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ON THE STABILITY OF THE LIONS-PEETRE METHOD OF REAL INTERPOLATION WITH FUNCTIONAL PARAMETER

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ABSTRACT. Let $\vec{X} = (X_0, X_1)$ be a compatible couple of Banach spaces, $1 \leq p \leq \infty$ and let φ be positive quasi-concave function. Denote by $\overline{X}_{\varphi,p} = (X_0, X_1)_{\varphi,p}$ the real interpolation spaces defined by S. Janson (1981). We give necessary and sufficient conditions on φ_0, φ_1 and φ for the validity of

$$\left(\overline{X}_{\varphi_0,1}, \overline{X}_{\varphi_1,1}\right)_{\varphi,p} = \left(\overline{X}_{\varphi_0,\infty}, \overline{X}_{\varphi_1,\infty}\right)_{\varphi,p}$$

for all $1 \leq p \leq \infty$, and all Banach couples $\overline{X}$.

1. Introduction

Let $\overline{X} = (X_0, X_1)$ be a compatible Banach couple. For $x \in \sum(\overline{X}) = X_0 + X_1$, Peetre's K-functional is defined by

$$K(t, x, \overline{X}) = \inf_{x=x_0+x_1, x_i \in X_i} (\|x_0\|_{X_0} + t\|x_1\|_{X_1}) \quad t > 0.$$

Let $0 < \theta < 1$, $1 \leq p \leq \infty$, then the Lions-Peetre spaces $\overline{X}_{\theta,p} = (X_0, X_1)_{\theta,p}$ are defined using the norm

$$\|x\|_{\theta,p} = \left(\int_0^\infty \left(\frac{K(t, x, \overline{X})}{t^\theta}\right)^p \frac{dt}{t}\right)^{\frac{1}{p}}.$$

One of the most important theoretical results for these spaces is the so-called reiteration theorem, which claims that

$$\left(\overline{X}_{\theta_0,p_0}, \overline{X}_{\theta_1,p_1}\right)_{\theta,p} = \overline{X}_{(1-\theta)\theta_0+\theta\theta_1,p}, \quad \theta_0 \neq \theta_1. \tag{1.1}$$

These definitions and properties can be found in any modern monograph on interpolation theory (e.g. [5], [6] and [19]). The statement (1.1) is the so called stability of the real method. The resulting space on the left-hand side of (1.1) does not depend on p_0, p_1.

The definition of real interpolation method was extended in different directions by a number of authors. For example, one can replace the function t^θ in (1.1) by a positive concave function φ, defined on $(0, \infty)$ (see T.F.Kalugina [13], J.Gustavsson

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[10], L.E.Persson [16]). In [12] S. Janson provided a different approach to these spaces using the discrete norm

$$\|x\|_{\varphi,p} = \left(\sum_{k\in\mathbb{Z}} \left(\frac{K(t_k, x, \overline{X})}{\varphi(t_k)}\right)^p\right)^{\frac{1}{p}},$$

where $\{t_k\}$ is a special discretizing sequence depending of φ (see Definition 2.3). In [12, Theorem 19] Janson proved that, if $\varphi(\varphi_0, \varphi_1)$ and $\frac{\varphi_1}{\varphi_0}$ are quasi-power functions (see Definition 2.4), then the following reiteration formula holds for any $p_0, p_1, p \in [1, \infty]$,

$$(1.2)\qquad \left(\overline{X}_{\varphi_0,p_0}, \overline{X}_{\varphi_1,p_1}\right)_{\varphi,p} = \overline{X}_{\varphi(\varphi_0,\varphi_1),p}$$
$$\left(\varphi(\varphi_0, \varphi_1)(t) = \varphi_0(t)\varphi\left(\frac{\varphi_1(t)}{\varphi_0(t)}\right)\right).$$

In [14], N.Krugljak gave a necessary and sufficient condition on φ_0, φ_1 and φ, so that (1.2) is true for any choice of $p_0, p_1, p \in [1, \infty]$.

A more general reiteration theorem for a real interpolation method was obtained by S. Astashkin [2, 3], Yu. A. Brudnyĭ and N. Ya. Krugljak [7].

It is clear that, from (1.2) we have

$$(1.3)\qquad \left(\overline{X}_{\varphi_0,1}, \overline{X}_{\varphi_1,1}\right)_{\varphi,p} = \left(\overline{X}_{\varphi_0,\infty}, \overline{X}_{\varphi_1,\infty}\right)_{\varphi,p}.$$

The inverse implication is not easy. The sufficient condition for (1.3) was obtained by E.Pustylnik [17] and E.Semenov [18] in case when $\varphi(t) = t^\theta$. V.Ovchinnikov [15] offered a new approach to study this problem. Semenov and Ovchinnikov used Krugljak 's result [1, Corollary 3]. Ovchinikov only considers the case $\varphi(t) = t^\theta$. In this paper we are going extend Ovchinnikov's theorem to te setting of non-degenerate quasi-concave function φ. In this context we will show (cf. Theorem 2.5) that the reiteration theorem (1.2) follows from the stability theorem (1.3).

In this paper we shall not consider the case of degenerate quasi-concave functions, which we leave as an open problem. We think that the study of the degenerate case could be of interest to experts in Extrapolation Theory (see e.g. [4].

We use the notation $A \lesssim B$ to indicate that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we shall write $A \approx B$.

2. Definitions and main result

We start with some basic definitions.

Definition 2.1. Let $\{a_k\}$ be a sequence of positive numbers. We shall say that $\{a_k\}$ is strongly increasing (resp. strongly decreasing) and write $a_k \uparrow\uparrow$ (resp. $a_k \downarrow\downarrow$) if

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \ge 2 \quad \left(\text{resp.} \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} \le \frac{1}{2},\right)$$

Definition 2.2. We shall say that φ is non-degenerate quasi-concave function on $(0,\infty)$, if φ is non-decreasing and $\frac{\varphi(t)}{t}$ is non-increasing on $(0,\infty)$ and, moreover,

$$\lim_{t\to 0+}\varphi(t)=\lim_{t\to+\infty}\frac{\varphi(t)}{t}=\lim_{t\to 0+}\frac{t}{\varphi(t)}=\lim_{t\to+\infty}\frac{1}{\varphi(t)}=0. \tag{2.1}$$

Definition 2.3. A strongly increasing sequence $\{t_k\}$ is a discretizing sequence for a non-degenerate quasi-concave function φ if

i) $\varphi(t_k)\uparrow\uparrow$ and $\frac{\varphi(t_{k+1})}{t_{k+1}}\downarrow\downarrow$;

ii) There exists a decomposition $\mathbb{Z}=\mathbb{Z}_1\cup\mathbb{Z}_2$, such that $\mathbb{Z}_1\cap\mathbb{Z}_2=\emptyset$, and

$$\begin{aligned}\varphi(t_{k+1})&\le 2\varphi(t_k) &&\text{if}\quad k\in\mathbb{Z}_1,\\ \frac{\varphi(t_k)}{t_k}&\le 2\frac{\varphi(t_{k+1})}{t_{k+1}} &&\text{if}\quad k\in\mathbb{Z}_2.\end{aligned}$$

Let us recall [9, Lemma 2.7], that if φ is non-degenerate quasi-concave function there always exists a discretizing sequence adapted to φ.

Definition 2.4. A quasi-concave function φ is a quasi-power function ($\varphi\in P^{+-}$), whenever $s_\varphi(t)\to 0$ as $t\to 0$, $s_\varphi(t)\to\infty$ as $t\to\infty$, where $s_\varphi(t)=\sup_u s_\varphi(ut)/s_\varphi(u)$ (cf. [11]).

It is known that (cf. [11]), any quasi-power function φ is equivalent to $t^{\theta_0}\psi(t^{\theta_1-\theta_0})$, for some quasi-concave function ψ and $0<\theta_0,\theta_1<1$. If φ is quasi-power function then $\{2^k\}$ is a discretizing sequence for φ.

It is easy to see that if φ_0, φ_1 and φ are non-degenerate positive quasi-concave functions on $(0,\infty)$, then the function $\varphi(\varphi_0,\varphi_1)(t)=\varphi_0(t)\varphi(\frac{\varphi_1(t)}{\varphi_0(t)})$ is a non-degenerate quasi-concave function. Throughout the paper the functions φ_0, φ_1 and φ will be assumed to be non-degenerate positive quasi-concave functions on $(0,\infty)$ and $\{t_k\}$ (resp. $\{\widetilde{t_k}\}$) will denote the discretizing sequence for φ (resp. the discretizing sequence for $\varphi(\varphi_0,\varphi_1)$).

Our main result now reads as follows.

Theorem 2.5. *Let φ_0, φ_1 and φ be positive non-degenerate quasi-concave functions on $(0,\infty)$. Let $\{t_k\}$ be discretizing sequence for φ and let $\{\widetilde{t_k}\}$ be discretizing sequence for $\varphi(\varphi_0,\varphi_1)$. The following assertions are equivalent:*

(i) (1.3) *holds for some* $p\in[1,\infty]$;

(ii) (1.3) *holds for every* $p\in[1,\infty]$;

(iii) (1.3) *holds for* $\overline{X}=(L_1(0,\infty),L_\infty(0,\infty))$ *and some* $p\in[1,\infty]$;

(iv) (1.2) *holds for any* $p_0,p_1,p\in[1,\infty]$;

(v) $\sup_{n\in\mathbb{Z}}$ *Card*$\{k\in\mathbb{Z}:\quad t_n\le\frac{\varphi_0(\widetilde{t_k})}{\varphi_1(\widetilde{t_k})}\le t_{n+1}\}<\infty$.

3. Descriptions of some special interpolation spaces

Definition 3.1. Suppose that X is an intermediate space for a compatible couple (X_0,X_1). The orbit of the space X relative to linear bounded operators mapping the couple $\{X_0,X_1\}$ to the couple $\{Y_0,Y_1\}$, which will be denoted by

$$\operatorname{Orb}\left(X,\{X_0,X_1\}\to\{Y_0,Y_1\}\right),$$

is the linear space of all $y \in Y_0 + Y_1$ that can be represented by

$$y = \sum_{j=1}^{\infty} T_j x_j, \quad \text{convergence in} \quad Y_0 + Y_1,$$

where

$$\sum_{j=1}^{\infty} \max(\|T_j\|_{X_0 \to Y_0}, \|T_j\|_{X_1 \to Y_1}) \|x_j\|_X < \infty.$$

Definition 3.2. Suppose X is an intermediate space for a compatible couple (X_0, X_1). The coorbit of the space $Y \in Y_0 + Y_1$ relative to linear bounded operators mapping the couple $\{X_0, X_1\}$ to the couple $\{Y_0, Y_1\}$, which will be denoted by

$$\operatorname{Corb}\left(Y, \{X_0, X_1\} \to \{Y_0, Y_1\}\right),$$

is the linear space of all $x \in X_0 + X_1$ such that

$$\sup\{\|T(x)\|_Y : \quad \|T\|_{\{X_0, X_1\} \to \{Y_0, Y_1\}} \leq 1\} < \infty.$$

If E is a sequence space and $\{w_k\}$ is a positive sequence (a weight), then $E(w)$ denotes the space of sequences $\{a_k\}$ such that $a_k w_k \in E$, provided with its natural norm $\|a\|_{E(w)} = \|a_k w_k\|_E$.

Let $\bar{l}_q = (l_q, l_q(\frac{1}{\tilde{t}_k}))$, $q \in [1, \infty]$ be a Banach couple consisting of two sided infinite sequences, where $\{\tilde{t}_k\}$ is discretizing sequence for $\varphi(\varphi_0, \varphi_1)$. The main property of real method in terms of Orbits can be formulated as follows (see [12])

$$\begin{aligned} \overline{X}_{\varphi,p} &= \operatorname{Orb}\left(l_p\left(\frac{1}{\varphi(\varphi_0, \varphi_1)(\tilde{t}_k)}\right), \bar{l}_1 \to \overline{X}\right) \\ &= \operatorname{Corb}\left(l_p\left(\frac{1}{\varphi(\varphi_0, \varphi_1)(\tilde{t}_k)}\right), \overline{X} \to \bar{l}_\infty\right). \end{aligned} \tag{3.1}$$

As it is known (see [12])

$$(\bar{l}_1)_{\varphi(\varphi_0,\varphi_1),p} = (\bar{l}_\infty)_{\varphi(\varphi_0,\varphi_1),p} = l_p\left(\frac{1}{\varphi(\varphi_0, \varphi_1)(\tilde{t}_k)}\right). \tag{3.2}$$

by embeddings $l_1 \subset l_q \subset l_\infty$ and $l_1\left(\frac{1}{\tilde{t}_k}\right) \subset l_q\left(\frac{1}{\tilde{t}_k}\right) \subset l_\infty\left(\frac{1}{\tilde{t}_k}\right)$ we get

$$(\bar{l}_q)_{\varphi(\varphi_0,\varphi_1),p} = l_p\left(\frac{1}{\varphi(\varphi_0, \varphi_1)(\tilde{t}_k)}\right). \tag{3.3}$$

Let l_q^M be the space of sequences with indices in the set M. So if $E = l_p\left(l_q^{M_k}\right)$ then

$$\|a\|_E = \left(\sum_{k \in \mathbb{Z}} \left(\sum_{i \in M_k} |a_i|^q\right)^{p/q}\right)^{1/p}.$$

The next lemma is a functional parameter version of Gilbert's interpolation theorem [8]

Lemma 3.3. *Let $p, q \in [1, \infty]$. Let $\{v_k\}$, $\{w_k\}$ be positive sequences and φ be non-degenerate quasi-concave function on $(0, \infty)$ then*

$$(l_q(v), l_q(w))_{\varphi,p} = l_p\left(l_q^{M_k}\left(\frac{v_i}{\varphi\left(\frac{v_i}{w_i}\right)}\right)\right), \tag{3.4}$$

where $M_k = \{i : t_{k-1} < \frac{v_i}{w_i} \leq t_k\}$ and $\{t_k\}$ is a discretizing sequence for φ.

Proof. Let $E = (l_q(v), l_q(w))_{\varphi,p}$. Assume that $p, q < \infty$. Since

$$K(t, \{a_k\}; \{l_q(v), l_q(w)\}) \approx \left(\sum_{k\in\mathbb{Z}} |a_k|^q \min(v_k, tw_k)^q\right)^{1/q}.$$

it follows that

$$\|\{a_k\}\|_E \approx \left(\sum_{i\in\mathbb{Z}} \frac{\left(\sum_{k\in\mathbb{Z}} |a_k|^q \min(v_k, t_i w_k)^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p}.$$

By the definition of of discretizing sequence (cf. Definition 2.3), we obtain

$$\begin{aligned}\|\{a_k\}\|_E &\gtrsim \left(\sum_{i\in\mathbb{Z}_1} \frac{\left(\sum_{t_{i-1}<\frac{v_k}{w_k}\leq t_i} |a_k|^q v_k^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p} \\ &\quad + \left(\sum_{i\in\mathbb{Z}_2} \frac{t_{i-1}^p\left(\sum_{t_{i-1}<\frac{v_k}{w_k}\leq t_i} |a_k|^q w_k^q\right)^{p/q}}{\varphi(t_{i-1})^p}\right)^{1/p} \\ &\approx \left(\sum_{i\in\mathbb{Z}}\left(\sum_{t_{i-1}<\frac{v_k}{w_k}\leq t_i} |a_k|^q\left(\frac{v_k}{\varphi\left(\frac{v_k}{w_k}\right)}\right)^q\right)^{p/q}\right)^{1/p}.\end{aligned} \tag{3.5}$$

To prove the reverse estimate we use Lemma 3.1 from [9].

$$\begin{aligned}\|\{a_k\}\|_E &\approx \left(\sum_{i\in\mathbb{Z}} \frac{\left(\sum_{j\in\mathbb{Z}}\sum_{t_{j-1}<\frac{v_k}{w_k}\leq t_j} |a_k|^q \min(v_k, t_i w_k)^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p} \\ &\lesssim \left(\sum_{i\in\mathbb{Z}} \frac{\left(\sum_{j\leq i}\sum_{t_{j-1}<\frac{v_k}{w_k}\leq t_j} |a_k|^q v_k^q\right)^{p/q}}{\varphi(t_i)^p}\right.\end{aligned} \tag{3.6}$$

$$+\left(\sum_{i\in\mathbb{Z}}\frac{t_i^p\left(\sum_{j>i}\sum_{t_{j-1}<\frac{v_k}{w_k}\le t_j}|a_k|^q w_k^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p}$$

$$\approx\left(\sum_{i\in\mathbb{Z}}\frac{\left(\sum_{t_{i-1}<\frac{v_k}{w_k}\le t_i}|a_k|^q v_k^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p}$$

$$+\left(\sum_{i\in\mathbb{Z}}\frac{t_i^p\left(\sum_{t_i<\frac{v_k}{w_k}\le t_{i+1}}|a_k|^q w_k^q\right)^{p/q}}{\varphi(t_i)^p}\right)^{1/p}$$

$$\lesssim\left(\sum_{i\in\mathbb{Z}}\left(\sum_{t_{i-1}<\frac{v_k}{w_k}\le t_i}|a_k|^q\left(\frac{v_k}{\varphi\left(\frac{v_k}{w_k}\right)}\right)^q\right)^{p/q}\right)^{1/p}.$$

Therefore from (3.5) and (3.6) we obtain (3.4). The case $p=\infty$ or $q=\infty$ can be obtained using a similar argument and we shall omit the details.

The proof is complete. □

Corollary 3.4. *Let $p,q\in[1,\infty]$. Let $\{\widetilde{t}_k\}$ be discretizing sequence for $\varphi(\varphi_0,\varphi_1)$ and let $\{t_k\}$ be discretizing sequence for φ. Then*

$$\left(l_q\left(\frac{1}{\varphi_0(\widetilde{t}_k)}\right),l_q\left(\frac{1}{\varphi_1(\widetilde{t}_k)}\right)\right)_{\varphi,p}=l_p\left(l_q^{M_k}\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right)\right),$$

where $M_k=\{i:t_k\le\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\le t_{k+1}\}$.

Corollary 3.5. *Let $p,q\in[1,\infty]$. Let $\{\tau_k\}$ be discretizing sequence for φ_0, let $\{z_k\}$ be discretizing sequence for φ_1 and let $\{\widetilde{t}_k\}$ be discretizing sequence for $\varphi(\varphi_0,\varphi_1)$. Then,*

$$(\bar{l}_q)_{\varphi_0,p}=l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right),\tag{3.7}$$

$$(\bar{l}_q)_{\varphi_1,p}=l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right),\tag{3.8}$$

where $M_k^0=\{i:\tau_k\le\widetilde{t}_i\le\tau_{k+1}\}$ and $M_k^1=\{i:z_k\le\widetilde{t}_i\le z_{k+1}\}$.

Corollary 3.6. *Let $p\in[1,\infty]$. Then*

$$\left((\bar{l}_p)_{\varphi_0,p},(\bar{l}_p)_{\varphi_1,p}\right)_{\varphi,p}=l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right).$$

Proof. Using Lemma 3.3 we get

$$\left((\bar{l}_p)_{\varphi_0,p},(\bar{l}_p)_{\varphi_1,p}\right)_{\varphi,p} = \left(l_p\left(\frac{1}{\varphi_0(\widetilde{t}_k)}\right), l_p\left(\frac{1}{\varphi_1(\widetilde{t}_k)}\right)\right)_{\varphi,p} = l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right).$$

□

Lemma 3.7. *Let $r \in \mathbb{R}_1 \setminus \{0\}$. Let $\{\tau_k\}$ be discretizing sequence for φ_0 and let $\{z_k\}$ be discretizing sequence for φ_1. Then we have the following estimates:*

$$\sum_{\tau_k \le \widetilde{t}_i \le \tau_{k+1}} \left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \sup_{\tau_k \le \widetilde{t}_i \le \tau_{k+1}} \left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r, \tag{3.9}$$

$$\sum_{z_k \le \widetilde{t}_i \le z_{k+1}} \left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \sup_{z_k \le \widetilde{t}_i \le z_{k+1}} \left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \tag{3.10}$$

with constants independent of τ_k and z_k.

Proof. Let us fix k. We consider two possibilities. Either

$$\varphi_0(\tau_{k+1}) \le 2\varphi_0(\tau_k) \tag{3.11}$$

or

$$\frac{\varphi_0(\tau_k)}{\tau_k} \le 2\frac{\varphi_0(\tau_{k+1})}{\tau_{k+1}}. \tag{3.12}$$

Suppose that (3.11) holds. In this case we will show that

$$\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})} \ge \frac{3}{2}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}$$

for every $\widetilde{t}_i, \widetilde{t}_{i+2} \in [\tau_k, \tau_{k+1}]$. Indeed, if we assume that

$$\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})} \le \frac{3}{2}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)},$$

by using (3.11) we get that

$$\varphi_0(\widetilde{t}_{i+2})\varphi\left(\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})}\right) \le 2\varphi_0(\widetilde{t}_i)\varphi\left(\frac{3}{2}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right) \le 3\varphi_0(\widetilde{t}_i)\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right),$$

as $\{\widetilde{t}_k\}$ is discretizing sequence for $\varphi(\varphi_0,\varphi_1)$ we obtain the contradiction. Suppose now that (3.12) holds. In this case we will show that

$$\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})} \le \frac{2}{3}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}$$

for every $\widetilde{t}_i, \widetilde{t}_{i+2} \in [\tau_k, \tau_{k+1}]$. Indeed, if we assume that

$$\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})} \ge \frac{2}{3}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)},$$

then using (3.12) we get that

$$\frac{\varphi_0(\widetilde{t}_i)}{\widetilde{t}_i}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right) \le 2\frac{\varphi_0(\widetilde{t}_{i+2})}{\widetilde{t}_{i+2}}\varphi\left(\frac{3}{2}\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})}\right) \le 3\frac{\varphi_0(\widetilde{t}_{i+2})}{\widetilde{t}_{i+2}}\varphi\left(\frac{\varphi_1(\widetilde{t}_{i+2})}{\varphi_0(\widetilde{t}_{i+2})}\right),$$

which, once again, is a contradiction. Therefore, we obtain

$$\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r\sum_{i=1}^{\infty}\left(\frac{2}{3}\right)^{|r|i} \lesssim \sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r.$$

In a similar fashion we can show the estimate (3.10). □

Lemma 3.8. *Let $r\in\mathbb{R}_1\setminus\{0\}$. Let $\{\tau_k\}$ be discretizing sequence for φ_0 and $\{z_k\}$ be discretizing sequence for φ_1. Then*

$$\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r, \tag{3.13}$$

$$\sum_{z_k\le\widetilde{t}_i\le z_{k+1}}\left(\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)\right)^r \lesssim \sup_{z_k\le\widetilde{t}_i\le z_{k+1}}\left(\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)\right)^r \tag{3.14}$$

with constants independent of τ_k and z_k.

Proof. Let us fix k. As it was mentioned during the course of the proof of Lemma 3.7. we have two cases either (3.11) or (3.12). If (3.11) holds we get

$$\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \frac{1}{\varphi_0(\tau_k)^r}\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi_0(\widetilde{t}_i)^r\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \frac{1}{\varphi_0(\tau_k)^r}\sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi_0(\widetilde{t}_i)^r\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r.$$

If (3.12) holds we get

$$\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \left(\frac{\tau_k}{\varphi_0(\tau_k)}\right)^r\sum_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\left(\frac{\varphi_0(\widetilde{t}_i)}{\widetilde{t}_i}\right)^r\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r \lesssim \left(\frac{\tau_k}{\varphi_0(\tau_k)}\right)^r\sup_{\tau_k\le\widetilde{t}_i\le\tau_{k+1}}\left(\frac{\varphi_0(\widetilde{t}_i)}{\widetilde{t}_i}\right)^r\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r$$

$$\lesssim \sup_{\tau_k \le \widetilde{t}_i \le \tau_{k+1}} \varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^r.$$

The proof of the estimate (3.13) is complete. The proof of (3.14) is similar and we omit the details. □

Lemma 3.9. *Let $q,p \in [1,\infty]$. Let $\{\tau_k\}$ be discretizing sequence for φ_0 and Let $\{z_k\}$ be discretizing sequence for φ_1. Let $M_k^0 = \{i : \tau_k \le \widetilde{t}_i \le \tau_{k+1}\}$ and $M_k^1 = \{i : z_k \le \widetilde{t}_i \le z_{k+1}\}$. Denote sets $\Omega_t = \{y : \frac{\varphi_1(y)}{\varphi_0(y)} \le t\}$ and $\Omega_t^c = \{y : \frac{\varphi_1(y)}{\varphi_0(y)} > t\}$. Then*

$$K\left(\{a_k\}, t; l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right), l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)\right)$$
$$\approx \|a_i\chi_{\Omega_t}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)} + t\|a_i\chi_{\Omega_t^c}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)}.$$

Proof. Let us consider sequences $b_i = a_i\chi_{\Omega_t}(\widetilde{t}_i)$ and $c_i = a_i\chi_{\Omega_t^c}(\widetilde{t}_i)$. Hence $a_i = b_i + c_i$. Using the definition of K-functional it easy to see that

$$K\left(\{a_k\}, t; l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right), l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)\right)$$
$$\lesssim \left(\sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i \in M_k^0, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t}\left(\frac{|a_i|}{\varphi_0(\widetilde{t}_i)}\right)^q\right)^{\frac{p}{q}}\right)^{1/p}$$
$$+ t\left(\sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i \in M_k^1, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} > t}\left(\frac{|a_i|}{\varphi_1(\widetilde{t}_i)}\right)^q\right)^{\frac{p}{q}}\right)^{1/p}$$
$$= \|a_i\chi_{\Omega_t}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)} + t\|a_i\chi_{\Omega_t^c}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)}.$$

So we have obtained an upper bound for the K-functional.

Let $\{a_k\} \in l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right) + l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)$ and let us consider any representation $a_i = b_i + c_i$, $i \in Z$ with $\{b_i\} \in l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)$ and $\{c_i\} \in l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)$. By using estimates (3.9) and (3.10) we obtain

$$\left(\sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i \in M_k^0, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t}\left(\frac{|c_i|}{\varphi_0(\widetilde{t}_i)}\right)^q\right)^{\frac{p}{q}}\right)^{1/p}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}} \left(\sum_{\widetilde{t}_i \in M_k^0, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t} \left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right)^q \right)^{\frac{p}{q}} \left(\sup_{\widetilde{t}_i \in M_k^0} \frac{|c_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}} \sum_{\tau_j \in M_k^1} \left(\sum_{\widetilde{t}_i \in M_j^0, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t} \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right)^p \left(\sup_{\widetilde{t}_i \in M_j^0} \frac{|c_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim \left(\sum_{k\in\mathbb{Z}} \sum_{\tau_j \in M_k^1} \left(\sup_{\widetilde{t}_i \in M_j^0, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t} \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right)^p \left(\sup_{\widetilde{t}_i \in M_j^0} \frac{|c_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim t \left(\sum_{k\in\mathbb{Z}} \sup_{\tau_j \in M_k^1} \left(\sup_{\widetilde{t}_i \in M_j^0} \frac{|c_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim t \left(\sum_{k\in\mathbb{Z}} \left(\sum_{\tau_i \in M_k^1} \left(\frac{|c_i|}{\varphi_1(\widetilde{t}_i)} \right)^q \right)^{p/q} \right)^{1/p}.$$

Therefore, we get

$$\|a_i \chi_{\Omega_t}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)} \lesssim \|b_i\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)} + t\|c_i\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)}. \tag{3.15}$$

Similarly by using estimates (3.10) and (3.9) we obtain

$$\left(\sum_{k\in\mathbb{Z}} \left(\sum_{\widetilde{t}_i \in M_k^1, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} > t} \left(\frac{|b_i|}{\varphi_1(\widetilde{t}_i)} \right)^q \right)^{\frac{p}{q}} \right)^{1/p}$$

$$\lesssim t \left(\sum_{k\in\mathbb{Z}} \left(\sum_{\widetilde{t}_i \in M_k^1, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} > t} \left(\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} \right)^q \right)^{\frac{p}{q}} \left(\sup_{\widetilde{t}_i \in M_k^1} \frac{|b_i|}{\varphi_0(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim t \left(\sum_{k\in\mathbb{Z}} \sum_{z_j \in M_k^0} \left(\sum_{\widetilde{t}_i \in M_j^1, \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} > t} \frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} \right)^p \left(\sup_{\widetilde{t}_i \in M_j^1} \frac{|b_i|}{\varphi_0(\widetilde{t}_i)} \right)^p \right)^{1/p}$$

$$\lesssim t\left(\sum_{k\in\mathbb{Z}}\sum_{z_j\in M_k^0}\left(\sup_{\widetilde{t}_i\in M_j^1,\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)}<1/t}\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)}\right)^p\left(\sup_{\widetilde{t}_i\in M_j^1}\frac{|b_i|}{\varphi_0(\widetilde{t}_i)}\right)^p\right)^{1/p}$$

$$\lesssim\left(\sum_{k\in\mathbb{Z}}\sup_{z_j\in M_k^0}\left(\sup_{\widetilde{t}_i\in M_j^1}\frac{|b_i|}{\varphi_0(\widetilde{t}_i)}\right)^p\right)^{1/p}$$

$$\lesssim\left(\sum_{k\in\mathbb{Z}}\left(\sup_{\widetilde{t}_i\in M_k^0}\left(\frac{|b_i|}{\varphi_0(\widetilde{t}_i)}\right)^q\right)^{p/q}\right)^{1/p}.$$

Hence

$$t\|a_i\chi_{\Omega_t^c}(\widetilde{t}_i)\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)} \lesssim \|b_i\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)}+t\|c_i\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)}. \tag{3.16}$$

Combining the estimates (3.15) and (3.16) we get

$$\|a_i\chi_{\Omega_t}(t_i)\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)}+t\|a_i\chi_{\Omega_t^c}(t_i)\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)} \lesssim \|b_i\|_{l_p\left(l_q^{M_k^0}\left(\frac{1}{\varphi_0(\widetilde{t}_i)}\right)\right)}+t\|c_i\|_{l_p\left(l_q^{M_k^1}\left(\frac{1}{\varphi_1(\widetilde{t}_i)}\right)\right)}.$$

If we take the infimum over all the representations $a_i=b_i+c_i$ we obtain the desired lower bound estimate for the K-functional. □

Lemma 3.10. *Let $q,p\in[1,\infty]$. Then*

$$((\bar{l}_q)_{\varphi_0,p},(\bar{l}_q)_{\varphi_1,p})_{\varphi,p}=l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right). \tag{3.17}$$

Proof. It is enough to show that

$$\left((\bar{l}_1)_{\varphi_0,p},(\bar{l}_1)_{\varphi_1,p}\right)_{\varphi,p}=\left((\bar{l}_\infty)_{\varphi_0,p},(\bar{l}_\infty)_{\varphi_1,p}\right)_{\varphi,p}. \tag{3.18}$$

Indeed, from the embeddings

$$l_1\subset l_q\subset l_\infty \quad\text{and}\quad l_1\left(\frac{1}{\widetilde{t}_k}\right)\subset l_q\left(\frac{1}{\widetilde{t}_k}\right)\subset l_\infty\left(\frac{1}{\widetilde{t}_k}\right),$$

we get

$$\left((\bar{l}_1)_{\varphi_0,p},(\bar{l}_1)_{\varphi_1,p}\right)_{\varphi,p}\subset\left((\bar{l}_q)_{\varphi_0,p},(\bar{l}_q)_{\varphi_1,p}\right)_{\varphi,p} \subset\left((\bar{l}_\infty)_{\varphi_0,p},(\bar{l}_\infty)_{\varphi_1,p}\right)_{\varphi,p}, \tag{3.19}$$

consequently, using (3.18), (3.19), and Corollary 3.6, we obtain

$$\left((\bar{l}_q)_{\varphi_0,p},(\bar{l}_q)_{\varphi_1,p}\right)_{\varphi,p}=\left((\bar{l}_p)_{\varphi_0,p},(\bar{l}_p)_{\varphi_1,p}\right)_{\varphi,p} =l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right).$$

To show (3.18) we only need to prove the embedding

$$((\bar{l}_\infty)_{\varphi_0,p},(\bar{l}_\infty)_{\varphi_1,p})_{\varphi,p} \subset ((\bar{l}_1)_{\varphi_0,p},(\bar{l}_1)_{\varphi_1,p})_{\varphi,p}. \tag{3.20}$$

By Lemma 3.9

$$\begin{aligned}
&\|\{a_i\}\|^p_{((\bar{l}_1)_{\varphi_0,p},(\bar{l}_1)_{\varphi_1,p})_{\varphi,p}} \\
&\quad\approx \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i\in M_k^0,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\le t_j}\frac{|a_i|}{\varphi_0(\widetilde{t}_i)}\right)^p\frac{1}{\varphi(t_j)^p} \\
&\qquad+\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i\in M_k^1,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\ge t_j}\frac{|a_i|}{\varphi_1(\widetilde{t}_i)}\right)^p\left(\frac{t_j}{\varphi(t_j)}\right)^p = I+II,
\end{aligned}$$

and

$$\begin{aligned}
&\|\{a_i\}\|^p_{((\bar{l}_\infty)_{\varphi_0,p},(\bar{l}_\infty)_{\varphi_1,p})_{\varphi,p}} \\
&\quad\approx \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\sup_{\widetilde{t}_i\in M_k^0,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\le t_j}\frac{|a_i|}{\varphi_0(\widetilde{t}_i)}\right)^p\frac{1}{\varphi(t_j)^p} \\
&\qquad+\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\left(\sup_{\widetilde{t}_i\in M_k^1,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\ge t_j}\frac{|a_i|}{\varphi_1(\widetilde{t}_i)}\right)^p\left(\frac{t_j}{\varphi(t_j)}\right)^p = III+IV.
\end{aligned}$$

Let A_k^n be defined $A_k^n := \sup_{\widetilde{t}_i\in M_k^n}\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}$, $\quad n=0,1$. Using (3.13) and (3.14) we obtain

$$\begin{aligned}
I &\lesssim \sum_{k\in\mathbb{Z}}\left(\sum_{\widetilde{t}_i\in M_k^0,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\le t_j}\frac{|a_i|}{\varphi_0(\widetilde{t}_i)}\right)^p\sum_{A_k^0\le t_j}\frac{1}{\varphi(t_j)^p} \\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\widetilde{t}_i\in M_k^0}\left(\frac{|a_i|}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right)^p\times \\
&\qquad\qquad\times\left(\sum_{\widetilde{t}_i\in M_k^0,\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\le t_j}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)^{p'}\right)^{p/p'}\frac{1}{\varphi(A_k^0)^p} \\
&\lesssim \sum_{k\in\mathbb{Z}}\sum_{\widetilde{t}_i\in M_k^0}\left(\frac{|a_i|}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)}\right)^p\left(\sup_{\widetilde{t}_i\in M_k^0}\varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)}\right)\right)^p\frac{1}{\varphi(A_k^0)^p}
\end{aligned}$$

$$
\begin{aligned}
&\lesssim \sum_{k\in\mathbb{Z}} \sum_{\widetilde{t}_i \in M_k^0} \left(\frac{|a_i|}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)} \right)^p \\
&= \sum_{j\in\mathbb{Z}_1} \sum_{k\in\mathbb{Z}} \sum_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \left(\frac{|a_i|}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)} \right)^p \\
&\quad + \sum_{j\in\mathbb{Z}_2} \sum_{k\in\mathbb{Z}} \sum_{\widetilde{t}_i \in M_k^1, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \left(\frac{|a_i|}{\varphi(\varphi_0,\varphi_1)(\widetilde{t}_i)} \right)^p \\
&\lesssim \sum_{j\in\mathbb{Z}_1} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_0(\widetilde{t}_i)} \right)^p \times \\
&\qquad\qquad \times \sum_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right)^{-p} \\
&+ \quad \sum_{j\in\mathbb{Z}_2} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^1, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \times \\
&\qquad\qquad \times \sum_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \left(\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} \varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right) \right)^{-p} \\
&\lesssim \sum_{j\in\mathbb{Z}_1} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_0(\widetilde{t}_i)} \right)^p \sup_{t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right)^{-p} \\
&\quad + \sum_{j\in\mathbb{Z}_2} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^1, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \times \\
&\qquad\qquad \times \sup_{t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \left(\frac{\varphi_0(\widetilde{t}_i)}{\varphi_1(\widetilde{t}_i)} \varphi\left(\frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \right) \right)^{-p} \\
&\lesssim \sum_{j\in\mathbb{Z}_1} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^0, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_0(\widetilde{t}_i)} \right)^p \frac{1}{\varphi(t_j)^p} \\
&\quad + \sum_{j\in\mathbb{Z}_2} \sum_{k\in\mathbb{Z}} \left(\sup_{\widetilde{t}_i \in M_k^1, t_j < \frac{\varphi_1(\widetilde{t}_i)}{\varphi_0(\widetilde{t}_i)} \le t_{j+1}} \frac{|a_i|}{\varphi_1(\widetilde{t}_i)} \right)^p \left(\frac{t_j}{\varphi(t_j)} \right)^p = III + IV.
\end{aligned}
$$

Similarly we see that

$$II \lesssim III + IV.$$

Finally, combining estimates we get (3.20). The proof is complete. □

Theorem 3.11. *Let $p \in [1,\infty]$. Then*

$$(\overline{X}_{\varphi_0,p}, \overline{X}_{\varphi_1,p})_{\varphi,p} = \overline{X}_{\varphi(\varphi_0,\varphi_1),p}.$$

Proof. Using (3.1) and (3.18) we get

$$\begin{aligned}
&\overline{X}_{\varphi(\varphi_0,\varphi_1),p} \\
&\quad = \operatorname{Orb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \bar{l}_1 \to \overline{X}\right) \\
&\quad \subset \operatorname{Orb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \{(\bar{l}_1)_{\varphi_0,p}, (\bar{l}_1)_{\varphi_1,p}\} \to \{\overline{X}_{\varphi_0,p}, \overline{X}_{\varphi_1,p}\}\right) \\
&\qquad \subset (\overline{X}_{\varphi_0,p}, \overline{X}_{\varphi_1,p})_{\varphi,p}.
\end{aligned}$$

Similarly, using (3.1) and (3.2) we get

$$\begin{aligned}
&(\overline{X}_{\varphi_0,p}, \overline{X}_{\varphi_1,p})_{\varphi,p} \\
&\quad \subset \operatorname{Corb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \{\overline{X}_{\varphi_0,p}, \overline{X}_{\varphi_1,p}\} \to \{(\bar{l}_\infty)_{\varphi_0,p}, (\bar{l}_\infty)_{\varphi_1,p}\}\right) \\
&\quad \subset \operatorname{Corb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \overline{X} \to \bar{l}_\infty\right) \\
&\quad = \overline{X}_{\varphi(\varphi_0,\varphi_1),p}.
\end{aligned}$$

The proof is complete. □

4. Proof of Theorem 2.5

(v) ⇒ (iv). If (v) holds, by Corollary 3.4 and 3.5 we have

$$\begin{aligned}
\big((\bar{l}_1)_{\varphi_0,1}, (\bar{l}_p)_{\varphi_1,p}\big)_{\varphi,p} &= \big((\bar{l}_\infty)_{\varphi_0,1}, (\bar{l}_\infty)_{\varphi_1,p}\big)_{\varphi,p} \\
&= l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right).
\end{aligned} \tag{4.1}$$

Using (3.1) and (4.1)we get

$$\begin{aligned}
&\overline{X}_{\varphi(\varphi_0,\varphi_1),p} \\
&\quad = \operatorname{Orb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \bar{l}_1 \to \overline{X}\right) \\
&\quad \subset \operatorname{Orb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right) :\ \{(\bar{l}_1)_{\varphi_0,1}, (\bar{l}_1)_{\varphi_1,1}\} \to \{\overline{X}_{\varphi_0,1}, \overline{X}_{\varphi_1,1}\}\right) \\
&\quad \subset (\overline{X}_{\varphi_0,1}, \overline{X}_{\varphi_1,1})_{\varphi,p}.
\end{aligned}$$

Similarly, using (3.1) and (3.2) we get

$$(\overline{X}_{\varphi_0,\infty}, \overline{X}_{\varphi_1,\infty})_{\varphi,p}$$

$$\subset \operatorname{Corb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right):\ \{\overline{X}_{\varphi_0,\infty},\overline{X}_{\varphi_1,\infty}\}\to\{(\bar{l}_\infty)_{\varphi_0,\infty},(\bar{l}_\infty)_{\varphi_1,\infty}\}\right)$$
$$\subset \operatorname{Corb}\left(l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right):\ \overline{X}\to\bar{l}_\infty\right)=\overline{X}_{\varphi(\varphi_0,\varphi_1),p}.$$

Since

$$(\overline{X}_{\varphi_0,1},\overline{X}_{\varphi_1,1})_{\varphi,p}\subset(\overline{X}_{\varphi_0,\infty},\overline{X}_{\varphi_1,\infty})_{\varphi,p},$$

we obtain

$$\begin{aligned}(\overline{X}_{\varphi_0,p_0},\overline{X}_{\varphi_1,p_1})_{\varphi,p}&=(\overline{X}_{\varphi_0,p_0},\overline{X}_{\varphi_1,p_1})_{\varphi,p}\\&=(\overline{X}_{\varphi_0,\infty},\overline{X}_{\varphi_1,\infty})_{\varphi,p}\\&=\overline{X}_{\varphi(\varphi_0,\varphi_1),p}.\end{aligned}$$

We now complete the proof of the implication (v) $\Rightarrow$ (iv).

The implications (iv) $\Rightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) are clear. We will show the implication (iii) $\Rightarrow$ (v).

Suppose that (1.3) holds for the couple $\overline{X}=(L_1(0,\infty),L_\infty(0,\infty))$ and for some $p\in(0,\infty)$. As the couple $(L_1(0,\infty),L_\infty(0,\infty))$ is complete couple then (1.3) holds for any couple (see [7]) and therefore for the couple $\bar{l}_q=(l_q,l_q(\frac{1}{\widetilde{t_k}}))$. By Corollary 3.4, (3.3) and (3.18) we get

$$\begin{aligned}l_p\left(l_q^{M_k}\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right)\right)&=\big((\bar{l}_q)_{\varphi_0,q},(\bar{l}_q)_{\varphi_1,q}\big)_{\varphi,p}\\&=\big((\bar{l}_q)_{\varphi_0,p},(\bar{l}_q)_{\varphi_1,p}\big)_{\varphi,p}\\&=(\bar{l}_q)_{\varphi(\varphi_0,\varphi_1),p}\\&=l_p\left(\frac{1}{\varphi(\varphi_0,\varphi_1)(\widetilde{t_i})}\right).\end{aligned}$$

It is easy to see that from here that (v) follows. The proof is complete.

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