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ATOMIC DECOMPOSITION IN SMALL LEBESGUE SPACE

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Abstract. We study grand $L^{p)}(\Omega)$ spaces, introduced by [12], and small $L^{(q}(\Omega)$ spaces, introduced by [8], and improve a result by [8] that if $\frac{1}{p}+\frac{1}{q}=1$, $L^{p)}$ is the dual of $L^{(q}$, giving a constructive characterization of $L^{(q}$ via *atomic decomposition* in the particular case $p=q=2$, $\Omega=]0,1[$. We notice that $L^{(q}(]0,1[)$ is isomorphic to the dual space of $L_b^{p)}$, the closure of $L^\infty(]0,1[)$ in $L^p(]0,1[)$. We also illustrate distance formula in $L^{p)}$ to L^∞ due to [3].

We incorporate the pair $L_b^{p)}, L^{p)}$ in the theory of o-O type pair of spaces (E_0,E) according to [17], [18]. Description of $(L^{p)})^\star$ is also provided in terms of $(L_b^{p)})^\star$ and $(L_b^{p)})^\perp$ (see [2]).

1. Introduction

Atomic decomposition of function spaces is a classical issue in Harmonic and Functional Analysis. A measurable function $a(x)$ supported in a cube Q in $\mathbb{R}^n$ is called an L^q– atom with defining cube Q if it satisfies the conditions

$$\left(\fint_Q |a(x)|^q\right)^{\frac{1}{q}} \leq \frac{1}{|Q|}$$

$$\int_Q a(x)\,dx = 0.$$

The importance of atoms lies in the fact that the classical Hardy space $\mathcal{H}^1$ consisting of L^1 functions f whose Riesz transform $R_j f \in L^1$, $j=1,\cdots n$, enjoys the property that any such f can be written as a (possible infinite) linear combination of $\mathcal{H}^1$–atoms: $f=\sum_{j=1}^\infty \lambda_j a_j$ with $\sum_{j=1}^\infty |\lambda_j|<\infty$. The norm being defined by

$$\|f\|_{\mathcal{H}^1} = \inf\left\{\sum_{j=1}^\infty |\lambda_j| : f=\sum_{j=1}^\infty \lambda_j a_j\right\}$$

where the infimum is taken over all atomic decompositions of f.

In 1974, R.R. Coifman proved this property of $\mathcal{H}^1$ and this fact was used to give another proof of the celebrated theorem by C. Fefferman that $(\mathcal{H}^1)^\star = BMO$.

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Actually, the fact that the elements of the predual of $BMO(\mathbb{R})$ have a representation

$$h = \sum_{j=1}^{\infty} \lambda_i g_i \qquad\qquad \sum_{j=1}^{\infty} |\lambda_i| < \infty$$

with $g_i \in L^2$, $\operatorname{supp} g_i \subset I$, $\int_I g_i = 0$, $\|g\|_{L^2(I)} \leq |I|^{-\frac{1}{2}}$ was noted in [13]. Very recently, in the paper [4], the function space JN_p, $p > 1$, based on a condition, introduced by John–Nirenberg (as a variant of BMO), on the supremum over collections of cubes was studied.

It was proved that JN_p is the dual space of a new "Hardy kind "space $HK_{p'}$ $\left(\frac{1}{p} + \frac{1}{p'} = 1\right)$ which enjoys atomic decomposition. However, reflecting the difference of a supremum over individual cubes in the definition of $BMO(Q_0)$, $Q_0 =]0,1[^n$

$$\|f\|_{BMO} = \sup_{Q \subset Q_0} ⨍_Q |f - f_Q|$$

$f_Q = ⨍_Q f$, and over collection of cubes in JN_p, $p > 1$

$$\|f\|_{JN_p} = \sup \left\{ \sum_i |Q_i| \left(⨍_{Q_i} |f - f_{Q_i}| \right)^p \right\}^{\frac{1}{p}}$$

where the supremum is taken over all collections of pairwise disjoint cubes $Q_i \subset Q_0$, the *atoms* of $\mathcal{H}^1$ are replaced here by more complicated structures called *polymer* in the definition of $HK_{p'}$.

Similar preduality results in terms of atoms pertain to the Garsia-Rodemich spaces, recently considered in [15], [1], [16] and introduced in [10] the space $GaRo_p$ whose norm is defined by

$$\|f\|_{GaRo_p} = \sup \frac{\sum_i \frac{1}{|Q_i|} \int_{Q_i} \int_{Q_i} |f(x) - f(y)|\, dx\, dy}{\left(\sum_i |Q_i|\right)^{\frac{1}{p'}}}$$

where the supremum is taken similarly as in JN_p case.

The main result by M. Milman is that

$$GaRo_p = \begin{cases} L^{p,\infty} & 1 < p < \infty \\ BMO & p = \infty \end{cases}$$

and since $L^{p,\infty}$ the Marcinkiewicz space is dual of $L^{q,1}$ $\left(\frac{1}{p} + \frac{1}{q} = 1\right)$ which enjoys atomic decomposition (see [5]) also $GaRo_p$ are dual spaces with atomic decomposition (see also [14]).

Let us notice that atomic decomposition can be an effective tool to prove the boundedness of operators such as the Hardy–Littlewood maximal operator, the Hilbert transform, the multiplication, the composition operator acting on these spaces. Actually, the boundedness of these operators is reduced to the boundedness on characteristic functions.

Our aim in this paper is to consider the grand Lebesgue space $L^{2)}$ which is dual of small Lebesgue space $L^{(2}$ and show that $L^{(2}$ enjoys atomic decomposition.

In all these examples the atomic decomposition of a non reflexive, separable Banach space H of functions is related to other fundamental properties of the dual $H^\star$ and of the predual $H_\star$ of H:

a) *characterization of $H^\star$ norm by big–O condition*
b) *characterization of $H_\star$ by little–o condition*
c) *distance formula to L^∞ in $H^\star$.*

In [3], a formula for the distance to $L^\infty(Q_0)$, $Q_0 =]0,1[^n$ in the grand Lebesgue space $L^{p)}(Q_0)$, $1 < p < \infty$, introduced in [12] by Iwaniec-Sbordone was given. The space $E = L^{p)}(Q_0)$ is naturally equipped with the "big-O"type norm

$$||u||_{L^{p)}(Q_0)} = \sup_{0<\varepsilon<p-1} [u]_\varepsilon \tag{1.1}$$

where

$$[u]_\varepsilon = \left(\varepsilon \fint_{Q_0} |u|^{p-\varepsilon} dx\right)^{\frac{1}{p-\varepsilon}}. \tag{1.2}$$

It turns out that L^∞ is not dense in $L^{p)}$; moreover in [3] the distance formula

$$dist_{L^{p)}}(u, L^\infty) = \limsup_{\varepsilon\to 0} [u]_\varepsilon \tag{1.3}$$

was established to characterize the closure $E_0 = L^{p)}_b$ of L^∞ in $L^{p)}$ with the "little-o"type condition

$$\limsup_{\varepsilon\to 0} [u]_\varepsilon = 0. \tag{1.4}$$

Let us recall some relations with Zygmund spaces (see [11]):

$$L^p \subset \frac{L^p}{\log L} \subset L^{p)} \subset \bigcap_{r>1} \frac{L^p}{(\log L)^r}$$

and

$$L^{p,\infty} \subset L^{p)}$$

In [8], the small Lebesgue space $L^{(p'}(Q_0)$ of $g \in L^{p'}(Q_0)$ such that

$$\|g\|_{L^{(p'}} = \inf_{g=\sum_{k=1}^\infty g_k} \left\{ \sum_{k=1}^\infty \inf_{0<\varepsilon<p-1} \varepsilon^{-\frac{1}{p-\varepsilon}} \left(\fint_{Q_0} |g_k|^{(p-\varepsilon)'} dx\right)^{\frac{1}{(p-\varepsilon)'}} \right\} < \infty \tag{1.5}$$

where $r' = \frac{r}{r-1}$, was introduced. Clearly, $L^{p'+\eta} \subset L^{(p'}$ for $\eta > 0$.

Let us note that properties of the small and grand Lebesgue space, using the interpolation-extrapolation theory of quasi-Banach spaces are deeply considered in [9] Section 2.

In [2], it is proved that $L^{p)}_b$ is M- ideal in $\left(L^{p)}_b\right)^{\star\star}$, that is

$$\left(L^{p)}_b\right)^{\star\star\star} \simeq \left(L^{p)}_b\right)^\star \oplus \left(L^{p)}_b\right)^\perp. \tag{1.6}$$

where $\oplus$ stands for the direct sum and $E^\perp$ is the orthogonal space of E.

In [8], [2], it was proved that, for $\frac{1}{p} + \frac{1}{q} = 1$,

$$\left(L^{(q}\right)^\star = L^{p)} \tag{1.7}$$

and

$$\left(L_b^{p)}\right)^\star \simeq L^{(q}. \tag{1.8}$$

Hence:

$$\left(L_b^{p)}\right)^{\star\star} \simeq L^{p)}. \tag{1.9}$$

Our aim here is to prove atomic decomposition of $L^{(2}(]0,1[)$. This can be obtained showing that the pair (E_0, E) with:

$$E = L^{2)}, \quad E_0 = L_b^{2)} \tag{1.10}$$

obeys the general framework of [6]. Indeed, as in [6], one can choose $X = L^{2-\delta}$, $0 < \delta < 1$ and $Y = L^2$, and introduce a family $L_\varepsilon : X \to Y$ of linear operators such that

$$L^{p)} = \{u \in X : \sup_{0<\varepsilon<1} ||L_\varepsilon u||_Y < \infty\}.$$

Then Theorem 3 of [6] (see Theorem 3.1 in Section 3) guarantees that if $Q_0 =]0,1[$ the elements φ of the predual $L^{(2}$ of $L^{2)}$ decompose as

$$\varphi = \sum_{j=1}^{\infty} \lambda_j g_j$$

where $\sum_{j=1}^{\infty} |\lambda_j| < \infty$, for suitable g_j.

2. Preliminaries

For $u \in L^1(Q_0)$ the distribution function of u is defined by

$$\mu_u(\lambda) = |\{x \in Q_0 : |u(x)| > \lambda\}|$$

$(\lambda > 0)$. The non-increasing rearrangement $u^\star$ of u is defined on $[0,1]$ by

$$u^\star(t) = \inf\{\lambda > 0 : \mu_u(\lambda) \le t\}.$$

We say that u and v are equimeasurable if $\mu_u(\lambda) = \mu_v(\lambda)$ for all $\lambda \ge 0$. The space $L^{p)}$ is a rearrangement invariant space, indeed $||u||_{L^{p)}} = ||v||_{L^{p)}}$ if u and v are equimeasurable (see (2.1), (2.3) below). We also define

$$u^{\star\star}(t) = \fint_0^t u^\star(s)ds,$$

and the fundamental function φ of $E = L^{p)}$ by

$$\varphi(t) = \varphi_E(t) = ||\chi_{A_t}||_E$$

for any A_t such that $|A_t| = t$. It turns out that

$$\varphi_E(t) \simeq t^{\frac{1}{p}} \left[\log\left(\frac{1}{t}\right)\right]^{-\frac{1}{p}}.$$

In [9] it was proved that the norm (1.1) is equivalent to another norm in which the nonincreasing rearrangement of u appears:

$$||u||_{L^{p)}} = \sup_{0<t<1} (1 - \log t)^{-\frac{1}{p}} \left(\int_t^1 [u^\star(s)]^p ds\right)^{\frac{1}{p}} \tag{2.1}$$

The small Lebesgue space $L^{(q}(Q_0)$, defined by (1.5), is the predual of $L^{p)}(Q_0)$ (see [7]) and can be equipped with the equivalent norm

$$||u||_{L^{(q}} = \int_0^1 (1-\log t)^{-\frac{1}{q}} \frac{\left(\int_0^t [u^\star(s)]^q ds\right)^{\frac{1}{q}}}{t} dt \tag{2.2}$$

The following results hold (see [2]):

Proposition 2.1. *The grand Lebesgue space $L^{p)}(Q_0)$, $p > 1$ is the dual space of $L^{(q}(Q_0)$*

$$(L^{(q}(Q_0))^\star = L^{p)}(Q_0).$$

Both spaces are not reflexive and not separable.

Proposition 2.2. *Every function $u \in L^{(q}(Q_0)$ has absolutely continuous norm, that is if $A_j \subset Q_0$ with $|A_j| \to 0$, then $||u\chi_{A_j}||_{L^{(q}(Q_0)} \to 0$.*

Proposition 2.3. *The space $L^\infty(Q_0)$ is dense in $L^{(q}(Q_0)$.*

As already observed Proposition 2.3 does not hold in the case of the grand Lebesgue space.

In [3] the following distance formula was obtained.

Theorem 2.4. *For $u \in L^{p)}(Q_0)$, we have*

$$dist_{L^{p)}}(u, L^\infty) = \limsup_{\varepsilon \to 0} \left(\varepsilon \fint_{Q_0} |u|^{p-\varepsilon}\right)^{\frac{1}{p-\varepsilon}}. \tag{2.3}$$

Moreover, the distance is attained: there exists $v \in L_b^{p)}$ such that the left hand side of (1.5) coincides with $\|u-v\|_{L^{p)}}$.

We conclude this Section by recalling a classical notion which will be useful in the sequel.

Definition 2.1. Let E be a Banach space and M a vectorial subspace of E. The orthogonal space $M^\perp$ of M is

$$M^\perp = \{f \in E^\star : \langle f, x\rangle = 0 \ \forall x \in M\}$$

where $\langle \cdot, \cdot \rangle$ is the duality inner product.

3. Atomic decomposition of $L^{(2}(]0,1[)$

Following [6], suppose that X is a reflexive Banach space and Y any Banach space. Let $(L_j)_{j=1}^\infty$ be a given sequence of operators $L_j : X \to Y$. Define

$$E = \{x \in X : \sup_j ||L_j x||_Y < \infty\}.$$

We suppose that E is a Banach space and that E is continuosly contained in X, and moreover, that E is dense in X in the X-norm. Note that we then have an isometric embedding

$$V : E \to l^\infty(Y), \ \ Vx(n) = L_n x.$$

Theorem 3.1. *E has a predual* $E_\star$,

$$E_\star = l^1(Y^\star)/E_\perp$$

where $E_\perp = V(E)^\perp \cap l^1(Y^\star)$. *Every* $x \in E$ *corresponds to a functional on* $l^1(Y^\star)/E_\perp$ *given by*

$$x(y_n) = \sum_{n=1}^{\infty} y_n(L_n x), \tag{3.1}$$

and conversely every bounded functional on $l^1(Y^\star)/E_\perp$ *is given by a unique* $x \in E$ *according to* (3.1).

Let us consider the grand Lebesgue space $L^{2)}(]0,1[)$ equipped with the norm (2.1). Let $0 < \delta < 1$, and consider $X = L^{2-\delta}(]0,1[)$ and $Y = L^2(]0,1[)$. In order to achieve atomic decomposition of small Lebesgue space $L^{(2}(]0,1[)$, which is the predual of $L^{2)}(]0,1[)$, we define a suitable family of linear operators

$$L_j : X \to Y.$$

Namely, for $u \in X = L^{2-\delta}(Q_0)$, for $j \in \mathbb{N}$, we set

$$L_j u(s) = \chi_{I_j}(s) \frac{1}{(1+\log j)^{\frac{1}{2}}} (u^\star(s) - (u^\star)_{I_j}), \text{ where } I_j = \left]\frac{1}{j}, 1\right[. \tag{3.2}$$

Using the Definition 2.1, and observing that

$$\int_{I_j} |u^\star(s) - \fint_{I_j} u^\star|^2 ds \le 4 \int_{I_j} (u^\star)^2 ds$$

and

$$\int_{I_j} |u^\star - \fint_{I_j} u^\star|^2 ds < \infty \Longrightarrow \int_{I_j} (u^\star)^2 ds < \infty,$$

we can identify

$$L^{2)}(]0,1[) = \left\{ u \in L^{2-\delta}(]0,1[) : \sup_j ||L_j u||_{L^2} < \infty \right\} \tag{3.3}$$

and moreover,$L^{2)}(]0,1[)$ is dense in $L^{2-\delta}(]0,1[)$. So we can apply Theorem 3.1 to obtain the following result

Theorem 3.2. *For every* $\varphi \in E_\star = L^{(2}(]0,1[)$, *there exist* $g_j \in L^2(]0,1[)$, $\lambda_j \in \mathbb{R}$, $j \in \mathbb{N}$, *such that*

$$\varphi = \sum_j \lambda_j g_j \tag{3.4}$$

with

$$||\varphi||_{L^{2)}} \backsimeq \sum_j |\lambda_j|. \tag{3.5}$$

Moreover,

$$supp\, g_j \subset I_j \tag{3.6}$$

$$\int_{I_j} g_j = 0 \tag{3.7}$$

$$||g_j||_{L^2} \leq 2\frac{1}{|I_j|^{\frac{1}{2}}} \tag{3.8}$$

Proof. For every $\varphi \in E_\star = L^{(2} = l^1(L^2)/(VE)^\perp \cap l^1(L^2)$ there exists $(y_j)_j \subset l^1(L^2)$ such that

$$||\varphi||_{E_\star} \backsimeq \sum_j ||y_j||_{L^2(]0,1[)}.$$

Since

$$L_j : L^{2-\delta}(]0,1[) \to L^2(]0,1[)$$

and

$$L_j^\star : L^2(]0,1[) \to L^{\frac{2-\delta}{1-\delta}}(]0,1[)$$

we observe that for all $f \in L^2(]0,1[)$

$$L_j^\star f = L_j f. \tag{3.9}$$

We will prove that $\varphi = \sum \lambda_j g_j$, that is

$$\langle x, \varphi\rangle = \sum_j \langle g_j, x\rangle.$$

where

$$g_j = \frac{L_j y_j}{||y_j||_{L^2}},\ \lambda_j = ||y_j||_{L^2}. \tag{3.10}$$

In fact, by Theorem 3.1, (3.9) and (3.10) we have

$$\begin{aligned}\langle x, \varphi\rangle = \ & \langle x, (y_j)_j\rangle \\ & = \textstyle\sum_j \langle y_j, L_j x\rangle \\ & = \textstyle\sum_j \langle L_j^\star y_j, x\rangle \\ & = \textstyle\sum_j ||y_j||_{L^2}\langle g_j, x\rangle \\ & = \textstyle\sum_j \lambda_j \langle g_j, x\rangle\end{aligned}$$

Since $g_j = \frac{L_j y_j}{||y_j||_{L^2}}$, we see that conditions (3.6) and (3.7) follow immediately from the definition of L_j.

To prove (3.8), we just need to observe that

$$\begin{aligned}||g_j||^2_{L^2} \ & = \tfrac{||L_j y_j||^2_{L^2}}{||y_j||^2_{L^2}} = \tfrac{1}{1+\log j}\int_{I_j} |y_j^\star - (y_j^\star)_{I_j}|^2 \tfrac{1}{||y_j||^2_{L^2}} \leq \\ & \leq 4\tfrac{1}{1+\log j}\int_{I_j}(y_j^\star)^2\tfrac{1}{||y_j||^2_{L^2}} \\ & \leq 4\tfrac{1}{|I_j|}.\end{aligned}$$

to conclude the proof. □

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