Pure and Applied Functional Analysis
Volume 6, Number 3, 2021, 631–649

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL
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OPTIMAL PAIRS OF SYMMETRIC SPACES FOR THE CALDERÓN TYPE OPERATORS

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Abstract. We identify the optimal pairs of symmetric spaces (X, Y) such that the discrete Hilbert transform, the Calderón operator and the triangular truncation operator act boundedly from X to Y.

1. Introduction

In 1967, D. Boyd described symmetric spaces in which the continuous Hilbert transform operator

$$(H^c f)(x) := \frac{1}{\pi}\text{v.p.}\int_{-\infty}^{+\infty} \frac{f(t)}{t-x}dt$$

acts boundedly [6]. In particular, [6, Theorem 2.1] states that if $X(\mathbb{R})$ and $Y(\mathbb{R})$ are symmetric spaces, then H^c is a bounded operator from $X(\mathbb{R})$ to $Y(\mathbb{R})$ if and only if the operators

$$(Rf)(t) := \frac{1}{t}\int_0^t f(s)ds, \quad (R'f)(t) := \int_t^\infty f(s)\frac{ds}{s}$$

are bounded from $X(0,\infty)$ to $Y(0,\infty)$. Here, $X(0,\infty)$ means the space of functions on semiaxis with norm $\|f\|_{X(0,\infty)} := \|f\chi_{(0,\infty)}\|_{X(\mathbb{R})}$. The same convention applies to other similar cases.

Later, K. Andersen considered the discrete version of the Hilbert transform

$$(H^d f)(n) := \frac{1}{\pi}\sum_{k=-\infty}^{+\infty}{}' \frac{f(k)}{k-n}, \ n \in \mathbb{Z},$$

where the prime symbol means the omission of the n-th term from the sum. In [1, Theorem 3] he proved that H^d acts boundedly from $X(\mathbb{Z})$ to $Y(\mathbb{Z})$ if and only if operators P and P' act boundedly from $X(\mathbb{Z}_+)$ to $Y(\mathbb{Z}_+)$, where

$$(Pf)(n) := \frac{1}{n}\sum_{k=1}^{n} f(k), \qquad (P'f)(n) = \sum_{k=n}^{\infty}\frac{f(k)}{k}. \tag{1.1}$$

2020 *Mathematics Subject Classification.* 46E30, 47B10, 46L51, 46L52, 44A15; Secondary 47L20, 47C15.

Key words and phrases. Symmetric sequence and operator spaces, discrete Hilbert transform, triangular truncation operator, optimal symmetric range.

In this note we describe the optimal pairs (X, Y) of sequence spaces on $\mathbb{Z}_+$ for the operator H^d. Also, we address the similar question for the Calderón operator (the operator closely related to the Hilbert transform) and the triangular truncation operator. We refer to [28] for a detailed discussion of these operators.

We would like to point out the interesting connection between our study and that in [26], where the authors described the optimal range for Hardy type operators such as Cesàro operator, its adjoint and the Calderón operator. The case of Hilbert transform was not studied in [26], and the setting of that paper was concerned with optimal domain for Cesàro operator in the special case of symmetric spaces with Fatou norm. The methods employed in [26] are different from those we employ here. In our present setting, we consider optimal range for the Calderón type operators among the interpolation spaces which do not necessarily have the Fatou property. Our techniques and approach also allow us to consider similar problems in the setting of ideals of compact operators (in particular, a special type of triangular truncation operator acting on such ideals). As an application of our methods we obtain various Lipschitz and commutator estimates.

The main technical tool of the paper is an extrapolation result for symmetric sequence spaces proved in Theorem 3.2. In the special case of the Marcinkiewicz space $M_{1,\infty}$, similar extrapolation estimates were proved in [8, Theorem 4.5]. Those results were motivated by the problems of computability of Dixmier traces in the A. Connes noncommutative geometry. More precisely, these estimates allowed the expression of Dixmier traces in terms of the residues of the operator zeta-function. These results were extended to the case of general Marcinkiewicz spaces M_ψ (see (3.1)) in [10].

2. Preliminaries

2.1. **Symmetric sequence spaces.** By $\mathbb{Z}_+$ we denote the set of all positive integers. By $\ell_\infty(\mathbb{Z}_+)$ we denote the Banach space of all real bounded sequences $x = (x(1), x(2), \ldots)$ with the usual partial order and equipped with the norm

$$\|x\|_{\ell_\infty(\mathbb{Z}_+)} = \sup_{n\in\mathbb{Z}_+} |x(n)|,$$

where $\mathbb{Z}_+$ is the set of positive integers.

For a strictly positive sequence $\{w(n)\}_{n\in\mathbb{Z}_+}$ and a sequence space E on $\mathbb{Z}_+$ by $E(\mathbb{Z}_+, w(n))$ (or simply by $E(w)$) we denote the space of all real sequences equipped with the norm

$$\|x\|_{E(\mathbb{Z}_+,w(n))} := \|xw(\cdot)\|_E.$$

For example,

$$\|x\|_{\ell_\infty(\mathbb{Z}_+,1/n)} = \sup_{n\in\mathbb{Z}_+} \frac{|x(n)|}{n},$$

For any $x \in \ell_\infty(\mathbb{Z}_+)$ by $\mu(x) = \{\mu(n, x)\}_{n\in\mathbb{Z}_+}$ we denote the non-increasing rearrangement of a sequence $|x| := (|x(1)|, |x(2)|, \ldots)$. We will write $\mu(y) \le \mu(x)$, if $\mu(n, y) \le \mu(n, x)$ for all $n \in \mathbb{Z}_+$.

Definition 2.1. A Banach subspace $(E, \|\cdot\|_E)$ of $\ell_\infty(\mathbb{Z}_+)$ is called a symmetric sequence space if, for every $x \in E$ and every $y \in \ell_\infty(\mathbb{Z}_+)$ such that $\mu(y) \le \mu(x)$ we have $y \in E$ and $\|y\|_E \le \|x\|_E$.

The classical examples of symmetric sequence spaces are $\ell_p(\mathbb{Z}_+)$, $1 \le p \le \infty$. The class of all symmetric spaces will be denoted by $\mathbb{S}$.

We say that a sequence y is submajorized by a sequence x in the sense of Hardy-Littlewood-Pólya (written $y \prec\prec x$) if

$$\sum_{k=1}^{n} \mu(k, y) \le \sum_{k=1}^{n} \mu(k, x), \quad n \ge 1. \tag{2.1}$$

(see [21]). Denote by $\mathbf{I}(X, Y)$ the set of all interpolation spaces between Banach spaces X and Y, and let $\mathbb{I} := \mathbf{I}(\ell_1(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+))$. For all unexplained terminology from interpolation theory we refer to [4, 7, 19].

Throughout this paper, we shall use the symbol $\mathcal{A} \lesssim \mathcal{B}$ to indicate that there exists a universal positive constant c_{abs}, independent of all important parameters, such that $\mathcal{A} \le c_{abs}\mathcal{B}$. The notation $\mathcal{A} \approx \mathcal{B}$ means that $\mathcal{A} \lesssim \mathcal{B}$ and $\mathcal{B} \lesssim \mathcal{A}$.

2.2. **Optimal pairs.** We explain here what we mean by the *optimal pair*.

A family of Banach spaces $\mathcal{X} = \{X_\alpha\}_{\alpha\in\mathcal{A}}$ is called *compatible* if there exists a linear Hausdorff space $\mathfrak{T} = \mathfrak{T}(\mathcal{X})$ such that for each $\alpha \in \mathcal{A}$ there is continuous embedding $X_\alpha \subset \mathfrak{T}$. For example, the class $\mathbb{S}$ is a compatible family. The *ambient* space $\mathfrak{T}(\mathbb{S})$ can be chosen to be the space $\mathcal{S}$ of all sequences with convergence in the counting measure. Of course, the choice of ambient space is not unique. Indeed, instead of $\mathcal{S}$ in the above example one can also choose the Banach space ℓ_∞. The situation is similar with the class $\mathbb{I}$ of all spaces which are interpolation with respect to the Banach pair $(\ell_1(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+))$.

Suppose now that $\mathcal{X}$ and $\mathcal{Y}$ are two compatible families with ambient spaces $\mathfrak{T}(\mathcal{X})$ and $\mathfrak{T}(\mathcal{Y})$. We will say that a linear operator T is *admissible* for the pair $(\mathcal{X}, \mathcal{Y})$ if it is correctly defined on the linear set $\mathcal{L} \subset \mathfrak{T}(\mathcal{X})$, and for each $x \in \mathcal{L}$ the condition $Tx \in \mathfrak{T}(\mathcal{Y})$ is satisfied. In this case, we will write $T \in [X, Y]$, where $X \in \mathcal{X}$ and $Y \in \mathcal{Y}$, if T is bounded operator from $X \subset \mathcal{L}$ to Y.

For example, the operator P, defined in (1.1), is admissible for a pair of families $(\mathbb{S}, \mathbb{S})$. Moreover, as $\mathcal{L}$ we can take the space $\mathcal{S}$ of all sequences. In the case $T = P'$, we can take $\mathcal{L} = \ell_1(\mathbb{Z}_+, 1/n)$.

Definition 2.2. Let $\mathcal{X}$ and $\mathcal{Y}$ be two compatible families (or classes), $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ and $T \in [X, Y]$.

(i) The space Y is said to be the optimal range for the operator T in the class $\mathcal{Y}$ if the fact that $T \in [X, Z]$, $Z \in \mathcal{Y}$, implies the inclusion $Y \subset Z$;

(ii) The space X is said to be the optimal domain for the operator T in the class $\mathcal{X}$ if the fact that $T \in [Z, Y]$, $Z \in \mathcal{X}$, implies the inclusion $Z \subset X$;

(iii) The pair (X, Y) is said to be the optimal pair for the operator T in the class $(\mathcal{X}, \mathcal{Y})$ if X is the optimal domain in the class $\mathcal{X}$ and Y is the optimal range in the class $\mathcal{Y}$.

2.3. **Symmetric operator spaces.** Let H denote a fixed separable Hilbert space and let $B(H)$ be the algebra of all bounded operators on H. Let us denote by $K(H)$

the ideal of compact operators on H and $\mu(A) := \{\mu(n, A)\}_{n\in\mathbb{Z}_+}$ is the sequence of singular values of a compact operator A (see [11, Chapter II]).

Definition 2.3. Let X be a linear subset in $K(H)$ equipped with a complete norm $\|\cdot\|_X$. We say that X is a *symmetric operator space* (in $K(H)$) if for $A \in X$ and for every $B \in K(H)$ with $\mu(B) \le \mu(A)$, we have $B \in X$ and $\|B\|_X \le \|A\|_X$.

Recall the construction of a symmetric Banach operator space (or non-commutative symmetric Banach space, or symmetric Banach ideal) $E(H)$. Let E be a symmetric Banach sequence space on $\mathbb{Z}_+$. Set

$$E(H) = \Big\{A \in K(H) : \ \mu(A) \in E(\mathbb{Z}_+)\Big\}.$$

We equip $E(H)$ with a natural quasi-norm

$$\|A\|_{E(H)} = \|\mu(A)\|_{E(\mathbb{Z}_+)}, \quad A \in E(H).$$

The following fundamental theorem was proved in [16] (see also [21, Question 2.5.5, p. 58]). It shows that the quasi-norm introduced above is, in fact, a norm.

Theorem 2.4. *Let E be a symmetric sequence space on $\mathbb{Z}_+$. Set*

$$E(H) = \Big\{A \in K(H) : \ \mu(A) \in E(\mathbb{Z}_+)\Big\}.$$

So defined $(E(H), \|\cdot\|_{E(H)})$ is a symmetric operator space.

An extensive discussion of the various properties of such spaces can be found in [16, 21]. If $E = \ell_p(\mathbb{Z}_+)$, $1 \le p < \infty$, then we obtain

$$L_p(H) := \ell_p(H) = \Big\{A \in K(H) : \ \mu(A) \in \ell_p(\mathbb{Z}_+)\Big\},$$

which is so called Schatten-von Neumann class of all compact operators $A : H \to H$ with finite norm

$$\|A\|_{L_p(H)} := \big(\sum_{k=1}^{\infty} \mu(k, A)^p\big)^{1/p}.$$

When $p = 2$ the space $L_2(H)$ (usually it is called Hilbert-Schmidt class) becomes Hilbert space with the inner product

$$< A, B >:= \tau(B^*A), \ A, B \in L_2(H),$$

where B^* is adjoint operator of B and τ is the canonical trace. Moreover, the space $K(H)$ will be considered with the uniform norm, i.e. $\|A\|_{K(H)} := \|A\|_{H\to H}$ for $A \in K(H)$.

We will also need one subclass of the class of symmetric operator spaces.

Definition 2.5. Let X be a linear subset in $K(H)$ equipped with a complete norm $\|\cdot\|_X$. We say that X is a *fully* symmetric operator space if for $A \in X$ and for every $B \in K(H)$ with $\mu(B) \prec\prec \mu(A)$, we have $B \in X$ and $\|B\|_X \le \|A\|_X$.

Using well-known inequality on the one hand

$$\sum_{k=1}^{n} \mu(k, A_0 + A_1) \le \sum_{k=1}^{n} \mu(k, A_0) + \sum_{k=1}^{n} \mu(k, A_1), \quad n \in \mathbb{Z}_+,$$

and selecting the appropriate optimal representation $A = A_0 + A_1$ on the other, it is easy to get the following formula for the K-functional in the Banach pair $(L_1(H), K(H))$:

$$K(n, A; L_1(H), K(H)) := \inf_{A=A_0+A_1} \{\|A_0\|_{L_1(H)} + n\|A_1\|_{K(H)}\} = \sum_{k=1}^{n} \mu(k, A)$$

(see also [23, formula (5.5), page 68] or [18, the proof of the Proposition 2.c.6]). Therefore, in view of the K-divisibility property [7], for each fully symmetric operator space X there is a Banach lattice F such that

$$\|A\|_X = \left\| \left\{ \sum_{k=1}^{n} \mu(k, A) \right\}_{n=1}^{\infty} \right\|_F .$$

Hence, without loss of generality, we can immediately assume that the norm in a fully symmetric operator space is given by the right side of the previous relation. Moreover, we can assume that $F \in \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$.

The class of all fully symmetric operator spaces will be denoted by $\mathbb{FS}(H)$.

2.4. **Lorentz spaces.** Let a function $\varphi :\ \mathbb{N} \to \mathbb{R}_+$ be nondecreasing, and let the sequence $\{\varphi(n+1) - \varphi(n)\}$ of its differences be nonincreasing. Then the Lorentz sequence space $\Lambda_\varphi(\mathbb{Z}_+)$ is defined as follows:

$$\Lambda_\varphi(\mathbb{Z}_+) := \left\{ a \in c_0 : \|a\|_{\Lambda_\varphi(\mathbb{Z}_+)} = \sum_{n=1}^{\infty} \mu(n, a)(\varphi(n+1) - \varphi(n)) < \infty \right\}, \tag{2.2}$$

where c_0 is the space of sequences converging to zero. These spaces are examples of symmetric Banach sequence spaces. For more details on Lorentz spaces, we refer the reader to [4, Chapter II.5] and [19, Chapter II.5].

The Lorentz (or Schatten-Lorentz) ideal $\Lambda_\varphi(H)$ (see [21, Example 1.2.7, p. 25]) is defined as follows:

$$\Lambda_\varphi(H) := \left\{ A \in K(H) : \|A\|_{\Lambda_\varphi(H)} = \sum_{n=1}^{\infty} \mu(n, A)(\varphi(n+1) - \varphi(n)) < \infty \right\}.$$

2.5. **Weak-L_1 and $M_{1,\infty}$ spaces.** The weak-ℓ_1 sequence space $\ell_{1,\infty}$ on $\mathbb{Z}_+$ is defined as

$$\ell_{1,\infty}(\mathbb{Z}_+) := \{a \in c_0 : \mu(n, a) = O(1/n)\}.$$

Define corresponding weak-L_1 ideal of compact operators on H as follows:

$$L_{1,\infty}(H) := \left\{ A \in K(H) : \{\mu(n, A)\}_{n\in\mathbb{Z}_+} \in \ell_{1,\infty}(\mathbb{Z}_+) \right\},$$

with the quasi-norm

$$\|A\|_{L_{1,\infty}(H)} := \sup_{n\in\mathbb{Z}_+} n \cdot \mu(n, A)$$

(see [21, Example 1.2.6, p. 24]). It is well-known (see e.g. [16, Section 7] or [27]) that the space $(L_{1,\infty}(H), \|\cdot\|_{L_{1,\infty}(H)})$ is quasi-Banach.

The non-commutative Marcinkiewicz space $M_{1,\infty}(H)$ is defined by setting

$$M_{1,\infty}(H) := \left\{ A \in K(H) : \|A\|_{M_{1,\infty}(H)} := \sup_{n\in\mathbb{Z}_+} \frac{1}{\log(en)} \sum_{k=0}^{n} \mu(k, A) < \infty \right\}.$$

This space is the dual of the Macaev ideal on a separable Hilbert space H.

It is easy to see that the following inclusion

$$L_{1,\infty}(H) \subset M_{1,\infty}(H)$$

holds and is strict (see [21, Lemma 1.2.8 and Example 1.2.9, pp. 25-26]).

2.6. **Calderón operator and a triangular truncation operator.** Define the discrete Calderón operator S^d formally as follows

$$(S^d x)(n) := \frac{1}{n}\sum_{k=1}^{n-1} x(k) + \sum_{k=n}^{+\infty} \frac{x(k)}{k}, \quad x \in E(\mathbb{Z}_+). \tag{2.3}$$

It is obvious that S^d is a linear operator. If x is nonnegative, it follows from the definition that $(S^d x)(n)$ is decreasing in $n \geq 1$. The operator S^d is often applied to the decreasing rearrangement $\mu(x)$ of a function x defined on some other measure space. Since $S^d\mu(x)$ is itself decreasing, it is easy to see that $\mu(S^d\mu(x)) = S^d\mu(x)$. For more information about this operator, we refer to [4, Chapter III] and [19, Chapter II].

The next proposition gives the exact domain of the operator S^d.

Proposition 2.6. *Let S^d be the operator defined in* (2.3)*. If $\varphi_0(n) = \log(en)$, then the Lorentz sequence space $\Lambda_{\varphi_0}(\mathbb{Z}_+)$ is the largest among all symmetric sequence spaces $\{E(\mathbb{Z}_+)\}$ such that*

$$S^d : E(\mathbb{Z}_+) \to \ell_\infty(\mathbb{Z}_+).$$

Proof. Since for each $n \geq 1$, the kernel $k_n(m) := \frac{1}{m} \cdot \min\left\{1, \frac{m}{n}\right\}$ is a decreasing sequence of $m > 0$, it follows from [4, Chapter II, Theorem 2.2, p. 44] that

$$\begin{aligned} |(S^d x)(n)| &\overset{(2.3)}{=} \Big| \sum_{k=1}^{\infty} x(k) \min\left\{1, \frac{k}{n}\right\} \frac{1}{k} \Big| \\ &\leq \sum_{k=1}^{\infty} |x(k)| \min\left\{1, \frac{k}{n}\right\} \frac{1}{k} \leq \sum_{k=1}^{\infty} \mu(k, x) \min\left\{1, \frac{k}{n}\right\} \frac{1}{k} \\ &\overset{(2.3)}{=} (S^d\mu(x))(n), \quad \forall n \in \mathbb{Z}_+. \end{aligned}$$

Therefore, to prove the theorem, we can restrict ourselves to the case $x = \mu(x)$.

Let $E(\mathbb{Z}_+)$ be a symmetric sequence space such that $S^d : E(\mathbb{Z}_+) \to \ell_\infty(\mathbb{Z}_+)$. If we have that

$$\|S^d\mu(x)\|_{\ell_\infty(\mathbb{Z}_+)} \approx \|x\|_{\Lambda_{\varphi_0}(\mathbb{Z}_+)}, \tag{2.4}$$

then, for any $x \in E(\mathbb{Z}_+)$, we have

$$\|x\|_{\Lambda_{\varphi_0}(\mathbb{Z}_+)} \lesssim \|S^d\mu(x)\|_{\ell_\infty(\mathbb{Z}_+)} \lesssim \|x\|_{E(\mathbb{Z}_+)}.$$

This shows that $E(\mathbb{Z}_+) \subset \Lambda_{\varphi_0}(\mathbb{Z}_+)$. Therefore, it is sufficient to show (2.4). Take $x \in \Lambda_{\varphi_0}(\mathbb{Z}_+)$. Since $\big(S^d\mu(x)\big)(n) \le \big(S^d\mu(x)\big)(1)$ for any $n \in \mathbb{Z}_+$, it follows that

$$\|S^d\mu(x)\|_{\ell_\infty(\mathbb{Z}_+)} = \sup_{n\in\mathbb{Z}_+} |(S^d\mu(x))(n)| = \big(S^d\mu(x)\big)(1) \overset{(2.3)}{=} \sum_{k=1}^{\infty} \frac{\mu(k,x)}{k}.$$

Using the fact that $\log(1+\frac{1}{k}) \approx \frac{1}{k}$ as $k \to +\infty$, we obtain

$$\|S^d\mu(x)\|_{\ell_\infty(\mathbb{Z}_+)} = \sum_{k=1}^{\infty} \frac{\mu(k,x)}{k} \approx \sum_{k=1}^{\infty} \mu(k,x)\log\big(\frac{k+1}{k}\big) = \|x\|_{\Lambda_{\varphi_0}(\mathbb{Z}_+)}.$$

This completes the proof. □

Our primary example is a triangular truncation operator on the Hilbert space $H = L_2(\mathbb{R})$. More precisely, let $\mathcal{K}$ be a fixed measurable function on $\mathbb{R}\times\mathbb{R}$. Let us consider an operator V with the integral kernel $\mathcal{K}$ on $L_2(\mathbb{R})$ defined by setting

$$(Vx)(t) = \int_{\mathbb{R}} \mathcal{K}(t,s)x(s)ds, \quad x \in L_2(\mathbb{R}). \tag{2.5}$$

Then for any $V \in E(H)$, we define the triangular truncation operator $\mathcal{T}(V)$ as follows (see [11, 12] for more details):

$$(\mathcal{T}(V)x)(t) = \int_{\mathbb{R}} \mathcal{K}(t,s)\mathrm{sgn}(t-s)x(s)ds, \quad x \in L_2(\mathbb{R}). \tag{2.6}$$

It was proved in [28, Theorem 11] that $\mathcal{T} : L_1(H) \to L_{1,\infty}(H)$ is bounded, i.e. $\mathcal{T}$ is a weak type (1,1) operator.

Remark 2.7. Since $\mathcal{T}$ is a weak type $(1,1)$ operator, it follows from [28, Theorem 14 (ii)] that $\mathcal{T}$ is dominated by the operator S^d in the following sense:

$$\mu(\mathcal{T}(A)) \lesssim S^d\mu(A), \quad \forall A \in \Lambda_{\varphi_0}(H),$$

where φ_0 is defined as in Proposition 2.6. Since the maximal domain of S^d is Lorentz space $\Lambda_{\varphi_0}(\mathbb{Z}_+)$ (see Proposition 2.6), it follows that $\mathcal{T}$ is defined on the Schatten-Lorentz ideal $\Lambda_{\varphi_0}(H)$.

2.7. **Double operator integrals.** Let A be a self-adjoint operator in $B(H)$ and ξ be a bounded Borel function on $\mathbb{R}^2$. Symbolically, a double operator integral is defined by the formula

$$T_\xi^{A,A}(V) = \int_{\mathbb{R}^2} \xi(\lambda,\mu)dE_A(\lambda)VE_A(\mu), \quad V \in L_2(H), \tag{2.7}$$

where $E_A(\lambda)$ and $E_A(\mu)$ spectral measures on $\mathbb{R}$ with values in the orthogonal projections in $B(H)$. For a more rigorous definition, consider projection valued measures on $\mathbb{R}$ acting on the Hilbert space $L_2(H)$ by the formulae $X \to E_A(\mathcal{B})X$ and $X \to XE_A(\mathcal{B})$. These spectral measures commute and, hence (see Theorem V.2.6 in [5]), there exists a countably additive (in the strong operator topology) projection-valued measure ν on $\mathbb{R}^2$ acting on the Hilbert space $L_2(H)$ by the formula

$$\nu(\mathcal{B}_1 \otimes \mathcal{B}_2) : X \to E_A(\mathcal{B}_1)XE_A(\mathcal{B}_2), \quad X \in L_2(H).$$

Integrating a bounded Borel function ξ on $\mathbb{R}^2$ with respect to the measure ν produces a bounded operator acting on the Hilbert space $L_2(H)$. In what follows, we denote the latter operator by $T_\xi^{A,A}$ (see also [24, Remark 3.1]).

We are mostly interested in the case $\xi = f^{[1]}$ for a Lipschitz function f on $\mathbb{R}$. Here,

$$f^{[1]}(\lambda,\mu) = \begin{cases} \frac{f(\lambda)-f(\mu)}{\lambda-\mu}, & \lambda \neq \mu \\ 0, \quad \lambda = \mu. \end{cases} \tag{2.8}$$

3. Commutative case

The following definition introduces a special class of Banach lattices of sequences on $\mathbb{Z}_+$.

Definition 3.1. We say that a Banach lattice F of sequences on $\mathbb{Z}_+$ belongs to the class $\mathbf{S}$ if the operator

$$S: \ \{f(n)\}_{n=1}^{\infty} \to \{f(n^2)\}_{n=1}^{\infty}$$

is bounded in F.

As an example of an element from $\mathbf{S}$ one can take a lattice F^ψ (with ψ being a positive function on $(0,\infty)$ such that $\psi(n^2) \lesssim \psi(n)$), equipped with the following norm:

$$\|f\|_{F^\psi} := \sup_{n\in\mathbb{Z}_+} \frac{|f(n)|}{\psi(n)}.$$

For an arbitrary Banach lattice of sequences F by X_F we denote a space of all sequences $x = \{x(n)\}_{n=1}^{\infty}$, for which the norm

$$\|x\|_{X_F} := \left\| \left\{ \sum_{k=1}^{n} \mu(k,x) \right\}_{n=1}^{\infty} \right\|_F$$

is finite.

For example, X_{F^ψ} is the Marcinkiewicz space (see e.g. [4,19]) with the following norm:

$$\sup_{n\in\mathbb{Z}_+} \frac{\sum_{k=1}^{n} \mu(k,x)}{\psi(n)}. \tag{3.1}$$

Note that any $X \in \mathbb{I}$ ($\mathbb{I} = \mathbf{I}(\ell_1(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+))$) can be written in the form X_F for some lattice F (see discussion at the beginning of Section 6 in [3]). Moreover, for each space $X \in \mathbb{I}$ there is a space $F \in \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$ such that $X = X_F$ [7, Corollaries 2.6.10 and 3.3.6].

If $F \in \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, then the operator

$$\tilde{f}(n) := \sup_{k\in\mathbb{Z}_+} \min\{1, n/k\}|f(k)|$$

is bounded in F [7, Remark 3.3.8].

Note that $\tilde{f}(n) \geq |f(n)|$ for every $n \in \mathbb{Z}_+$ and $\tilde{f}$ is a non-negative, non-decreasing, concave sequence. Thus, one can find a non-negative, non-increasing sequence y

such that

$$\tilde{f}(n) = \sum_{k=1}^{n} y(k).$$

We shall use these simple facts in the proof of Theorem 3.6 below.

The following result is proved in [3] as a consequence of a more general theorem. The direct proof is given in [22]. In special cases $F = \ell_\infty(\mathbb{Z}_+, w)$ and $F = \ell_p(\mathbb{Z}_+, w)$, this result can be obtained from earlier papers [13, 14, 23] and [17], respectively.

Theorem 3.2. *If $F \in \mathbf{S}$, then*

$$\|x\|_{X_F} \approx \left\| \left\{ \|x\|_{\ell_{p(n)}} \right\}_{n=1}^{\infty} \right\|_F,$$

where $p(n) = \frac{2\log(en)}{2\log(en)-1}$.

Note that for $n \in \mathbb{Z}_+$ values $p(n)$ from Theorem 3.2 belong to the interval $(1, 2]$.

By $F(\log^{-1})$ we denote the space of all sequences f with finite norm

$$\|f\|_{F(\log^{-1})} := \left\| \left\{ \frac{f(n)}{\log(en)} \right\}_{n=1}^{\infty} \right\|_F.$$

Note that if $F \in \mathbf{S}$, then $F(\log^{-1}) \in \mathbf{S}$ as well. Indeed, if $f \in F(\log^{-1})$, then $\{f(n)/\log(en)\} \in F$. Since $F \in \mathbf{S}$, it follows that $\{f(n^2)/\log(en^2)\} \in F$, as well. Since

$$|f(n^2)|/\log(en) \le 2|f(n^2)|/\log(en^2),$$

it follows that $\{f(n^2)/\log(en)\} \in F$ and, so $\{f(n^2)\} \in F(\log^{-1})$. In particular, if $F \in \mathbf{S}$, then it follows from Theorem 3.2 that

$$\|x\|_{X_{F(\log^{-1})}} \approx \left\| \left\{ \frac{\|x\|_{\ell_{p(n)}}}{\log(en)} \right\}_{n=1}^{\infty} \right\|_F,$$

where $p(n) = \frac{2\log(en)}{2\log(en)-1}$.

Theorem 3.3. *Let $F \in \mathbf{S}$, an operator T is bounded from $\ell_p(\mathbb{Z}_+)$ to $\ell_p(\mathbb{Z}_+)$ for some $p \in (1, 2]$ and*

$$\|T\|_{\ell_p(\mathbb{Z}_+)\to\ell_p(\mathbb{Z}_+)} \lesssim \frac{1}{p-1}.$$

Then T is bounded as an operator from X_F to $X_{F(\log^{-1})}$.

Proof. For every $n \in \mathbb{Z}_+$ and $p(n) = \frac{2\log(en)}{2\log(en)-1}$ we have

$$(p(n) - 1)\log(en) = \frac{\log(en)}{2\log(en) - 1} \in (1/2; 1].$$

Hence,

$$\begin{aligned}\|Tx\|_{F(\log^{-1})} &\lesssim \left\|\left\{\frac{\|Tx\|_{\ell_{p(n)}}}{\log(en)}\right\}_{n=1}^{\infty}\right\|_F \lesssim \left\|\left\{\frac{\|x\|_{\ell_{p(n)}}}{(p(n)-1)\log(en)}\right\}_{n=1}^{\infty}\right\|_F \\ &\lesssim \left\|\left\{\|x\|_{\ell_{p(n)}}\right\}_{n=1}^{\infty}\right\|_F \lesssim \|x\|_{X_F}.\end{aligned}$$

□

Remark 3.4. Conditions of Theorem 3.3 are satisfied by a wide class of operators. In particular, this is the case for operators of weak type $(1,1)$ and strong type $(2,2)$. This follows from the estimates on the norm of an operator, which are obtained in the standard proof of the Marcinkiewicz interpolation theorem.

It is easy to show that the operator P defined in (1.1) is of weak type $(1,1)$ and strong type $(2,2)$ and thus, satisfies the conditions of Theorem 3.3. This is also the case for the discrete Hilbert transform H^d [15].

Example 3.5. Consider the Lorentz space defined in (2.2) and denote by $\{d(n)\}_{n=1}^{\infty}$ is nonincreasing nonnegative sequence $d(n) := \varphi(n+1) - \varphi(n)$, $n \geq 1$. If $d(n) \to 0$ then

$$\|x\|_{\Lambda_\varphi(\mathbb{Z}_+)} = \sum_{n=1}^{\infty} \mu(n,x)d(n) = \sum_{k=1}^{\infty} w(k) \sum_{n=1}^{k} \mu(n,x),$$

where $w(k) = d(k) - d(k+1) \geq 0$, and, hence,

$$\sum_{k=n}^{\infty} w(k) = d(n).$$

Consider the space $\tilde{\ell}_1(\mathbb{Z}_+, w(n))$ with norm

$$\|f\|_{\tilde{\ell}_1(\mathbb{Z}_+,w(n))} := \sum_{n=1}^{\infty} \tilde{f}(n)w(n),$$

where $\tilde{f}(n) = \sup_{k\in\mathbb{Z}_+} \min\{1, n/k\}|f(k)|$. Since for a concave function $f(n)$ one has

$$\tilde{f}(n) = f(n),$$

it follows that

$$\|x\|_{\Lambda_\varphi(\mathbb{Z}_+)} = \left\|\{\sum_{j=1}^{n} \mu(j,x)\}_{n=1}^{\infty}\right\|_{\tilde{\ell}_1(\mathbb{Z}_+,w(n))}$$

(we cannot use the space $\ell_1(\mathbb{Z}_+, w(n))$ without tilde, since $w(n)$ may be zero for some n, and thus $\ell_1(\mathbb{Z}_+, w(n))$ may not be a Banach space). Since $\tilde{f}$ is the K-functional of f in the pair $(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, it follows that the space $\tilde{\ell}_1(\mathbb{Z}_+, w(n))$ is interpolation space with respect to the pair $(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$.

Let $\alpha > 0$ and

$$w(n) = \begin{cases} 2^{-\alpha k} & \text{if} \quad n = 2^{2^k},\ k \in \mathbb{Z}_+ \\ 0 & \text{otherwise.} \end{cases} \tag{3.2}$$

We shall prove that the operator $S:\{f(n)\}\to\{f(n^2)\}$ is bounded in the space $\tilde\ell_1(\mathbb{Z}_+,w(n))$. Since $\tilde f\ge|f|$, and the operator S is monotone, it suffices to consider the case $f=\tilde f$. Let $f=\tilde f$ be such that $\|f\|_{\tilde\ell_1(\mathbb{Z}_+,w(n))}=1$. Then

$$\sum_{n=1}^{\infty} f(n)w(n)=1,$$

and therefore

$$f(2^{2^k})\le 1/w(2^{2^k})\le 2^{\alpha k}.$$

So,

$$\begin{aligned}
\|Sf\|_{\tilde\ell_1(\mathbb{Z}_+,w(n))} &= \sum_{n=1}^{\infty}\sup_{k\in\mathbb{Z}_+}\left(\min\{1,n/k\}f(k^2)\right)w(n)\\
&= \sum_{l=1}^{\infty}\sup_{k\in\mathbb{Z}_+}\left(\min\{1,2^{2^l}/k\}f(k^2)\right)w(2^{2^l})\\
&= \sum_{l=1}^{\infty}\max\{f(2^{2^{l+1}}),\ \sup_{k\ge 2^{2^l}} 2^{2^l}f(k^2)/k\}2^{-\alpha l}\\
&\le \sum_{l=1}^{\infty} f(2^{2^{l+1}})2^{-\alpha l}+\sum_{l=1}^{\infty}\sup_{m\ge l}\left[\max_{2^{2^m}\le k<2^{2^{m+1}}} 2^{2^l}f(k^2)/k\right]2^{-\alpha l}\\
&\le 2^{\alpha}\sum_{l=1}^{\infty} f(2^{2^{l+1}})2^{-\alpha(l+1)}+\sum_{l=1}^{\infty}\sup_{m\ge l}\left(2^{2^l-2^m}f(2^{2^{m+2}})\right)2^{-\alpha l}\\
&\le 2^{\alpha}+\sum_{l=1}^{\infty} f(2^{2^{l+2}})2^{-\alpha l}+\sum_{l=1}^{\infty}\sup_{m\ge l+1}\left(2^{2^l-2^m}2^{\alpha(m+2)}\right)2^{-\alpha l}\\
&\le 2^{\alpha}+2^{2\alpha}+C_1,
\end{aligned}$$

where

$$C_1=\sum_{l=1}^{\infty}\sup_{m\ge l+1}\left(2^{2^l-2^m}2^{\alpha(m+2)}\right)2^{-\alpha l}<\infty,$$

since $2^l-2^m+\alpha(m+2)<-2^{l-1}$ for all $m\ge l+1$ and $l>l_0$ for some large enough l_0.

If $w(n)$ is given by (3.2), then one can take l_0 to be $\lceil\log_2(\log_2 n)\rceil$ and

$$d(n)=\sum_{k=n}^{\infty} w(k)=\sum_{l:\,2^{2^l}\ge n}^{\infty} 2^{-\alpha l}=\frac{2^{-\alpha l_0}}{1-2^{-\alpha}}\approx \log^{-\alpha}(en).$$

Hence,

$$\|x\|_{\Lambda_\varphi(\mathbb{Z}_+)}\approx\sum_{n=1}^{\infty}\mu(n,x)\log^{-\alpha}(en).$$

Denote by $\Lambda^{\alpha}_{\log}$ the space with the norm

$$\|x\|_{\Lambda^{\alpha}_{\log}}=\sum_{n=1}^{\infty}\mu(n,x)\log^{-\alpha}(en).$$

Further, if $w(n)$ is given by (3.2) and $F = \tilde{\ell}_1(\mathbb{Z}_+, w(n))$, then $F(\log^{-1})$ is the space $\tilde{\ell}_1(\mathbb{Z}_+, w_1(n))$ with

$$\begin{aligned} w_1(n) &= \begin{cases} 2^{-\alpha k}/\log(en), & n = 2^{2^k},\ k \in \mathbb{Z}_+ \\ 0 \quad \text{otherwise} \end{cases} \\ &\approx \begin{cases} 2^{-(\alpha+1)k}, & n = 2^{2^k},\ k \in \mathbb{Z}_+ \\ 0 \quad \text{otherwise,} \end{cases} \end{aligned}$$

and the corresponding Lorentz space $\Lambda_{\varphi_1}(\mathbb{Z}_+)$ is the space $\Lambda^{\alpha+1}_{\log}$.

Theorem 3.6. *If $F \in \mathbf{S} \bigcap \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, then $(X_F, X_{F(\log^{-1})})$ is optimal pair in the class $(\mathbb{S}, \mathbb{I})$ for the operator P, defined in* (1.1).

Proof. Denote, for brevity $X := X_F$, $X_1 := X_{F(\log^{-1})}$. It follows from Theorem 3.3 and Remark 3.4 that P acts boundedly from X to X_1. To show that the space X_1 is the optimal range for P, it is sufficient to prove that for every $x \in X_1$ there exists $z \in X$ such that for some $C > 0$ one has

$$\sum_{k=1}^{n} \mu(k, x) \leq C \sum_{k=1}^{n} \mu(k, Pz),\ \forall n \in \mathbb{Z}_+.$$

Let $x \in X_1$. It follows from the definition of the space F and $F(\log^{-1})$ that the sequence

$$g(n) := \frac{\sum_{k=1}^{n} \mu(k, x)}{\log(en)},\ n \geq 1.$$

belongs to F. Since the operator $g \mapsto \tilde{g}$ is bounded in F (see the notation and discussion before Theorem 3.2), it follows that $\tilde{g} \in F$. Hence,

$$\tilde{g}(n) = \sum_{k=1}^{n} \mu(k, y),$$

where $y \in X$. Without loss of generality, we can assume that $\mu(y) = y$. Set

$$f(n) := \sum_{k=1}^{n^2} y(k),\ n \geq 1.$$

Since $F \in \mathbf{S}$, $f(n) = \tilde{g}(n^2)$ and $\tilde{g} \in F$, it follows that $f \in F$ and, so, $\tilde{f} \in F$. Thus,

$$\tilde{f}(n) = \sum_{k=1}^{n} \mu(k, z),$$

where $z \in X$. Again we can assume that $\mu(z) = z$. We have

$$\begin{aligned}
\sum_{k=1}^{n} \mu(k, Pz) &= \sum_{k=1}^{n} \frac{1}{k} \sum_{l=1}^{k} z(l) = \sum_{k=1}^{n} \frac{1}{k} \tilde{f}(k) \geq \sum_{k=1}^{n} \frac{1}{k} f(k) = \sum_{k=1}^{n} \frac{1}{k} \sum_{l=1}^{k^2} y(l) \\
&= \sum_{l=1}^{n^2} y(l) \sum_{k=\lceil\sqrt{l}\rceil}^{n} \frac{1}{k} \geq \sum_{l=1}^{n} y(l) \sum_{k=\lceil\sqrt{l}\rceil}^{n} \frac{1}{k} \geq \sum_{l=1}^{n} y(l) \sum_{k=\lceil\sqrt{n}\rceil}^{n} \frac{1}{k} \\
&\geq c \log(en) \sum_{l=1}^{n} y(l) = c \log(en) \tilde{g}(n) \geq c \log(en) g(n) \\
&= c \log(en) \frac{\sum_{k=1}^{n} \mu(k, x)}{\log(en)} = c \sum_{k=1}^{n} \mu(k, x).
\end{aligned}$$

This means that every element $x \in X_1$ is majorised (in the Hardy-Littlewood-Pólya sense (2.1)) by an element Pz for some $z \in X$. Hence, every space $Y \in \mathbb{I}$, containing the image $P(X)$, must contain the space $X_1 = X_{F(\log^{-1})}$ as well.

To show that the space X is the optimal domain for P supposing $P \in [Z, X_1]$. Take $x \in Z$, $x = \mu(x)$. Then $Px \in X_{F(\log^{-1})}$, and, since $F(\log^{-1}) \in \mathbf{S}$ (see reasoning before Theorem 3.3),

$$\frac{\sum_{k=1}^{n^2} \mu(k, Px)}{\log(en)} \in F.$$

But

$$\begin{aligned}
\sum_{k=1}^{n^2} \mu(k, Px) &= \sum_{k=1}^{n^2} \frac{1}{k} \sum_{l=1}^{k} x(l) = \sum_{l=1}^{n^2} x(l) \sum_{k=l}^{n^2} \frac{1}{k} \\
&\geq \sum_{l=1}^{n} x(l) \sum_{k=n}^{n^2} \frac{1}{k} \geq c \log(en) \sum_{k=1}^{n} x(k).
\end{aligned}$$

Hence

$$\sum_{k=1}^{n} x(k) \in F, \quad x \in X, \quad \text{and} \quad Z \subset X.$$

This shows that the pair (X, X_1) is optimal. □

Example 3.7. We conclude from Example 3.5 that $(\Lambda_{\log}^{\alpha}, \Lambda_{\log}^{\alpha+1})$ is the optimal pair for the operator P. The same is also true for the pair $(M_{1,\infty}^{\alpha}, M_{1,\infty}^{\alpha+1})$, where $M_{1,\infty}^{\alpha}$ is the Marcinkiewicz space with norm

$$\|x\|_{M_{1,\infty}^{\alpha}} = \sup_{n \in \mathbb{Z}_+} \frac{\sum_{k=1}^{n} \mu(k, x)}{\log^{\alpha}(en)}.$$

Note that for $\alpha = 0$ the space $M_{1,\infty}^{\alpha}$ coincides with the space ℓ_1. Thus, the pair $(\ell_1, M_{1,\infty})$ is optimal for operator P.

Theorems 3.3 and 3.6 can be generalised to the case of operators T with the following norm estimate:

$$\|T\|_{\ell_p(\mathbb{Z}_+)\to\ell_p(\mathbb{Z}_+)} \lesssim \frac{1}{(p-1)^\alpha}.$$

This should be compared to [2, Theorem 5.7].

The following result follows from Theorems 3.3, 3.6 and Remark 3.4.

Corollary 3.8. *If $F \in \mathbf{S}$, then a pair $(X_F, X_{F(\log^{-1})})$ is optimal for the operator H^d in the class $(\mathbb{S}, \mathbb{I})$.*

The following result describes the optimal pair for the operator P'.

Theorem 3.9. *If $F \in \mathbf{S}$, then the pair (X_F, X_F) is optimal for the operator P', defined in (1.1), in the class $(\mathbb{S}, \mathbb{I})$.*

Proof. Denote, for brevity $X := X_F$. We first prove that P' is bounded in X. For $x \in X$ we have

$$\|P'x\|_{\ell_1} = \sum_{n=1}^{\infty} |(P'x)(n)| \le \sum_{n=1}^{\infty}\sum_{k=n}^{\infty} \frac{|x(k)|}{k} = \sum_{k=1}^{\infty} \frac{|x(k)|}{k} \sum_{n=1}^{k} 1 = \|x\|_{\ell_1},$$

and

$$\begin{aligned}\|P'x\|_{\ell_2}^2 &= \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty} \frac{x(k)}{k^{1/4}}\cdot\frac{1}{k^{3/4}}\right)^2 \le \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{(x(k))^2}{k^{1/2}}\cdot\sum_{k=n}^{\infty}\frac{1}{k^{3/2}}\right)\\ &\lesssim \sum_{n=1}^{\infty}\left(\sum_{k=n}^{\infty}\frac{(x(k))^2}{k^{1/2}}\cdot\frac{1}{n^{1/2}}\right) \lesssim \sum_{k=1}^{\infty}\frac{(x(k))^2}{k^{1/2}}\cdot\sum_{n=1}^{k}\frac{1}{n^{1/2}}\\ &\lesssim \sum_{k=1}^{\infty}\frac{(x(k))^2}{k^{1/2}}\cdot k^{1/2} = \sum_{k=1}^{\infty}(x(k))^2 = \|x\|_{\ell_2}^2.\end{aligned}$$

Hence, P' is bounded in ℓ_1 and in ℓ_2, and, by the real version of Riesz-Thorin theorem, P' is uniformly bounded in ℓ_p for all $p \in [1,2]$. Then, by Theorem 3.2 we obtain

$$\|P'x\|_X \lesssim \left\|\left\{\|P'x\|_{\ell_{p(n)}}\right\}_{n=1}^{\infty}\right\|_F \lesssim \left\|\left\{\|x\|_{\ell_{p(n)}}\right\}_{n=1}^{\infty}\right\|_F \lesssim \|x\|_X.$$

To prove optimality we note that if $\mu(x) = x$, then

$$\sum_{k=1}^{n}\mu(k, P'x) = \sum_{k=1}^{n}\sum_{l=k}^{\infty}\frac{\mu(l,x)}{l} \ge \sum_{k=1}^{n}\mu(k,x).$$

Hence, $P'x$ majorises x in the sense of Hardy-Littlewood-Pólya. Thus, the facts that P' acts boundedly from X to Y and $Y \in \mathbb{I}$ imply that $X \subset Y$. Similarly, the facts that P' acts boundedly from Z to X imply that $\sum_{k=1}^n \mu(k, P'x) \in F$ for $x \in Z$. Hence $\sum_{k=1}^n \mu(k,x) \in F$, $x \in X$ and $Z \subset X$.

□

Remark 3.10. Since

$$\begin{aligned}\sum_{n=1}^{\infty}(Px)(n)y(n) &= \sum_{n=1}^{\infty}\frac{1}{n}\sum_{k=1}^{n}x(k)\,y(n)\\ &= \sum_{k=1}^{\infty}x(k)\sum_{n=k}^{\infty}\frac{y(n)}{n} = \sum_{k=1}^{\infty}x(k)(P'y)(k),\end{aligned}$$

then P' is conjugate operator to P, and

$$\|P'\|_{\ell_p\to\ell_p} = \|P\|_{\ell_{p'}\to\ell_{p'}}, \quad \text{where } p' = \frac{p}{p-1},\ p>1.$$

As it was noted in Remark 3.4, the operator P is bounded in ℓ_2. Thus, the boundedness of the operator P' could also be obtained from Remark 3.4. Conversely, since we have proved that the operator P' is bounded in ℓ_2, it follows that the operator P is bounded, too.

The following theorem shows that the pair $(X_F, X_{F(\log^{-1})})$ is optimal in the class of interpolation spaces (and even in a slightly wider class $(\mathbb{S},\mathbb{I})$) for the discrete Calderón operator S^d.

Theorem 3.11. *If Banach lattice $F \in \mathbf{S}\cap\mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, then the pair $(X_F, X_{F(\log^{-1})})$ is optimal in the class $(\mathbb{S},\mathbb{I})$ for the operator S^d defined in* (2.3).

Proof. Let $F\in\mathbf{S}$. Since S^d is bounded from $\ell_p(\mathbb{Z}_+)$ into $\ell_p(\mathbb{Z}_+)$ for $p\in(1,2]$ by the Hardy's inequality (3.18) and (3.19) in [4, Lemma 3.9, p. 124], i.e.

$$\|S^d\|_{\ell_p(\mathbb{Z}_+)\to\ell_p(\mathbb{Z}_+)} \lesssim \frac{1}{p-1},\ p\in(1,2],$$

it follows from Theorem 3.3 that the operator S^d acts boundedly from X_F into $X_{F(\log^{-1})}$. Therefore, similar proof as Theorem 3.6 one shows that the pair $(X_F, X_{F(\log^{-1})})$ is optimal in the class $(\mathbb{S},\mathbb{I})$ for the operator S^d. $\square$

4. Non-commutative case

Let $\mathbf{F}$ denote the symmetric operator space

$$\mathbf{F} := \{A\in K(H) : \|A\|_{\mathbf{F}} := \|\mu(A)\|_{X_F} < \infty\}, \tag{4.1}$$

where X_F is defined in Section 3. Note that each fully symmetric operator space admits such a representation, i.e. if $\mathbf{Y}\in\mathbb{FS}(H)$ then there is a Banach lattice Y such that $\|A\|_{\mathbf{Y}} = \|\mu(A)\|_{X_Y}$, where

$$\|x\|_{X_Y} := \left\|\left\{\sum_{k=1}^{n}\mu(k,x)\right\}_{n=1}^{\infty}\right\|_Y.$$

In addition, it is clear that $X_Y\in\mathbb{I}$ (recall that $\mathbb{I} = \mathbf{I}(\ell_1(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+))$).

It was proved in [3, Theorem 10] (see also [22, Theorem 2]) that if $F\in\mathbf{S}$, then

$$\|A\|_{\mathbf{F}} \approx \left\|\{\|A\|_{L_{p(n)}(H)}\}_{n\in\mathbb{Z}_+}\right\|_F, \tag{4.2}$$

where $p(n) = \frac{2\log(en)}{2\log(en)-1}$. In the particular case of $F = \ell_\infty(\log^{-1}(en))$, this relation was considered earlier in [14, Section 7] and [23, Section 5.5.2]. Define $\mathbf{F}(\log^{-1})$ to be the class of all $A \in K(H)$ such that

$$\|A\|_{\mathbf{F}(\log^{-1})} := \big\|\mu(A)\big\|_{X_{F(\log^{-1})}} < \infty,$$

where $X_{F(\log^{-1})}$ is also defined in Section 3. Moreover, if $F \in \mathbf{S}$, then, as shown in Section 3, $F(\log^{-1}) \in \mathbf{S}$. Therefore, applying [3, Theorem 10] (see also [22, Theorem 2]) again, we obtain

$$\|A\|_{\mathbf{F}(\log^{-1})} \approx \big\|\big\{\frac{1}{\log(en)}\|A\|_{L_{p(n)}(H)}\big\}_{n\in\mathbb{Z}_+}\big\|_F < \infty. \tag{4.3}$$

Similar to the proof of [3, Theorem 11] (see also [22, Theorem 3]), we obtain the following result for the triangular truncation operator $\mathcal{T}$ defined in (2.6).

Theorem 4.1. *Let $\mathcal{T}$ be the triangular truncation operator defined in* (2.6)*. If $F \in \mathbf{S}$, then $\mathcal{T}$ is bounded from $\mathbf{F}$ into $\mathbf{F}(\log^{-1})$.*

Proof. First, since $\mathcal{T}$ is a weak type $(1,1)$ operator by [28, Theorem 11], we have the following estimate for $p \in (1,2]$

$$\|\mathcal{T}\|_{L_p(H)\to L_p(H)} \lesssim \frac{1}{p-1} \tag{4.4}$$

(see also [28, Theorem 14 (ii)]). Let $p(n) = \frac{2\log(en)}{2\log(en)-1}$, $n \in \mathbb{Z}_+$, and $A \in \mathbf{F}$. Since

$$p(n) \in (1,2] \text{ and } (p(n)-1)\log(en) = \frac{\log(en)}{2\log(en)-1} \in (\frac{1}{2}, 1] \tag{4.5}$$

for $n \in \mathbb{Z}_+$, for every $A \in \mathbf{F}$ one has the following estimates:

$$\begin{aligned} \|\mathcal{T}(A)\|_{\mathbf{F}(\log^{-1})} &\overset{(4.3)}{\lesssim} \big\|\big\{\frac{1}{\log(en)}\|\mathcal{T}(A)\|_{L_{p(n)}(H)}\big\}_{n\in\mathbb{Z}_+}\big\|_F \\ &\overset{(4.4)}{\lesssim} \big\|\big\{\frac{1}{(p(n)-1)\log(en)}\|A\|_{L_{p(n)}(H)}\big\}_{n\in\mathbb{Z}_+}\big\|_F \\ &\overset{(4.5)}{\lesssim} \big\|\big\{\|A\|_{L_{p(n)}(H)}\big\}_{n\in\mathbb{Z}_+}\big\|_F \\ &\overset{(4.2)}{\lesssim} \|A\|_{\mathbf{F}}. \end{aligned} \tag{4.6}$$

Since A is arbitrary, this concludes the proof. $\square$

Theorem 4.2. *If $F \in \mathbf{S} \cap \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, then $(\mathbf{F}, \mathbf{F}(\log^{-1}))$ is optimal pair in the class $(\mathbb{FS}(H), \mathbb{FS}(H))$ for the operator $\mathcal{T}$ defined in* (2.6)*, i.e. if $\mathbf{Y}, \mathbf{Z}$ are fully symmetric operator spaces and*

$$\mathcal{T} : \mathbf{F} \to \mathbf{Y} \text{ and } \mathcal{T} : \mathbf{Z} \to \mathbf{F}(\log^{-1}))$$

then $\mathbf{F}(\log^{-1}) \subset \mathbf{Y}$ and $\mathbf{Z} \subset \mathbf{F}$.

Proof. If $F \in \mathbf{S} \cap \mathbf{I}(\ell_\infty(\mathbb{Z}_+), \ell_\infty(\mathbb{Z}_+, 1/n))$, then boundedness of the operator $\mathcal{T}$ from $\mathbf{F}$ into $\mathbf{F}(\log^{-1}))$ follows from the Theorem 4.1. Let us show that the pair $(\mathbf{F}, \mathbf{F}(\log^{-1}))$ is optimal in the class of interpolation spaces for the operator $\mathcal{T}$.

Suppose that $\mathbf{Y} := \{A \in K(H) : \mu(A) \in X_Y\}$ is an symmetric operator space such that $\mathcal{T} : \mathbf{F} \to \mathbf{Y}$ is bounded. We shall show that $\mathbf{F}(\log^{-1}) \subset \mathbf{Y}$. If $a \in X_F$, then by [28, Theorem 21] there exists an operator A such that $\mu(a) = \mu(A)$ and

$$S^d\mu(a) = S^d\mu(A) \lesssim \mu(\mathcal{T}(A)).$$

Since $\mathcal{T}(A) \in \mathbf{Y}$ by assumption, i.e. $\mu(\mathcal{T}(A)) \in X_Y$, it follows that $S^d\mu(a) \in X_Y$, it means $S^d : X_F \to X_Y$ is bounded. But, Theorem 3.11 states that the pair $(X_F, X_{F(\log^{-1})})$ is optimal in the class of interpolation spaces for the operator S^d. Therefore, we have $X_{F(\log^{-1})} \subset X_Y$, i.e. $\mathbf{F}(\log^{-1}) \subset \mathbf{Y}$. Embedding $\mathbf{Z} \subset \mathbf{F}$, under condition $\mathcal{T} : \mathbf{Z} \to \mathbf{F}(\log^{-1}))$, is proved in a similar way. $\square$

Theorem 4.3. *Let $\mathbf{F}$ and $\mathbf{F}(\log^{-1})$ be as in Theorem 4.2. The following assertions hold:*

(i) *If $A = A^*$ is a self-adjoint operator in $B(H)$, then the double operator integral (associated with the function $f^{[1]}$ defined in* (2.8)*) $T^{A,A}_{f^{[1]}} : \mathbf{F} \to \mathbf{F}(\log^{-1})$ is bounded and*

$$\|T^{A,A}_{f^{[1]}}\|_{\mathbf{F}\to\mathbf{F}(\log^{-1})} \le c_{\mathbf{F}}\|f'\|_{L_\infty(\mathbb{R})},$$

where $c_{\mathbf{F}}$ is the constant depending on $\mathbf{F}$ only.

(ii) *For all self-adjoint operators $A, B \in B(H)$ such that $[A, B] \in \mathbf{F}$ and for every Lipschitz function $f : \mathbb{R} \to \mathbb{C}$, we have*

$$\|[f(A), B]\|_{\mathbf{F}(\log^{-1})} \lesssim \|f'\|_{L_\infty(\mathbb{R})}\|[A, B]\|_{\mathbf{F}},$$

where $[A, B] := AB - BA$. For all self-adjoint operators $X, Y \in B(H)$ such that $X - Y \in \mathbf{F}$ and for every Lipschitz function $f : \mathbb{R} \to \mathbb{C}$, we have

$$\|f(X) - f(Y)\|_{\mathbf{F}(\log^{-1})} \lesssim \|f'\|_{L_\infty(\mathbb{R})}\|X - Y\|_{\mathbf{F}}.$$

Proof. (i). By Theorem 1.2 in [9], we have

$$\|T^{A,A}_{f^{[1]}}(V)\|_{L_{1,\infty}(H)} \lesssim \|f'\|_{L_\infty(\mathbb{R})}\|V\|_{L_1(H)}, \quad V \in L_1(H).$$

Thus, the operator $T^{A,A}_{f^{[1]}}$ (see Subsection 2.7) satisfies conditions of [28, Theorem 14 (ii)], and

$$\mu(T^{A,A}_{f^{[1]}}(V)) \lesssim \|f'\|_{L_\infty(\mathbb{R})}S^d\mu(V), \quad V \in \Lambda_{\varphi_0}(H),$$

where φ_0 is defined as in Proposition 2.6.

By Theorem 3.11, the operator S^d acts boundedly from X_F into $X_{F(\log^{-1})}$. Thus

$$\begin{aligned}\|T^{A,A}_{f^{[1]}}(V)\|_{\mathbf{F}(\log^{-1})} &= \|\mu\big(T^{A,A}_{f^{[1]}}(V)\big)\|_{X_{F(\log^{-1})}} \lesssim \|f'\|_{L_\infty(\mathbb{R})}\|S^d\mu(V)\|_{X_{F(\log^{-1})}} \\ &\lesssim \|f'\|_{L_\infty(\mathbb{R})}\|\mu(V)\|_{X_F} = c_{\mathbf{F}}\|f'\|_{L_\infty(\mathbb{R})}\|V\|_{\mathbf{F}}, \ \ V \in \mathbf{F}.\end{aligned}$$

In other words, $T^{A,A}_{f^{[1]}} : \mathbf{F} \to \mathbf{F}(\log^{-1})$ is bounded.

(ii). The double operator integral $T^{A,A}_{f^{[1]}}([A, B])$ is equal to $[f(A), B]$ for the operators $A, B \in B(H)$ such that $[A, B] \in \mathbf{F}$ (see [25, Proposition 2.6]). Therefore, the commutator estimate follows from part (i). Finally, applying the commutator estimate to the operators

$$A = \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

we obtain Lipschitz estimate. □

5. Acknowledgment

The first author was partially supported by the RFBR grants 17-01-00138 and 18-01-00414. The second author was partially supported by Australian Research Council. The third author was partially supported by the grant No. AP08052004 of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan. The fourth author was partially supported by the Swedish Research Council Grant 2015-00137 and Marie Sklodowska Curie Actions, Cofund, Project INCA 600398.

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Manuscript received February 25 2019
revised July 2 2019

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