that Calderón also understood that one could formulate

the results through the use of K -functionals, as was explicitly displayed in the thesis of his Ph.D. student E. Oklander ([61], [62]). In a nutshell, the idea, expressed in modern terminology, is simply a consequence of

$$(4.1) \quad T : \vec{X} \rightarrow \vec{Y} \text{ with norm } M \Leftrightarrow \text{ for all } t > 0, K(t, Tf; \vec{Y}) \leq MK(t, f; \vec{X}).$$

In particular, when dealing with weak type interpolation the abstract formulation can be made very explicit. Indeed, if we let $\vec{X} = (L(p_0, 1), L(p_1, 1))$ and $\vec{Y} = (L(q_0, \infty), L(q_1, \infty))$, the corresponding K -functionals are known. In fact, this result was already contained in Oklander [61, Theorem 3, page 51; and Theorem 4, page 52], [62], where these K -functionals are computed exactly, and was also done by Sharpley [67, Lemma 6.8, page 504], in the slightly more general context of general Lorentz spaces¹⁸ and Marcinkiewicz spaces. As a consequence, in all these cases one can obtain an explicit characterization which, in turn, can be reformulated in terms of what nowadays is called the Calderón operator. We refer to Sharpley [67, Theorem 6.9, page 511], Bennett-Sharpley [15] and the references therein.

Returning to Yano's theorem, as we have seen in the previous Section, the natural formulation of the result is in terms of weak type operators. More generally, we are led to consider the following problem in the setting of real interpolation scales. Suppose that \vec{X} and \vec{Y} are Banach pairs and suppose that T satisfies

$$(4.2) \quad T : \vec{X}_{\theta,1;J}^{\blacktriangleleft} \rightarrow \vec{Y}_{\theta,\infty;K}^{\blacktriangleleft}, \text{ with } \|T\|_{\vec{X}_{\theta,1;J}^{\blacktriangleleft} \rightarrow \vec{Y}_{\theta,\infty;K}^{\blacktriangleleft}} \leq M(\theta), \text{ for all } \theta \in (0, 1).$$

Problem 4.1. Provide an intrinsic characterization of (4.2).

Jawerth-Milman [43] took up this problem and formulated the solution as an extension of Calderón's characterization of weak type interpolation (4.1). We now reformulate their solution introducing the notion of K -functional for a scale of real interpolation spaces¹⁹. In fact, formally we can consider the K -functional for any scale of interpolation spaces. Maybe, it is worth to stress that the more important issue here is the dependence of the K -functional on the weight $M(\theta)$. Indeed, if a scale of interpolation functors is complete, then when $M(\theta) = 1$ we recover the usual K -functional (cf. Remark 4.4 below).

¹⁸For a concave function ϕ let

$$\Lambda_\phi = \{f : \|f\|_{\Lambda_\phi} = \int_0^\infty f^*(s) d\phi(s) < \infty\},$$

then if ϕ_1, ϕ_2 are concave functions we have

$$K(t, f; \Lambda_{\phi_1}, \Lambda_{\phi_2}) = \int_0^\infty f^*(s) d \min(\phi_1(s), t\phi_2(s)).$$

In particular,

$$\langle \Lambda_{\phi_1}, \Lambda_{\phi_2} \rangle_{0,1;K} = \{f : \int_0^1 \int_0^\infty f^*(s) \left(\frac{d}{ds} \min(\phi_1(s), t\phi_2(s)) \right) \frac{dt}{t}.$$

¹⁹ K -functionals for many spaces or for families of spaces have been defined and studied before but there were hardly ever explicitly computed (cf. [23]). The special setting of extrapolation allows us to do explicit computations

Definition 4.2. Let $\{\rho_\theta\}_{\theta \in (0,1)}$ be a family of (quasi)-concave functions, and let $\{F_{\rho_\theta}\}_{\theta \in (0,1)}$ be a family of interpolation functors such that the characteristic function (cf. Definition 2.9) of each F_{ρ_θ} is ρ_θ^{20} , $\theta \in (0,1)$. Let M be a weight, that is $M : (0,1) \rightarrow (0,\infty)$. Then, for any Banach pair \vec{X} we let

$$K(t, f; \{M(\theta)F_{\rho_\theta}(\vec{X})\}) := \|f\|_{\sum_\theta \rho_\theta(t)M(\theta)F_{\rho_\theta}(\vec{X})}.$$

Remark 4.3. In particular, when dealing with a family of interpolation methods $\{F_{t^\theta}\}_{\theta \in (0,1)}$ of exact type θ , then we shall usually write F_θ instead of F_{t^θ} , and therefore in this case we have

$$K(t, f; \{M(\theta)F_\theta(\vec{X})\}) = \|f\|_{\sum_\theta t^\theta M(\theta)F_\theta(\vec{X})}.$$

Remark 4.4. Note that if $M(\theta) \equiv 1$, then $K(t, f; F_\theta(\vec{X}))$ is essentially $K(t, f; \vec{X})$ (cf. [43, page 25, formula (3.9)]).

Likewise, we can introduce the concept of J -functional for a scale of spaces.

Definition 4.5. Let $\{\rho_\theta\}_{\theta \in (0,1)}$ be a family of (quasi)-concave functions, and let $\{F_{\rho_\theta}\}_{\theta \in (0,1)}$ be a family of interpolation functors such that the characteristic function of each F_{ρ_θ} is ρ_θ , $\theta \in (0,1)$. Then, for any Banach pair \vec{X} and a given weight M , we let

$$J(t, f; \{M(\theta)F_{\rho_\theta}(\vec{X})\}) := \|f\|_{\Delta(\rho_\theta(t)M(\theta)F_{\rho_\theta}(\vec{X}))}.$$

Theorem 4.6 (cf. [43]). *Let \vec{X} and \vec{Y} be mutually closed Banach pairs and let $M(\theta)$ be a tempered weight. Then (4.2) holds if and only if there exists a constant $c > 0$ such that*

$$(4.3) \quad K(t, Tf; \{\vec{Y}_{\theta, \infty; K}^\blacktriangleleft\}) \leq cK(t, f; \{M(\theta)\vec{X}_{\theta, 1; J}^\blacktriangleleft\}), \quad t > 0.$$

In fact, we can also write down a “Calderón operator type formulation” of this result by making explicit the K -functionals for scales that are involved. The extrapolation version of Calderón’s result then reads: (4.3) holds if and only if²¹

$$(4.4) \quad K(t, Tf, \vec{Y}^\circ) \leq c \int_0^\infty K\left(\frac{t}{r}, f; \vec{X}\right) d\mu(r),$$

where

$$(4.5) \quad \tau(r) = \inf_\theta \{r^\theta M(\theta)\}, \quad r > 0,$$

and μ is the representing measure of τ :

$$(4.6) \quad \tau(x) = \int_0^\infty \min\{1, \frac{x}{r}\} d\mu(r), \quad x > 0.$$

²⁰For example the functors $\vec{X} \rightarrow \vec{X}_{\rho_\theta, q; J}^\blacktriangleleft$ or $\vec{X} \rightarrow \vec{X}_{\rho_\theta, q; K}^\blacktriangleleft$.

²¹For a Banach pair \vec{Y} we let Y_i° to be the closure of $Y_0 \cap Y_1$ in Y_i .

Let us recall the details. Suppose that (4.3) holds. Recall that with absolute constants we can write (cf. [43])

$$\begin{aligned} K(t, Tf; \{\vec{Y}_{\theta, \infty; K}^{\blacktriangleleft}\}) &\approx K(t, Tf; \{\vec{Y}_{\theta, 1; J}^{\blacktriangleleft}\}) \\ &= \|Tf\|_{\sum_{\theta} t^{\theta} \vec{Y}_{\theta, 1; J}^{\blacktriangleleft}} \\ &\approx K(t, Tf, \vec{Y}^{\circ}). \end{aligned}$$

It remains to establish the equivalence between the right hand side of (4.4) and

$$K(t, f; \{M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft}\}) := \|f\|_{\sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft}}.$$

To compute the indicated norm on the right hand side observe that the characteristic function, $C_{\theta}(s)$ say, of $t^{\theta} M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft}$, is given by $C_{\theta}(s) = \frac{s^{\theta}}{t^{\theta} M(\theta)}$, and therefore by Theorem 2.10, for each $t > 0$,

$$(4.7) \quad \sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft} = \vec{X}_{\rho_t, 1; J}$$

where $\rho_t(s) = \sup_{0 < \theta < 1} \{\frac{s^{\theta}}{t^{\theta} M(\theta)}\}$. Consequently, rewriting $\rho_t(s)$ in terms of τ (cf. (4.5)), we have

$$\begin{aligned} \|f\|_{\sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft}} &= \|f\|_{\vec{X}_{\rho_t, 1; J}} = \inf_{f = \int_0^{\infty} u(s) \frac{ds}{s}} \left\{ \int_0^{\infty} \frac{J(s, u(s); \vec{X})}{\rho_t(s)} \frac{ds}{s} \right\} \\ &= \inf_{f = \int_0^{\infty} u(s) \frac{ds}{s}} \left\{ \int_0^{\infty} J(s, u(s); \vec{X}) \tau\left(\frac{t}{s}\right) \frac{ds}{s} \right\} \\ (4.8) \quad &= \inf_{f = \int_0^{\infty} u(s) \frac{ds}{s}} \left\{ \int_0^{\infty} \int_0^{\infty} J(s, u(s); \vec{X}) \min\{1, \frac{t}{rs}\} d\mu(r) \frac{ds}{s} \right\}. \end{aligned}$$

Using the strong form of the fundamental Lemma, we can find a special decomposition $f = \int_0^{\infty} u_f(s) \frac{ds}{s}$, such that, with constants independent of f , we have

$$K(t, f; \vec{X}) \approx \int_0^{\infty} J(s, u_f(s); \vec{X}) \min\{1, \frac{t}{s}\} \frac{ds}{s}.$$

Therefore, combining with (4.8) we obtain

$$\begin{aligned} \|f\|_{\sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta, 1; J}^{\blacktriangleleft}} &\leq \int_0^{\infty} \int_0^{\infty} J(s, u_f(s); \vec{X}) \min\{1, \frac{t}{rs}\} d\mu(r) \frac{ds}{s} \\ &= \int_0^{\infty} \int_0^{\infty} J(s, u_f(s); \vec{X}) \min\{1, \frac{t}{rs}\} \frac{ds}{s} d\mu(r) \\ &\leq \int_0^{\infty} K\left(\frac{t}{r}, f; \vec{X}\right) d\mu(r). \end{aligned}$$

The last inequality can be reversed. In preparation to prove this claim we let

$$N(\theta) := \int_0^{\infty} r^{-\theta} d\mu(r), \theta \in (0, 1),$$

where μ is the representing measure of τ (cf. (4.6)).

It is shown in [43, Lemma 3.9] that if $M(\theta)$ is tempered then with constants independent of θ ,

$$N(\theta) \preceq M(\theta).$$

We will also use the fact that $\sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta,1;J}^{\blacktriangleleft} = \sum_{\theta} t^{\theta} M(\theta) \vec{X}_{\theta,\infty;K}^{\blacktriangleleft}$. Now, for each $\theta \in (0, 1)$, and for each $f \in \vec{X}_{\theta,\infty;K}^{\blacktriangleleft}$, we have

$$\begin{aligned} \int_0^{\infty} K\left(\frac{t}{r}, f; \vec{X}\right) d\mu(r) &= \int_0^{\infty} \left(K\left(\frac{t}{r}, f; \vec{X}\right) \left(\frac{t}{r}\right)^{-\theta} \left(\frac{t}{r}\right)^{\theta} d\mu(r)\right) \\ &\leq \int_0^{\infty} \left(\sup_x \{K(x, f; \vec{X}) x^{-\theta}\}\right) \left(\frac{t}{r}\right)^{\theta} d\mu(r) \\ &= t^{\theta} \|f\|_{\vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \int_0^{\infty} r^{-\theta} d\mu(r) \\ &= N(\theta) t^{\theta} \|f\|_{\vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \\ &\preceq M(\theta) t^{\theta} \|f\|_{\vec{X}_{\theta,\infty;K}^{\blacktriangleleft}}. \end{aligned}$$

Let $f = \sum f_{\theta}$ be a decomposition such that

$$\sum \|f_{\theta}\|_{M(\theta) t^{\theta} \vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \approx \|f\|_{\sum M(\theta) t^{\theta} \vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \approx \|f\|_{\sum t^{\theta} M(\theta) \vec{X}_{\theta,1;J}^{\blacktriangleleft}}.$$

Then formally (by Fatou's Lemma),

$$\begin{aligned} \int_0^{\infty} K\left(\frac{t}{r}, f; \vec{X}\right) d\mu(r) &= \int_0^{\infty} K\left(\frac{t}{r}, \sum f_{\theta}; \vec{X}\right) d\mu(r) \\ &\leq \sum \int_0^{\infty} K\left(\frac{t}{r}, f_{\theta}; \vec{X}\right) d\mu(r) \\ &\preceq \sum_{\theta} M(\theta) t^{\theta} \|f_{\theta}\|_{\vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \\ &\approx \|f\|_{\sum t^{\theta} M(\theta) \vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} \\ &\approx \|f\|_{\sum t^{\theta} M(\theta) \vec{X}_{\theta,1;J}^{\blacktriangleleft}}, \end{aligned}$$

concluding the proof of the equivalence between (4.3) and (4.4).

Problem 4.7. More generally carry out explicit calculations of the K -functionals for interpolation scales of interpolation functors $\{F_{\rho_{\theta}}\}_{\theta \in I}$.

Remark 4.8. In connection with the previous Problem it is important to recall the concept of “complete families of interpolation functors” introduced in [43]. We say that a family $\{F_{\rho_{\theta}}\}_{\theta \in I}$ is *complete* if there exists a constant $C > 0$ such that

$$\inf_{\theta \in I} \left\{ \frac{\rho_{\theta}(t)}{\rho_{\theta}(s)} \right\} \leq C \min\left\{1, \frac{t}{s}\right\}, s, t > 0.$$

In this context the import of this notion is explained in the next Example.

Example 4.9. Suppose that $\{F_{\rho_\theta}\}_{\theta \in I}$ is complete. Then, if \vec{X} is mutually closed,

$$K(t, f; \{F_{\rho_\theta}(\vec{X})\}) := \|f\|_{\sum_{\theta} \rho_\theta(t) F_{\rho_\theta}(\vec{X})} \approx K(t, f; \vec{X}^\circ)$$

and

$$J(t, f; \{F_{\rho_\theta}(\vec{X})\}) := \|f\|_{\Delta_\theta(\rho_\theta(t) F_{\rho_\theta}(\vec{X}))} \approx J(t, f; \vec{X}^\circ)$$

(cf. [43, page 15: formulae in line 9 and (2.15)], [8, Theorem 2.2].)

Problem 4.10. Let $\{\rho_\theta\}_{\theta \in I}$ be a family of quasi-concave functions, let $\{F_{\rho_\theta}\}_{\theta \in I}$ be a complete family of interpolation functors and let $M(\theta)$ be a tempered weight. Compute $K(t, f; \{M(\theta)F_{\rho_\theta}(\vec{X})\})$ and $J(t, f; \{M(\theta)F_{\rho_\theta}(\vec{X})\})$.

Remark 4.11. When dealing with ordered pairs we can replace the usual real interpolation spaces by the modified ones $\langle \vec{X} \rangle_{\theta, 1; J}^\blacktriangleleft$; $\langle \vec{X} \rangle_{\theta, q(\theta); K}^\blacktriangleleft$, etc. (cf. [43], [11]).

Example 4.12. Let us consider the weak type Yano condition: $T : L(p, 1) \subset L^p \rightarrow L^p \subset L(p, \infty)$ with $\|T\|_{L(p, 1) \rightarrow L(p, \infty)} \leq cp$, $p > p_0 > 1$. Here all spaces are based on $[0, 1]$. Then²²,

$$\langle L^1, L^\infty \rangle_{1/p', \infty; K}^\blacktriangleleft = L(p, \infty) := \{f : \|f\|_{L(p, \infty)} := \sup_{0 < t < 1} \{f^{**}(t)t^{1/p}\} < \infty\},$$

$L^p = \langle L^1, L^\infty \rangle_{1/p', p; K}^\blacktriangleleft$, and since $\langle L^1, L^\infty \rangle_{1/p', 1; J}^\blacktriangleleft \subset \langle L^1, L^\infty \rangle_{1/p', p; K}^\blacktriangleleft$, we have (cf. (4.3) and (4.4))

$$\begin{aligned} K(t, Tf; \{\langle L^1, L^\infty \rangle_{1/p', \infty; K}\}_{1 < p < \infty}) &\preceq K(t, f; \{p \langle L^1, L^\infty \rangle_{1/p', 1; J}\}_{1 < p < \infty}) \\ &\approx \int_0^\infty K\left(\frac{t}{r}, f; L^1, L^\infty\right) w(r) dr, \end{aligned}$$

where $w(r)dr$ is the measure representing the function

$$\tau(t) = \inf_{p > 1} \{pt^{1-1/p}\} = t \inf_{p > 1} \{pt^{-1/p}\} = t \inf_{0 < u < 1} \left\{ \frac{e^{u \log \frac{1}{t}}}{u} \right\} := t \inf_{0 < u < 1} g(u).$$

We can compute directly $\frac{dg(u)}{du} = \frac{g(u)}{u}(u \log \frac{1}{t} - 1)$, and

$$\frac{d^2 g(u)}{du^2} = g'(u) \frac{(u \log \frac{1}{t} - 1)}{u} + \frac{g(u)}{u^2} = \frac{g(u)}{u^2} \left((u \log \frac{1}{t} - 1)^2 + 1 \right).$$

Then, for $t < 1/e$, $u_* = \frac{1}{\log \frac{1}{t}} \in (0, 1)$, $g'(u_*) = 0$, $g''(u_*) > 0$ and we see that for $t < 1/e$, $\tau(t) = et \log \frac{1}{t}$. If $t \leq 1/e$, then $g'(u) \leq 0$, and $g(u)$ decreases. Hence, $\tau(t) = 1$ in this range. So all in all,

$$\begin{aligned} \tau(t) &= et \log \frac{1}{t} \chi_{(0, 1/e)}(t) + \chi_{(1/e, \infty)}(t), \\ w(t) &\approx -t\tau''(t) = \chi_{(0, 1/e)}(t). \end{aligned}$$

Consequently,

$$\int_0^\infty K\left(\frac{t}{r}, f; L^1, L^\infty\right) w(r) dr = \int_0^{1/e} K\left(\frac{t}{r}, f; L^1, L^\infty\right) dr = t \int_{et}^\infty K(u, f; L^1, L^\infty) \frac{du}{u^2}.$$

²² $f^{**}(t) = \frac{1}{t} \int_0^t f^*(s) ds$

Thus, for $0 < t < 1$, we can write

$$\begin{aligned} (Tf)^{**}(t) &\leq \int_{et}^e \frac{K(u, f; L^1, L^\infty)}{u} \frac{du}{u} + \int_e^\infty K(u, f; L^1, L^\infty) \frac{du}{u^2} \\ &\leq f^{**}(et) \log \frac{1}{t} + \|f\|_{L^1} e^{-1}. \end{aligned}$$

Therefore,

$$(4.9) \quad \frac{(Tf)^{**}(t)}{\log \frac{2}{t}} \leq f^{**}(et) + \frac{\|f\|_{L^1} e^{-1}}{\log \frac{2}{t}}$$

and we obtain

$$\begin{aligned} \sup_{t \in (0,1)} \frac{(Tf)^{**}(t)}{\log \frac{2}{t}} &\leq \|f\|_{L^\infty} + \frac{\|f\|_{L^1}}{\log 2} \\ &\leq \|f\|_{L^\infty}. \end{aligned}$$

But (4.9) implies more general results. Indeed, let X be a rearrangement invariant (briefly, r.i.) space²³ such that the Hardy operator $Pf(t) = \frac{1}{t} \int_0^t f(s)ds$, is bounded on X , and let $X(\log^{-1})$ be defined by the norm (cf. [6])

$$\|f\|_{X(\log^{-1})} = \left\| \frac{f^{**}(t)}{\log \frac{2}{t}} \right\|_X.$$

Then applying the X -norm to (4.9) we obtain the extrapolation theorem (cf. [6, Theorem 4.3]):

$$T : X \rightarrow X(\log^{-1}).$$

Example 4.13. A similar result can be obtained if we consider operators such that

$$\|T\|_{L(p,1) \rightarrow L(p,\infty)} \leq cp^\alpha \text{ as } p \rightarrow \infty,$$

where $\alpha > 0$. Indeed, in this case we have $\tau(t) = t \inf_{p>1} \{p^\alpha t^{-1/p}\}$, $0 < t < 1$. A computation shows that $\tau(t) \approx t(\log \frac{1}{t})^\alpha$ near zero, and

$$w(t) \approx -t\tau''(t) \approx (\log \frac{1}{t})^{\alpha-1} \text{ as } t \rightarrow 0.$$

which leads to estimates of the form (cf. [43])

$$(Tf)^{**}(t) \leq \int_{et}^e \frac{K(u, f; L^1, L^\infty)}{u} (\log \frac{u}{t})^{\alpha-1} \frac{du}{u}, \quad 0 < t < 1.$$

Example 4.14. (Yano type extrapolation theorem) The result of Example 4.12 can be extended to scales. We illustrate this considering deteriorating norms when $p \rightarrow 1$. Let \vec{A}, \vec{B} be mutually closed ordered Banach pairs. Let X be a r.i. space on $(0,1)$. Suppose that T is a bounded operator, $T : \vec{A}_{\theta,1;J}^\blacktriangleleft \rightarrow \vec{B}_{\theta,\infty;K}^\blacktriangleleft$, with $\|T\|_{\vec{A}_{\theta,1;J}^\blacktriangleleft \rightarrow \vec{B}_{\theta,\infty;K}^\blacktriangleleft} \leq \theta^{-1}$, $\theta \in (0,1)$. Then,

$$\left\| \frac{d}{dt} (K(t, Tf; \vec{B})) \right\|_X \leq c \left\| \frac{K(t, f; \vec{A})}{t} \right\|_X.$$

²³For the definition see e.g. [15].

In particular, if $\vec{A} = \vec{B} = (L^1(0, 1), L^\infty(0, 1))$ and $T : L(p, 1) \rightarrow L(p, \infty)$, with norm less or equal than $cp' = \frac{cp}{p-1}$, $1 < p < \infty$, then

$$\|(Tf)^*\|_X \leq c \|f^{**}\|_X.$$

Yano's theorem (i) corresponds to the case $X = L^1$.

Proof. By extrapolation, via the K -functional for scales, we have

$$K(t, Tf; \{\vec{B}_{\theta, \infty; K}^\blacktriangleleft\}) \leq cK(t, f; \{\theta^{-1}\vec{A}_{\theta, 1; J}^\blacktriangleleft\}).$$

Therefore, by a calculation similar to that of Example 4.12 (cf. [43]), there exists an absolute constant $c > 0$, such that

$$K(t, Tf; \vec{B}) \leq c \int_0^t K(s, f; \vec{A}) \frac{ds}{s}.$$

Since $\lim_{t \rightarrow 0} K(t, Tf; \vec{B}) = 0$, we can rewrite the last inequality as

$$\int_0^t \frac{d}{ds}(K(s, Tf; \vec{B})) ds \leq c \int_0^t K(s, f; \vec{A}) \frac{ds}{s}.$$

Therefore, since $\frac{d}{ds}(K(s, Tf; \vec{B}))$ and $\frac{K(s, f\vec{A})}{s}$ are decreasing then, by the Calderón-Mityagin principle for any r.i. space we have

$$\left\| \frac{d}{dt}(K(t, Tf; \vec{B})) \right\|_X \leq c \left\| \frac{K(t, f; \vec{A})}{t} \right\|_X,$$

as desired. \square

Example 4.15. (Rearrangement inequalities associated to Yano type extrapolation) Let \vec{A}, \vec{B} be mutually closed Banach pairs. Suppose that T is a bounded operator, $T : \vec{A}_{\theta, 1; J}^\blacktriangleleft \rightarrow \vec{B}_{\theta, \infty; K}^\blacktriangleleft$, with $\|T\|_{\vec{A}_{\theta, 1; J}^\blacktriangleleft \rightarrow \vec{B}_{\theta, \infty; K}^\blacktriangleleft} \leq \theta^{-1}(1 - \theta)^{-1}$, $\theta \in (0, 1)$. Then, there exists a constant $c > 0$ such that

$$(4.10) \quad K(t, Tf; \vec{B}) \leq c \left(\int_0^t K(s, f; \vec{A}) \frac{ds}{s} + t \int_t^\infty K(s, f; \vec{A}) \frac{ds}{s^2} \right).$$

In particular, if $\vec{A} = \vec{B} = (L^1, L^\infty)$, we have $\vec{A}_{1/p', 1; J}^\blacktriangleleft = L(p, 1)$, $\vec{B}_{1/p', \infty; K}^\blacktriangleleft = L(p, \infty)$, $1 < p < \infty$. Then (4.10) can be written as

$$(4.11) \quad t(Tf)^{**}(t) \leq c \left(\int_0^t f^{**}(s) ds + t \int_t^\infty f^{**}(s) \frac{ds}{s} \right).$$

For the Hilbert transform (4.11) is a classical inequality due to O'Neil-Weiss [63]. In the same paper O'Neil-Weiss also prove (4.11) for Calderón-Zygmund operators, a result which they credit to A. Calderón and E. Stein. In [19, Appendix], Calderón extended these results to operators T that, together with their adjoints, are of weak type $(1, 1)$ and strong type $(2, 2)$. It has been pointed out by several authors, for example, Bennett-Rudnick (cf. [15, Theorem 4.7, page 134]), Jawerth-Milman [43, Proposition 5.2.2, page 50], Semenov [66]), that for the Hilbert transform or C-Z operators, one can improve (4.11) by means of replacing ** by * throughout, using the fact that C-Z singular integral operators are of weak type

(1,1). The weak type (1,1) assumption apparently cannot be dispensed with, and in general does not follow from the assumptions $\|T\|_{L(p,1) \rightarrow L(p,\infty)} \preceq \frac{p^2}{p-1}$, $p \in (1, \infty)$ (cf. [43, First paragraph, page 2], [6, Remark 4.5], [35]). In [14, Theorem 2.2, page 603], the authors define operators of weak type (∞, ∞) as those that map L^∞ into $L(\infty, \infty)$, and using this definition show that operators of weak type (1,1) and (∞, ∞) satisfy (4.11). Earlier, in [27], DeVore-Riemenschneider-Sharpley define an abstract notion of generalized weak type (1,1), (∞, ∞) by asking that condition (4.10) be satisfied. Therefore, the notion of weak type (1,1), (∞, ∞) of DeVore-Riemenschneider-Sharpley is equivalent to the assumption $T : \vec{A}_{\theta,1;J}^\blacktriangleleft \rightarrow \vec{B}_{\theta,\infty;K}^\blacktriangleleft$, with $\|T\|_{\vec{A}_{\theta,1;J}^\blacktriangleleft \rightarrow \vec{B}_{\theta,\infty;K}^\blacktriangleleft} \preceq \theta^{-1}(1-\theta)^{-1}$, $\theta \in (0, 1)$.

One can prove (4.10) via the K -functional for scales. Equivalently, we note (cf. Example 9.4 below) that for

$$\tau(t) = \inf_{0 < \theta < 1} \{ \theta^{-1}(1-\theta)^{-1} t^\theta \}$$

we have

$$\tau(t) \approx \int_0^\infty \min(1, \frac{t}{n}) \min(1, n) \frac{dn}{n}.$$

Then, since under current assumptions T satisfies the K/J inequality (cf. the next section)

$$K(t, Tf; \vec{B}) \leq c\tau(t/s)J(s, f; \vec{A}), \quad s, t > 0,$$

it follows that selecting a decomposition of $f = \int_0^\infty u(s) \frac{ds}{s}$, such that

$$\int_0^\infty \min(1, \frac{t}{s}) J(s, u(s); \vec{A}) \frac{ds}{s} \leq cK(t, f; \vec{A})$$

leads to

$$\begin{aligned} K(t, Tf; \vec{B}) &\leq c \int_0^\infty \min(1, n) \int_0^\infty \min(1, \frac{t}{sn}) J(s, u(s); \vec{A}) \frac{ds}{s} \frac{dn}{n} \\ &= c \int_0^1 K(\frac{t}{n}, f; \vec{A}) \min(1, n) \frac{dn}{n} \\ &= c \left(\int_0^1 K(\frac{t}{n}, f; \vec{A}) dn + \int_1^\infty K(\frac{t}{n}, f; \vec{A}) \frac{dn}{n} \right) \\ &= c \left(t \int_t^\infty K(s, f; \vec{A}) \frac{ds}{s^2} + \int_0^t K(s, f; \vec{A}) \frac{ds}{s} \right), \end{aligned}$$

as we wished to show.

5. K/J INEQUALITIES AND EXTRAPOLATION

In extrapolation we start from a family of inequalities for a given operator, but we do not have *a priori* end point spaces where the estimates are valid. It is then natural to collect estimates on elements that belong to the intersection of all the domain spaces on which the operators act and use this information to derive a basic inequality. This is the idea of the K/J inequalities of [43], a particular case of which we now review.

Indeed, it turns out to be useful for further developments (e.g. Bilinear Extrapolation treated in Section 9) to bring to the forefront some of the key arguments underlying the proof of the equivalence between (5.8) and (5.9). This leads in two steps to the K/J inequalities: (i) Associated to the extrapolation information there is a natural concave function that allows one to establish an inequality between the K - and J -functionals; and (ii) Extend this inequality to a K -functional inequality using the strong form of the fundamental Lemma of Interpolation. As it turns out this approach is essentially equivalent to the use of the K -functional of scales discussed in the previous section, as we shall now show.

It is instructive to see how K/J inequalities arise in classical setting of interpolation theory. Let \vec{A}, \vec{B} , be Banach pairs, and suppose that $T : \vec{A} \rightarrow \vec{B}$ is a bounded operator. Then,

$$(5.1) \quad K(t, Tf; \vec{B}) \preceq K(t, f; \vec{A}), \quad t > 0,$$

which combined with the elementary inequality (cf. [16, Lemma 3.2.1])

$$(5.2) \quad K(t, f; \vec{A}) \leq \min(1, \frac{t}{s}) J(s, f; \vec{A}), \quad t, s > 0$$

leads to the “mother” of all K/J inequalities

$$(5.3) \quad K(t, Tf; \vec{B}) \preceq \min(1, \frac{t}{s}) J(s, f; \vec{A}), \quad t, s > 0.$$

Conversely, if (5.3) holds then we can return to (5.1) via the strong form of the fundamental Lemma, as we now explain. Select a decomposition $f = \int_0^\infty u_f(s) \frac{ds}{s}$, such that $\int_0^\infty \min(1, \frac{t}{s}) J(s, u_f(s); \vec{A}) \frac{ds}{s} \leq c K(t, f; \vec{A})$. Then,

$$(5.4) \quad \begin{aligned} K(t, Tf; \vec{B}) &\leq c \int_0^\infty K(t, Tu_f(s); \vec{B}) \frac{ds}{s} \\ &\leq c \int_0^\infty \min(1, \frac{t}{s}) J(s, u_f(s); \vec{A}) \frac{ds}{s} \\ &\leq c K(t, f; \vec{A}). \end{aligned}$$

Now we turn to the context of extrapolation. Here (5.3) is not available but there is natural concave function that will replace $\phi(u) = \min\{1, u\}$ (cf. Example 5.1 below). We develop this point in detail.

Let \vec{A}, \vec{B} , be mutually closed Banach pairs, and let F_θ, G_θ , be exact interpolation functors of type θ . Consider a bounded operator $T : F_\theta(\vec{A}) \rightarrow G_\theta(\vec{B})$, with norm $M(\theta), \theta \in (0, 1)$. Now (5.2) is not available to us anymore and we search for a substitute. For this purpose we note that for all $\theta \in (0, 1)$, $\vec{A}_{\theta,1;J}^\blacktriangleleft \stackrel{1}{\subset} F_\theta(\vec{A})$, and $G_\theta(\vec{B}) \stackrel{1}{\subset} \vec{B}_{\theta,\infty;K}^\blacktriangleleft$, consequently, for all $s, t > 0$,

$$(5.5) \quad K(t, Tf; \vec{B}) t^{-\theta} \leq \|Tf\|_{\vec{B}_{\theta,\infty;K}^\blacktriangleleft} \leq M(\theta) \|f\|_{\vec{A}_{\theta,1;J}^\blacktriangleleft} \leq M(\theta) s^{-\theta} J(s, f; \vec{A}),$$

where the rightmost inequality is elementary (cf. [16, Theorem 3.11.2 (4), page 64]).

Hence, we arrive to

$$(5.6) \quad \begin{aligned} K(t, Tf; \vec{B}) &\leq \inf_{\theta} \{M(\theta) \left(\frac{t}{s}\right)^{\theta}\} J(s, f; \vec{A}) \\ &= \phi\left(\frac{t}{s}\right) J(s, f; \vec{A}), s, t > 0, \end{aligned}$$

where $\phi(u) = \inf_{\theta \in (0,1)} \{M(\theta)u^{\theta}\}$. Inequalities (5.6) and (5.3) are examples of K/J inequalities. Let Ψ be a concave function, such that $\lim_{t \rightarrow 0} \Psi(t) = \lim_{t \rightarrow \infty} \frac{\Psi(t)}{t} = 0$, e.g. $\Psi(u) = \min(1, u)$, then we say that T satisfies a $\Psi - K/J$ inequality if there exists a constant $c > 0$ such that²⁴

$$K(t, Tf; \vec{B}) \leq c\Psi\left(\frac{t}{s}\right) J(s, f; \vec{A}), s, t > 0.$$

When Ψ is understood we simply drop it and talk about K/J inequalities.

Example 5.1. Note that if $M(\theta) \equiv 1$, then $\inf_{\theta \in (0,1)} \{u^{\theta}\} = \inf_{\theta \in (0,1)} \{e^{\theta \log u}\} = \min\{1, u\}$, and we are “back to interpolation”.

Then, when properly interpreted the equivalence between K/J inequalities and K -functional inequalities persists in the setting of extrapolation theory.

Theorem 5.2. Let \vec{A}, \vec{B} , be mutually closed Banach pairs, and let F_{θ}, G_{θ} , be exact interpolation functors of type θ . Consider a bounded operator $T : F_{\theta}(\vec{A}) \rightarrow G_{\theta}(\vec{B})$, with norm $M(\theta), \theta \in (0, 1)$. Then, the following are equivalent:

(i) There exists $c > 0$ such that

$$(5.8) \quad K(t, Tf; \{\vec{A}_{\theta, \infty; K}^{\blacktriangleleft}\}_{\theta \in (0,1)}) \leq cK(t, f; \{M(\theta)\vec{B}_{\theta, 1; J}^{\blacktriangleleft}\}_{\theta \in (0,1)}).$$

²⁴These inequalities can be formulated without the use of the J -functional. Indeed, note that since (5.6) is valid for all $s > 0$, then choosing $s = \frac{\|f\|_{A_0}}{\|f\|_{A_1}}$, gives $J(s, f; \vec{A}) = \|f\|_{A_1} \max\{\frac{\|f\|_{A_0}}{\|f\|_{A_1}}, s\} = \|f\|_{A_0}$ and (5.6) implies

$$(5.7) \quad K(t, Tf; \vec{B}) \leq \|f\|_{A_0} \phi\left(t \frac{\|f\|_{A_1}}{\|f\|_{A_0}}\right).$$

Conversely, suppose that (5.7) holds. Then, since for any $s > 0$, $\|f\|_{A_0} \leq J(s, f; \vec{A})$, and $\frac{\phi(u)}{u}$ decreases, we have

$$\frac{\phi\left(t \frac{\|f\|_{A_1}}{\|f\|_{A_0}}\right)}{t \frac{\|f\|_{A_1}}{\|f\|_{A_0}}} \leq \frac{\phi\left(t \frac{\|f\|_{A_1}}{J(s, f; \vec{A})}\right)}{t \frac{\|f\|_{A_1}}{J(s, f; \vec{A})}}.$$

Consequently,

$$\|f\|_{A_0} \phi\left(t \frac{\|f\|_{A_1}}{\|f\|_{A_0}}\right) \leq J(s, f; \vec{A}) \phi\left(t \frac{\|f\|_{A_1}}{J(s, f; \vec{A})}\right).$$

But $\|f\|_{A_1} \leq \frac{J(s, f; \vec{A})}{s}$ and ϕ increases. Hence, the right hand side is smaller than

$$J(s, f; \vec{A}) \phi\left(\frac{t}{s}\right)$$

as we wished to show.

(ii) *There exists a constant $c > 0$ such that*

$$(5.9) \quad K(t, Tf; \vec{B}^\circ) \leq c \int_0^\infty K\left(\frac{t}{r}, f; \vec{A}\right) d\mu(r),$$

where μ is the representing measure of the concave function $\phi(u) = \inf_\theta \{u^\theta M(\theta)\}$,

$$\phi(t) = \int_0^\infty \min\left\{1, \frac{t}{r}\right\} d\mu(r).$$

(iii) *T satisfies the $\phi - K/J$ inequality, that is there exists a constant $c > 0$ such that*

$$(5.10) \quad K(t, Tf; \vec{B}) \leq c \phi\left(\frac{t}{s}\right) J(s, f; \vec{A}).$$

Proof. The equivalence between (5.8) and (5.9) was shown in Section 4 (cf. (4.3) and 4.4). Furthermore, by Theorem 4.6, (5.8) is equivalent to

$$(5.11) \quad T : \vec{A}_{\theta,1;J}^\blacktriangleleft \rightarrow \vec{B}_{\theta,\infty;K}^\blacktriangleleft, \text{ with norm } M(\theta), \theta \in (0, 1).$$

The argument provided before the statement of this theorem shows that (5.11) implies (5.10). Finally, suppose that (5.10) holds. Using the strong form of the fundamental Lemma, we can write

$$f = \int_0^\infty u_f(s) \frac{ds}{s}$$

such that

$$\int_0^\infty \min\left(1, \frac{t}{s}\right) J(s, u_f(s); \vec{A}) \frac{ds}{s} \preceq K(t, f; \vec{A}).$$

Therefore

$$\begin{aligned} K(t, Tf; \vec{B}) &\leq \int_0^\infty K(t, Tu_f(s); \vec{B}) \frac{ds}{s} \\ &\leq c \int_0^\infty \phi\left(\frac{t}{s}\right) J(s, u_f(s); \vec{A}) \frac{ds}{s} \\ &= c \int_0^\infty \int_0^\infty \min\left(1, \frac{t}{sr}\right) J(s, u_f(s); \vec{A}) \frac{ds}{s} d\mu(r) \\ &\leq c \int_0^\infty K\left(\frac{t}{r}, f; \vec{A}\right) d\mu(r), \end{aligned}$$

as we wished to show. □

Let us comment that one key point of the argument is the fact that the concave functions $\min(1, \frac{t}{s})$ are *extremal* (in a suitable “Krein-Milman sense”) and that in fact we can build “all” concave functions from them. Indeed, by the representation theorem (cf. [16, Lemma 5.4.3, page 117]), to each concave function ϕ such that $\lim_{s \rightarrow 0} \phi(s) = \lim_{s \rightarrow \infty} \phi(s)/s = 0$, there corresponds a measure (representing measure) such that²⁵

$$(5.12) \quad \phi(t) = \int_0^\infty \min\left(1, \frac{t}{r}\right) d\mu(r).$$

²⁵In case $d\mu(r) = w(r)dr$, there is a simple algorithm to find w , namely $d\mu(t) = -td\phi'(t)$.

More generally,

$$\phi(t) = \alpha + \beta t + \int_0^\infty \min(1, \frac{t}{r}) d\mu(r)$$

where $\alpha = \lim_{s \rightarrow 0} \phi(s)$, and $\beta = \lim_{s \rightarrow \infty} \phi(s)/s$.

One of the difficulties in the treatment of bilinear extrapolation is the lack of such formulae for concave functions of two variables. This leads to

Problem 5.3. Find an analogue of (5.12) for concave functions of two variables (cf. Section 9 below).

Example 5.4. (cf. [42]) We consider an elementary approach to the underlying K/J inequalities associated with Yano's theorem²⁶. Suppose that T is a bounded operator $T : L^p(0, 1) \rightarrow L^p(0, 1)$, with

$$(5.13) \quad \|Tf\|_{L^p(0,1)} \leq c \frac{p}{p-1} \|f\|_{L^p(0,1)}, \quad 1 < p < \infty.$$

Let us first remark that since $L(\text{Log}L)(0, 1)$ is a Lorentz space, by [49, Lemma II,5,2], to prove Yano's theorem we only need to establish

$$\|Tf\|_{L^1(0,1)} \leq c \|f\|_{L\text{Log}L(0,1)},$$

for functions of the form $f(x) = \gamma \chi_A$, where $\gamma > 0$ and χ_A is the characteristic function of a measurable set $A \subset [0, 1]$. Now, taking limits when $p \rightarrow \infty$ in (5.13) we see that

$$(5.14) \quad \|Tf\|_{L^\infty(0,1)} \leq c \|f\|_{L^\infty(0,1)}.$$

Let $t > s = m(A)$ ($m(A)$ is Lebesgue measure of A), then for any $p > 1$,

$$\begin{aligned} \int_s^t (Tf)^*(u) du &\leq \left\{ \int_s^t [(Tf)^*(u)]^p du \right\}^{1/p} (t-s)^{1/p'} \\ &\leq cp' \|f\|_{L^p(0,1)} s^{1/p'} \left(\frac{t}{s} - 1\right)^{1/p'} \\ &= cp' \gamma s^{1/p} s^{1/p'} \left(\frac{t}{s} - 1\right)^{1/p'} \\ &= cp' \gamma s \left(\frac{t}{s} - 1\right)^{1/p'}. \end{aligned}$$

Suppose that $\frac{t}{s} > e$, then we can select $p' = \log \frac{t}{s}$, and we get

$$(5.15) \quad t(Tf)^{**}(t) - s(Tf)^{**}(s) = \int_s^t (Tf)^*(u) du \leq ce\gamma s \log \frac{t}{s}.$$

Moreover, by (5.14) we have for any $u > 0$

$$(5.16) \quad u(Tf)^{**}(u) \leq u \|Tf\|_{L^\infty} \leq uc \|f\|_{L^\infty} = cu\gamma.$$

Therefore, if we let $u = s$, we get

$$s(Tf)^{**}(s) \leq cs\gamma = c \|f\|_{L^1}.$$

²⁶For simplicity we let $\alpha = 1$.

Inserting this last inequality in (5.15), we find that for $t > es$,

$$t(Tf)^{**}(t) \leq ce \|f\|_{L^1} (1 + \log \frac{t}{s}),$$

while if $t \leq es$, then we can apply (5.16) to find

$$t(Tf)^{**}(t) \leq c\gamma t = ct \|f\|_{L^\infty}.$$

Note that since $t(Tf)^{**}(t) = K(t, Tf; L^1, L^\infty)$, and

$$ces\gamma(1 + \log \frac{t}{s}) = ce \|f\|_{L^1} (1 + \log \frac{t \|f\|_{L^\infty}}{\|f\|_{L^1}})$$

we have the K/J inequality

$$K(t, Tf; L^1, L^\infty) \leq ce \|f\|_{L^1} \phi(\frac{t \|f\|_{L^\infty}}{\|f\|_{L^1}}),$$

where

$$\phi(u) = \begin{cases} e(1 + \log u), & u \geq e \\ eu, & u \leq e \end{cases}.$$

The representing measure for ϕ is given by

$$d\mu(r) \simeq w(r)dr, \text{ where } w(r) = \begin{cases} \frac{1}{r}, & u \geq e \\ 0, & u \leq e \end{cases}$$

so that

$$\begin{aligned} K(t, Tf; L^1, L^\infty) &\leq C \int_e^\infty K(\frac{t}{r}, f; L^1, L^\infty) \frac{dr}{r} \\ &\leq C \int_0^t K(u, f; L^1, L^\infty) \frac{du}{u}. \end{aligned}$$

Therefore,

$$(5.17) \quad (Tf)^{**}(t) \leq \frac{C}{t} \int_0^t f^{**}(u)du.$$

Yano's theorem then follows letting $t = 1$, which yields

$$\begin{aligned} \|Tf\|_{L^1(0,1)} &\leq C \int_0^1 f^{**}(u)du \\ &\approx C \|f\|_{L\text{Log}L(0,1)}. \end{aligned}$$

The result also holds when dealing with infinite measure spaces, in which case

$$(Tf)^{**}(1) = \|Tf\|_{L^1+L^\infty}, \text{ and } \int_0^1 f^{**}(u)du \approx \|f\|_{L\text{Log}L+L^\infty},$$

yielding

$$\|Tf\|_{L^1+L^\infty} \leq C \|f\|_{L\text{Log}L+L^\infty}.$$

Note that (5.17) gives us back (5.13). Indeed, since (5.17) can be rewritten as

$$\int_0^t (Tf)^*(s)ds \leq C \int_0^t f^{**}(u)du, \text{ for all } t > 0,$$

by the Calderón-Mityagin principle we have that for all $p \geq 1$

$$\|Tf\|_{L^p} \leq C \|f^{**}\|_{L^p}.$$

In particular, if $p > 1$, we can continue the estimate of the right hand side using Hardy's inequality to obtain

$$\|Tf\|_{L^p} \leq C \frac{p}{p-1} \|f\|_{L^p}.$$

Problem 5.5. We ask for a systematic "elementary" treatment of extrapolation for general weights $M(p)$.

6. \mathbf{F} -FUNCTORS AND EXTRAPOLATION R.I. SPACES

The $e^{L^{1/\alpha}}$ spaces and the $L(\text{Log}L)^\alpha$ spaces are prototypes of "extrapolation" or limiting spaces for the scale of L^p -spaces. Let us also remark that the $e^{L^{1/\alpha}}$ spaces belong to the class of Marcinkiewicz spaces, while the $L(\text{Log}L)^\alpha$ spaces can be seen to be Lorentz spaces. More generally, it is easy to see that the spaces obtained by applying the Δ -method of extrapolation to $\{L^p\}$ scales can be described as Marcinkiewicz spaces (cf. Theorem 2.11), on the other hand, the corresponding Σ -spaces can be described as Lorentz spaces (cf. Theorem 2.10). It is then natural to ask for a characterization of all the Lorentz or Marcinkiewicz spaces that can be obtained by extrapolation methods applied to scales of L^p -spaces. More generally, one would like to describe all the r.i. spaces that can be obtained by extrapolation of L^p spaces. For definiteness, we shall only consider here r.i. spaces on $[0, 1]$. In this setting, L^∞ and L^1 are the smallest and the largest r.i. spaces, respectively. The prototype scale associated with the pair (L^1, L^∞) is, of course,

$$L^p = \langle L^1, L^\infty \rangle_{1/p', p; K}, \quad 1 < p < \infty.$$

with norm equivalence independent of p [56, Example 7]. So our prototype problem in this section is to characterize certain subclasses of r.i. spaces X that are in a suitable sense "close" to either L^∞ or to L^1 , and whose norms can be obtained by *extrapolation*, that is, by using the \mathbf{F} -functors of extrapolation that are described briefly in Appendix 13.3.

Let us consider, for example, the case of spaces X "close" to L^∞ , in the sense that $X \subset L^p$, for all $p < \infty$. Our aim is to describe all r.i. spaces X such that

$$X = \mathbf{F}(\{L^p\}_{1 < p < \infty}),$$

for some extrapolation functor \mathbf{F} , in which case we shall say that X is an *extrapolation space (at ∞)*. This can be reformulated as follows. For each Banach function lattice F on $[1, \infty)$, we let

$$\mathcal{L}_F = \mathbf{F}(\{L^p\}_{1 < p < \infty}) = \{f : [0, 1] \rightarrow \mathbb{R}, \text{ such that } \xi_f(p) := \|f\|_p \in F\},$$

$$\|f\|_{\mathcal{L}_F} := \|\xi_f\|_F.$$

Since L^p is r.i., $\xi_f = \xi_{f^*}$ and consequently \mathcal{L}_F is a r.i. space. Our aim then is to characterize the class of r.i. spaces X , which we denote by \mathcal{E}_∞ , such that there exists F so that

$$X = \mathcal{L}_F \text{ (with equivalence of norms).}$$

Clearly, this construction is a natural generalization of the Δ -functor, which corresponds to choosing F to be a weighted L^∞ -space.

6.1. Extrapolation characterization of Marcinkiewicz, Orlicz and Lorentz spaces. Let φ be a quasi-concave function on $[0, 1]$. The Marcinkiewicz space $M(\varphi)$ consists of all measurable functions $f(t)$ on $[0, 1]$, such that

$$(6.1) \quad \|f\|_{M(\varphi)} = \sup_{0 < s \leq 1} \frac{\varphi(s)}{s} \cdot \int_0^s f^*(t) dt = \sup_{0 < s \leq 1} \varphi(s) f^{**}(s) < \infty.$$

We shall now consider the problem of identifying the Marcinkiewicz spaces $M(\varphi)$ that belong to \mathcal{E}_∞ . Suppose then that $M(\varphi) \subset L^p$ for all $1 \leq p < \infty$. It follows readily that $\lim_{t \rightarrow 0+} \tilde{\varphi}(t) = 0$, where $\tilde{\varphi}(t) := t/\varphi(t)$. Consequently, the function $\tilde{\varphi}'$ is absolutely continuous on $[0, 1]$, $(\tilde{\varphi}')^{**}(s) = \tilde{\varphi}(s)/s$ and therefore we have $\|\tilde{\varphi}'\|_{M(\varphi)} = 1$. The assumption that $M(\varphi) \subset L^p$ for all $p < \infty$, therefore implies that $\tilde{\varphi}' \in L^p$ for all $p < \infty$. Moreover, from the definition (6.1) we see that for all $0 < s \leq 1$,

$$\int_0^s f^*(t) dt \leq \|f\|_{M(\varphi)} \tilde{\varphi}(s) = \|f\|_{M(\varphi)} \int_0^s \tilde{\varphi}'(t) dt.$$

Therefore, by the Calderón-Mityagin theorem (see e.g. [49, Theorem II.4.3]), we conclude that for all $f \in M(\varphi)$ and $1 \leq p < \infty$

$$\|f\|_p \leq \|f\|_{M(\varphi)} \cdot \|\tilde{\varphi}'\|_p.$$

In other words,

$$(6.2) \quad M(\varphi) \stackrel{1}{\subset} \mathcal{L}_{F^\varphi},$$

where F^φ is the weighted Banach lattice $L^\infty(1/\|\tilde{\varphi}'\|_p)$.

Remark 6.1. Recall that the fundamental function of a r.i. space X is defined by $\phi_X(t) := \|\chi_{[0,t]}\|_X$, $0 < t \leq 1$. In particular, $\phi_{M(\varphi)}(t) = \varphi(t)$. It follows readily that $M(\varphi)$ is the largest among all r.i. spaces with the fundamental function $\varphi(t)$ (cf. [49, Theorem II.5.7]), the fact which we will need to prove the next proposition.

Proposition 6.2. *Let φ be a quasi-concave function on $[0, 1]$. The following conditions are equivalent:*

- (i) $M(\varphi) \in \mathcal{E}_\infty$;
- (ii) $M(\varphi) = \mathcal{L}_{F^\varphi}$;
- (iii) *there exists $C > 0$ such that*

$$(6.3) \quad \varphi(t) \leq C \cdot \sup_{p \geq 1} \frac{t^{\frac{1}{p}}}{\|\tilde{\varphi}'\|_p}, \quad 0 < t \leq 1.$$

Proof. First of all, each of the conditions (i), (ii), (iii) implies that $M(\varphi) \subset L^p$ for all $p < \infty$. Therefore, in all three cases we have embedding (6.2).

[(ii) \leftrightarrow (iii)]. Suppose that (iii) holds. Then,

$$\begin{aligned}\varphi(t) &\leq C \cdot \sup_{p \geq 1} \frac{t^{\frac{1}{p}}}{\|\tilde{\varphi}'\|_p} \\ &= C \cdot \sup_{p \geq 1} \frac{1}{\|\tilde{\varphi}'\|_p} \|\chi_{[0,t]}\|_p \\ &= C \cdot \|\chi_{[0,t]}\|_{\mathcal{L}_{F^\varphi}} \\ &= C \cdot \varphi_{\mathcal{L}_{F^\varphi}}(t).\end{aligned}$$

It follows (cf. the Remark preceding the statement of this Proposition) that $\mathcal{L}_{F^\varphi} \subset M(\varphi_{\mathcal{L}_{F^\varphi}}) \subset M(\varphi)$, which combined with (6.2) yields that $\mathcal{L}_{F^\varphi} = M(\varphi)$. Conversely, if $\mathcal{L}_{F^\varphi} = M(\varphi)$, then $\varphi \approx \varphi_{\mathcal{L}_{F^\varphi}}$ and (iii) trivially holds.

[(i) \leftrightarrow (ii)]. Since the implication (ii) \Rightarrow (i) is obvious, we only need to prove the converse. Suppose then that $M(\varphi) = \mathcal{L}_{F_1}$ for some Banach function lattice F_1 . We will now show that $\mathcal{L}_{F_1} = \mathcal{L}_{F^\varphi}$. Indeed, by (6.2), $\mathcal{L}_{F_1} = M(\varphi) \stackrel{1}{\subset} \mathcal{L}_{F^\varphi}$. On the other hand, suppose that $g \in \mathcal{L}_{F^\varphi}$, then for all $p > 1$

$$\xi_g(p) \leq \|g\|_{\mathcal{L}_{F^\varphi}} \|\tilde{\varphi}'\|_p.$$

Therefore, applying the lattice norm F_1 to the previous inequality, we get

$$\begin{aligned}\|\xi_g\|_{F_1} &\leq \|g\|_{\mathcal{L}_{F^\varphi}} \left\| \|\tilde{\varphi}'\|_p \right\|_{F_1} \\ &= \|g\|_{\mathcal{L}_{F^\varphi}} \|\tilde{\varphi}'\|_{\mathcal{L}_{F_1}} \\ &\approx \|g\|_{\mathcal{L}_{F^\varphi}} \|\tilde{\varphi}'\|_{M(\varphi)} \\ &= \|g\|_{\mathcal{L}_{F^\varphi}}.\end{aligned}$$

Consequently,

$$\|g\|_{\mathcal{L}_{F_1}} = \|\xi_g\|_{F_1} \leq C \|g\|_{\mathcal{L}_{F^\varphi}},$$

and therefore $g \in \mathcal{L}_{F_1}$, concluding the proof. \square

Problem 6.3. Let $w(p)$ be a bounded positive function on $[1, \infty)$. We set $X_w := \Delta_{1 \leq p < \infty}(w(p)L^p)$. Then, X_w is a r.i. space with the fundamental function $\phi_{X_w}(t) = \sup_{1 \leq p < \infty}(w(p)t^{1/p})$. By Proposition 6.2, from $M(\varphi) \in \mathcal{E}_\infty$ it follows that $M(\varphi) = X_w$ with $w(p) = 1/\|\tilde{\varphi}'\|_p$. We ask: What other r.i. spaces may be represented as spaces of the X_w -type? It is known [7, Theorem 4.7] that if an Orlicz space L_M coincides with some Marcinkiewicz space, then $L_M \in \mathcal{E}_\infty$ and therefore it coincides with the space X_w for some w . In contrast to that, in [10, Proposition 3.4], one can find examples of the Orlicz spaces of the X_w -type that do not coincide with Marcinkiewicz spaces. Thus, it is natural to ask, which Orlicz spaces are spaces of the X_w -type?

Problem 6.4. In connection with Problem 6.3, maybe it could be useful to take into account that every space X_w is D -convex (for the definitions, we refer to [59] or [12]). Moreover, it is known (see [59, Theorem 23] or [12, Corollary 4.10]) that a r.i. space X with the Fatou property coincides with an Orlicz space if and only

if X is D -convex and D -concave (some authors refer to the latter property as D^* -convexity). Therefore, we ask: Under what conditions on a weight w , is the space X_w D -concave?

Problem 6.5. (Open ended) Let us remark that, with minor modifications, the framework we are discussing here could be used to derive a generalized theory of “Grand Lebesgue spaces” (cf. Section 8 below for definitions and background). In fact, a natural setting for generalized “Grand Lebesgue spaces” could be have by means of replacing $\Delta_{1 \leq p < \infty}(w(p)L^p)$ by $\Delta_{\theta \in I}(w(\theta)L^{p(\theta)})$.

Problem 6.6. Give a characterization of Lorentz spaces from the class \mathcal{E}_∞ . Recall that the norm in the Lorentz space $\Lambda_p(\varphi)$, where φ is an increasing concave function on $[0, 1]$, $\varphi(0) = 0$, and $1 \leq p < \infty$, is defined as follows:

$$\|f\|_{\Lambda_p(\varphi)} := \left(\int_0^1 (f^*(t))^p d\varphi(t) \right)^{1/p}.$$

Moreover, we ask to introduce in a similar way a notion of extrapolation r.i. spaces at 1, i.e., as $p \rightarrow 1+$, which would therefore generalize the Σ -functor, and then using this notion to give a description of Marcinkiewicz, Lorentz, Orlicz spaces that are “extrapolation at 1”.

Remark 6.7. Concerning Problem 6.6 we note that some partial results related to a description of Lorentz spaces from the class \mathcal{E}_∞ were obtained in [5, Theorem 3].

Problem 6.8. The same type of questions can be formulated in the non-commutative setting. In this context instead of L^p -spaces, we deal with the scale of Schatten ideals \mathfrak{S}^p , $1 < p < \infty$, of compact operators acting in a separable complex Hilbert space (see Example 2.8). It is natural to ask similar questions in connection with an extrapolation description of Schatten ideals. Some partial results can be found in [11].

Problem 6.9. We are asking whether any r.i. space X on $[0, 1]$ such that the X -norm of every function is determined by the family of its L^p -norms coincides with a space of the form \mathcal{L}_F for a suitable Banach function lattice F on $[1, \infty)$? More formally, let X be a r.i. space on $[0, 1]$ such that $X \subset L_p$ for all $p < \infty$. Suppose that there is $p_0 > 0$ such that from the inequality $\|x\|_p \leq C\|y\|_p$, for some $C > 0$ and all $p \geq p_0$, it follows that $\|x\|_X \leq \|y\|_X$. Does this imply that $X = \mathcal{L}_F$ for some parameter F ?

Problem 6.10. (Open ended) In connection with Problems 6.5 and 6.8 we are led to ask for the corresponding theory of Non-Commutative Grand L^p spaces. We believe that abstract Extrapolation theory provides the right tools to develop this project.

6.2. Tempered F-parameters and strong extrapolation r.i. spaces. The following definition, introduced in [7], could be considered as a natural generalization of the notion of a tempered weight.

Definition 6.11. We shall say that a parameter space F (i.e. F is a Banach function lattice on $[1, \infty)$) of an extrapolation \mathbf{F} -method is *tempered* if the operator $Df(p) := f(2p)$ is bounded on F .

It turns out that the spaces \mathcal{L}_F with tempered parameters F form a very special subclass of the extrapolation at ∞ r.i. spaces.

Let X be a r.i. space on $[0, 1]$. We denote by \tilde{X} the Banach lattice of all the measurable functions f on $(1, \infty)$ such that

$$\|f\|_{\tilde{X}} := \|f(\log(e/t))\|_X < \infty.$$

Definition 6.12. We shall say that a r.i. space X is a *strong extrapolation space with respect to the L^p -scale* (in which case we shall write $X \in \mathcal{SE}_\infty$) if $X = \mathcal{L}_{\tilde{X}}$ (with equivalence of norms).

By definition, if $X \in \mathcal{SE}_\infty$, then the corresponding extrapolation parameter F is explicitly determined by X . More precisely,

$$\|f\|_X \approx \|\|f\|_{\log(e/t)}\|_X,$$

where, consistently with our notation throughout this paper, for each $t \in (0, 1)$, we let $\|f\|_{\log(e/t)} := \|f\|_{L^{\log(e/t)}}$.

The class \mathcal{SE}_∞ admits a simple characterization (see [8, Theorem 4.3]).

Theorem 6.13. *Let X be a r.i. space on $[0, 1]$. The following conditions are equivalent:*

- (1) $X = \mathcal{L}_F$ for some tempered extrapolation parameter F ;
- (2) $X \in \mathcal{SE}_\infty$;
- (3) the operator $Sf(t) = f(t^2)$ is bounded on X .

The class \mathcal{SE}_∞ is rather wide. In particular, the Zygmund $\text{Exp } L^\alpha$ spaces, with $\alpha > 0$, lie in this class. Moreover, a Marcinkiewicz space $\mathcal{M}(\varphi)$ (resp. a Lorentz space $\Lambda(\varphi)$) belongs to the class \mathcal{SE}_∞ if and only if $\varphi(t) \approx \varphi(t^2)$, $0 < t \leq 1$ (cf. [6, Theorem 2.10]). In this connection it is worth to note that, in the definition of the space $X(\log^{-1})$ (see Example 4.12), the $**$ operation may be replaced by $*$ operation, whenever $X \in \mathcal{SE}_\infty$ (cf. [6, Proposition 4.1]).

One can verify that, for every $1 < p < \infty$, we have

$$L(2p, \infty) \stackrel{2}{\subset} L(p, 1) \stackrel{1}{\subset} L(p, \infty).$$

Hence, if F is a tempered parameter, we have the following generalized version of the second relation from (2.8) (as it applies to the pair $\vec{A} = (L^1, L^\infty)$):

$$(6.4) \quad \mathbf{F}(\{L(p, 1)\}_{1 < p < \infty}) = \mathbf{F}(\{L(p, \infty)\}_{1 < p < \infty}),$$

where $\mathbf{F}(\{L(p, q)\}_{1 < p < \infty})$, $1 \leq q \leq \infty$, is defined exactly as the space $\mathcal{L}_F = \mathbf{F}(\{L^p\}_{1 < p < \infty})$, replacing L^p by $L(p, q)$.

Problem 6.14. (The multiplier problem for \mathbf{F} -functor) Characterize the parameters for the \mathbf{F} -method of extrapolation that have the property (6.4).

7. OPERATORS WITH A QUASI-BANACH TARGET SPACE

The classical extrapolation theorems deal mainly with linear or sublinear operators taking values on quasi-Banach spaces. For example we may want to extrapolate estimates for maximal operators, e.g. the maximal operator M of Hardy-Littlewood, or the Carleson maximal operator \mathfrak{C} defined by

$$(7.1) \quad \mathfrak{C}f(e^{i\theta}) := \sup_{N=1,2,\dots} |S_N f(e^{i\theta})|,$$

where $S_N f(e^{i\theta}) := \sum_{n=-N}^N \hat{f}(n) e^{in\theta}$, and $\hat{f}(n)$ is the n -th Fourier coefficient of f . Note that for these examples the "natural target space" is the quasi-normed space $L(1, \infty)$ (see e.g. [2]).

By construction, the \sum -methods crucially use linearity and the triangle inequality. But it is possible to modify the construction of \sum -methods in order to be able to deal with some of these difficulties, although we shall not discuss the issues in detail here (cf. [43, Section 4, pages 35-44]).

Here an operator T defined on a Banach space X and taking values in the set of all measurable functions $f : [0, 1] \rightarrow \mathbb{R} \cup \{\pm\infty\}$, is *sublinear* if for some $B > 0$ and an arbitrary expansion $x = \sum_{j=1}^{\infty} x_j$, as a convergent series in X , we have

$$|Tx(t)| \leq B \sum_{j=1}^{\infty} |Tx_j(t)| \quad \text{a.e. on } [0, 1].$$

Another important class of non-linear operators acting on function spaces are those for which we have $T(f+g) = Tf + Tg$, whenever functions f and g have disjoint supports. It turns out that in the context of lattices one can find suitable versions of the strong form of the fundamental Lemma (cf. Appendix 13.1.1, including Remark 13.3) that guarantee the existence of good representations $f = \sum f_n$ such that the f_n 's are disjointly supported (cf. [26]).

The weak L^1 -space, $L(1, \infty)$, is quasi-normed but it barely misses to be normable. It belongs to the class of logconvex quasi-Banach lattices. We shall say that a quasi-Banach space Y is called *logconvex* if there is a constant $C > 0$ such that for all $y_j \in Y$ we have

$$\left\| \sum_{j=1}^{\infty} y_j \right\|_Y \leq C \sum_{j=1}^{\infty} (1 + \log j) \|y_j\|_Y,$$

(see e.g. [45]). $L(1, \infty)$ is *logconvex* (cf. [69, Lemma 2.3] or [45, Theorem 3.4]).

The following result presents a version²⁷ of Yano's extrapolation theorem for operators that take values in a logconvex space.

Theorem 7.1 ([52, Theorem 9]). *Let \mathcal{M} be the set of all measurable functions $f : [0, 1] \rightarrow \mathbb{R}$, and let $Y \subset \mathcal{M}$ be a logconvex quasi-Banach lattice. Let $\alpha > 0$, and let T be a sublinear operator defined on the Lorentz space $\Lambda(\psi_\alpha)$, where*

$$\psi_\alpha(t) \approx t \log^\alpha(b/t) \log \log \log(b/t), \quad 0 < t \leq 1, \quad \text{where } b > e^e.$$

²⁷We refer to [52] for more general results.

Furthermore, suppose that T is bounded, $T : L(p, 1) \rightarrow Y$, $p > 1$, and for some $C > 0$ and all $p > 1$ we have

$$(7.2) \quad \|T\|_{L(p,1) \rightarrow Y} \leq C \left(\frac{p}{p-1} \right)^\alpha.$$

Then, T is a bounded operator

$$T : \Lambda(\psi_\alpha) \rightarrow Y.$$

Problem 7.2. To understand Theorem 7.1 note that, if Y had been a Banach space, by the classical extrapolation theorem (cf. Section 3) we would get that T is bounded, $T : L(\text{Log} L)^\alpha \rightarrow Y$. In other words, the penalty we pay for having only logconvexity is the extra "triple logarithm" factor that appears in the norm of $\Lambda(\psi_\alpha)$. In this connection we therefore ask if this result is sharp or if it is possible to enlarge the domain space?

7.1. A.e. convergence of Fourier series and extrapolation. A classical extrapolation due to Carleson–Sjölin theorem [68] states that if T is a continuous sublinear operator on L^p such that for every measurable set $A \subset [0, 1]$ and all $1 < p \leq 2$, $t > 0$

$$t \cdot m\{x \in [0, 1] : |T\chi_A(x)| > t\}^{1/p} \leq C(p-1)^{-1} m(A)^{1/p},$$

with some constant $C > 0$ independent of A , p , and t (m is the Lebesgue measure), then T maps the space $L(\text{Log} L)\text{LogLog} L$ into $L(1, \infty)$. Applying this result to the Carleson maximal operator M (see (7.1)), we immediately get that the Fourier series of each function from the space $L(\text{Log} L)\text{LogLog} L$ converges a.e. If $\alpha > 0$, and $(p-1)^{-1}$ is replaced with $(p-1)^{-\alpha}$ then an analogous result holds replacing $L(\text{Log} L)\text{LogLog} L$ by $L(\text{Log} L)^\alpha \text{LogLog} L$, (cf. [43, Theorem 5.7.1]), but the domain space provided by Theorem 7.1 is wider than the space $L(\text{Log} L)\text{LogLog} L$.

Problem 7.3. Do the conditions of the Carleson–Sjölin extrapolation theorem imply that a sublinear operator T maps the larger space $L(\text{Log} L)\text{LogLogLog} L$ into $L(1, \infty)$, thus implying a well-known result for Fourier series due to Antonov [1] ?

8. GRAND LEBESGUE SPACES AND THEIR VERSIONS VIA EXTRAPOLATION

Let $1 < p < \infty$. The *Grand Lebesgue $L^{(p)}$ space* introduced by Iwaniec and Sbordone [37], consists of all measurable functions f on $[0, 1]$ such that

$$(8.1) \quad \|f\|_{L^{(p)}} := \sup_{0 < \varepsilon < p-1} \varepsilon^{\frac{1}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}} < \infty.$$

These spaces have found many applications in analysis, including the study of maximal operators, PDEs, interpolation theory, etc (see [29, 38] and the references therein). On the other hand, the expression (8.1) is somewhat difficult to work with. In this context, Fiorenza–Karadzhov [32, Theorem 4.2] gave the following more explicit description of the Grand Lebesgue spaces $L^{(p)}$ in the terms of the decreasing rearrangement of the function f :

$$\|f\|_{L^{(p)}} \approx \sup_{0 < t < 1} (\log(e/t))^{-\frac{1}{p}} \left(\int_t^1 f^*(s)^p ds \right)^{\frac{1}{p}},$$

with universal constants of equivalence. The proof given in [32, Theorem 4.2] is based on the extrapolation methods of [46]. A simpler proof of this result was obtained recently in [11, Theorem 12] exploiting an extrapolation description of suitable limiting interpolation spaces. More generally the same method yields that for every $\alpha > 0$,

$$\|f\|_{L^p, \alpha} := \sup_{0 < \varepsilon < p-1} \varepsilon^{\frac{\alpha}{p-\varepsilon}} \|f\|_{L^{p-\varepsilon}} \approx \sup_{0 < t < 1} \log^{-\alpha/p}(e/t) \left(\int_t^1 (f^*(s))^p ds \right)^{\frac{1}{p}}$$

(see [11, Remark 11]), a result obtained for the first time in [31, Theorem 1.1 and Theorem 3.1].

Let $p > 1$ and let $\psi : (0, p-1) \rightarrow [0, \infty)$ be a nondecreasing function. We shall say that $\psi \in \Delta_2$ if $\psi(2t) \leq \psi(t)$, for small t . In [29, Theorem 1] it is shown that the equivalence

$$(8.2) \quad \sup_{0 < \varepsilon < p-1} \psi(\varepsilon) \|f\|_{L^{p-\varepsilon}} \approx \sup_{0 < t < 1} \psi\left(\frac{p-1}{1-\log t}\right) \left(\int_t^1 (f^*(s))^p ds \right)^{\frac{1}{p}}$$

holds if and only if $\psi \in \Delta_2 \cap L^\infty$.

Problem 8.1. Prove equivalence (8.2) in the case of non-power functions $\psi \in \Delta_2 \cap L^\infty$, using extrapolation methods. What is the corresponding equivalence formula when the Δ_2 -assumption does not hold?

9. BILINEAR EXTRAPOLATION: CALDERÓN'S OPERATOR REVISITED

The K/J inequalities can be extended to bilinear operators and used to prove extrapolation theorems following the scheme used to treat the linear case (cf. [44]). We briefly review the story here and present a number of open problems.

We start by recalling that, in the classical setting, the interpolation of bilinear operators can be effected using K/J inequalities (cf. [44]). In this case starting with weak type inequalities for a bilinear operator T we control the K -functional of $K(t, T(f, g))$ via an analog of the Calderón operator that this time is expressed as the multiplicative convolution of the K -functionals of f and g . We recall the details. Let $\vec{A}, \vec{B}, \vec{C}$ be pairs of mutually closed spaces and let T be a bounded bilinear operator $T : \vec{A} \times \vec{B} \rightarrow \vec{C}$. Then it is easy to see that there exists $c > 0$ such that (cf. [44])

$$J(rs, T(f, g); \vec{C}) \leq cJ(r, f; \vec{A})J(s, g; \vec{B}), \quad r, s > 0.$$

If we combine this fact with the basic elementary K/J inequality: For any pair \vec{X} , and $f \in \Delta(\vec{X})$,

$$K(t, h; \vec{X}) \leq \min\left\{1, \frac{t}{s}\right\} J(s, h; \vec{X}),$$

we obtain for $f \in \Delta(\vec{A})$ and $g \in \Delta(\vec{B})$,

$$(9.1) \quad \begin{aligned} K(t, T(f, g); \vec{C}) &\leq \min\left\{1, \frac{t}{rs}\right\} J(rs, T(f, g); \vec{C}) \\ &\leq c \min\left\{1, \frac{t}{rs}\right\} J(r, f; \vec{A})J(s, g; \vec{B}). \end{aligned}$$

Suppose now that γ_a, γ_b and γ_c are quasi-concave functions such that

$$\frac{1}{\gamma_c(uv)} \leq \frac{1}{\gamma_a(u)} \frac{1}{\gamma_b(v)}, \quad u, v > 0$$

and furthermore suppose that

$$1 \leq q_i \leq \infty, i = 1, 2, 3, \text{ and } \frac{1}{q_3} = \frac{1}{q_1} + \frac{1}{q_2} - 1.$$

Then,

$$T : \vec{A}_{\gamma_a, q_1; K} \times \vec{B}_{\gamma_b, q_2; K} \rightarrow \vec{C}_{\gamma_c, q_3; K}.$$

We go over the proof. Represent $f = \int_0^\infty u_f(s) \frac{ds}{s}$, and $g = \int_0^\infty u_g(s) \frac{ds}{s}$, so that $J(r, u_f(r); \vec{A}) \leq K(r, f; \vec{A})$, and $\int_0^\infty \min\{1, \frac{t}{s}\} J(s, u_g(s); \vec{B}) \frac{ds}{s} \leq K(t, g; \vec{B})$. Now applying the K -functional to the representation

$$T(f, g) = \int_0^\infty \int_0^\infty T(u_f(r), u_g(s)) \frac{dr}{r} \frac{ds}{s},$$

by (9.1), yields

$$\begin{aligned} \frac{K(t, T(f, g); \vec{C})}{\gamma_c(t)} &\leq \int_0^\infty \int_0^\infty \frac{K(t, T(u_f(r), u_g(s)); \vec{C})}{\gamma_c(t)} \frac{dr}{r} \frac{ds}{s} \\ &\leq c \int_0^\infty \int_0^\infty \frac{\min\{1, \frac{t}{rs}\}}{\gamma_c(\frac{t}{r})} J(r, u_f(r); \vec{A}) J(s, u_g(s); \vec{B}) \frac{dr}{r} \frac{ds}{s} \\ &\leq c \int_0^\infty \frac{K(r, f; \vec{A})}{\gamma_a(r)} \frac{1}{\gamma_b(\frac{t}{r})} \int_0^\infty \min\{1, \frac{t}{rs}\} J(s, u_g(s); \vec{B}) \frac{ds}{s} \frac{dr}{r} \\ &\leq \int_0^\infty \frac{K(r, f; \vec{A})}{\gamma_a(r)} \frac{K(\frac{t}{r}, g; \vec{B})}{\gamma_b(\frac{t}{r})} \frac{dr}{r} \\ &= \frac{K(\circ, f; \vec{A})}{\gamma_a(\circ)} \blacklozenge \frac{K(\circ, g; \vec{B})}{\gamma_b(\circ)}, \text{ where } \blacklozenge \text{ is convolution w.r. to } (\mathbb{R}^+, \frac{dt}{t}). \end{aligned}$$

Consequently, by Young's convolution inequality for the multiplicative group, there exists a constant $c > 0$ such that

$$\|T(f, g)\|_{\vec{C}_{\gamma_c, q_3; K}} \leq c \|f\|_{\vec{A}_{\gamma_a, q_1; K}} \|g\|_{\vec{B}_{\gamma_b, q_2; K}}.$$

We now consider the extrapolation case. Let T be a bilinear operator, let $M(\theta), N(\theta)$ be weights such that for all $\theta \in (0, 1)$,

$$(9.2) \quad T : M(\theta) \vec{A}_{\theta, 1; J}^{\blacktriangleleft} \times N(\theta) \vec{B}_{\theta, 1; J}^{\blacktriangleleft} \rightarrow \vec{C}_{\theta, \infty; K}^{\blacktriangleleft}, \text{ with norm 1, for all } \theta \in (0, 1).$$

Then we have the following bilinear Calderón type extrapolation version of Theorem 4.6 (cf. the discussion that follows it), and Theorem 5.2.

Theorem 9.1. *The following are equivalent:*

- (i) (9.2) holds.
- (ii) (bilinear K/J inequality) *There exists a constant $c > 0$, such that T satisfies the following (compare with (9.1) above): for all $f \in \Delta(\vec{A}), g \in \Delta(\vec{B}), t, s, h > 0$,*

$$(9.3) \quad K(t, T(f, g); \vec{C}) \leq c \tau\left(\frac{t}{sh}\right) J(s, f; \vec{A}) J(h, g; \vec{B})$$

where

$$\tau(x) = \inf_{0 < \theta < 1} \{x^\theta M(\theta)N(\theta)\}.$$

(iii) There exists a constant $c > 0$ such that

$$K(t, T(f, g); \vec{C}) \leq c \inf \left\{ \int_0^\infty \int_0^\infty \tau\left(\frac{t}{sh}\right) J(s, u_f(s); \vec{A}) J(h, u_g(h); \vec{B}) \frac{ds}{s} \frac{dh}{h} \right\}$$

where the infimum is taken over all the J -decompositions of $f = \int_0^\infty u_f(s) \frac{ds}{s}$, and $g = \int_0^\infty u_g(h) \frac{dh}{h}$.

Proof. If (i) holds then for all $f \in \Delta(\vec{A}), g \in \Delta(\vec{B}), \theta \in (0, 1)$,

$$\begin{aligned} K(t, T(f, g), \vec{C}) &\leq t^\theta \|T(f, g)\|_{\vec{C}_{\theta, \infty; K}} \\ &\leq t^\theta M(\theta) \|f\|_{\vec{A}_{\theta, 1; J}} N(\theta) \|g\|_{\vec{B}_{\theta, 1; J}} \\ &\leq t^\theta h^{-\theta} s^{-\theta} M(\theta) N(\theta) J(s, f; \vec{A}) J(h, g; \vec{B}). \end{aligned}$$

Consequently,

$$\begin{aligned} K(t, T(f, g), \vec{C}) &\leq \inf_\theta \left(\left(\frac{t}{sh} \right)^\theta M(\theta) N(\theta) \right) J(s, f; \vec{A}) J(h, g; \vec{B}) \\ &= \tau\left(\frac{t}{sh}\right) J(s, f; \vec{A}) J(h, g; \vec{B}). \end{aligned}$$

This takes care of the implication (i) \rightarrow (ii). The implication (ii) \rightarrow (iii) follows from the triangle inequality. Indeed, suppose that $f = \int_0^\infty u_f(s) \frac{ds}{s}, g = \int_0^\infty u_g(h) \frac{dh}{h}$, then

$$\begin{aligned} T(f, g) &= T\left(\int_0^\infty u_f(s) \frac{ds}{s}, \int_0^\infty u_g(h) \frac{dh}{h}\right) \\ &= \int_0^\infty \int_0^\infty T(u_f(s), u_g(h)) \frac{ds}{s} \frac{dh}{h} \end{aligned}$$

and therefore, applying (9.3), we find that for all $t > 0$,

$$\begin{aligned} K(t, T(f, g), \vec{C}) &\leq \int_0^\infty \int_0^\infty K(t, T(u_f(s), u_g(h)); \vec{C}) \frac{ds}{s} \frac{dh}{h} \\ (9.4) \quad &\leq \int_0^\infty \int_0^\infty \tau\left(\frac{t}{sh}\right) J(s, u_f(s); \vec{A}) J(h, u_g(h); \vec{B}) \frac{ds}{s} \frac{dh}{h}. \end{aligned}$$

Finally, if (iii) holds then for each $\theta \in (0, 1)$ and $\varepsilon > 0$ we can select decompositions $f = \int_0^\infty u_f(s) \frac{ds}{s}, g = \int_0^\infty u_g(h) \frac{dh}{h}$, such that

$$\begin{aligned} \int_0^\infty J(s, u_f(s); \vec{A}) s^{-\theta} \frac{ds}{s} &\leq (1 + \varepsilon) \|f\|_{\vec{A}_{\theta, 1; J}}, \\ \int_0^\infty J(h, u_g(h); \vec{A}) h^{-\theta} \frac{dh}{h} &\leq (1 + \varepsilon) \|g\|_{\vec{B}_{\theta, 1; J}}. \end{aligned}$$

Moreover, since by definition, for each $\theta \in (0, 1), t, s, h > 0$

$$\tau\left(\frac{t}{sh}\right) \leq M(\theta) N(\theta) \frac{t^\theta}{s^\theta h^\theta},$$

it follows that

$$\begin{aligned} & \int_0^\infty \int_0^\infty \tau\left(\frac{t}{sh}\right) J(s, u_f(s); \vec{A}) J(h, u_g(h); \vec{B}) \frac{ds}{s} \frac{dh}{h} \\ & \leq t^\theta M(\theta) \int_0^\infty J(s, u_f(s); \vec{A}) s^{-\theta} \frac{ds}{s} N(\theta) \int_0^\infty J(h, u_g(h); \vec{B}) h^{-\theta} \frac{dh}{h} \\ & \leq (1 + \varepsilon)^2 t^\theta M(\theta) \|f\|_{\vec{A}_{\theta,1;J}^\blacktriangleleft} N(\theta) \|g\|_{\vec{B}_{\theta,1;J}^\blacktriangleleft} \end{aligned}$$

Combining this inequality with (9.4), and letting $\varepsilon \rightarrow 0$, we get

$$\|T(f, g)\|_{\vec{C}_{\theta,\infty;K}^\blacktriangleleft} = \sup_{t>0} K(t, T(f, g), \vec{C}) t^{-\theta} \leq M(\theta) \|f\|_{\vec{A}_{\theta,1;J}^\blacktriangleleft} N(\theta) \|g\|_{\vec{B}_{\theta,1;J}^\blacktriangleleft}.$$

□

Extrapolation theorems follow from assumptions about τ . In particular, to the assumptions of the previous theorem we add the following property that is particularly suitable for our method.

Definition 9.2. Let us say that a concave function $\tau : (0, \infty) \rightarrow R_+$, is adequate if there exists a measure ν on $(0, \infty)$ such that τ can be represented by

$$(9.5) \quad \tau(t) = \int_0^\infty \int_0^\infty \min(1, \frac{t}{n}) \min(1, \frac{n}{r}) d\nu(r) \frac{dn}{n}, t > 0.$$

Theorem 9.3. Let $\vec{A}, \vec{B}, \vec{C}$, be mutually closed Banach pairs and let T be a bilinear operator such that (9.2) holds where, moreover, M, N are weights such that the function

$$\tau(t) = \inf_{0 < \theta < 1} \{t^\theta M(\theta) N(\theta)\}.$$

is adequate. Then,

$$K(t, T(f, g); \vec{C}) \leq c \int_0^\infty \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K\left(\frac{u}{r}, g; \vec{B}\right) d\nu(r) \frac{du}{u}.$$

Proof. Using the strong form of fundamental Lemma we select J -decompositions of $f = \int_0^\infty u_f(s) \frac{ds}{s}$ and $g = \int_0^\infty u_g(h) \frac{dh}{h}$ such that

$$\begin{aligned} & \int_0^\infty \min(1, \frac{t}{s}) J(s, u_f(s); \vec{A}) \frac{ds}{s} \preceq K(t, f; \vec{A}); \\ & \int_0^\infty \min(1, \frac{t}{h}) J(h, u_g(h); \vec{B}) \frac{dh}{h} \preceq K(t, g; \vec{B}). \end{aligned}$$

In view of Theorem 9.1 we know that (9.4) above holds for the above decompositions. To estimate the resulting right hand side of (9.4) we use the fact that τ is adequate.

Then, we can estimate $K(t, T(f, g), \vec{C})$ by

$$\begin{aligned}
& \int_0^\infty \int_0^\infty \tau\left(\frac{t}{sh}\right) J(s, u_f(s); \vec{A}) J(h, u_g(h); \vec{B}) \frac{ds}{s} \frac{dh}{h} \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \int_0^\infty \min\left(1, \frac{t}{nsh}\right) \min\left(1, \frac{n}{r}\right) \\
&\quad J(s, u_f(s); \vec{A}) J(h, u_g(h); \vec{B}) \frac{ds}{s} \frac{dh}{h} d\nu(r) \frac{dn}{n} \\
&= \int_0^\infty \int_0^\infty \int_0^\infty \min\left(1, \frac{n}{r}\right) \\
&\quad J(h, u_g(h); \vec{B}) \left(\int_0^\infty \min\left(1, \frac{t}{nsh}\right) J(s, u_f(s); \vec{A}) \frac{ds}{s} \right) \frac{dh}{h} d\nu(r) \frac{dn}{n} \\
&\preceq \int_0^\infty \int_0^\infty J(h, u_g(h); \vec{B}) \int_0^\infty \min\left(1, \frac{n}{r}\right) K\left(\frac{t}{nh}, f; \vec{A}\right) \frac{dn}{n} \frac{dh}{h} d\nu(r) \\
&= \int_0^\infty \int_0^\infty \int_0^\infty J(h, u_g(h); \vec{B}) \min\left(1, \frac{u}{rh}\right) K\left(\frac{t}{u}, f; \vec{A}\right) \frac{du}{u} \frac{dh}{h} d\nu(r) \quad (\text{let } u = nh) \\
&= \int_0^\infty \int_0^\infty \left(\int_0^\infty J(h, u_g(h); \vec{B}) \min\left(1, \frac{u}{rh}\right) \frac{dh}{h} \right) K\left(\frac{t}{u}, f; \vec{A}\right) \frac{du}{u} d\nu(r) \\
&\preceq \int_0^\infty \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K\left(\frac{u}{r}, g; \vec{B}\right) \frac{du}{u} d\nu(r),
\end{aligned}$$

Thus,

$$K(t, T(f, g), \vec{C}) \preceq \int_0^\infty \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K\left(\frac{u}{r}, g; \vec{B}\right) \frac{du}{u} d\nu(r),$$

as we wished to show. \square

Example 9.4. Suppose that

$$M(\theta)N(\theta) \approx \theta^{-1}(1 - \theta)^{-1}$$

and let

$$\tau(t) = \inf_{\theta} \{t^\theta M(\theta)N(\theta)\}.$$

Then, by direct computation we see that (cf. [43, page 50], Example 4.12 for similar computations)

$$\begin{aligned}
\int_0^\infty \min\left(1, \frac{t}{n}\right) \min(1, n) \frac{dn}{n} &\approx \begin{cases} 2t + t \log \frac{1}{t} & 0 < t < 1 \\ 2 + \log t & t > 1 \end{cases} \\
&\approx \inf_{\theta} \{t^\theta \theta^{-1}(1 - \theta)^{-1}\} \\
&= \tau(t).
\end{aligned}$$

Consequently, (9.5) holds if we select ν to be the delta-function at 1, $\nu = \delta_1$. Indeed,

$$\begin{aligned}
\int_0^\infty \int_0^\infty \min\left(1, \frac{t}{n}\right) \min\left(1, \frac{n}{r}\right) \frac{dn}{n} d\nu(r) &= \int_0^\infty \int_0^\infty \min\left(1, \frac{t}{n}\right) \min\left(1, \frac{n}{r}\right) \frac{dn}{n} d\delta_{r=1}(r) \\
&= \int_0^\infty \min\left(1, \frac{t}{n}\right) \min(1, n) \frac{dn}{n}.
\end{aligned}$$

Example 9.5. (Yano's bilinear extrapolation) Let $\vec{A}, \vec{B}, \vec{C}$, be mutually closed ordered (with constant 1) Banach pairs, moreover, let M, N and τ be as in the previous Example. Suppose that T is a bilinear operator such that (9.2) holds. Then

(i)

$$T : \langle \vec{A} \rangle_{0,1;K} \times \langle \vec{B} \rangle_{0,1;K} \rightarrow C_0.$$

(ii)

$$T : A_1 \times B_1 \rightarrow \text{Exp}(\vec{C}) = \{f : \|f\|_{\text{Exp}(\vec{C})} = \sup_{0 < t < 1} \frac{K(t, f; \vec{C})}{t(1 + \log \frac{1}{t})} < \infty\}.$$

Proof. (i) By Theorem 9.3 and the previous example

$$\begin{aligned} K(t, T(f, g), \vec{C}) &\preceq \int_0^\infty \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K\left(\frac{u}{r}, g; \vec{B}\right) d\nu(r) \frac{du}{u} \\ &= \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u}. \end{aligned}$$

Letting $t = 1$, we get

$$\begin{aligned} \|T(f, g)\|_{C_0} &= K(1, T(f, g), \vec{C}) \\ &\preceq \int_0^\infty K\left(\frac{1}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} \\ &= \int_0^1 K\left(\frac{1}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} + \int_1^\infty K\left(\frac{1}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u}. \end{aligned}$$

Now,

$$K\left(\frac{1}{u}, f; \vec{A}\right) \chi_{(0,1)}(u) \leq \|f\|_{A_0}, \text{ while } K(u, g; \vec{B}) \chi_{(1,\infty)}(u) \leq \|g\|_{B_0}.$$

Consequently,

$$\begin{aligned} \|T(f, g)\|_{C_0} &\preceq \|f\|_{A_0} \int_0^1 K(u, g; \vec{B}) \frac{du}{u} + \|g\|_{B_0} \int_1^\infty K\left(\frac{1}{u}, f; \vec{A}\right) \frac{du}{u} \\ &\preceq \|f\|_{A_0} \|g\|_{\langle \vec{B} \rangle_{0,1;K}} + \|g\|_{B_0} \|f\|_{\langle \vec{A} \rangle_{0,1;K}} \\ &\preceq \|f\|_{\langle \vec{A} \rangle_{0,1;K}} \|g\|_{\langle \vec{B} \rangle_{0,1;K}}, \end{aligned}$$

as we wished to show.

(ii) For $t \in (0, 1)$ we can write

$$\begin{aligned} K(t, T(f, g), \vec{C}) &\preceq \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} \\ &\leq \int_0^t \dots \frac{du}{u} + \int_t^1 \dots \frac{du}{u} + \int_1^\infty \dots \frac{du}{u} \\ &= (I) + (II) + (III). \end{aligned}$$

We estimate the terms on the right hand side as follows,

$$\begin{aligned}
(I) &\leq \int_0^t K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} \\
&\leq \|f\|_{A_0} \int_0^t \|g\|_{B_1} du \\
&\leq t \|f\|_{A_0} \|g\|_{B_1}.
\end{aligned}$$

Likewise,

$$\begin{aligned}
(II) &= \int_t^1 K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} \\
&\leq \|f\|_{A_1} \int_t^1 \frac{t}{u} K(u, g; \vec{B}) \frac{du}{u} \\
&\leq \|f\|_{A_1} \|g\|_{A_1} t \int_t^1 \frac{du}{u} \\
&\leq \|f\|_{A_1} \|g\|_{A_1} t \log \frac{1}{t},
\end{aligned}$$

$$\begin{aligned}
(III) &= \int_1^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u} \\
&\leq \|f\|_{A_1} \|g\|_{A_0} t \int_1^\infty \frac{du}{u^2} \\
&= \|f\|_{A_1} \|g\|_{A_0} t.
\end{aligned}$$

Consequently,

$$\begin{aligned}
\frac{K(t, T(f, g), \vec{C})}{t(1 + \log \frac{1}{t})} &\preceq \frac{t \|f\|_{A_0} \|g\|_{B_0}}{t(1 + \log \frac{1}{t})} + \frac{\|f\|_{A_1} \|g\|_{A_1} t \log \frac{1}{t}}{t(1 + \log \frac{1}{t})} + \frac{\|f\|_{A_1} \|g\|_{A_0} t}{t(1 + \log \frac{1}{t})} \\
&\preceq \|f\|_{A_1} \|g\|_{B_1}.
\end{aligned}$$

□

Example 9.6. For a finite measure space we let $\vec{A} = \vec{B} = \vec{C} = (L^1, L^\infty)$. Then, if $T : L(p, 1) \times L(p, 1) \rightarrow L^1$, with norm $\sim \frac{p}{p-1}$, $1 < p < \infty$, then

(i)

$$T : L(\log L) \times L(\log L) \rightarrow L^1.$$

(ii)

$$T : L^\infty \times L^\infty \rightarrow e^L.$$

Proof. (i) Follows from the previous Example if we recall that $\langle L^1, L^\infty \rangle_{0,1;K} = L(\log L)$.

(ii) Since $K(t, f; L^1, L^\infty) = t f^{**}(t)$, we have

$$\|f\|_{Exp(L^1, L^\infty)} = \sup_{0 < t < 1} \frac{K(t, f; \vec{C})}{t(1 + \log \frac{1}{t})} = \sup_{0 < t < 1} \frac{f^{**}(t)}{(1 + \log \frac{1}{t})}$$

and therefore (cf. [43])

$$Exp(L^1, L^\infty) = e^L.$$

□

Remark 9.7. Let us note that part (ii) of Example 9.5 can be obtained directly by linear extrapolation. Indeed, freezing one variable, say by letting $f \in A_1$ be fixed, then we have a linear operator $T_f := T(f, \circ)$:

$$T_f : M(\theta)N(\theta)\vec{B}_{\theta,1;J}^{\blacktriangleleft} \rightarrow \vec{C}_{\theta,\infty;K}^{\blacktriangleleft}$$

yielding

$$\|T_f(g)\|_{Exp(\vec{C})} \preceq \|f\|_{A_1} \|g\|_{B_1}.$$

Example 9.8. The case where the norm decays as $\sim \left(\frac{p}{p-1}\right)^\alpha$, $\alpha > 1$, was treated in [44, Theorem 4.10].

Problem 9.9. Develop the corresponding extrapolation results for the general case. That is suppose now that $\{\rho_{\theta,a}\}_{\theta \in (0,1)}$, $\{\rho_{\theta,b}\}_{\theta \in (0,1)}$, $\{\rho_{\theta,c}\}_{\theta \in (0,1)}$ are families of quasi-concave functions, let $M(\theta), N(\theta)$ be weights, and let T be a bilinear operator such that for all $\theta \in (0,1)$,

$$T : M(\theta)\vec{A}_{\rho_{\theta,a},1;J}^{\blacktriangleleft} \times N(\theta)\vec{B}_{\rho_{\theta,b},1;J}^{\blacktriangleleft} \rightarrow \vec{C}_{\rho_{\theta,c},\infty;K}^{\blacktriangleleft}, \text{ with norm } 1, \theta \in (0,1).$$

Then the following K/J inequality holds (compare with (9.3)): for all $t, u, s > 0$, $f \in \Delta(\vec{A}), g \in \Delta(\vec{B})$,

$$(9.6) \quad K(t, T(f, g); \vec{C}) \leq J(s, f; \vec{A})J(u, g; \vec{B}) \inf \left\{ \frac{\rho_{\theta,c}(t)}{\rho_{\theta,a}(s)\rho_{\theta,b}(u)} M(\theta)N(\theta) \right\},.$$

Develop the corresponding K/J inequalities and extrapolation results (cf. [44]).

Problem 9.10. (Open ended) Using Theorem 9.1 and the results of the previous Problem develop a complete theory of bilinear extrapolation.

Problem 9.11. Part of the difficulties of dealing with extrapolation of bilinear operators lies with the theory of representation of concave functions of two variables. We ask for a representation formula for concave functions of two variables.

Problem 9.12. Is it possible to eliminate the restriction for τ to be adequate (cf. (9.5))?

Problem 9.13. Arrange conditions on the function τ so that to be able to control the decay in each component separately and in this fashion prove bilinear results where the resulting spaces are different in each component.

Problem 9.14. In view of Example 4.15, Theorem 9.1 and Example 9.5 one may extend the definition of weak type $(1,1), (\infty, \infty)$ of [27] to the bilinear case by demanding that a bilinear operator satisfies

$$(9.7) \quad K(t, T(f, g), \vec{C}) \preceq \int_0^\infty K\left(\frac{t}{u}, f; \vec{A}\right) K(u, g; \vec{B}) \frac{du}{u}.$$

For example, one can then ask for an extension to bilinear operators of the result in [14]: Do bilinear mappings such that $T : L^1 \times L^1 \rightarrow L(1,1)$, and $L^\infty \times L^\infty \rightarrow L(\infty, \infty)$ satisfy (9.7)?

10. CONVERSE TO YANO'S THEOREM: TAO'S THEOREM

It is natural to ask if one can prove a converse to Yano's theorem. In other words, suppose that an operator T is bounded, $T : L(\text{Log} L)^\alpha(0, 1) \rightarrow L^1(0, 1)$, then we ask: does there exist a constant c such that $\|T\|_{L^p \rightarrow L^p} \leq \frac{c}{(p-1)^\alpha}$, as $p \rightarrow 1$?

It is well known and easy to see that, for general operators, the answer is negative. and therefore one needs to impose more assumptions on the operators. A positive result in this direction was obtained by Tao [71], who considers translation invariant operators. More precisely, Tao considered translation invariant operators T defined on compact symmetric spaces X that are provided with a compact symmetry group G . In this context Tao [71] shows that

$$(10.1) \quad \|T\|_{L^p(X) \rightarrow L^p(X)} \leq \frac{c}{(p-1)^\alpha}, 1 < p < 2 \Leftrightarrow T : L(\text{Log} L)^\alpha(X) \rightarrow L^1(X).$$

So we ask

Problem 10.1. What other types of conditions guarantee the validity of a Yano-Tao theorem? In particular, we ask this question in the context of non-commutative L^p spaces.

10.1. Multiplier Problem II: Equivalence of K -functional inequalities. In extrapolation, constants can create what, at first, may appear to be unexpected effects. The situation that we shall describe now already appears in some form or other in previous discussions but here we shall consider only a typical concrete example connected with the extrapolation of $L^p := L^p(0, 1)$ spaces. Let us consider the "Yano situation":

$$(10.2) \quad T : \left\{ \frac{p}{p-1} L^p \right\}_{p>1} \rightarrow \{L^p\}_{p>1},$$

which, as we know, yields, via the \sum -method:

$$(10.3) \quad T : L \log L = \sum_{p>1} \left\{ \frac{p}{p-1} L^p \right\}_{p>1} \rightarrow \sum_{p>1} \{L^p\}_{p>1} = L^1.$$

Now, multiplying the underlying inequalities of (10.2) by $\{\frac{p}{p-1}\}$ yields

$$(10.4) \quad T : \left\{ \left(\frac{p}{p-1} \right)^2 L^p \right\}_{p>1} \rightarrow \left\{ \left(\frac{p}{p-1} \right) L^p \right\}_{p>1}$$

which, once again by the \sum -method, gives

$$(10.5) \quad T : L(\log L)^2 = \sum_{p>1} \left\{ \left(\frac{p}{p-1} \right)^2 L^p \right\}_{p>1} \rightarrow \sum_{p>1} \left\{ \left(\frac{p}{p-1} \right) L^p \right\}_{p>1} = L \text{Log} L.$$

Conversely, starting from (10.4) we can, by multiplication by $\{\frac{p-1}{p}\}$ return (10.2). On the other hand, once we have applied the \sum -functor we cannot claim the equivalence between the pair of estimates $\{(10.2) \text{ and } (10.3)\}$ or the equivalence between $\{(10.4) \text{ and } (10.5)\}$, unless we have extra assumptions (e.g. Tao's theorem (cf. Section 10)). On the other hand, if we apply the corresponding K -functionals for scales, we know that (10.2) is equivalent to (cf. (2.8) and (4.3))

$$(10.6) \quad K(t, Tf; \{L^p\}_{p>1}) \leq cK(t, f; \left\{ \frac{p}{p-1} L^p \right\}_{p>1}),$$

and likewise (10.4) is equivalent to

$$(10.7) \quad K(t, Tf; \{\frac{p}{p-1} L^p\}_{p>1}) \leq cK(t, f; \{\left(\frac{p}{p-1}\right)^2 L^p\}_{p>1}).$$

Therefore, since (10.2) and (10.4) are equivalent we see that (10.6) and (10.7) are equivalent. These K -functional estimates can be made explicit, (cf. Examples 4.12, 4.14, 4.15)

$$\begin{aligned} K(t, Tf; \{L^p\}_{p>1}) &\approx K(t, Tf; \{(L^1, L^\infty)_{1/p', \infty; K}^\blacktriangleleft\}_{p>1}) \\ &\approx K(t, Tf; L^1, L^\infty) = \int_0^t (Tf)^*(s) ds, \\ K(t, f; \{\frac{p}{p-1} L^p\}_{p>1}) &\approx K(t, f; \{\frac{p}{p-1} (L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft\}_{p>1}) \\ &\approx \int_0^t f^*(s) \log \frac{t}{s} ds. \end{aligned}$$

Likewise,

$$\begin{aligned} K(t, f; \{\left(\frac{p}{p-1}\right)^2 L^p\}_{p>1}) &\approx K(t, f; \{\left(\frac{p}{p-1}\right)^2 (L^1, L^\infty)_{1/p', 1; J}^\blacktriangleleft\}_{p>1}) \\ &\approx \int_0^t f^*(s) (\log \frac{t}{s})^2 ds. \end{aligned}$$

So, we get the equivalence of the rearrangement inequalities

$$(10.8) \quad \int_0^t (Tf)^*(s) ds \preceq \int_0^t f^*(s) \log \frac{t}{s} ds \quad \text{and} \quad \int_0^t (Tf)^*(s) \log \frac{t}{s} ds \preceq \int_0^t f^*(s) (\log \frac{t}{s})^2 ds.$$

Usually in the classical papers such results are described as a *gain* or *loss* of logarithms.

Here is another example that comes from classical interpolation. Consider informally the ultra classical situation (Calderón's Theorem): Let T be a linear operator such that $T : L^1 \rightarrow L^1$, and $T : L^\infty \rightarrow L^\infty$. These conditions are equivalent to

$$K(t, Tf; L^1, L^\infty) \preceq K(t, f; L^1, L^\infty)$$

which yields

$$(10.9) \quad \int_0^t (Tf)^*(s) ds \preceq \int_0^t f^*(s) ds.$$

Comparing with (10.8) shows the lack of logarithms on the right hand side, reflecting that the corresponding deterioration *weight* is a constant: $M(p) \approx 1$. Now from (10.9) we see that, by the Calderón-Mityagin theorem, with constants independent of p

$$T : L^p \rightarrow L^p, \quad p \geq 1.$$

In particular, multiplication by $\{\frac{p}{p-1}\}$ yields

$$T : \{\frac{p}{p-1} L^p\}_{p>1} \rightarrow \{\frac{p}{p-1} L^p\}_{p>1},$$

which as we have seen is equivalent²⁸ now to

$$\int_0^t (Tf)^*(s) \log \frac{t}{s} ds \preceq \int_0^t f^*(s) \log \frac{t}{s} ds.$$

So in this case there is no *gain* of logarithm, as indeed it should be, since the assumption that $T : L^1 \rightarrow L^1$ and $T : L^\infty \rightarrow L^\infty$ is essentially stronger. We also note that once explicit inequalities are written down they can be proved by more direct methods. This is certainly the case of rearrangement inequalities.

Problem 10.2. Give a direct proof of the equivalence of rearrangement inequalities (10.8).

Problem 10.3. Let T be a bounded operator, $T : L^p[0, 1] \rightarrow L^p[0, 1]$, $1 < p < \infty$, and let $\Phi(p) = \|T\|_{p \rightarrow p}$. Characterize in terms of Φ the functions $\varphi : [1, \infty) \rightarrow [0, \infty)$, $\varphi(1) = 0$, that make the inequalities

$$\begin{aligned} \int_0^t (Tf)^*(s) ds &\preceq \int_0^t f^*(s) \varphi(t/s) ds, \quad 0 < t \leq 1, \\ \int_0^t (Tf)^*(s) \varphi(t/s) ds &\preceq \int_0^t f^*(s) (\varphi(t/s))^2 ds, \quad 0 < t \leq 1, \end{aligned}$$

equivalent for all $f \in L^\infty[0, 1]$.

11. NON-COMMUTATIVE CALDERÓN OPERATOR AND EXTRAPOLATION

Let \mathcal{N} be a semifinite von Neumann algebra on a Hilbert space \mathcal{H} equipped with a faithful normal semifinite trace τ , $\mathfrak{S}^p(\mathcal{N})$ be the corresponding Schatten-von Neumann classes, $\mathcal{M}^1(\mathcal{N})$ be the Matsaev ideal, $\mu = \mu(t, A)$ be the *-operation in the non-commutative setting (see for the definitions [51] and Example 2.8). Moreover, let S be the Calderón operator, defined by

$$(11.1) \quad Sf(t) := \frac{1}{t} \int_0^t f(s) ds + \int_t^\infty \frac{f(s)}{s} ds, \quad t > 0.$$

Let $\mathcal{T} : \mathfrak{S}^2(\mathcal{N}) \rightarrow \mathfrak{S}^2(\mathcal{N})$ be a selfadjoint contraction. Suppose that \mathcal{T} admits a bounded linear extension on $\mathfrak{S}^p(\mathcal{N})$, for all $1 < p \leq 2$. If

$$(11.2) \quad \|\mathcal{T}\|_{\mathfrak{S}^p(\mathcal{N}) \rightarrow \mathfrak{S}^p(\mathcal{N})} \leq \frac{Cp}{p-1}, \quad 1 < p \leq 2,$$

then it is shown in [70] that with some absolute constant C

$$(11.3) \quad \frac{1}{t} \int_0^t \mu(s, \mathcal{T}(A)) ds \leq C \frac{1}{t} \int_0^t S(\mu(\cdot, A))(s) ds, \quad t > 0, \quad A \in \mathcal{M}^1(\mathcal{N}).$$

Let us show how this result can be obtained by extrapolation. Indeed, we will show that (11.3) is the exact non-commutative analogue of Calderón's result, whose abstract extrapolation extension was formulated in [43] and discussed at length in Example 4.15. Let us present the details. For this purpose it will be convenient to let $Pf(t) := \frac{1}{t} \int_0^t f(s) ds$, $Qf(t) = \int_t^\infty f(s) \frac{ds}{s}$. Then, we can rewrite (11.3) as

$$(11.4) \quad P(\mu(\cdot, \mathcal{T}(A)))(t) \leq CP(S(\mu(\cdot, A)))(t).$$

²⁸This equivalence in principle is valid for functions in $L^1 \cap L^\infty$.

Moreover, from the definitions of P and Q , (11.1), and a simple computation, we have that

$$\begin{aligned} S &= P + Q \\ &= PQ \\ &= QP. \end{aligned}$$

Therefore

$$PS = P(QP) = PQP = QPP = SP$$

and we can rewrite (11.4) as

$$(11.5) \quad P(\mu(\cdot, \mathcal{T}(A))(t) \leq C(SP(\mu(\cdot, A)))(t).$$

On the other hand, if we combine the assumption (11.2) with the fact that \mathcal{T} is selfadjoint, and the duality formula $\mathfrak{S}^p(\mathcal{N})^* = \mathfrak{S}^{p'}(\mathcal{N})$, we see that

$$(11.6) \quad \|\mathcal{T}\|_{\mathfrak{S}^p(\mathcal{N}) \rightarrow \mathfrak{S}^p(\mathcal{N})} \leq \frac{Cp^2}{p-1}, \quad 1 < p < \infty.$$

Now taking into account the classical computation of K -functionals for non-commutative L^p spaces (cf. [65], and Example 2.8)

$$(11.7) \quad K(t, A, \mathfrak{S}^1(\mathcal{N}), \mathfrak{S}^\infty(\mathcal{N})) = \int_0^t \mu(s, A) ds$$

yields, just like in the commutative case (cf. [56]), that

$$(11.8) \quad \mathfrak{S}^p(\mathcal{N}) = (\mathfrak{S}^1(\mathcal{N}), \mathfrak{S}^\infty(\mathcal{N}))_{1/p', p; K}^\blacktriangleleft.$$

Given (11.6) and (11.8) we can apply the extrapolation theorem of [43] (discussed extensively in Example 4.15 above) to obtain an absolute constant C such that

$$\frac{K(t, \mathcal{T}(A), \mathfrak{S}^1(\mathcal{N}), \mathfrak{S}^\infty(\mathcal{N}))}{t} \leq C S\left(\frac{K(s, A, \mathfrak{S}^1(\mathcal{N}), \mathfrak{S}^\infty(\mathcal{N}))}{s}\right)(t).$$

Finally, combining the last inequality with the formula for the K -functional provided by (11.7) yields (11.5), as we wished to show.

Likewise, it is shown in [70, Theorem 14 (ii)] that if \mathcal{T} is assumed to be of weak type $(1, 1)$, then one can replace the non-commutative averaging operator P by the corresponding non-commutative $*$ -operation. Again from the previous discussion and Example 2.8 we see that this last result follows from Jawerth-Milman [43, Proposition 5.2.2, page 50].

In particular, our extrapolation method allows to treat more general type of norm decays, e.g. $\sim \left(\frac{p}{p-1}\right)^\alpha$, $\alpha > 1$, weak type versions, etc.

Problem 11.1. In view of the previous discussion and Example 2.8 we are asking for the non-commutative (∞, ∞) version of the Bennett-DeVore-Sharpely theorem. In this connection we ask: Formulate interpolation/extrapolation theorems for operators assuming properties of their adjoints. For example, what can be said about operators T such that T and T^* are weak type $(1, 1)$?

12. MORE OPEN ENDED PROBLEMS

12.1. Gagliardo coordinate spaces and Extrapolation.

Problem 12.1. This project asks to incorporate the “Gagliardo coordinate spaces” (cf. [57]) to extrapolation theory. Let $\theta \in [0, 1]$, $q \in (0, \infty]$. We define the spaces

$$(X_1, X_2)_{\theta, q}^{(1)} = \left\{ f \in X_1 + X_2 : \|f\|_{(X_1, X_2)_{\theta, q}^{(1)}} < \infty \right\},$$

where

$$\|f\|_{(X_1, X_2)_{\theta, q}^{(1)}} = \left\{ \int_0^\infty \left(t^{1-\theta} \left[\frac{K(t, f; X_1, X_2)}{t} - K'(t, f; X_1, X_2) \right] \right)^q \frac{dt}{t} \right\}^{1/q},$$

and

$$(X_1, X_2)_{\theta, q}^{(2)} = \left\{ f \in X_1 + X_2 : \|f\|_{(X_1, X_2)_{\theta, q}^{(2)}} < \infty \right\},$$

where

$$\|f\|_{(X_1, X_2)_{\theta, q}^{(2)}} = \left\{ \int_0^\infty (t^{-\theta} t K'(t, f; X_1, X_2))^q \frac{dt}{t} \right\}^{1/q}.$$

The Gagliardo coordinate spaces in principle are not linear, and the corresponding functionals, $\|f\|_{(X_1, X_2)_{\theta, q}^{(i)}}$, $i = 1, 2$, are not norms. However, it turns out that, when $\theta \in (0, 1)$, $q \in (0, \infty]$, we have, with *norm* equivalence (cf. [36], [42], [57]),

$$(12.1) \quad (X_1, X_2)_{\theta, q}^{(1)} = (X_1, X_2)_{\theta, q}^{(2)} = (X_1, X_2)_{\theta, q}.$$

More precisely, the “norm” equivalence depends only on θ , and q . On the other hand, at the end points, $\theta = 0$ or $\theta = 1$, the resulting spaces can be very different.

Let $(X_1, X_2) = (L^1, L^\infty)$. Then, if $\theta = 1$, $q = \infty$, we obtain the space introduced by Bennett-DeVore-Sharpely [14]

$$(12.2) \quad \|f\|_{(L^1, L^\infty)_{1, \infty}^{(1)}} = \|f\|_{L(\infty, \infty)} = \sup\{f^{**}(t) - f^*(t)\}, \quad (L^1, L^\infty)_{0, \infty}^{(2)} = L(1, \infty).$$

It was shown in [14] that in the case of finite measure $L(\infty, \infty)$ is the rearrangement invariant hull of BMO. The corresponding space that one obtains when $q < \infty$, $L(\infty, q)$, also makes sense, and was first introduced by Bastero-Milman-Ruiz [13] who showed a sharp end point for the Sobolev embedding theorem

$$\|f\|_{L(\infty, n)} \leq c \|\nabla f\|_{W_n^1(\mathbb{R}^n)}, \quad f \in C_0^\infty(\mathbb{R}^n).$$

More generally these spaces play an important rôle in the theory of Sobolev inequalities (cf. [58]). This justifies the interest in the following

Problem 12.2. (Open Ended) We ask to incorporate the Gagliardo coordinate spaces to Extrapolation Theory. In particular, find the constants of basic interpolation inequalities connected with Gagliardo coordinate spaces. In this direction some results were obtained in [57].

Problem 12.3. $L(\infty, \infty)$ vs e^L ? Here is a concrete extrapolation problem concerning the space $L(\infty, \infty)$. What extra conditions are needed to be able to extrapolate weak type (∞, ∞) from the usual extrapolation assumptions? More concretely,

suppose that $\|Tf\|_{L^p} \leq c_p \|f\|_{L^p}$ with $c_p = c \frac{p^2}{p-1}$, for large p , under what extra conditions can we conclude that restricted to simple functions, $T : L^\infty \rightarrow L(\infty, \infty)$?

12.2. Calderón-Mityagin Scales. The characterization of Calderón-Mityagin pairs has been extensively studied in the context of interpolation theory (cf. [18] and the references therein). The concept can be extended to scales of spaces. Here we consider only one of the simplest possible definitions (cf. [43, page 71-72] for a more general formulation). Let $\{F_\theta\}_{\theta \in I}$, $\{G_\theta\}_{\theta \in I}$ be two families of interpolation functors of exact type θ , and let $M(\theta)$ be a tempered weight. Let \vec{A}, \vec{B} be pairs of mutually closed Banach spaces. We shall say that the pair of scales $(\{F_\theta(\vec{A})\}_{\theta \in I}, \{G_\theta(\vec{B})\}_{\theta \in I})$ is a *relative Calderón-Mityagin M -pair of scales*, if given $a \in \sum A_\theta$ and $b \in \sum B_\theta$, such that

$$K(t, b; \{G_\theta(\vec{B})\}_{\theta \in I}) \leq cK(t, a; \{M(\theta)F_\theta(\vec{A})\}_{\theta \in I}), \quad t > 0,$$

where $c > 0$ is a constant independent of $t > 0$, it follows that there exist an operator T and constant $C > 0$ such that

$$T : \{M(\theta)F_\theta(\vec{A})\}_{\theta \in I} \xrightarrow{C} \{G_\theta(\vec{B})\}_{\theta \in I}, \quad \text{with } Ta = b.$$

Example 12.4 (cf. [43, Theorem 6.4]). Let $M(\theta)$ be such that $\tau(t) = \inf_{0 < \theta < 1} (M(\theta)t^\theta)$ is a C^2 -function with $-t^2\tau''(t)$ quasi-concave. Let \vec{A}, \vec{B} be mutually closed pairs, and let $F_\theta(\vec{A}) = \vec{A}_{\theta,1;J}^\blacktriangleleft$, and $G_\theta(\vec{B}) = \vec{B}_{\theta,\infty;K}^\blacktriangleleft$. Then $(\{\vec{A}_{\theta,1;J}^\blacktriangleleft\}_{\theta \in I}, \{\vec{B}_{\theta,\infty;K}^\blacktriangleleft\}_{\theta \in I})$ is a Calderón-Mityagin M -pair of scales.

Example 12.5 (cf. [43, Corollary 6.6]). Under the same assumptions as in the previous example, suppose that $1 \leq q(\theta) \leq \infty$. Let $F_\theta(\vec{A}) = \vec{A}_{\theta,q(\theta);J}^\blacktriangleleft$, and $G_\theta(\vec{B}) = \vec{B}_{\theta,q(\theta);K}^\blacktriangleleft$. Then, $(\{\vec{A}_{\theta,q(\theta);J}^\blacktriangleleft\}_{\theta \in I}, \{\vec{B}_{\theta,q(\theta);K}^\blacktriangleleft\}_{\theta \in I})$ is a Calderón-Mityagin M -pair of scales.

Example 12.6. If $F_{1/p'}(\vec{A}) = G_{1/p'}(\vec{A}) = \vec{A}_{1/p',p;K}$, then the previous Example implies that $(\{L^p\}_{p>1}, \{L^p\}_{p>1})$ is a Calderón-Mityagin 1-pair of scales (this is essentially a reformulation of the classical result of Calderón-Mityagin).

Problem 12.7. Let $\{F_\theta\}_{\theta \in I}$, $\{G_\theta\}_{\theta \in I}$ be two families of interpolation functors of exact type θ , and let $M(\theta)$ be a weight. Find sharp conditions on M so that for all \vec{A}, \vec{B} be mutually closed pairs $(\{F_\theta(\vec{A})\}_{\theta \in I}, \{G_\theta(\vec{B})\}_{\theta \in I})$ is a Calderón-Mityagin M -pair of scales.

12.3. Complex Extrapolation (open ended project). Given the central rôle of complex methods in Interpolation theory it is somewhat surprising that so far there has been little progress in the direction of developing connections between complex methods and Extrapolation theory. The general interpolation methods introduced in [22] provide a unification of real and complex interpolation. We now give a brief summary of the basic definitions.

The spaces introduced in [22] are based on an extension of the concept of “lattice”. Let **Ban** be the class of all Banach spaces over the complex numbers. Then we say that a mapping $\mathcal{X} : \mathbf{Ban} \rightarrow \mathbf{Ban}$ is a pseudolattice (or a pseudo- Z -lattice), if it satisfies the following conditions

(i) for each $B \in \mathbf{Ban}$ the space $\mathcal{X}(B)$ consists of B -valued sequences $\{b_n\}_{n \in \mathbb{Z}}$;
(ii) whenever A is a closed subspace of B it follows that $\mathcal{X}(A)$ is a closed subspace of $\mathcal{X}(B)$;

(iii) there exists a positive constant $C = C(\mathcal{X})$ such that, for all $A, B \in \mathbf{Ban}$ and all bounded linear operators $T : A \rightarrow B$ and every sequence $\{a_n\}_{n \in \mathbb{Z}} \in \mathcal{X}(A)$, the sequence $\{Ta_n\}_{n \in \mathbb{Z}} \in \mathcal{X}(B)$ and satisfies the estimate

$$\|\{Ta_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}(B)} \leq C(\mathcal{X})\|T\|_{A \rightarrow B}\|\{a_n\}\|_{\mathcal{X}(A)};$$

(iv)

$$\|b_m\|_B \leq \|\{b_n\}\|_{\mathcal{X}(B)}$$

for all $m \in \mathbb{Z}$, all $\{b_n\}_{n \in \mathbb{Z}} \in \mathcal{X}(B)$ and all Banach spaces B .

Example of pseudo-lattices: Lattices; the Fourier spaces $FL^1, FC; UC$; the space of unconditionally convergent series; WUC , the space of weakly unconditionally convergent sequences. We refer to [22] for complete details.

For each Banach pair \vec{B} and each pair $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1)$ of pseudolattices, let $\mathcal{J}(\mathbf{X}, \vec{B})$ to be the space of all $B_0 \cap B_1$ valued sequences $\{b_n\}_{n \in \mathbb{Z}}$ for which the sequence $\{e^{jn}b_n\}_{n \in \mathbb{Z}}$ is in $\mathcal{X}_j(B_j)$ for $j = 0, 1$. This space is normed by

$$\|\{b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(\mathbf{X}, \vec{B})} := \max_{j=0,1} \|\{e^{jn}b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{X}_j(B_j)}.$$

For each Banach pair \vec{B} , each pair of pseudolattices \mathbf{X} as above, and each fixed $s \in \mathbb{A} = \{z \in \mathbb{C} : 1 < |z| < e\}$, the spaces $\vec{B}_{\mathbf{X},s}$ consist of all elements of the form $b = \sum_{n \in \mathbb{Z}} s^n b_n$ where $\{b_n\}_{n \in \mathbb{Z}} \in \mathcal{J}(\mathbf{X}, \vec{B})$, with the natural quotient norm

$$(12.3) \quad \|b\|_{\vec{B}_{\mathbf{X},s}} := \inf \left\{ \|\{b_n\}_{n \in \mathbb{Z}}\|_{\mathcal{J}(\mathbf{X}, \vec{B})} : b = \sum_{n \in \mathbb{Z}} s^n b_n \right\}.$$

Examples: Let $s = e^\theta$ for some $\theta \in (0, 1)$. (i) If $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1)$, with $\mathcal{X}_0 = \mathcal{X}_1 = \ell^p$, then space $\vec{B}_{\mathbf{X},s}$ coincides with the Lions-Peetre real method space $\vec{B}_{\theta,p} = (B_0, B_1)_{\theta,p}$; (ii) If $\mathbf{X} = (\mathcal{X}_0, \mathcal{X}_1)$, with $\mathcal{X}_0 = \mathcal{X}_1 = FC$, then $\vec{B}_{\mathbf{X},s}$ coincides, to within equivalence of norm, with the Calderón complex method space $\vec{B}_{[\theta]} = [B_0, B_1]_\theta = [\vec{B}]_\theta$; (iii) Likewise, if $\mathcal{X}_0 = \mathcal{X}_1 = UC$, then $\vec{B}_{\mathbf{X},s}$ is the Peetre \pm method space $\vec{B}_{\langle \theta \rangle} = \langle B_0, B_1 \rangle_\theta$; If we replace UC by WUC , in (iii) we obtain the Gustavsson-Peetre variant of $\langle B_0, B_1 \rangle_\theta$ which is denoted by $\langle \vec{B}, \rho_\theta \rangle$ (cf. [39]).

Problem 12.8. We ask to incorporate the interpolation spaces $\vec{B}_{\mathbf{X},s}$ of [22] to Extrapolation theory. In particular, we ask for an extrapolation version of the classical interpolation theorem for analytic families of operators (cf. [24]). For a different possible connection between complex methods and extrapolation we refer to [50] and the references therein.

13. APPENDIX

Let us start recalling some definitions from interpolation theory (cf. [16]). The classical theory of interpolation deals with pairs of compatible Banach spaces (“Banach pairs” or simply “pairs”), $\vec{A} = (A_0, A_1)$, which are contained in a suitable

larger Hausdorff topological space. We say that a Banach space H is intermediate with respect to \vec{A} , if $A_0 \cap A_1 \subset H \subset A_0 + A_1$. Given two Banach pairs \vec{A} and \vec{B} , let H be intermediate for the pair \vec{A} and let G be intermediate for the pair \vec{B} , we then say that the Banach spaces H, G , are interpolation spaces with respect to the pairs \vec{A} and \vec{B} if any operator T that is bounded from \vec{A} to \vec{B} ²⁹ also defines a bounded operator $T : H \rightarrow G$. An “interpolation method” is a functor F that assigns to each pair \vec{A} an interpolation space $F(\vec{A})$ so that for all linear operators T that are bounded from \vec{A} to \vec{B} we have that $T : F(\vec{A}) \rightarrow F(\vec{B})$ is bounded. An interpolation method is exact if $\|T\|_{F(\vec{A}) \rightarrow F(\vec{B})} \leq \max\{\|T\|_{A_0 \rightarrow B_0}, \|T\|_{A_1 \rightarrow B_1}\} := \|T\|_{\vec{A} \rightarrow \vec{B}}$.

13.1. The K and J methods of interpolation. It can be argued that the most successful method of real interpolation is the one based on using the K -functional of Peetre³⁰. The method explicitly provides a penalty problem on the splitting of elements that underlies the Lions-Peetre method of interpolation.

Recall that given a compatible pair of Banach spaces $\vec{X} = (X_0, X_1)$ we let, for $f \in X_0 + X_1, t > 0$,

$$K(t, f; \vec{X}) := \inf_{f=f_0+f_1, f_i \in X_i} \{\|f_0\|_{X_0} + t\|f_1\|_{X_1}\}.$$

It follows immediately that if T is a bounded operator, $T : \vec{X} \rightarrow \vec{Y}$, then

$$K(t, Tf; \vec{Y}) \preceq K(t, f; \vec{X}), \quad t > 0.$$

If ρ is any function norm on measurable functions on $(0, \infty)$ then³¹

$$\rho(K(\cdot, Tf; \vec{Y})) \preceq \rho(K(\cdot, f; \vec{X})).$$

In particular, let $0 < \theta < 1$, and $1 \leq q \leq \infty$, and consider the function norms

$$\Phi_{\theta, q}(f) := \begin{cases} \left\{ \int_0^\infty (s^{-\theta} |f(s)|)^q \frac{ds}{s} \right\}^{1/q} & \text{if } q < \infty \\ \sup_{s>0} \{s^{-\theta} |f(s)|\} & \text{if } q = \infty. \end{cases}$$

The Lions-Peetre interpolation spaces $\vec{X}_{\theta, q; K}$ consist of the elements $f \in X_0 + X_1$, such that $\|f\|_{\vec{X}_{\theta, q; K}} < \infty$, where

$$\|f\|_{\vec{X}_{\theta, q; K}} := \Phi_{\theta, q}(K(\cdot, f; \vec{X})).$$

We normalize the norms so that the interpolation functor $\vec{X} \rightarrow \vec{X}_{\theta, q; K}$ is of exact type θ , and for each Banach pair \vec{X} we denote the corresponding normalized spaces by $\vec{X}_{\theta, q; K}^\blacktriangleleft$:

$$(13.1) \quad \|f\|_{\vec{X}_{\theta, q; K}^\blacktriangleleft} := (q\theta(1-\theta))^{\frac{1}{q}} \|f\|_{\vec{X}_{\theta, q; K}},$$

²⁹That is $T : A_i \rightarrow B_i, i = 0, 1$.

³⁰Calderón and his student Oklander (cf. [62], [61]) independently also defined K -functionals for Banach pairs and implemented some of the early applications of K -functionals to interpolation theory. In particular, to weak type interpolation (cf. Section 4).

³¹It is often more convenient to write the inequalities in terms of decreasing functions and thus use the expression $\frac{K(t, f; \vec{X})}{t}$.

with the convention that $(q\theta(1-\theta))^{\frac{1}{q}} = 1$ when $q = \infty$. Our convention means that

$$\|\circ\|_{\vec{X}_{\theta,\infty;K}^{\blacktriangleleft}} = \|\circ\|_{\vec{X}_{\theta,\infty;K}}.$$

There is a dual construction associated with the J -functional which is defined for $g \in X_0 \cap X_1, t > 0$, by

$$J(t, g; \vec{X}) := \max\{\|g\|_{X_0}, t\|g\|_{X_1}\}.$$

The corresponding $\vec{X}_{\theta,q;J}$ spaces consist of all $g \in X_0 + X_1$ that can be represented as

$$(13.2) \quad g = \int_0^\infty u(t) \frac{dt}{t} \quad (\text{convergence in } X_0 + X_1),$$

for some strongly measurable function $u : (0, \infty) \rightarrow X_0 \cap X_1$ such that $\Phi_{\theta,q}(J(s, u(s); \vec{X})) < \infty$. We let

$$\|g\|_{\vec{X}_{\theta,q;J}} := \inf\{\Phi_{\theta,q}(J(s, u(s); \vec{X})) : g = \int_0^\infty u(t) \frac{dt}{t}\}.$$

The interpolation functor $\vec{X} \rightarrow \vec{X}_{\theta,q;J}$ can be normalized so that it becomes of exact type θ . This is achieved using the norms

$$(13.3) \quad \|g\|_{\vec{X}_{\theta,q;J}^{\blacktriangleleft}} := (q'\theta(1-\theta))^{-1/q'} \|g\|_{\vec{X}_{\theta,q;J}},$$

with the convention that if $q = 1$ we set $(q'\theta(1-\theta))^{-1/q'} = 1$. Thus,

$$\|\circ\|_{\vec{X}_{\theta,1;J}^{\blacktriangleleft}} = \|\circ\|_{\vec{X}_{\theta,1;J}}.$$

We will consider also the modified spaces $\langle \vec{X} \rangle_{\theta,q;K}$, defined by

$$(13.4) \quad \langle \vec{X} \rangle_{\theta,q;K} := \{f \in X_0 + X_1 : \|f\|_{\langle \vec{X} \rangle_{\theta,q;K}} := \Phi_{\theta,q}(\chi_{(0,1)} K(s, f; \vec{X})) < \infty\}$$

and the similarly constructed $\langle \vec{X} \rangle_{\theta,q;J}$ spaces (cf. [11]).

Example 13.1. From $K(t, f; L^1, L^\infty) = tf^{**}(t) = \int_0^t f^*(s)ds$, by Hardy's inequality and reverse Hardy's inequality for decreasing functions (cf. [56, Example 7]), it follows that

$$(L^1, L^\infty)_{1/p', p; K}^{\blacktriangleleft} = L^p, 1 < p < \infty,$$

with constants of norm equivalence independent of p .

13.1.1. The strong form of the fundamental Lemma. Underlying the equivalence of these methods is the fundamental Lemma of Interpolation theory (cf. [16]). The strong form of the fundamental Lemma can be found in [25] and states that there exists a constant γ such that if \vec{X} is a mutually closed pair then, for all $f \in X_0 + X_1$, such that $\lim K(t, f; \vec{X}) \min\{1, \frac{1}{t}\} = 0$ when $t \rightarrow 0$ and $t \rightarrow \infty$, and for all $\varepsilon > 0$, there exists $u : (0, \infty) \rightarrow X_0 \cap X_1$, strongly measurable, such that $f = \int_0^\infty u(s) \frac{ds}{s}$ and

$$\int_0^\infty \min(1, \frac{t}{s}) J(s, u(s); \vec{X}) \frac{ds}{s} \leq (\gamma + \varepsilon) K(t, f; \vec{X}).$$

It follows that there exists a decomposition $f = \int_0^\infty u(s) \frac{ds}{s}$ such that

$$K(t, f; \vec{X}) \approx \int_0^\infty \min(1, \frac{t}{s}) J(s, u(s); \vec{X}) \frac{ds}{s}, \quad t > 0.$$

Example 13.2. Let us show that for a mutually closed Banach pair \vec{X} we have

$$\vec{X}_{\theta,1;K}^\triangleleft = \vec{X}_{\theta,1;J}.$$

We shall use the elementary inequality (cf. [16, Lemma 3.2.1])

$$K(t, f; \vec{X}) \leq \min(1, \frac{t}{s}) J(s, f; \vec{X}), \quad t, s > 0.$$

Then, for any decomposition

$$f = \int_0^\infty u(s) \frac{ds}{s},$$

we have

$$K(t, f; \vec{X}) \leq \int_0^\infty \min(1, \frac{t}{s}) J(s, u(s); \vec{X}) \frac{ds}{s}.$$

Therefore,

$$\begin{aligned} \int_0^\infty K(t, f; \vec{X}) t^{-\theta} \frac{dt}{t} &\leq \int_0^\infty J(s, u(s); \vec{X}) \int_0^\infty \min(1, \frac{t}{s}) t^{-\theta} \frac{dt}{t} \frac{ds}{s} \\ &= \frac{1}{\theta(1-\theta)} \int_0^\infty J(s, u(s); \vec{X}) s^{-\theta} \frac{ds}{s}. \end{aligned}$$

Taking infimum over all such decompositions we find

$$\|f\|_{\vec{X}_{1/p',1;K}^\triangleleft} \leq \|f\|_{\vec{X}_{1/p',1;J}}.$$

On the other hand, applying the strong form of the fundamental Lemma, we can find a special decomposition such that

$$\int_0^\infty \min(1, \frac{t}{s}) J(s, u(s); \vec{X}) \frac{ds}{s} \leq \gamma K(t, f; \vec{X}).$$

Consequently,

$$\begin{aligned} \frac{1}{\theta(1-\theta)} \int_0^\infty J(s, u(s); \vec{X}) s^{-\theta} \frac{ds}{s} &= \int_0^\infty J(s, u(s); \vec{X}) \int_0^\infty \min(1, \frac{t}{s}) t^{-\theta} \frac{dt}{t} \frac{ds}{s} \\ &\leq \gamma \int_0^\infty K(t, f; \vec{X}) t^{-\theta} \frac{dt}{t} \end{aligned}$$

and the desired result follows.

Remark 13.3. For some problems it is useful to replace integrals by series. We refer to [25] for complete details.

13.2. Extreme extrapolation functors. In extrapolation the starting point are families of interpolation spaces, and we are trying to find the end point spaces of them. Here is the basic set up. We are given compatible families $\{A_\theta\}_{\theta \in (0,1)}$ and $\{B_\theta\}_{\theta \in (0,1)}$ of Banach spaces (in the sense that there exist two Banach spaces \mathbb{A}_0 and \mathbb{A}_1 , such that for each $\theta \in (0,1)$, we have with continuous inclusions³² $\mathbb{A}_1 \subset A_\theta \subset \mathbb{A}_0$). We shall say that A and B are extrapolation spaces with respect to the compatible families $\{A_\theta\}_{\theta \in (0,1)}, \{B_\theta\}_{\theta \in (0,1)}$ if $\mathbb{A}_1 \subset A \subset \mathbb{A}_0, \mathbb{A}_1 \subset B \subset \mathbb{A}_0$, and every operator T that is bounded, $T : A_\theta \xrightarrow{1} B_\theta$ for all $\theta \in (0,1)$, has an extension that is bounded, $T : A \rightarrow B$. An extrapolation method \mathfrak{E} assigns to each compatible family an extrapolation space $\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)})$ with the following interpolation property. If T is an operator such that $T : A_\theta \xrightarrow{1} B_\theta$, for each $\theta \in (0,1)$, then T can be extended to $T : \mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) \rightarrow \mathfrak{E}(\{B_\theta\}_{\theta \in (0,1)})$. Given a compatible family $\{A_\theta\}_{\theta \in (0,1)}$, we let $M_\Sigma(\theta)$ denote the norm of the inclusions $A_\theta \subset \mathbb{A}_0$, and let $M_\Delta(\theta)$ denote the norm of the corresponding inclusions $\mathbb{A}_1 \subset A_\theta$. We shall say that the family is *strongly compatible* if these inclusions are uniformly bounded, that is if $\sup_{\theta \in (0,1)} \{M_\Sigma(\theta), M_\Delta(\theta)\} < \infty$.

Here we shall restrict ourselves to consider strongly compatible families³³. There are two natural constructions of strongly compatible families: the Σ - and Δ -methods of extrapolation. Given a family $\{A_\theta\}_{\theta \in (0,1)}$, the space $\Sigma(\{A_\theta\}_{\theta \in (0,1)})$ consists of all the elements $x \in \mathbb{A}_0$ that can be represented by $x = \sum_{0 < \theta < 1} a_\theta$, $a_\theta \in A_\theta$, with $\sum_{0 < \theta < 1} \|a_\theta\|_{A_\theta} < \infty$. We endow $\Sigma(\{A_\theta\}_{\theta \in (0,1)})$ with the corresponding quotient norm. It is customary to write $\Sigma_{\theta \in (0,1)} A_\theta$ rather than $\Sigma(\{A_\theta\}_{\theta \in (0,1)})$.

Likewise, let the space $\Delta\{A_\theta\}_{\theta \in (0,1)}$ consist of all elements $x \in \cap_{\theta \in (0,1)} A_\theta$, such that

$$\|x\|_{\Delta\{A_\theta\}_{\theta \in (0,1)}} := \sup_{\theta \in (0,1)} \|x\|_{A_\theta} < \infty.$$

It is customary to write $\Delta_{\theta \in (0,1)} A_\theta$ rather than $\Delta\{A_\theta\}_{\theta \in (0,1)}$. It is easy to verify that the Σ and Δ are extrapolation functors, and moreover they are **exact** in the sense that, if \mathfrak{E} denotes either the Σ - or Δ -method and $T : A_\theta \xrightarrow{1} B_\theta$, $\theta \in (0,1)$, then

$$\|T\|_{\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) \rightarrow \mathfrak{E}(\{B_\theta\}_{\theta \in (0,1)})} \leq \sup_{0 < \theta < 1} \{\|T\|_{A_\theta \rightarrow B_\theta}\}.$$

The Σ -method exhibits a behavior analogous to the $A_{\theta,1,J}$ spaces, while the Δ -method is closely related to the $A_{\theta,\infty,K}$ -construction of classical interpolation theory. In the setting of rearrangement invariant spaces these constructions are related to the Lorentz spaces (Σ -method) and the Marcinkiewicz spaces (Δ -method). Thus, the Σ - and Δ -methods are, in a suitable sense, extremal extrapolation functors on the class of exact extrapolation functors. To see this let us first show that

³²In practice these families consist of interpolation spaces, e.g. $A_\theta = [A_0, A_1]_\theta$, $A_\theta = (A_0, A_1)_{\theta,q(\theta)}$, and in this case we can take $\mathbb{A}_0 = A_0 + A_1$, and $\mathbb{A}_1 = A_0 \cap A_1$. In particular, if the pair (A_0, A_1) is ordered, $A_1 \subset A_0$, we can take $\mathbb{A}_0 = A_0, \mathbb{A}_1 = A_1$. More generally, the construction of the Δ -method makes sense if we consider families of spaces $\{A_\theta\}_{\theta \in \gamma}$, where γ is a lattice (cf. [54]).

³³Which we will also refer to as “families”

Lemma 13.4. *An extrapolation method applied to a constant family, i.e. a family where all its elements are equal to a given Banach space, reproduces this space. In other words, if given a Banach space A we consider the family $\{A_\theta\}_{\theta \in (0,1)}$, where $A_\theta = A$, for all $\theta \in (0,1)$, then if \mathfrak{E} is an extrapolation functor we have,*

$$\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) = A.$$

Proof. In fact, by definition

$$A = A_0 \subset \mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) \subset A_1 = A,$$

which forces (with equivalent norms)

$$\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) = A,$$

as we wished to show³⁴ □

We shall now compare any exact extrapolation functor \mathfrak{E} with the \sum and Δ functors.

Lemma 13.5. *Let \mathfrak{E} be an exact extrapolation functor, then for all strongly compatible families, $\{A_\theta\}_{\theta \in (0,1)}$ we have*

$$\Delta(\{A_\theta\}_{\theta \in (0,1)}) \stackrel{1}{\subset} \mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) \stackrel{1}{\subset} \sum(\{A_\theta\}_{\theta \in (0,1)}).$$

Proof. Let $\{A_\theta\}_{\theta \in (0,1)}$ be a strongly compatible family. Let $\theta_0 \in (0,1)$. Then since any $a \in A_{\theta_0}$, can be represented by a sum $a = \sum_{\theta \in (0,1)} a_\theta$, where all the terms are zero except for $a_{\theta_0} = a$, we see that

$$(13.5) \quad A_{\theta_0} \stackrel{1}{\subset} \sum_{\theta \in (0,1)} A_\theta, \text{ for all } \theta_0 \in (0,1).$$

Consider the family of strongly compatible spaces that is defined for all $\theta \in (0,1)$ by $B_\theta = \sum_{\theta \in (0,1)} A_\theta$, then applying the extrapolation functor \mathfrak{E} to (13.5) and Lemma 13.4 yield

$$\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}) \stackrel{1}{\subset} \mathfrak{E}(\{B_\theta\}_{\theta \in (0,1)}) = \sum_{\theta \in (0,1)} A_\theta.$$

Likewise, since

$$\Delta_{\theta \in (0,1)} A_\theta \stackrel{1}{\subset} A_\theta, \text{ for all } \theta \in (0,1),$$

it follows that for any exact extrapolation functor \mathfrak{E}

$$\Delta_{\theta \in (0,1)} A_\theta = \mathfrak{E}(\Delta_{\theta \in (0,1)} A_\theta) \subset \mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)}),$$

as we wished to show. □

³⁴At this point it will be convenient to agree on the following notation. Let A be a Banach space and let \mathfrak{E} be an extrapolation functor. By abuse of notation we shall write $\mathfrak{E}(A)$ to denote the space $\mathfrak{E}(\{A_\theta\}_{\theta \in (0,1)})$, where $A_\theta = A, \theta \in (0,1)$. Thus, with this notation the previous discussion shows that we have $\mathfrak{E}(A) = A$.

13.3. Abstract extrapolation methods. The Σ - and Δ -methods are part of more general families of extrapolation functors that were introduced in [3] (cf. also [47]), and then were studied in [4, 6–8]. Let F be a Banach function lattice on the interval $[0, 1]$ (with respect to the usual Lebesgue measure). A given family $\{A_\theta\}_{\theta \in (0,1)}$ of compatible Banach spaces, we define the Banach space $\mathbf{F}(\{A_\theta\}_{\theta \in (0,1)})$, consisting of all $a \in \bigcap_{\theta \in (0,1)} A_\theta$ such that the function $\xi_a(\theta) := \|a\|_{A_\theta}$ defined on $(0, 1)$ belongs to F , endowed with the norm $\|a\| := \|\xi_a\|_F$. In particular, if $F = L^\infty[0, 1]$, we arrive at the definition of the Δ -functor. In analogous way one can define a family of extrapolation functors generalizing the Σ -functor (see the definition of the $\bar{A}_{\xi,q,G}^J$ spaces in [9, 11]).

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S. ASTASHKIN

Faculty of Mathematics, Samara National Research University, 443086 Samara, Russia

E-mail address: `astash56@mail.ru`

M. MILMAN

Instituto Argentino de Matematica, Buenos Aires, Argentina

E-mail address: `mario.milman@icloud.com`