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EXACT CALCULATION OF THE SUM AND INTERSECTION OF APPROXIMATION SPACES

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Abstract. Exact formulas for calculating the norms of the sum and the intersection of approximation spaces through the norms of the sum and intersection of the spaces of sequences of number provided by the parameter in the definition of approximation spaces are given. The existence of such a reduction allows one to obtain new extrapolation theorems for approximation spaces with exact constants.

1. Introduction

The problem of calculating the sum and intersection of collections of spaces, which play an important role in the theory of extrapolation of linear operators [12, 13, 15], has become popular recently. So, grand-Lebesgue spaces $L^{p)}$ and small-Lebesgue spaces $L^{(p}$ have found a wide application in the theory of partial differential equations [11, 17] in the study of maximal and other typical operators in harmonic analysis [8]. Similar constructions appeared for other collections of spaces in [9]. Unfortunately, there is no explicit description of even spaces $L^{p)}$ and $L^{(p}$ in standard terms. In fact, only for a collection of Lorentz spaces Λ^{α} in the works [4, 5] was proposed a method for the exact calculation of the space $\sum_{0\leq\underline{\beta}<\alpha<\overline{\beta}\leq 1}\xi(\alpha)\Lambda^{\alpha}$ and some cones in these spaces (with the equality of norms). On the basis of these facts, it was possible to obtain extrapolation theorems for cones in Lorentz, Lebesgue and Marcinkiewicz spaces with exact constants.

In this paper, we have found another class of spaces – the so-called approximation spaces – for which we managed to calculate the exact sum and the intersection of spaces from this class (with the equality of norms). More precisely, it is shown that the calculation of the sum of approximation spaces can be reduced to the calculation of the sum of sequence spaces, included by the parameter in the definition of approximation spaces. The presence of such a reduction allows one to obtain new extrapolation theorems for approximation spaces with exact constants.

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2. Preliminary Information

Let $\{B_i\}$ be a sequence of Banach spaces such that each of them is continuously embedded into a complete topological separable space V. Such a sequence is called a V-sequence. If the sequence of positive numbers $\{\gamma_i\}$ is additionally given, then, as usual, by $\Sigma\gamma_i B_i$ we denote a new Banach space, the norm of which is given by

$$\|b|\Sigma\gamma_i B_i\| = \inf\{\Sigma\gamma_i\|b_i|B_i\| : b = \Sigma b_i \text{ the series converges in } V\},$$

Now suppose that, instead of sequence, a collection of Banach spaces $\{B_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta})$, $-\infty \leq \underline{\beta} < \overline{\beta} \leq \infty)$, continuously embedded in a complete topological separable space V is given. Analogously, we refer to such a collection as a V-collection.

For a V-collection of Banach spaces $\{B_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta}))$ and a measurable the function $\{\xi : (\underline{\beta}, \overline{\beta}) \to R_+\}$ we denote a new space $\sum\{\xi, B_\beta\}$ by norm

$$\|b|\sum\{\xi, B_\beta\}\| = \inf\{\|b|\sum\xi(\beta_i)B_{\beta_i}\| :$$
$$\underline{\beta} < ... < \beta_{-i} < \beta_{-i+1} ... < \beta_0 < \beta_1 < ... < \beta_i < ... < \overline{\beta}\}.$$

The main properties of the space $\sum\{\xi, B_\beta\}$ are given in the following lemma.

Lemma 2.1 ([4]). *The space $\sum\{\xi, B_\beta\}$ is a Banach space. If a linear subset $U \subseteq \bigcap B_\beta$ is dense in each B_β, then U is dense in $\sum\{\xi, B_\beta\}$.*

Analogously, by $\bigcap\{\xi, B_\beta\}$ we denote a new Banach space, the norm of which is given by $\|b|\bigcap\{\xi, B_\beta\| = \sup_{\beta\in(\underline{\beta},\overline{\beta})}\xi(\beta)\|b|B_\beta\|$.

Let $S(Z)$ be the space of number sequences $a = \{a_i\}_{-\infty}^{\infty}$ with coordinate-wise convergence, $\{e^i\}_{-\infty}^{\infty}$ is a standard basis in $S(Z)$. A Banach space $l \subset S(Z)$ is called ideal [14], if $x = \sum_{-\infty}^{\infty} e^i x_i \in l$, $y = \sum_{-\infty}^{\infty} e^i y_i \in S(Z)$ and for all $i \in Z$ the inequality $|y_i| \leq |x_i|$ implies that $y \in l$ and $\|y|l\| \leq \|x|l\|$ (by $\|x|l\|$ we denote the norm of element x in the space l). Operator τ is the right-shift operator acting on a sequence $a = \sum_{-\infty}^{\infty} e^i a_i$ as

$$\tau(\sum_{-\infty}^{\infty} e^i a_i) = \sum_{-\infty}^{\infty} e^i a_{i-1}.$$

If the sequence a is such that for any $i \in Z$ the inequality $\sum_{j=i}^{\infty}|a_j| < \infty$ holds, then we define the operator Λ by the formula

$$\Lambda a = \sum_{-\infty}^{\infty} e^i(\sum_{j=i+1}^{\infty} a_j). \tag{2.1}$$

Let l be a Banach ideal sequence space in $S(Z)$. In what follows, we will consider only those l that satisfy the following condition:

$$\text{if} \quad \sum_{-\infty}^{\infty} a_i e^i \in l, \quad \text{then} \quad \lim_{i\to\infty} a_i = 0. \tag{2.2}$$

Lemma 2.2. *If for each $\beta \in (\underline{\beta}, \overline{\beta})$ space l_β is ideal, then the space $\sum\{\xi, l_\beta\}$ is also an ideal. If for each $\beta \in (\underline{\beta}, \overline{\beta})$ space l_β satisfies condition (2.2), then the space*

$\sum\{\xi, l_\beta\}$ *also satisfies condition (2.2), if for each* $\beta \in (\underline{\beta}, \overline{\beta})$ *in* l_β *the operator* τ *is bounded and satisfies the condition*

$$\sup\{\|\tau|l_\beta \to l_\beta\| : \beta \in (\underline{\beta}, \overline{\beta})\} = c_0 < \infty, \tag{2.3}$$

then in the space $\sum\{\xi, l_\beta\}$ *is also bounded the operator* τ *and its norm does not exceed* c_0. *If for each* $\beta \in (\underline{\beta}, \overline{\beta})$ *in the space* l_β *the operator* Λ *is bounded and*

$$\sup\{\|\Lambda|l_\beta \to l_\beta\| : \beta \in (\underline{\beta}, \overline{\beta})\} = c_1 < \infty, \tag{2.4}$$

then in the space $\sum\{\xi, l_\beta\}$ *the operator* Λ *is also bounded and the inequality* $\|\Lambda| \sum\{\xi, l_\beta\} \to \sum\{\xi, l_\beta\}\| \le c_1$ *holds.*

Proof. Let $V = S(Z)$. Then, the collection of ideal spaces l_β is a V-collection and the space $\sum\{\xi, l_\beta\}$ is well defined. We check the implication (2.2). The remaining statements of this lemma can be readily verified.

Let $a \in \sum\{\xi, l_\beta\}$ with $\|a| \sum\{\xi, l_\beta\}\| = 1$. This means that there are $\{\beta_i\}$ such that $\underline{\beta} < ... < \beta_{-i} < \beta_{-i+1} ... < \beta_0 < \beta_1 < ... < \beta_i < ... < \overline{\beta}$ and $b_j = \sum b_{j,i} e^i$ such that

$$a = \sum b_j; \quad \sum \xi(\beta_i)\|b_i|l_{\beta_i}\| < 2.$$

Fix $\varepsilon > 0$. We first choose n so that the inequality holds

$$\sum_{i=n+1}^{\infty} \xi(\beta_i)\|b_i|B_{\beta_i}\| + \sum_{i=-n-1}^{-\infty} \xi(\beta_i)\|b_i|B_{\beta_i}\| < \frac{\varepsilon}{2},$$

and, then for each $i = -n, -n+1, ..., n-1, n$ for b_i choose $n_i \in N$ so that

$$\sup_{i \ge n_j} |b_{j,i}| < \frac{\varepsilon}{2n},$$

Define the number $n_\varepsilon = \max\{n_j : \quad j = -n, -n+1, ..., n-1, n\}$. Then, for any $i \ge n_\varepsilon$ inequality $|a_i| \le \varepsilon$ holds, whence it follows that for $\sum\{\xi, l_\beta\}$ the implication (2.2) is satisfied. □

The main objects considered in this article are the approximation spaces. Roughly speaking, an approximation space is a functional class whose elements are completely determined by the behavior of its best approximations. As a typical example, we indicate the space $Lip\beta$, $(0 < \beta < 1)$, of periodic functions of smoothness β, which, according to the classical approximation theory (see [1, 7]), is isomorphic to the approximation space generated by the sequence $e_n(x)$ of the best approximations of the function x by trigonometric polynomials of degree at most 2^n in the uniform norm. Links to other examples and the calculation of interpolation functors on pairs of approximation spaces can be found in [3].

Recall the definition of an approximation space. Let an ideal sequence space l be given, a separable topological space B in which the normalizing function $\alpha : B \to [0, \infty]$ is given (i.e. α satisfies all the conditions for the norm, but can take values of ∞) for which B is complete space. Suppose that in B a sequence of subspaces $A = \{A_i\}_{-\infty}^{\infty}$ is given and

$$\bigcap_{i \in Z} A_i = 0; \quad \forall i \in Z \Rightarrow A_i \subset A_{i+1}. \tag{2.5}$$

For each $u \in B$, we define the best approximations of u, using the subspace A_i, by

$$\alpha_i(u) = \inf\{\alpha(u - v_i) : v_i \in A_i\}.$$

The approximation space $E(l; \alpha, A, B)$ consists of such $u \in B$ for which the norm is finite

$$\|u|E(l; \alpha, A, B)\| = \|\sum_{-\infty}^{\infty} \alpha_i(u)e^i|l\|.$$

The relation (2.5) implies that for each $u \in B$ the inequalities $\alpha_i(u) \geq \alpha_{i+1}(u)$ hold. If $\bigcup_{i\in Z} A_i$ is dense in B, then for each $u \in B$, the equality $\lim_{i\to\infty} \alpha_i(u) = 0$ is true.

Along with the class of approximation spaces $E(l; \alpha, A, B)$ we introduce another class of approximation spaces $E_0(l; \alpha, A, B)$. The space $E_0(l; \alpha, A, B)$ consists of those $u \in B$, for which the norm is finite

$$\|u|E(l; \alpha, A, B)\| = \inf\{\|\sum_{-\infty}^{\infty} \alpha(v_i)e^i|l\| : u = \sum_{-\infty}^{\infty} v_i, \text{series converges in} \quad B\}.$$

The spaces $E(l; \alpha, A, B)$ are generated by the direct approximation theorems, and the spaces $E_0(l; \alpha, A, B)$ are closely connected with the inverse approximation theorems. Note that, in applications, for all $i \leq i_0$, the relation $A_i = 0$ is often fulfilled. It will be exactly the same when describing the Lipschitz space $Lip\ \beta$, $(0 < \beta < 1)$, as approximation. Therefore, in what follows, we assume that $A_i = 0$ for all $i \leq 0$.

It is known [3] that the space $E(l; \alpha, A, B)$ is continuously embedded in the space $E_0(l; \alpha, A, B)$ with a embedding constant $2\|\tau|l \to l\|$. If the operator Λ is bounded in the space l, then the space $E_0(l; \alpha, A, B)$ is continuously embedded in the space $E(l; \alpha, A, B)$ with a embedding constant $\|\Lambda|l \to l\|$.

We illustrate our definitions with an example.

Example 1. *Let $B = C[0, 2\pi]$ be the space of continuous periodic functions with uniform norm. For each $x \in C[0, 2\pi]$, the modulus of continuity of the function x is defined by*

$$\upsilon(x, \delta) = \sup_{\tau,s:|\tau-s|\leq\delta} |x(\tau) - x(s)|.$$

Let the modulus of continuity ν be given. Without loss of generality, we can assume that $\nu(1) = 1$. Let $Lip(\nu)$ denote the space of functions for which the norm is finite

$$\|x|Lip(\nu)\| = \max\{\|x|C[0, 2\pi]\|, \sup_{\delta} \frac{\upsilon(x, \delta)}{\nu(\delta)}\}.$$

Let $A_i = 0$ $(i = 0, -1, -2, ...)$ and $A_i = T_i$ is the space of trigonometric polynomials of degree at most 2^i $(i \in N)$. In this case, the operator Λ defined by the equation (2.1) has the form

$$\Lambda(\sum_{0}^{\infty} a_i e^i) = \sum_{0}^{\infty} e^i(\sum_{j=i+1}^{\infty} a_j). \tag{2.6}$$

Now, let the normalizing function $\alpha : B \to R_+$ be given by $\alpha(x) = \max\{|x(s)| : s \in [0, 2\pi]\}$. Moreover, for each $x \in C[0, 2\pi]$ the formula $e_i(x) = \inf_{t_i\in T_i} \alpha(x(.) - t_i(.))$ defines the best approximations by polynomials of degree at most 2^i. As an *ideal sequence space l we will consider spaces l^p_ω, ($1 \le p \le \infty$), with the following norm*

$$\|\Sigma_{i=0}^{\infty} a_i e^i | l^p_\omega\| = \begin{cases} \Sigma_{i=0}^{\infty}(|a_i|\omega(i))^p\}^{1/p}, & \text{if } 1 \le p < \infty \\ \sup_i |a_i|\omega(i), & \text{if } p = \infty. \end{cases}$$

For the modulus of continuity ν we define the weight function $\omega_\nu(.)$ by the formula $\omega_\nu(i) = 1/\nu(2^{-i})$, ($i = 0, 1, 2, ...$).

From the direct theorems of the theory of approximation of periodic functions by trigonometric polynomials (see [1, 7]), for all n follows the inequality $e_n(x) \le c_2 \upsilon(x, 2^{-i})$, ($c_2$ does not depend on i and $x \in C[0, 2\pi]$). Therefore, the embedding constant

$$Lip(\nu) \subseteq E(l^\infty_{\omega_\nu}; \alpha, A, B)$$

does not exceed c_2.

The inverse theorems of the theory of approximation of periodic functions by trigonometric polynomials (see [1, 7]) for all i imply the inequality

$$\upsilon(x, \delta) \le c_3 \inf_i \{\delta \sum_0^i 2^j e_j(x) + \sum_i^\infty e_j(x)\}, \tag{2.7}$$

where c_3 does not depend on i and $x \in C[0, 2\pi]$.

Suppose additionally that the modulus of continuity ν or that, in any case, the weight ω_ν for all $i = 0, 1, ...$ satisfies the conditions

$$\Sigma_{j=0}^{i} 2^j \nu(2^{-j}) \le c_4 2^i \nu(2^{-i}), \quad \Sigma_{j=i}^{\infty} \nu(2^{-j}) \le c_5 \nu(2^{-i}), \tag{2.8}$$

where c_4 and c_5 do not depend on i.

Then, for each function $x \in C[0, 2\pi]$ from (2.7) - (2.8) that

$$\upsilon(x, 2^{-i}) \le c_3(c_4 + c_5)\frac{1}{\omega_\nu(i)}, \quad i = 0, 1, 2, ...$$

Therefore, the embedding constant

$$E_0(l^\infty_{\omega_\nu}; \alpha, A, B) \subseteq Lip(\nu)$$

does not exceed $c_3(c_4 + c_5)$.

Conditions (2.8) imply the inequalities

$$\|\tau | l^\infty_{\omega_\nu} \to l^\infty_{\omega_\nu}\| = \sup_i \frac{\nu(2^{-i})}{\nu(2^{-i-1})} \le 2c_4;$$

$$\|\Lambda | l^\infty_{\omega_\nu} \to l^\infty_{\omega_\nu}\| = \sup_i \frac{1}{\nu(2^{-i})}\{\Sigma_{j=i}^{\infty} \nu(2^{-j})\} \le c_5.$$

Therefore, up to equivalent norms, the following equalities hold

$$E(l^\infty_{\omega_\nu}; \alpha, A, B) = E_0(l^\infty_{\omega_\nu}; \alpha, A, B) = Lip(\nu).$$

In particular, for any $\beta \in (0, 1)$, the approximation spaces are Lip β.

If we replace the uniform norm by the norm in the space L^p, ($1 \le p < \infty$), then, by a similar scheme, it can be shown that the Lipschitz spaces defined by the modulus of continuity calculated in the norm of L^p are also approximation space. Details and generalizations can be found in [1, 7].

3. Main results

The following result is the main one.

Theorem 3.1. *Fix a separable topological space B that is complete with respect to the normalizing function α and a sequence of subspaces $\{A_i\}_{-\infty}^{\infty}$, satisfying the conditions (2.5). Let a collection of ideal sequence spaces $\{l_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta}))$ and a measurable function $\xi : (\underline{\beta}, \overline{\beta}) \to R_+$ are given. For each $\{l_\beta\}$ we construct an approximation space $Y_\beta = E_0(l_\beta; \alpha, A, B)$, $(\beta \in (\underline{\beta}, \overline{\beta}))$. Set $\bar{l} = \sum\{\xi, l_\alpha\}$.*

Then,

$$\sum\{\xi, E_0(l_\beta; \alpha, A, B)\} = E_0(\bar{l}; \alpha, A, B), \tag{3.1}$$

and the norms in these spaces coincide.

Proof. It follows from the lemma 2.2 that the space $\bar{l}$ is well-defined and can be a parameter for the approximation space.

We now show that the space $\sum\{\xi, E_0(l_\beta; \alpha, A, B)\}$ is well-defined. Indeed, if we take B as a complete topological separable space V, then the collection of $\{Y_\beta\}$ spaces will be a B-collection. Therefore, the space $Y = \sum\{\xi, Y_\beta\}$ is well-defined.

First we show that the embedding is satisfied

$$\sum\{\xi, Y_\beta\} \subset E_0(\bar{l}; \alpha, A, B), \tag{3.2}$$

with a embedding constant unit.

We note that if $\beta \in (\underline{\beta}, \overline{\beta})$ is fixed, then the inequality holds

$$\xi(\beta) \| \sum_{-\infty}^{\infty} a_i e^i | l_\beta \| \geq \| \sum_{1}^{\infty} a_i e^i | \bar{l} \|. \tag{3.3}$$

Fix $\varepsilon > 0$ and set positive numbers $\{\varepsilon_j\}_{j=1}^{\infty}$, such that $\sum_j \varepsilon_j < \varepsilon$. Let $b \in \sum\{\xi, Y_\beta\}$ be a sequence of numbers $\beta_i \in (\underline{\beta}, \overline{\beta})$ and the sequence of elements $b_i \in E_0(l_{\beta_i}; \alpha, A, B)$, such that $b = \sum_i b_i$ (the series converges in B). For each $b_i \in E_0(l_{\beta_i}; \alpha, A, B)$ we consider the representation $b_i = \sum_{j=1}^{\infty} u_{i,j}$, where $u_{i,j} \in A_j$, $(j \in N)$ and the series converges in B, such that at all i are executed inequalities

$$\xi(\beta_i) \| b_i | E_0(l_{\beta_i}; \alpha, A, B) \| > \xi(\beta_i) \| \sum_{j=1}^{\infty} \alpha(u_{i,j}) e^j | l_{\beta_i} \| - \varepsilon_i. \tag{3.4}$$

From (3.3) - (3.4) we obtain

$$\sum_i \xi(\beta_i) \| b_i | E_0(l_{\beta_i}; \alpha, A, B) \| > \sum_i (\xi(\beta_i) \| \sum_{j=1}^{\infty} \alpha(u_{i,j}) e^j | l_{\beta_i} \| - \varepsilon_i) \geq$$

$$\sum_i \xi(\beta_i) \| \sum_{j=1}^{\infty} \alpha(u_{i,j}) e^j | l_{\beta_i} \| - \varepsilon \geq \sum_i \| \sum_{j=1}^{\infty} \alpha(u_{i,j}) e^j | \bar{l} \| - \varepsilon \geq$$

$$\| \sum_i (\sum_{j=1}^{\infty} \alpha(u_{i,j}) e^j) | \bar{l} \| - \varepsilon = \| \sum_{j=1}^{\infty} (\sum_i \alpha(u_{i,j})) e^j) | \bar{l} \| - \varepsilon \geq$$

$$\|\sum_{j=1}^{\infty}(\alpha(\sum_i u_{i,j}))e^j)|\bar{l}\| - \varepsilon \geq \|(\sum_i b_i)|E_0(\bar{l};\alpha,A,B)\| - \varepsilon =$$

$$\|b|E_0(\bar{l};\alpha,A,B)\| - \varepsilon,$$

from which it follows (3.2).

We now show that the converse to (3.2) embedding is also satisfied

$$E_0(\bar{l};\alpha,A,B) \subset \sum\{\xi, Y_\beta\}, \tag{3.5}$$

with a embedding constant unit.

Fix $\varepsilon > 0$. Let $u \in E_0(\bar{l};\alpha,A,B)$:

$$u = \sum_1^{\infty} u_n,\ \ u_n \in A_n,\ \ (\forall n \in N),\ \ x = \sum_1^{\infty}\alpha(u_n)e^n,\ \ \|x|\bar{l}\| \leq \|u|E_0(\bar{l};\alpha,A,B)\| + \varepsilon.$$

Choose a sequence $\{\beta_i\}$, $\beta_i \in (\underline{\beta},\overline{\beta})$, so that the relations are satisfied

$$x \equiv \sum_i y_i, \text{ where } \ y_i = \sum_{j=1}^{\infty} y_{i,j}e^j \in l_{\beta_i}, \quad \text{and} \quad \sum_i \xi(\beta_i)\|y_i|l_{\beta_i}\| \leq \|x|\bar{l}\| + \varepsilon.$$

For each i define

$$w_{i,n} \equiv \frac{u_n}{\alpha(u_n)}y_{i,n}, \qquad w_i = \sum_{n=1}^{\infty} w_{i,n}. \tag{3.6}$$

Then, directly from (3.6), it follows that the following relations are true

$$u = \sum_i w_i, \quad \sum_{n=1}^{\infty}\alpha(w_{i,n})e^n \equiv y_i.$$

Therefore

$$\|u|\sum\{\xi,Y_\beta\}\| \leq \sum_i \xi(\beta_i)\|w_i|E_0(l_{\beta_i};\alpha,A,B)\| \leq \sum_i \xi(\beta_i)\|\sum_{n=1}^{\infty}\alpha(w_{i,n})e^n|l_{\beta_i}\| =$$

$$\sum_i \xi(\beta_i)\|y_i|l_{\beta_i}\| \leq \|x|\bar{l}\| + \varepsilon \leq \|u|E_0(\bar{l};\alpha,A,B)\| + 2\varepsilon.$$

The last relation implies (3.5), and the embedding constant does not exceed one. Combining (3.2) and (3.5), we obtain (3.1) with the equality of norms. ☐

From lemma 2.2 and theorem 3.1, we obtain the following.

Theorem 3.2. *Fix a separable topological space B that is complete with respect to the normalizing function α, a sequence of subspaces $\{A_i\}_{-\infty}^{\infty}$, satisfying the conditions (2.5), a collection of ideal sequence spaces $\{l_\beta\}$, $(\beta \in (\underline{\beta},\overline{\beta}))$, each of which satisfies the condition (2.2), and a measurable function $\xi : (\underline{\beta},\overline{\beta}) \to R_+$. Let the operators τ and Λ satisfy conditions (2.3) and (2.4). For each $\{l_\beta\}$ we construct an approximation space $Y_\beta = E(l_\beta;\alpha,A,B)$, $(\beta \in (\underline{\beta},\overline{\beta}))$.*

Then, up to equivalent norms,

$$\sum\{\xi, E(l_\beta;\alpha,A,B)\} = E(\bar{l};\alpha,A,B).$$

The dual fact to theorem 3.1 is as follows.

Theorem 3.3. *Fix a separable topological space B that is complete with respect to the normalizing function α and a sequence of subspaces $\{A_i\}_{-\infty}^{\infty}$, satisfying the conditions (2.5). Let a collection of ideal sequence spaces $\{l_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta}))$ and a measurable function $\xi : (\underline{\beta}, \overline{\beta}) \to R_+$ are given. For each $\{l_\beta\}$ we construct an approximation space $Y_\beta = E_0(l_\beta; \alpha, A, B)$, $(\beta \in (\underline{\beta}, \overline{\beta}))$. Set $\underline{l} = \bigcap_\beta \{\xi, l_\alpha\}$.*

Then

$$\bigcap_\beta \{\xi, E(l_\beta; \alpha, A, B)\} = E(\underline{l}; \alpha, A, B), \tag{3.7}$$

and the norms in these spaces coincide.

Proof. The proof (3.7) is contained in the equality

$$\|u|\xi(\beta)E(l_\beta; \alpha, A, B)\| = \|u|E(\xi(\beta)l_\beta; \alpha, A, B)\|.$$

□

From lemma 2.2 and theorem 3.3 we get the following.

Theorem 3.4. *Fix a separable topological space B that is complete with respect to the normalizing function α, a sequence of subspaces $\{A_i\}_{-\infty}^{\infty}$, satisfying the conditions (2.5), a collection of ideal sequence spaces $\{l_\beta\}$, $(\beta \in (\underline{\beta}, \overline{\beta}))$, each of which satisfies the condition (2.2), and a measurable function $\xi : (\underline{\beta}, \overline{\beta}) \to R_+$. Let the operators τ and Λ satisfy conditions (2.3) and (2.4). For each $\{l_\beta\}$ we construct an approximation space $Y_\beta = E(l_\beta; \alpha, A, B)$, $(\beta \in (\underline{\beta}, \overline{\beta}))$.*

Then, the following equality holds

$$\bigcap_\beta \{\xi, E_0(l_\beta; \alpha, A, B)\} = E_0(\underline{l}; \alpha, A, B),$$

and the norms in these spaces are equivalent.

The classical formulation of the problems of extrapolation theory is of the form.

Problem 1.

Let $\{A_\beta\}$ be a V-collection of Banach spaces, X a Banach space, $T : A_\beta \to X$ a quasilinear operator, and $\xi(\beta) = \|T|A_\beta \to X\|$, $(\beta \in (\underline{\beta}, \overline{\beta}))$. It is necessary to construct from the collection $\{A_\beta\}$ taking into account the behavior of the function $\xi(\beta)$ "maximal" space $\overline{A}$, such that $T : \overline{A} \to X$ and $\|T|\overline{A} \to X\| < \infty$.

Problem 2. Let $\{A_\beta\}$ be a V-collection of Banach spaces $\{A_\beta\}$, X a Banach space, $S : X \to A_\beta$ a quasilinear operator, and $\xi(\beta) = \|S|X \to A_\beta\|$, $(\beta \in (\underline{\beta}, \overline{\beta}))$. It is necessary to construct from the collection $\{A_\beta\}$ taking into account the behavior of the function $\xi(\beta)$ "minimal" space $\underline{A}$, such that $S : X \to \underline{A}$ and $\|S|X \to \underline{A}\| < \infty$.

Let $b \in \Sigma\{\xi, A_\beta\}$ and

$$\sum b_i = b; \quad \sum \xi(\beta_i)\|b_i|A_{\beta_i}\| < 1.$$

From inequality

$$\|Tb|Y\| = \|T\sum b_i|Y\| \le \sum \|Tb_i|Y\| \le \sum \xi(\beta_i)\|b_i|Y\|$$

it follows that $\Sigma\{\xi, A_\beta\}$ can act as $\overline{A}$.

Let $b \in X$ and $\|b|X\| \leq 1$.

From inequality

$$\|Sb|A_\beta\| \leq \xi(\beta_i)\|b|X\|$$

it follows that the space $\bigcap\{\xi, A_\beta\}$ can act as $\underline{A}$.

For definiteness, it is usually assumed that $\lim_{\beta\to\overline{\beta}-0}\xi(\beta) = \infty$, or $\lim_{\beta\to\underline{\beta}+0}\xi(\beta) = \infty$, or it is assumed that a combination of singularities at the boundaries of the interval $(\underline{\beta}, \overline{\beta})$. That is why the spaces $\overline{A}$ and $\underline{A}$ are defined.

Therefore, from the theorems 3.1 and 3.3 we obtain the following extrapolation theorems for approximation spaces.

Theorem 3.5. *Let some Banach space Y be given, and let $T : E(l_\beta; \alpha, A, B) \to Y$ be some linear operator whose norms for all $\beta \in (\underline{\beta}, \overline{\beta})$ are admissible*

$$\|T|E_0(l_\beta; \alpha, A, B) \to Y\| \leq \xi(\beta).$$

Set $\bar{l} = \Sigma\{\xi, l_\beta\}$. Then, T is bounded as an operator from $E_0(\bar{l}; \alpha, A, B)$ to Y and its norm does not exceed 1.

Theorem 3.6. *Let some Banach space Y be given, and let $S : X \to E(l_\beta; \alpha, A, B)$ be some linear operator whose norms for all $\beta \in (\underline{\beta}, \overline{\beta})$ are admissible*

$$\|S|X \to E(l_\beta; \alpha, A, B)\| \leq \xi(\beta).$$

Set $\underline{l} = \bigcap_\beta\{\xi, l_\beta\}$. Then, S is bounded as an operator from X to $E(\underline{l}; \alpha, A, B)$ and its norm does not exceed 1.

We illustrate the theorems obtained for calculating concrete spaces.

Let W be the set of increasing sequences $\{\psi_k\}_0^\infty$, each of which satisfies the conditions: $\psi(0) = 0$ and, for any $k \in N$, the inequality holds $2\psi_k \geq \psi_{k+1} + \psi_{k-1}$. For every $\psi \in W$, Lorentz space $\lambda(\psi)$ (Marcinkiewicz space $m(\psi)$) consists of those $\{x_k\}_0^\infty$, each of which has finite norm

$$\|\Sigma x_k e^k|\lambda(\psi)\| = \Sigma_{i=1}^\infty x_i^*(\psi_i - \psi_{i-1}), \quad (\|x|m(\psi)\| = \sup_k \frac{\psi(k)}{k}\Sigma_{i=1}^k x_i^*).$$

Here, the sequence $\{x_k^*\}$ is a permutation of the sequence $\{|x_k|\}$ in non-increasing order.

If $\psi_k = k^\alpha$, $(\alpha \in (0,1))$, then the Lorentz space $\lambda(\psi_\alpha)$ (Marcinkevich space $m(\psi_\alpha)$) is denoted by λ^α (m^α). For uniformity, the Lebesgue space l^p will also be denoted by l^α, setting $p = 1/\alpha$ $(1/0 = \infty)$.

Lorentz and Marcinkiewicz spaces, along with Lebesgue spaces, are classical examples of Banach symmetric spaces. For more information on symmetric spaces and permutations, see [6, 14].

As in the case of a continuous measure ([2, 4]), the following theorem can be proved.

Theorem 3.7. *Fix a pair of numbers $0 \leq \underline{\beta} < \overline{\beta} \leq 1$, a measurable function $\xi : (\underline{\beta}, \overline{\beta}) \to R_+$, a collection of Lorentz spaces λ^β and a collection of Marcinkiewicz spaces m^β, $(\beta \in (\underline{\beta}, \overline{\beta}))$.*

For each $i \in N$, we define a function

$$\overline{\psi}_\xi(i) = \inf_{\beta\in(\underline{\beta},\overline{\beta})} \xi(\beta)\cdot i^\beta, \quad \underline{\psi}_\xi(i) = \sup_{\beta\in(\underline{\beta},\overline{\beta})} \xi(\beta)\cdot i^\beta. \tag{3.8}$$

Then, the equalities hold

$$\Sigma\{\xi, \lambda^\beta\} = \lambda(\overline{\psi}_\xi), \quad \bigcap\{\xi, m^\beta\} = m(\underline{\psi}_\xi),$$

and the norms on these pairs spaces coincide.

Therefore, from the theorems 3.5 - 3.7 we obtain the following theorem.

Theorem 3.8. *Fix a pair of numbers $0 \le \underline{\beta} < \overline{\beta} \le 1$, a separable topological space B that is complete with respect to the normalizing function α and a sequence of subspaces $\{A_i\}_{-\infty}^{\infty}$, satisfying the conditions (2.5). Let a collection of Lorentz spaces λ^β, a collection of Marcinkiewicz spaces m^β, ($\beta \in (\underline{\beta},\overline{\beta})$) and a measurable function $\xi : (\underline{\beta},\overline{\beta}) \to R_+$ are given. For each $\{l_\beta\}$ we construct an approximation space $E_0(\lambda^\beta;\alpha,A,B)$ and $E(m^\beta;\alpha,A,B)$.*

For each $i \in N$, we define the functions $\overline{\psi}_\xi(i)$ and $\underline{\psi}_\xi(i)$ by the equation (3.8).

Then, the equalities hold

$$\Sigma\{\xi, E_0(\lambda^\beta;\alpha,A,B)\} = E_0(\lambda(\overline{\psi}_\xi));\alpha,A,B), \tag{3.9}$$

$$\bigcap\{\xi, E(m^\beta;\alpha,A,B)\} = E(m(\underline{\psi}_\xi);\alpha,A,B), \tag{3.10}$$

and the norms on pairs of spaces in (3.9) and (3.10) coincide.

We show how the extrapolation theorem for classical Lipschitz spaces $Lip\ \beta$. The problem of extrapolation of Lipschitz spaces has been considered several times before (see [10, 16]). We demonstrate a new approach that allows us to consider arbitrary growth of the operator norms. In a similar way, one can obtain theorems of extrapolation for other approximation spaces, for example, for Besov spaces.

Theorem 3.9. *Fix $0 < \underline{\beta} < \overline{\beta} < 1$. Now, let Y be some Banach space and $T : Lip\ \beta \to Y$ some linear operator for all $\beta \in (\underline{\beta},\overline{\beta})$ the following relations hold*

$$\|T|Lip\ \beta \to Y\| \le \xi(\beta),$$

$$\lim_{\beta\to\underline{\beta}+0} \xi(\beta) = \infty.$$

We define the weight function $\omega_{\nu(\beta,.)}$ by the formula $\omega_{\nu(\beta,.)}(n) = 2^{n\beta}$, ($n = 0,1,2,...$), and the function $\overline{\psi}_\xi(i)$ by

$$\overline{\psi}_\xi(i) = \inf\{\xi(\beta)2^{\beta i} : \quad \beta \in (\underline{\beta},\overline{\beta})\} \tag{3.11}$$

for every $i \in N$.

Moreover, we define the space

$$\widetilde{\lambda}(\overline{\psi}_\xi) = \{a = \sum_{k=1}^{\infty} a_i e^i : \sum_{i=1}^{\infty} (a_i^* - a_{i+1}^*)\overline{\psi}_\xi(i) < \infty\}.$$

Finally, let α, A, B be parameters for constructing approximation spaces defined as in Example 1. We employ these parameters to construct an approximation space $E(\widetilde{\lambda}(\overline{\psi}_\xi); \alpha, A, B)$.

Then, T is bounded as an operator from $E(\widetilde{\lambda}(\overline{\psi}_\xi); \alpha, A, B)$ to Y.

Proof. Fix $\beta \in (\underline{\beta}, \overline{\beta})$, the modulus of continuity $\nu(\beta, \delta) = \delta^\beta$ and $Lip\ \beta$. By analogy with example 1, we construct approximation spaces $E(l^\infty_{\omega_{\nu(\beta,.)}}; \alpha, A, B)$ and $E_0(l^\infty_{\omega_{\nu(\beta,.)}}; \alpha, A, B)$. Since the modulus of continuity is $\nu(\beta, \delta)$ satisfies the conditions (2.8) with

$$c_4 = \frac{1}{1 - 2^{-\beta}}, \quad c_5 = \frac{1}{2^{1-\beta} - 1},$$

for any $\beta \in (\underline{\beta}, \overline{\beta})$ the equalities hold

$$E(l^\infty_{\omega_{\nu(\beta,.)}}; \alpha, A, B) = E_0(l^\infty_{\omega_{\nu(\beta,.)}}; \alpha, A, B) = Lip\ \beta.$$

The restrictions on β, the lemma 2.2 and the theorem 3.1 imply

$$E_0(\Sigma\{\xi, l^\infty_{\omega_{\nu(\beta,.)}}\}; \alpha, A, B) = E(\Sigma\{\xi, l^\infty_{\omega_{\nu(\beta,.)}}\}; \alpha, A, B).$$

For each $k \in N$, we define the vectors d^k by $d^k = \Sigma^k_{i=1} e^i$. We prove that for any $k \in N$

$$\|d^k | \Sigma\{\xi, l^\infty_{\omega_\beta}\}\| = \|d^k | l^\infty_{\overline{\psi}_\xi}\}\| = \overline{\psi}_\xi(k). \tag{3.12}$$

For any $\beta \in (\underline{\beta}, \overline{\beta})$, from (3.11) follows the embedding

$$\xi(\beta) l^\infty_{\omega_\beta} \subseteq l^\infty_{\overline{\psi}_\xi}$$

with a the embedding constant 1. Therefore, follows the embedding

$$\sum\{\xi, l^\infty_{\omega_\beta}\} \subseteq l^\infty_{\overline{\psi}_\xi}$$

with a the embedding constant 1. From this relation it follows that

$$\|d^k | l^\infty_{\overline{\psi}_\xi}\}\| \leq \|d^k | \Sigma\{\xi, l^\infty_{\omega_\beta}\}\|.$$

To prove the inverse inequality, we define $b_1 = d^k; b_i = 0,\ i = 2, 3, 4.....$ Then, the inequality holds

$$\|d^k | \Sigma \xi(\beta_i) l^\infty_{\omega_{\beta_i}}\}\| \leq \xi(\beta_1) \|d_1 | l^\infty_{\omega_{\beta_1}}\| = \xi(\beta_1) \omega_{\beta_1}(k).$$

This inequality holds for any $\beta_1 \in (\underline{\beta}, \overline{\beta})$. Therefore,

$$\|d^k | \Sigma\{\xi, l^\infty_{\omega_\beta}\}\| \leq \inf\{\xi(\beta)\omega_\beta(k) : \beta \in (\underline{\beta}, \overline{\beta})\} = \overline{\psi}_\xi(k).$$

The last formula implies

$$\|d^k | \Sigma\{\xi, l^\infty_{\omega_\beta}\}\| \leq \|d^k | l^\infty_{\overline{\psi}_\xi}\}\|.$$

Thus, (3.12) is proved.

Let $a = \sum_{i=1}^{\infty} a_i e^i$, $a_i \geq 0$, $a_i \downarrow$ be given and $\sum_{i=1}^{\infty}(a_i - a_{i+1})\overline{\psi}_\xi(i) < \infty$. Then, $a = \sum_{k=1}^{\infty}(a_k - a_{k+1})d^k$ and

$$\|a|\Sigma\{\xi, l^{\infty}_{\omega_\beta}\}\| \leq \sum_{k=1}^{\infty}(a_k - a_{k+1})\|d^k|\Sigma\{\xi, l^{\infty}_{\omega_\beta}\}\| =$$

$$\sum_{k=1}^{\infty}(a_k - a_{k+1})\|d^k|l^{\infty}_{\overline{\psi}_\xi}\| = \sum_{k=1}^{\infty}(a_k - a_{k+1})\overline{\psi}_\xi(k). \tag{3.13}$$

From the inequality (3.13) follows the embedding

$$\widetilde{\lambda}(\overline{\psi}_\xi) \subseteq \Sigma\{\xi, l^{\infty}_{\omega_{\nu(\beta,.)}}\} \tag{3.14}$$

with a the embedding constant 1. From the theorem 3.5 it follows that T is bounded as an operator from $E_0(\Sigma\{\xi, l^{\infty}_{\omega_{\nu(\beta,.)}}\}; \alpha, A, B)$ to Y and from $E(\Sigma\{\xi, l^{\infty}_{\omega_{\nu(\beta,.)}}\}; \alpha, A, B)$ to Y. From the embedding (3.14) we obtain that T is bounded as an operator from $E_0(\widetilde{\lambda}(\overline{\psi}_\xi); \alpha, A, B)$ to Y and from $E(\widetilde{\lambda}(\overline{\psi}_\xi); \alpha, A, B)$ to Y. □

Theorem 3.10. *Fix $0 < \underline{\beta} < \overline{\beta} < 1$. Let X be some Banach space, and $S : X \to Lip\ \beta$ some linear operator for all $\beta \in (\underline{\beta}, \overline{\beta})$ the following relations hold*

$$\|S|X \to Lip\ \beta\| \leq \xi(\beta),$$

$$\lim_{\beta \to \overline{\beta} - 0} \xi(\beta) = \infty.$$

We define the weight function $\omega_{\nu(\beta,.)}$ by $\omega_{\nu(\beta,.)}(n) = 2^{n\beta}$, $n = 0, 1, 2, ...,$ and the function $\underline{\psi}_\xi(i)$ by

$$\underline{\psi}_\xi(i) = \sup\{\xi(\beta)2^{\beta i} : \quad \beta \in (\underline{\beta}, \overline{\beta})\},$$

for every $i \in N$. Moreover, we define the space

$$\widetilde{m}(\underline{\psi}_\xi) = \{a = \sum_{i=1}^{\infty} a_i e^i : \sup_i a_i^* \underline{\psi}_\xi(i) < \infty\}.$$

Finally, let α, A, B be parameters for constructing approximation spaces defined as in theorem 3.9. We construct an approximation space $E(\widetilde{m}(\underline{\psi}_\xi); \alpha, A, B)$.

Then, S is bounded as an operator from X to $E(\widetilde{m}(\underline{\psi}_\xi); \alpha, A, B)$.

Proof. The proof is similar to one of theorem 3.9, if for $a = \sum_{i=1}^{\infty} a_i e^i$, $a_i \geq 0$, $a_i \downarrow$ use the equality

$$\|\sum_{k=1}^{\infty} a_i e^i| \bigcap_{\beta \in (\underline{\beta}, \overline{\beta})} \xi(\beta) l^{\infty}_{\omega_{\nu(\beta,.)}}\| = \sup_{\beta \in (\underline{\beta}, \overline{\beta})} \{\xi(\beta) \sup_i a_i 2^{i\beta}\} =$$

$$\sup_i \{a_i \sup_{\beta \in (\underline{\beta}, \overline{\beta})} \xi(\beta) 2^{i\beta}\} = \sup_i a_i \underline{\psi}_\xi(i).$$

□

Examples of calculating the functions $\overline{\psi}_\xi$, $\underline{\psi}_\xi$ can be found in ([2–4]).

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