Pure and Applied Functional Analysis
Volume 6, Number 3, 2021, 511–529

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

EXTRAPOLATION DESCRIPTION OF LIMITING INTERPOLATION J-SPACES

SERGEY V. ASTASHKIN AND KONSTANTIN V. LYKOV

Abstract. In this paper, we propose a generalized version of the extrapolation Σ-functor, which was studied in detail earlier by Jawerth and Milman. Our main result shows that, under some non-restrictive conditions, limit interpolation spaces of the real J-method admit a rather simple extrapolation description with respect to the classical Lions-Peetre interpolation scale.

1. Introduction

It is well-known that the starting result for extrapolation of operators is the following theorem [16].

Yano's extrapolation theorem. *Let an operator T be defined on the space $L_1[0,1]$ with values in the set of measurable functions on $[0,1]$, and let T satisfy the sublinearity condition: for some $B>0$ and all $x_j \in L_1[0,1]$ such that the series $\sum_{j=1}^{\infty} x_j$ converges in $L_1[0,1]$ we have*

$$\Big|T\Big(\sum_{j=1}^{\infty} x_j\Big)(t)\Big| \leqslant B\sum_{j=1}^{\infty} |Tx_j(t)| \quad a.e.\ on\ [0,1].$$

Suppose also that T is bounded in $L_p[0,1]$ for every $p\in(1,p_0)$, $p_0>1$, and

$$\|T\|_{L_p\to L_p} \leqslant C(p-1)^{-\beta}, \ \ p\in(1,p_0), \tag{1.1}$$

for some $\beta>0$, with a constant $C>0$ independent of p. Then,

$$T:\ L(\log L)^{\beta} \to L_1,$$

where the Zygmund space $L(\log L)^{\beta}$ consists of all measurable on $[0,1]$ functions $x(t)$ such that

$$\|x\|_{L(\log L)^{\beta}} = \int_0^1 x^*(t)\log^{\beta}(e/t)\,dt < \infty$$

($x^*(t)$) stands for the left-continuous, non-increasing rearrangement of $|x(t)|$).

As it is well known (see e.g. [9–11, 13]), Yano's theorem and a number of other related results can be obtained as a consequence of the modern extrapolation theory. One of the most important constructions (or extrapolation functors) that is widely used in this theory is the classical Σ-method.

2020 *Mathematics Subject Classification.* 46B70, 46B42, 46B45, 46E30.
Key words and phrases. J-method of real interpolation, Banach sequence lattice, extrapolation.

Let $\{A_\theta\}_{\theta\in\Theta}$ be a family of Banach spaces such that for some Banach space $\mathcal{A}$ the continuous inclusions $A_\theta \subset \mathcal{A}$, $\theta \in \Theta$, hold with uniformly bounded norms. Then we set

$$\Sigma(A_\theta) = \Sigma_\Theta(A_\theta) := \Big\{a \in \mathcal{A} :\ a = \sum_{k=1}^{\infty} a_k : a_k \in A_{\theta_k},\ \sum_k \|a_k\|_{A_{\theta_k}} < \infty\Big\},$$

with the norm $\|a\|_{\Sigma(A_\theta)} := \inf \sum_k \|a_k\|_{A_{\theta_k}}$, where the infimum is taken over all admissible representations of $a \in \Sigma(A_\theta)$. One can calculate (see e.g. [10, page 83]) that for every $\beta > 0$ and $p_0 > 1$ we have

$$\Sigma_{(1,p_0)}\left(L_p[0,1]\right) = L_1 \text{ and } \Sigma_{(1,p_0)}\Big((p-1)^{-\beta}L_p[0,1]\Big) = L(\log L)^\beta. \tag{1.2}$$

Clearly, Yano's theorem follows easily from these relations, showing the usefulness of the Σ-method as an extrapolation construction. Later on, in [11] Karadzhov and Milman have introduced and developed as a generalization of the Σ-method the functors $\Sigma^{(p)}$, $1 \le p \le \infty$, which allowed them to expand the scope of extrapolation methods and to obtain new results. Meanwhile, in the same year, 2005, Astashkin [1] defined a new family of extrapolation methods, including the functors Σ and $\Sigma^{(p)}$ as very special cases. Basing on this approach, he has obtained in [1] (see also [2]) an extrapolation description of all interpolation spaces between the spaces A_0 and $\Lambda_\varphi(\vec{A})$, where (A_0, A_1) is an ordered Banach couple (i.e., $A_1 \subset A_0$) and $\Lambda_\varphi(\vec{A})(= \vec{A}^J_{\varphi,1})$ is the generalized Lorentz space [15, p. 430] constructed by a function φ satisfying some growth conditions. More recently, by using another extrapolation construction, similar results were proved, provided that a parameter G is separable and A_1 is dense in A_0, for limiting interpolation J-spaces $\vec{A}^J_G$ [3, Theorem 4].

It is worth to emphasize that all the above-mentioned results have been obtained only in the case of ordered Banach couples. The main aim of the present paper is to give an extrapolation description of limiting interpolation J-spaces $\vec{A}^J_G$ for an arbitrary Banach couple (A_0, A_1). Moreover, we replace the above restrictions imposed on a parameter G with a more natural condition of the boundedness of a simple doubling operator acting in the underlying parameter G (see Section 3). The proof of the main results, Theorems 5.2 and 5.3, is based on using some special sparse interpolation and extrapolation constructions (see Section 4), which may be hopefully of independent interest.

Finally, in Section 6, we present some applications of our results to an extrapolation description of real interpolation spaces parameterized by weighted ℓ_1-spaces.

Acknowledgement. We are very grateful to Mario Milman for helpful discussions and valuable suggestions concerning some results of this paper.

2. Preliminaries

In this section, we briefly list definitions and notions from interpolation theory, which will be used later on. For more detailed information, we refer to the monographs [4–6].

Let $\vec{A} = (A_0, A_1)$ be an arbitrary Banach couple. Suppose that G is a Banach sequence lattice modelled on $\mathbb{N} := \{1, 2, \ldots\}$ such that $\ell_1(2^k) \subset G \subset \ell_1$. Denote by

$\vec{A}^J_G$ the space of all $a \in A_0 + A_1$ representable in the form

$$a = \sum_{k=1}^{\infty} a_k, \quad a_k \in A_0 \cap A_1, k = 1, 2, \dots \text{ (the series converges in } A_0 + A_1), \tag{2.1}$$

with the norm

$$\|a\|_{\vec{A}^J_G} := \inf \left\| \{J(2^{-k}, a_k; \vec{A})\}_{k=1}^{\infty} \right\|_G, \tag{2.2}$$

where the infimum is taken over all representations (2.1). Here and next, $J(t, a; \vec{A})$ is the Peetre J-functional defined for all $a \in A_0 \cap A_1$ by

$$J(t, a; \vec{A}) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \quad t > 0.$$

In the same manner we can also define the real interpolation K-spaces. For every Banach couple $\vec{A} = (A_0, A_1)$ and for all $a \in A_0 + A_1$ we set

$$K(t, a; \vec{A}) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_i \in A_i, i = 0, 1\}, \quad t > 0.$$

Then, if F is a Banach sequence lattice modelled on $\mathbb{N}$ such that $\ell_\infty(2^k) \subset F \subset \ell_\infty$, by $\vec{A}^K_F$ we denote the space of all $a \in A_0 + A_1$ such that

$$\{K(2^{-k}, a; \vec{A})\}_{k=1}^{\infty} \in F$$

with the norm

$$\|a\|_{\vec{A}^K_F} := \left\| \{K(2^{-k}, a; \vec{A})\}_{k=1}^{\infty} \right\|_F.$$

Let us emphasize that, in contrast to the usual real method, here we use as parameters Banach lattices of *one*-sided sequences of real numbers. However, it is easy to see that the mappings $\vec{A} \mapsto \vec{A}^J_G$ and $\vec{A} \mapsto \vec{A}^K_G$ are still exact interpolation functors and, moreover, this definition leads to the usual real interpolation J- and K-spaces whenever $A_1 \subset A_0$ [3, Propositions 1 and 2].

Let $0 < \theta \le 1$ and $1 \le q \le \infty$. Next, by $\vec{A}^J_{\theta,q}$ we denote the modified Lions-Peetre interpolation J-spaces consisting of all $a \in A_0 + A_1$ representable in the form (2.1) with the norm

$$\|a\|_{\theta,q} := \theta^{-1/q'} \inf \left(\sum_{k=1}^{\infty} \left(2^{k\theta} J(2^{-k}, a_k; \vec{A}) \right)^q \right)^{1/q}, \tag{2.3}$$

where $q' = q/(q-1)$ and the infimum is taken over all representations (2.1). In what follows, the notation $\|a\|_{\theta,q}$ is understood as it is defined in (2.3).

Clearly, for all $0 < \theta \le 1$ we have

$$\|a\|_{\theta,1} \le 2^{n\theta} J(2^{-n}, a; \vec{A}), \quad n \in \mathbb{N}, \tag{2.4}$$

and, in view of the embedding $\ell_1 \subset \ell_q$, $1 \le q \le \infty$,

$$\|a\|_{\theta,q} \le \theta^{-1/q'} \|a\|_{\theta,1}. \tag{2.5}$$

Moreover, if $\theta_1 < \theta_2$, then

$$\|a\|_{\theta_1,q} \le \|a\|_{\theta_2,q}. \tag{2.6}$$

The main aim of this paper is to identify the space $\vec{A}^J_G$, under a certain condition imposed on a parameter G, as the extrapolation space $\vec{A}^{q(\theta)}_{G,ext}$, consisting of all $a \in A_0 + A_1$ representable in the form (2.1), with the norm

$$\|a\|_{\vec{A}^{q(\theta)}_{G,ext}} := \inf \left\|\{\|a_k\|_{1/k,q(1/k)}\}_{k=1}^\infty\right\|_G,$$

where the infimum is taken over all representations (2.1), provided that a continuous function $q(\theta):\ (0,1] \to [1,\infty)$ tends to 1 in appropriate way as $\theta \to 0$. In particular, if $q(\theta) \equiv q$, the latter space will be denoted by $\vec{A}^q_{G,ext}$.

A Banach couple (A_0, A_1) is said to be *Gagliardo complete* if the condition $x = \lim_{n\to\infty} x_n$ in the A_0+A_1-norm for some bounded sequence $\{x_n\} \subset A_i$ implies that $x \in A_i$ for $i = 0$ and $i = 1$.

If X is a sequence space modelled on $\mathbb{N}$ and $w = w(k)$ is a positive function on $\mathbb{N}$, then we denote by $X(w)$ the sequence space equipped with the norm

$$\|\{x_k\}_{k=1}^\infty\|_{X(w)} := \|\{x_k \cdot w(k)\}_{k=1}^\infty\|_X.$$

If $\{a_k\}_{k=1}^\infty$ is a sequence of elements from a Banach space A, then

$$\operatorname{supp}(\{a_k\}) := \{k \in \mathbb{N} :\ a_k \neq 0\}.$$

The notation $A \asymp B$ means that there exist constants $C > 0$ and $c > 0$ not depending on the arguments of A and B such that $c{\cdot}A \le B \le C{\cdot}A$. Finally, for a Banach couple (A_0, A_1) we denote by $\mathbf{I}(A_0, A_1)$ the class of all interpolation spaces between A_0 and A_1.

3. Auxiliary results

Let $\{e_k\}_{k=1}^\infty$ be the unit vector basis in sequence spaces modelled on $\mathbb{N}$. Denote by R (resp. L, D) the right shift (resp. left shift, doubling) operator defined as follows

$$R\Big(\sum_{k=1}^\infty x_k e_k\Big) := \sum_{k=1}^\infty x_k e_{k+1}$$

(resp.

$$L\Big(\sum_{k=1}^\infty x_k e_k\Big) := \sum_{k=1}^\infty x_{k+1} e_k \ \text{ and } \ D\Big(\sum_{k=1}^\infty x_k e_k\Big) := \sum_{k=1}^\infty x_k (e_{2k-1} + e_{2k})).$$

Moreover, let

$$P\Big(\sum_{k=1}^\infty x_k e_k\Big) := \sum_{k=1}^\infty x_k e_{2^k} \ \text{ and } \ S\Big(\sum_{k=1}^\infty x_k e_k\Big) := \sum_{k=0}^\infty \Big(\sum_{l=2^k+1}^{2^{k+1}} x_l\Big) e_{2^k}.$$

Lemma 3.1. *Let G be a sequence Banach lattice such that $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$. Then the operators R, L and S are bounded in G.*

Proof. It is clear that R is an isometry in ℓ_1. Since

$$\|R(x)\|_{\ell_1(2^k)} = \sum_{k=2}^\infty |x_{k-1}| 2^k = 2 \sum_{k=1}^\infty |x_k| 2^k = 2\|x\|_{\ell_1(2^k)},$$

we get the desired result for R. Similarly, $\|L(x)\|_{\ell_1} \le \|x\|_{\ell_1}$, $\|L(x)\|_{\ell_1(2^k)} \le \frac{1}{2}\|x\|_{\ell_1(2^k)}$, and, therefore, L is bounded on G.

For the operator S we have

$$\|S(x)\|_{\ell_1} = \sum_{k=0}^{\infty} \left| \sum_{l=2^k+1}^{2^{k+1}} x_l \right| \le \sum_{l=2}^{\infty} |x_l| \le \|x\|_{\ell_1}$$

and

$$\|S(x)\|_{\ell_1(2^k)} = \sum_{k=0}^{\infty} \left| \sum_{l=2^k+1}^{2^{k+1}} x_l \right| 2^{2^k} \le \sum_{l=2}^{\infty} |x_l| 2^l \le \|x\|_{\ell_1(2^k)}.$$

□

Let $m, n \in \mathbb{N}$, $q \in [1, \infty]$ and $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$. Define the space $\vec{A}^{q,m,n}_{G,ext}$, consisting of all $a \in A_0 + A_1$ representable in the form (2.1) and equipped with the norm

$$\|a\|_{\vec{A}^{q,m,n}_{G,ext}} := \inf \left\| \{\|a_k\|_{1/(k+m),q}\}_{k=1}^{\infty} \right\|_G,$$

where the infimum is taken over all representations (2.1) with $\operatorname{supp}(\{a_k\}) \subset [n, \infty)$.

Proposition 3.2. *Suppose $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$. Then, we have*

$$\vec{A}^{q,m,n}_{G,ext} = \vec{A}^{q}_{G,ext} \quad \text{and} \quad \|a\|_{\vec{A}^{q,m,n}_{G,ext}} \asymp \|a\|_{\vec{A}^{q}_{G,ext}}.$$

Proof. First, due to inequality (2.6), $\vec{A}^{q}_{G,ext} \subset \vec{A}^{q,m,1}_{G,ext}$.

Conversely, let $a \in \vec{A}^{q,m,1}_{G,ext}$ be represented as $a = \sum_{k=1}^{\infty} a_k$, with

$$2\|a\|_{\vec{A}^{q,m,1}_{G,ext}} \ge \left\| \{\|a_k\|_{1/(k+m),q}\}_{k=1}^{\infty} \right\|_G.$$

Putting $b_k = a_{k-m}$, $k \in \mathbb{N}$, where $a_0 = a_{-1} = \ldots = a_{-m+1} = 0$, by Lemma 3.1, we obtain

$$\begin{aligned} \|a\|_{\vec{A}^{q}_{G,ext}} &\le \left\| \{\|b_k\|_{1/k,q}\}_{k=1}^{\infty} \right\|_G = \left\| R^m(\{\|a_k\|_{1/(k+m),q}\}_{k=1}^{\infty}) \right\|_G \\ &\le 2\|R\|^m_{G\to G} \cdot \|a\|_{\vec{A}^{q,m,1}_{G,ext}}. \end{aligned}$$

Furthermore, $\vec{A}^{q,m,n}_{G,ext} \subset \vec{A}^{q,m,1}_{G,ext} \subset \vec{A}^{q}_{G,ext}$ for $n \in \mathbb{N}$. On the other hand,

$$\begin{aligned} \|a\|_{\vec{A}^{q,m,n}_{G,ext}} &= \inf \left\| R^{n-1}(\{\|a_k\|_{1/(k+m+n-1),q}\}_{k=1}^{\infty}) \right\|_G \\ &\le \|R\|^{n-1}_{G\to G} \cdot \|a\|_{\vec{A}^{q,m+n-1,1}_{G,ext}} \le \|R\|^{n-1}_{G\to G} \cdot \|a\|_{\vec{A}^{q}_{G,ext}}, \end{aligned}$$

and the proof is completed. □

Let G be a Banach lattice such that $\ell_1(2^k) \subset G \subset \ell_1$. Define by $P(G)$ the set of all $x \in \ell_1$ such that $P(x) \in G$. Then $P(G)$ is the Banach lattice with the norm

$$\|x\|_{P(G)} := \|P(x)\|_G.$$

Moreover, it is easy to see that we have $\ell_1(2^{2^k}) \subset P(G) \subset \ell_1$.

Lemma 3.3. *Let G be a sequence Banach lattice such that the operator D is bounded in G. Then, the operator R is bounded in $P(G)$ and*

$$\|R\|_{P(G)\to P(G)} \le \|D\|_{G\to G}.$$

Proof. In fact, if L is the left shift operator, then

$$\begin{aligned}\|R(x)\|_{P(G)} &= \|P(R(x))\|_G \le \|P(R(x)) + L(P(R(x)))\|_G = \|D(P(x))\|_G \\ &\le \|D\|_{G\to G}\|P(x)\|_G = \|D\|_{G\to G}\|x\|_{P(G)}.\end{aligned}$$

□

We proceed with establishing an extrapolation description of a certain sparse version of the interpolation J-method.

4. Extrapolation description of sparse interpolation J-spaces

Let $\vec{A} = (A_0, A_1)$ be an arbitrary Banach couple and let H be a Banach lattice such that $\ell_1(2^{2^k}) \subset H \subset \ell_1$. Denote by $\vec{A}_H^{J,*}$ the space of all $a \in A_0 + A_1$ representable in the form (2.1), equipped with the norm

$$\|a\|_{\vec{A}_H^{J,*}} := \inf \left\| \{J(2^{-2^k}, a_k; \vec{A})\}_{k=1}^{\infty} \right\|_H, \tag{4.1}$$

where the infimum is taken over all representations (2.1) of $a \in A_0 + A_1$. One can easily check that the mapping $\vec{A} \mapsto \vec{A}_H^{J,*}$ is an exact interpolation functor. Show that $\vec{A}_H^{J,*}$ coincides, under some non-restrictive conditions imposed on H, with the sparse extrapolation space $\vec{A}_{H,ext}^{1,*}$ endowed with the norm

$$\|a\|_{\vec{A}_{H,ext}^{1,*}} := \inf \left\| \{\|a_k\|_{2^{-k},1}\}_{k=1}^{\infty} \right\|_H,$$

where the infimum is taken, as above, over all representations (2.1).

Theorem 4.1. *Let H be a Banach sequence lattice such that $\ell_1(2^{2^k}) \subset H \subset \ell_1$, and let the operator R be bounded in H. Then, $\vec{A}_H^{J,*} = \vec{A}_{H,ext}^{1,*}$ for each Banach couple $\vec{A}$.*

In the proof of Theorem 4.1 we shall use a certain description of a sparse interpolation J-space. To this end, we introduce one more interpolation construction.

Let $\vec{A} = (A_0, A_1)$ be a Banach couple. Define on the sum $A_0 + A_1$ the following sequence of norms:

$$\|b\|_k := \inf \left(\sup_{m\in\mathbb{N},\, m\ge k} 2^{2^{m-k}} J(2^{-2^m}, b_m; \vec{A}) \right), \quad k \in \mathbb{N},$$

where the infimum is taken over all representations

$$b = \sum_{m=k}^{\infty} b_m, \quad b_m \in A_0 \cap A_1, m \ge k \text{ (the series converges in } A_0 + A_1). \tag{4.2}$$

Then, the space $\vec{A}_H^{J,**}$ consists of all $a \in A_0 + A_1$ representable in the form (2.1) and it is equipped with the norm

$$\|a\|_{\vec{A}_H^{J,**}} := \inf \|\{\|a_k\|_k\}_{k=1}^{\infty}\|_H,$$

where the infimum is taken over all such representations.

Proposition 4.2. *Suppose the conditions of Theorem 4.1 to be fulfilled. Then,* $\vec{A}_H^{J,*} = \vec{A}_H^{J,**}$ *for each Banach couple* $\vec{A}$.

Proof. Firstly, let $a \in \vec{A}_H^{J,*}$ and let $a = \sum_{k=1}^{\infty} a_k$, $a_k \in A_0 \cap A_1$. Representing $a_k = \sum_{m=k}^{\infty} b_m^k$, with $b_k^k = a_k$ and $b_m^k = 0$ for $m > k$, we see that

$$\|a_k\|_k \leq 2J(2^{-2^k}, a_k; \vec{A}),\ \ k = 1, 2, \ldots$$

Hence,

$$\vec{A}_H^{J,*} \subset \vec{A}_H^{J,**} \ \text{ and } \ \|a\|_{\vec{A}_H^{J,**}} \leq 2\|a\|_{\vec{A}_H^{J,*}}.$$

Conversely, suppose $a \in \vec{A}_H^{J,**}$. Let

$$a = \sum_{k=1}^{\infty} a_k \quad \text{and} \quad a_k = \sum_{m=k}^{\infty} a_m^k$$

be "almost optimal" representations of a and a_k, $k = 1, 2, \ldots$, or more specifically

$$2\|a\|_{\vec{A}_H^{J,**}} \geq \|\{\|a_k\|_k\}_{k=1}^{\infty}\|_H$$

and

$$2\|a_k\|_k \geq \sup_{m \geq k} 2^{2^{m-k}} J(2^{-2^m}, a_m^k; \vec{A}),\quad k = 1, 2, \ldots.$$

Then, by the definition of the operator R, for every $m = 1, 2, \ldots$ we have

$$\begin{aligned}
\sum_{k=1}^{m} J(2^{-2^m}, a_m^k; \vec{A}) &= \sum_{k=1}^{m} 2^{-2^{m-k}} \cdot 2^{2^{m-k}} J(2^{-2^m}, a_m^k; \vec{A}) \\
&\leq 2 \cdot \sum_{k=1}^{m} 2^{-2^{m-k}} \|a_k\|_k \\
&\leq 2 \cdot \sum_{k=1}^{m} 2^{-2^{m-k}} \left[R^{m-k}\left(\{\|a_j\|_j\}_{j=1}^{\infty}\right)\right]_m \\
&\leq 2 \cdot \sum_{i=0}^{\infty} 2^{-2^i} \left[R^{i}\left(\{\|a_j\|_j\}_{j=1}^{\infty}\right)\right]_m,
\end{aligned}$$

where $[\{y_k\}_{k=1}^{\infty}]_m$ denotes the m-th coordinate of the sequence $\{y_k\}_{k=1}^{\infty}$, i.e., y_m. Hence, from the hypothesis of the proposition it follows

$$\begin{aligned}
\left\|\left\{\sum_{k=1}^{m} J(2^{-2^m}, a_m^k; \vec{A})\right\}_{m=1}^{\infty}\right\|_H &\leq 2\left\|\sum_{i=0}^{\infty} 2^{-2^i} R^i\left(\{\|a_j\|_j\}_{j=1}^{\infty}\right)\right\|_H \\
&\leq 2\sum_{i=0}^{\infty} 2^{-2^i} \|R\|_{H\to H}^i \left\|\{\|a_j\|_j\}_{j=1}^{\infty}\right\|_H \\
&\leq C\left\|\{\|a_j\|_j\}_{j=1}^{\infty}\right\|_H \\
&\leq 2C\|a\|_{\vec{A}_H^{J,**}}.
\end{aligned} \tag{4.3}$$

Since the lattice H is continuously embedded into ℓ_1, this inequality implies that

$$\begin{aligned}\sum_{m=1}^{\infty}\sum_{k=1}^{m}\|a_m^k\|_{A_0} &\leq \sum_{m=1}^{\infty}\sum_{k=1}^{m} J(2^{-2^m}, a_m^k; \vec{A})\\ &\leq C\left\|\left\{\sum_{k=1}^{m} J(2^{-2^m}, a_m^k; \vec{A})\right\}_{m=1}^{\infty}\right\|_H < \infty,\end{aligned}$$

i.e., the series $\sum_{m=1}^{\infty}\sum_{k=1}^{m} a_m^k$ absolutely converges in A_0. Therefore, denoting $b_m = \sum_{k=1}^{m} a_m^k$, $m = 1, 2, \ldots$, we obtain

$$\sum_{m=1}^{\infty} b_m = \sum_{m=1}^{\infty}\sum_{k=1}^{m} a_m^k = \sum_{k=1}^{\infty}\sum_{m=k}^{\infty} a_m^k = a \ \text{ (in } A_0 + A_1).$$

Moreover, since

$$J(2^{-2^m}, b_m; \vec{A}) \leq \sum_{k=1}^{m} J(2^{-2^m}, a_m^k; \vec{A}), \quad m = 1, 2, \ldots,$$

by (4.3), we have

$$\left\|\{J(2^{-2^m}, b_m; \vec{A})\}_{m=1}^{\infty}\right\|_H \leq 2C\|a\|_{\vec{A}_H^{J,**}}.$$

Combining this inequality with the definition of the $\vec{A}_H^{J,*}$-norm, we conclude that $\vec{A}_H^{J,**} \subset \vec{A}_H^{J,*}$, and the proof is completed. $\square$

Proof of Theorem 4.1. Let us consider a representation of $a \in \vec{A}_H^{J,*}$ in the form (2.1). Then, inserting into inequality (2.4) $\theta = 2^{-k}$ and $n = 2^k$, we obtain

$$\|a_k\|_{2^{-k},1} \leq 2J(2^{-2^k}, a_k; \vec{A}), \quad k = 1, 2, \ldots$$

Therefore, by the definition of $\vec{A}_H^{J,*}$ and $\vec{A}_{H,ext}^{1,*}$, we have

$$\vec{A}_H^{J,*} \subset \vec{A}_{H,ext}^{1,*} \quad \text{and} \quad \|a\|_{\vec{A}_{H,ext}^{1,*}} \leq 2\|a\|_{\vec{A}_H^{J,*}}.$$

Keeping in mind future applications of Proposition 4.2, we prove for every $a \in A_0 \cap A_1$ the following estimate:

$$\|a\|_k \leq 4\|a\|_{2^{-k+1},1}, \quad k = 2, 3, \ldots. \tag{4.4}$$

Given $k = 2, 3, \ldots$ we choose a representation $a = \sum_{l=1}^{\infty} a_l$ so that

$$2\|a\|_{2^{-k+1},1} \geq \sum_{l=1}^{\infty} 2^{l2^{1-k}} J(2^{-l}, a_l; \vec{A}).$$

Now, putting $b_k := \sum_{l=1}^{2^k} a_l$ and $b_m := \sum_{l=2^{m-1}+1}^{2^m} a_l$, $m \geq k+1$, we have $a = \sum_{m=k}^{\infty} b_m$. Then, by Minkowski's inequality,

$$\begin{aligned}
2\|a\|_{2^{-k+1},1} &\geq \sum_{l=1}^{2^k} 2^{l2^{1-k}} J(2^{-l}, a_l; \vec{A}) + \sum_{m=k+1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} 2^{l2^{1-k}} J(2^{-l}, a_l; \vec{A}) \\
&\geq \sum_{l=1}^{2^k} J(2^{-2^k}, a_l; \vec{A}) + \sum_{m=k+1}^{\infty} \sum_{l=2^{m-1}+1}^{2^m} 2^{2^{m-k}} J(2^{-2^m}, a_l; \vec{A}) \\
&\geq J(2^{-2^k}, \sum_{l=1}^{2^k} a_l; \vec{A}) + \sum_{m=k+1}^{\infty} 2^{2^{m-k}} J(2^{-2^m}, \sum_{l=2^{m-1}+1}^{2^m} a_l; \vec{A}) \\
&\geq \frac{1}{2} \left(\sum_{m=k}^{\infty} 2^{2^{m-k}} J(2^{-2^m}, b_m; \vec{A}) \right) \\
&\geq \frac{1}{2} \left(\sup_{m \geq k} 2^{2^{m-k}} J(2^{-2^m}, b_m; \vec{A}) \right) \geq \frac{1}{2} \|a\|_k,
\end{aligned}$$

and (4.4) is proved.

Further, let a representation (2.1) of $a \in A_0 + A_1$ be "almost optimal" for the norm $\|a\|_{\vec{A}^{1,*}_{H,ext}}$, i.e.,

$$2\|a\|_{\vec{A}^{1,*}_{H,ext}} \geq \left\| \{\|a_k\|_{2^{-k},1}\}_{k=1}^{\infty} \right\|_H .$$

Setting $a_0 = 0$ and applying (4.4), we have

$$\begin{aligned}
\|a\|_{\vec{A}^{J,**}_H} &\leq \|\{\|a_{k-1}\|_k\}_{k=1}^{\infty}\|_H \leq 4 \left\| \{\|a_{k-1}\|_{2^{1-k},1}\}_{k=1}^{\infty} \right\|_H \\
&= 4 \left\| R\left(\{\|a_k\|_{2^{-k},1}\}_{k=1}^{\infty} \right) \right\|_H \leq 4\|R\|_{H \to H} \left\| \{\|a_k\|_{2^{-k},1}\}_{k=1}^{\infty} \right\|_H \\
&\leq 8\|R\|_{H \to H} \|a\|_{\vec{A}^{1,*}_{H,ext}} .
\end{aligned}$$

Combining this with Proposition 4.2, we infer

$$\vec{A}^{1,*}_{H,ext} \subset \vec{A}^{J,*}_H \quad \text{and} \quad \|a\|_{\vec{A}^{J,*}_H} \leq C\|a\|_{\vec{A}^{1,*}_{H,ext}} .$$

Summing up, we get $\vec{A}^{1,*}_{H,ext} = \vec{A}^{J,*}_H$ (with equivalence of norms), and so everything is done. □

The following result is an immediate consequence of Theorem 4.1 and Lemma 3.1.

Corollary 4.3. *If H is a Banach sequence lattice such that $H \in \mathbf{I}(\ell_1, \ell_1(2^k))$ then we have $\vec{A}^{J,*}_H = \vec{A}^{1,*}_{H,ext}$ for every Banach couple $\vec{A}$.*

5. Main results

We suppose first that $q(\theta) \equiv 1$ and establish a direct link between extrapolation spaces $\vec{A}^{1}_{G,ext}$ and their sparse versions.

Theorem 5.1. *Let G be a Banach sequence lattice, $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$, and let the operator D be bounded in G. Then, for every Banach couple $\vec{A}$ we have $\vec{A}^1_{G,ext} = \vec{A}^{1,*}_{H,ext}$ (with equivalence of norms), where $H := P(G)$.*

Proof. Denote by $\mathbb{D}$ the set $\{2^i,\, i \in \mathbb{N}\}$. From the estimate

$$\begin{aligned}
\|a\|_{\vec{A}^1_{G,ext}} &= \inf_{a=\sum a_k} \left\|\{\|a_k\|_{1/k,1}\}_{k=1}^\infty\right\|_G \\
&\le \inf_{a=\sum a_k,\, \mathrm{supp}(\{a_k\})\subset\mathbb{D}} \left\|\{\|a_k\|_{1/k,1}\}_{k=1}^\infty\right\|_G \\
&= \inf_{a=\sum a_{2^k}} \left\|P\left(\{\|a_{2^k}\|_{2^{-k},1}\}_{k=1}^\infty\right)\right\|_G \\
&= \inf_{a=\sum a_{2^k}} \left\|\{\|a_{2^k}\|_{2^{-k},1}\}_{k=1}^\infty\right\|_{P(G)} \\
&= \|a\|_{\vec{A}^{1,*}_{H,ext}}
\end{aligned}$$

it follows immediately that $\vec{A}^{1,*}_{H,ext} \subset \vec{A}^1_{G,ext}$. Therefore, it is left to prove only the opposite embedding.

Observe that the norm $\|a\|_{\vec{A}^1_{G,ext}}$ is equivalent to the infimum taken over all representations (2.1) of a with the additional property $a_1 = 0$. Indeed, since $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$, then from Lemma 3.1 it follows that the operator R is bounded in G. Hence, if $a_1 = 0$, by inequality (2.6), we obtain

$$\begin{aligned}
\left\|\{\|a_k\|_{1/k,1}\}_{k=1}^\infty\right\|_G &\le \left\|R\left(\{\|a_{k+1}\|_{1/k,1}\}_{k=1}^\infty\right)\right\|_G \\
&\le \|R\|_{G\to G}\left\|\{\|a_{k+1}\|_{1/k,1}\}_{k=1}^\infty\right\|_G,
\end{aligned}$$

whence

$$\inf_{a=\sum a_k,\, a_1=0} \left\|\{\|a_k\|_{1/k,1}\}_{k=1}^\infty\right\|_G \le \|R\|_{G\to G}\|a\|_{\vec{A}^1_{G,ext}}.$$

Thus, we can find a representation of the form (2.1) with $a_1 = 0$ such that

$$\left\|\{\|a_k\|_{1/k,1}\}_{k=1}^\infty\right\|_G \le 2\|R\|_{G\to G}\|a\|_{\vec{A}^1_{G,ext}}. \tag{5.1}$$

Clearly, $a = \sum_{k=1}^\infty b_k$, where

$$b_k = \sum_{l=2^{k-1}+1}^{2^k} a_l, \quad k = 1, 2, \dots.$$

Moreover, by (2.6),

$$\begin{aligned}
\|a\|_{\vec{A}^{1,*}_{H,ext}} &\le \left\|\{\|b_k\|_{2^{-k},1}\}_{k=1}^\infty\right\|_H = \left\|P(\{\|b_k\|_{2^{-k},1}\}_{k=1}^\infty)\right\|_G \\
&\le \left\|P\left(\left\{\sum_{l=2^{k-1}+1}^{2^k} \|a_l\|_{2^{-k},1}\right\}_{k=1}^\infty\right)\right\|_G \\
&\le \left\|P\left(\left\{\sum_{l=2^{k-1}+1}^{2^k} \|a_l\|_{1/l,1}\right\}_{k=1}^\infty\right)\right\|_G.
\end{aligned}$$

On the other hand, taking into account the definition of the operators D and S and the fact that $a_1 = 0$, we have

$$\begin{aligned} S(D(\{\|a_l\|_{1/l,1}\}_{l=1}^{\infty})) &= S\Big(\sum_{l=1}^{\infty} \|a_l\|_{1/l,1}(e_{2l-1}+e_{2l})\Big) \\ &= 2\sum_{k=1}^{\infty}\Big(\sum_{l=2^{k-1}+1}^{2^k} \|a_l\|_{1/l,1}\Big)e_{2^k} \\ &= 2P\bigg(\bigg\{\sum_{l=2^{k-1}+1}^{2^k} \|a_l\|_{1/l,1}\bigg\}_{k=1}^{\infty}\bigg) \end{aligned}$$

Combining this with the preceding estimate, Lemma 3.1, the hypothesis of the theorem and inequality (5.1), we obtain

$$\begin{aligned} \|a\|_{\vec{A}^{1,*}_{H,ext}} &\le \frac{1}{2}\|S\|_{G\to G}\cdot \big\|D\big(\{\|a_k\|_{1/k,1}\}_{k=1}^{\infty}\big)\big\|_G \\ &\le \frac{1}{2}\|S\|_{G\to G}\|D\|_{G\to G}\cdot \big\|\{\|a_k\|_{1/k,1}\}_{k=1}^{\infty}\big\|_G \\ &\le \|S\|_{G\to G}\|D\|_{G\to G}\|R\|_{G\to G}\|a\|_{\vec{A}^1_{G,ext}}. \end{aligned}$$

As a result, $\vec{A}^1_{G,ext} \subset \vec{A}^{1,*}_{H,ext}$, which completes the proof. □

Theorem 5.2. *Suppose that $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$ and the operator D is bounded in G. Then, for every Banach couple $\vec{A}$ the interpolation space $\vec{A}^J_G$ can be identified as the extrapolation space $\vec{A}^1_{G,ext}$. In particular,*

$$\|a\|_{\vec{A}^J_G} \asymp \inf \big\|\{\|a_k\|_{1/k,1}\}_{k=1}^{\infty}\big\|_G,$$

where the infimum is taken over all representations (2.1) *of $a \in A_0 + A_1$.*

Proof. Arguing in the same way as in the beginning of the proof of Theorem 4.1 by using inequality (2.4), we get the embedding $\vec{A}^J_G \subset \vec{A}^1_{G,ext}$ (with some constant ≤ 2).

Next, according to Lemma 3.3, the operator R is bounded in the lattice $H := P(G)$. Therefore, the spaces H and G satisfy the conditions of Theorems 4.1 and 5.1, respectively. Thus,

$$\vec{A}^1_{G,ext} = \vec{A}^{1,*}_{H,ext} = \vec{A}^{J,*}_H \text{ (with equivalence of norms).} \tag{5.2}$$

At the same time, precisely as in the proof of Theorem 5.1, for every $a \in \vec{A}^J_G$ we have

$$\begin{aligned} \|a\|_{\vec{A}^J_G} &= \inf_{a=\sum a_k} \left\| \{J(2^{-k}, a_k; \vec{A})\}_{k=1}^\infty \right\|_G \\ &\le \inf_{a=\sum a_k,\, \mathrm{supp}(\{a_k\})\subset \mathbb{D}} \left\| \{J(2^{-k}, a_k; \vec{A})\}_{k=1}^\infty \right\|_G \\ &= \inf_{a=\sum a_{2^k}} \left\| P\left(\{J(2^{-2^k}, a_{2^k}; \vec{A})\}_{k=1}^\infty \right) \right\|_G \\ &= \inf_{a=\sum a_{2^k}} \left\| \{\{J(2^{-2^k}, a_{2^k}; \vec{A})\}_{k=1}^\infty \right\|_{P(G)} \\ &= \|a\|_{\vec{A}^{J,*}_H}, \end{aligned}$$

whence $\vec{A}^{J,*}_H \subset \vec{A}^J_G$ (with some constant ≤ 1). Combining this embedding with (5.2), we conclude that $\vec{A}^1_{G,ext} \subset \vec{A}^J_G$, and the proof is completed. □

Show that the result of Theorem 5.2 still holds if we replace the norms $\|\cdot\|_{\theta,1}$ with the norms $\|\cdot\|_{\theta,q(\theta)}$ provided that a function $q(\theta) : (0,1) \to [1,\infty)$ tends to 1 in appropriate way as $\theta \to 0$.

Theorem 5.3. *Let G be a Banach sequence lattice, $G \in \mathbf{I}(\ell_1, \ell_1(2^k))$, and let the operator D be bounded in G. Then, for every Banach couple $\vec{A}$ and each continuous function $q(\theta) : (0,1) \to [1,\infty)$ such that $q(\theta) \le \frac{1}{1-\theta}$ it holds*

$$\vec{A}^J_G = \vec{A}^{q(\theta)}_{G,ext} \ \ (\textit{with equivalence of norms}). \tag{5.3}$$

Firstly, we prove that replacing $\{1/k\}$ with the sequence $\{1/(2k)\}$ in the definition of the norm $\|a\|_{\vec{A}^1_{G,ext}}$ gives an equivalent norm.

Lemma 5.4. *If the operator D is bounded in a Banach lattice G, then for every Banach couple $\vec{A}$ we have*

$$\|a\|_{\vec{A}^1_{G,ext}} \asymp \inf \left\| \{\|a_k\|_{1/(2k),1}\}_{k=1}^\infty \right\|_G,$$

where the infimum is taken over all representations (2.1).

Proof. From inequality (2.6) it follows immediately

$$\inf \left\| \{\|a_k\|_{1/(2k),1}\}_{k=1}^\infty \right\|_G \le \|a\|_{\vec{A}^1_{G,ext}}.$$

Conversely, denote by $\mathbb{E}$ the set of all even positive integers. Then, we have

$$\begin{aligned} \|a\|_{\vec{A}^1_{G,ext}} &= \inf_{a=\sum a_k} \left\| \{\|a_k\|_{1/k,1}\}_{k=1}^\infty \right\|_G \\ &\le \inf_{a=\sum a_k,\, \mathrm{supp}(\{a_k\})\subset \mathbb{E}} \left\| \{\|a_k\|_{1/k,1}\}_{k=1}^\infty \right\|_G \\ &\le \inf_{a=\sum a_{2k}} \left\| D\left(\{\|a_{2k}\|_{1/(2k),1}\}_{k=1}^\infty \right) \right\|_G \\ &\le \|D\|_{G\to G} \inf_{a=\sum a_{2k}} \left\| \{\|a_{2k}\|_{1/(2k),1}\}_{k=1}^\infty \right\|_G. \end{aligned}$$

□

Proof of Theorem 5.3. By Hölder's inequality, for every $0 < \theta \le 1/2$ and $1 \le q < \infty$ we have

$$\begin{aligned}\sum_{k=1}^{\infty} 2^{k\theta} J(2^{-k}, a_k; \vec{A}) &= \sum_{k=1}^{\infty} (2^{2\theta k} J(2^{-k}, a_k; \vec{A})) \cdot 2^{-\theta k} \\ &\le \left(\sum_{k=1}^{\infty} (2^{2\theta k} J(2^{-k}, a_k; \vec{A}))^q\right)^{1/q} \cdot \left(\sum_{k=1}^{\infty} 2^{-\theta k q'}\right)^{1/q'} \\ &= (2^{\theta q'} - 1)^{-1/q'} \left(\sum_{k=1}^{\infty} (2^{2\theta k} J(2^{-k}, a_k; \vec{A}))^q\right)^{1/q} \\ &\le 4(2\theta)^{-1/q'} \left(\sum_{k=1}^{\infty} (2^{2\theta k} J(2^{-k}, a_k; \vec{A}))^q\right)^{1/q},\end{aligned}$$

whence

$$\|a\|_{\theta,1} \le 4\|a\|_{2\theta,q}. \tag{5.4}$$

Therefore, applying Lemma 5.4, we get

$$\begin{aligned}\|a\|_{\vec{A}^q_{G,ext}} &= \inf_{a=\sum a_k} \left\|\{\|a_k\|_{1/k,q}\}_{k=1}^{\infty}\right\|_G \\ &\ge \frac{1}{4} \inf_{a=\sum a_k} \left\|\{\|a_k\|_{1/(2k),1}\}_{k=1}^{\infty}\right\|_G \ge c\|a\|_{\vec{A}^1_{G,ext}}.\end{aligned}$$

Thus, $\vec{A}^q_{G,ext} \subset \vec{A}^1_{G,ext}$ for each $1 \le q < \infty$. Moreover, the same reasoning shows that $\vec{A}^{q(\theta)}_{G,ext} \subset \vec{A}^1_{G,ext}$, where $q = q(\theta) : (0,1) \to [1,\infty)$ is an arbitrary continuous function.

On the other hand, from inequality (2.5) with $q = q(\theta) \le 1/(1-\theta)$ it follows that

$$\|a\|_{\theta,q(\theta)} \le e^{1/e}\|a\|_{\theta,1}.$$

Hence, the converse embedding $\vec{A}^1_{G,ext} \subset \vec{A}^{q(\theta)}_{G,ext}$ holds as well. Finally, applying Theorem 5.2, we complete the proof. $\square$

Remark 5.5. In view of Proposition 3.2, the space $\vec{A}^1_{G,ext}$ in Theorem 5.2 can be replaced by the space $\vec{A}^{1,m,n}_{G,ext}$ for every $m, n \in \mathbb{N}$. Similarly, under the conditions of Theorem 5.3, we have

$$\|a\|_{\vec{A}^J_G} \asymp \inf \left\|\{\|a_k\|_{1/(k+m),q(1/(k+m))}\}_{k=1}^{\infty}\right\|_G,$$

where the infimum is taken over all representations (2.1) of $a \in A_0 + A_1$ with $\operatorname{supp}(\{a_k\}) \subset [n,\infty)$. This observation shows that the above results lead to extrapolation theorems applicable to operators $T : \vec{A} \to \vec{B}$ with the prescribed behaviour of norms $\|T\|_{\vec{A}_{\theta,q(\theta)} \to \vec{B}_{\theta,q(\theta)}}$ only for sufficiently small values of θ (cf. the discussion related to equalities (1.2) and Yano's theorem in the Introduction).

We proceed now with considering the following partial case. Let $L_p = L_p[0,1]$, $1 \le p \le \infty$ and $\vec{A} = (L_1, L_\infty)$. Then, we have

$$\|x\|_{1/k,k/(k-1)} \asymp \|x\|_{L_{k/(k-1)}}, \tag{5.5}$$

with constants independent of $k \ge 2$ (as above, the norm $\|x\|_{\theta,p}$ is defined by (2.3)). Equivalence (5.5) can be obtained by an inspection of the proof of Theorem 3.7.1 in [5] combined with using duality and the fact that

$$\|x\|_{(L_1,L_\infty)^K_{1/q',q}} \asymp \|x\|_{L_q}$$

with constants independent of $q \ge 2$ (cf. [14, Example 7]), where $q' = q/(q-1)$. An alternative way to prove (5.5) is to exploit a connection between the real method of interpolation and the so-called methods of constants and means (see [12, Theorem IV.2.14 and Section IV.5]).

Applying now Theorem 5.3 to the couple $\vec{A} = (L_1, L_\infty)$ and the parameter $G = \ell_1$ (resp. $G = \ell_1(k^\beta)$, $\beta > 0$) together with equivalence (5.5) and taking into account Remark 5.5, for arbitrary $k_0 \ge 1$ we get

$$(L_1, L_\infty)^J_{\ell_1} = \Sigma_{k\ge k_0}(L_{1+1/k}) \quad (\text{resp. } (L_1, L_\infty)^J_{\ell_1(k^\beta)} = \Sigma_{k\ge k_0}(k^\beta L_{1+1/k})). \tag{5.6}$$

Moreover, one can easily check that

$$(L_1, L_\infty)^J_{\ell_1} = L_1 \tag{5.7}$$

Indeed, if $a = \sum_{k=1} a_k$, then

$$\left\| \{J(2^{-k}, a_k; \vec{A})\}_{k=1}^\infty \right\|_{\ell_1} \ge \sum_{k=1}^\infty \|a_k\|_{L_1} \ge \|a\|_{L_1},$$

and therefore

$$\|a\|_{(L_1,L_\infty)^J_{\ell_1}} \ge \|a\|_{L_1}.$$

On the other hand, let $E_l := \{t \in [0,1] : l-1 \le |a(t)| < l\}$, $l = 1, 2, \dots$ Then, setting

$$a_k := \begin{cases} a \cdot \chi_{E_l}, & \text{if } k = n_l, \\ 0, & \text{otherwise}, \end{cases}$$

where positive integers $n_1 < n_2 < \dots$ are chosen in such a way that

$$\|a \cdot \chi_{E_l}\|_{L_\infty} \le 2^{n_l} \|a \cdot \chi_{E_l}\|_{L_1},$$

we have $a = \sum_{k=1}^\infty a_k$ and

$$\|a\|_{(L_1,L_\infty)^J_{\ell_1}} \le \left\| \{J(2^{-k}, a_k; \vec{A})\}_{k=1}^\infty \right\|_{\ell_1} = \sum_{k=1}^\infty \|a_k\|_{L_1} = \|a\|_{L_1}.$$

The identification of the spaces $(L_1, L_\infty)^J_{\ell_1(k^\beta)}$, $\beta > 0$, is somewhat more complicated and this will be done in the next section.

6. Extrapolation description of real interpolation spaces generated by $\ell_1(w)$-parameters

Let $w(t)$ be a quasi-concave function defined on the interval $[1,\infty)$ (i.e., $w(t)>0$, $w(t)$ increases and the function $w(t)/t$ decreases) such that for some $M>0$ we have

$$w(t^2)\le Mw(t),\quad t\ge 1. \tag{6.1}$$

Then, the weighted space $\ell_1(w)$, consisting of all sequences $x=(x_k)_{k=1}^\infty$, for which the norm

$$\|x\|_{\ell_1(w)}:=\sum_{k=1}^\infty |x_k|w(2^k)$$

is finite, is an interpolation space with respect to the couple $(\ell_1,\ell_1(2^k))$ (see e.g. [5, Theorem 5.4.4]). Moreover, condition (6.1) assures that the operator D is bounded in $\ell_1(w)$ and $\|D\|_{\ell_1(w)\to\ell_1(w)}\le 2M$. Therefore, by Lemma 3.3, the operator R is bounded in $P(\ell_1(w))$ and

$$\|R\|_{P(\ell_1(w))\to P(\ell_1(w))}\le 2M.$$

In particular, condition (6.1) is fulfilled for the weights $w_\alpha(t)=\log^\alpha(ct)$, $\alpha\ge 0$, $c=e^\alpha$. We shall denote the spaces $\ell_1(w_\alpha)$ by $\ell_1(\alpha)$. Clearly,

$$\|x\|_{\ell_1(\alpha)}\asymp\sum_{k=1}^\infty |x_k|k^\alpha.$$

Applying Theorem 5.3, we obtain the following result.

Corollary 6.1. *Let $w(t)$ be a quasi-concave function on the interval $[1,\infty)$ satisfying condition (6.1). Then, for every Banach couple $\vec{A}$ and each continuous function $q(\theta):\ (0,1)\to[1,\infty)$ such that $q(\theta)\le\frac{1}{1-\theta}$ it holds*

$$\vec{A}^J_{\ell_1(w)}=\vec{A}^{q(\theta)}_{\ell_1(w),ext}\ \ (\textit{with equivalence of norms}).$$

In particular, for every $\alpha\ge 0$ we have $\vec{A}^J_{\ell_1(\alpha)}=\vec{A}^{q(\theta)}_{\ell_1(\alpha),ext}$.

Let us show that in the case of ordered Banach couples a similar result holds also for the corresponding interpolation K-spaces. First, condition (6.1) guarantees that $\ell_1(w)$ is an intermediate space with respect to the couple $(\ell_\infty,\ell_\infty(2^k))$, i.e.,

$$\ell_\infty(2^k)\subset\ell_1(w)\subset\ell_\infty. \tag{6.2}$$

Indeed, it is clear that the left-hand side embedding will be proved once we show that the sequence $\{2^{-k}\}$ belongs to $\ell_1(w)$. The latter follows from (6.1) because

$$\begin{aligned}\|\{2^{-k}\}\|_{\ell_1(w)} &= \sum_{k=1}^\infty 2^{-k}w(2^k)=\sum_{i=0}^\infty\sum_{j=2^i}^{2^{i+1}-1}2^{-j}w(2^j)\\ &\le \sum_{i=0}^\infty 2^i2^{-2^i}w(2^{2^{i+1}})\le w(2)\sum_{i=0}^\infty 2^iM^{i+1}2^{-2^i}<\infty.\end{aligned}$$

Since the right-hand side embedding in (6.2) is obvious, we get the desired result and therefore interpolation K-spaces $\vec{A}^K_{\ell_1(w)}$ are well-defined. The next lemma, which shows that these spaces coincide with suitable interpolation J-spaces, is in fact a version of known results (cf. [7, Theorem 7.6] and [3, Lemma 2]). We include its proof below for the completeness.

For every weighted sequence $v = (v_k)_{k=1}^{\infty}$ and arbitrary Banach couple $\vec{A}$ by $\vec{A}_v^J$ (resp. $\vec{A}_v^K$) we denote the space $\vec{A}_{\ell_1(v_k)}^J$ (resp. $\vec{A}_{\ell_1(v_k)}^K$), where $\ell_1(v_k)$ is the Banach sequence space with the norm

$$\|(x_k)\|_{\ell_1(v_k)} := \sum_{k=1}^{\infty} |x_k| v_k.$$

Lemma 6.2. *Let $v = (v_k)_{k=1}^{\infty}$ be a weighted sequence such that*

$$\tilde{v}_k := \sum_{i=1}^{\infty} \min(1, 2^{k-i}) v_i < \infty$$

for each $k = 1, 2, \dots$ Then for every Gagliardo complete couple $\vec{A} = (A_0, A_1)$ such that $A_1 \subset A_0$ and A_1 is dense in A_0 it holds $\vec{A}_v^K = \vec{A}_{\tilde{v}}^J$.

Proof. Let $a \in \vec{A}_{\tilde{v}}^J$. Represent a in the form (2.1) so that

$$2\|a\|_{\vec{A}_{\tilde{v}}^J} \geq \sum_{k=1}^{\infty} J(2^{-k}, a_k; \vec{A}) \tilde{v}_k.$$

From Minkowski's inequality and the fact that

$$K(2^{-i}, a_k; \vec{A}) \leq \min(1, 2^{k-i}) J(2^{-k}, a_k; \vec{A}) \text{ for all } k, i = 1, 2, \dots$$

[5, Lemma 3.2.1], it follows that for each $i = 1, 2, \dots$

$$K(2^{-i}, a; \vec{A}) \leq \sum_{k=1}^{\infty} K(2^{-i}, a_k; \vec{A}) \leq \sum_{k=1}^{\infty} \min(1, 2^{k-i}) J(2^{-k}, a_k; \vec{A}).$$

Hence, using the definition of $\tilde{v}$, we get

$$\begin{aligned}
\|a\|_{\vec{A}_v^K} &= \sum_{i=1}^{\infty} K(2^{-i}, a; \vec{A}) v_i \\
&\leq \sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \min(1, 2^{k-i}) J(2^{-k}, a_k; \vec{A}) v_i \\
&= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} \min(1, 2^{k-i}) v_i J(2^{-k}, a_k; \vec{A}) \\
&= \sum_{k=1}^{\infty} J(2^{-k}, a_k; \vec{A}) \tilde{v}_k \\
&\leq 2\|a\|_{\vec{A}_{\tilde{v}}^J}.
\end{aligned}$$

Conversely, let $a \in \vec{A}_v^K$. Since A_1 is dense in A_0, we have $\lim_{t \to 0} K(t, a; \vec{A}) = 0$. Therefore, by the strong form of the fundamental lemma (cf. [8, Theorem 1.4]), we can find a representation $a = \sum_{k=1}^{\infty} a_k$, $a_k \in A_1$, such that

$$\sum_{k=1}^{\infty} \min(1, 2^{k-i}) J(2^{-k}, a_k; \vec{A}) \leq \gamma K(2^{-i}, a; \vec{A}), \quad i = 1, 2, \ldots$$

with a universal constant γ (observe that in the case of ordered couples the proof of this result given in [8] can be easily modified to get the above one-sided representation instead of a representation in the form $a = \sum_{k=-\infty}^{\infty} a_k$). Multiplying the last inequality by v_i and then summing over all $i = 1, 2, \ldots,$ we have

$$\sum_{i=1}^{\infty} \sum_{k=1}^{\infty} \min(1, 2^{k-i}) J(2^{-k}, a_k; \vec{A}) v_i \leq \gamma \sum_{i=1}^{\infty} K(2^{-i}, a; \vec{A}) v_i = \gamma \|a\|_{\vec{A}_v^K},$$

or

$$\sum_{k=1}^{\infty} J(2^{-k}, a_k; \vec{A}) \tilde{v}_k \leq \gamma \|a\|_{\vec{A}_v^K}.$$

Thus,

$$\|a\|_{\vec{A}_{\tilde{v}}^J} \leq \gamma \|a\|_{\vec{A}_v^K},$$

as we wished to show. □

Corollary 6.3. *Let $w(t)$ be a quasi-concave function on the interval $[1, \infty)$ satisfying condition (6.1). Then, for every Gagliardo complete couple $\vec{A}$ such that $A_1 \subset A_0$ and A_1 is dense in A_0, and each continuous function $q(\theta) : (0,1) \to [1, \infty)$ such that $q(\theta) \leq \frac{1}{1-\theta}$ we have*

$$\vec{A}^K_{\ell_1(w)} = \vec{A}^{q(\theta)}_{\ell_1(\tilde{w}),ext},$$

where

$$\tilde{w}(t) := \int_1^{\infty} \min(1, \frac{t}{s}) w(s) \, \frac{ds}{s}.$$

In particular, for every $\alpha \geq 0$

$$\vec{A}^K_{\ell_1(\alpha)} = \vec{A}^{q(\theta)}_{\ell_1(\alpha+1),ext}.$$

Proof. First of all, $\tilde{w}$ is clearly a quasi-concave function on $[1, \infty)$, and hence $\ell_1(\tilde{w}) \in \mathbf{I}(\ell_1, \ell_1(2^k))$. Let us check that the function $\tilde{w}$ satisfies condition (6.1).

Indeed, representing

$$\tilde{w}(t^2) = \int_1^{t^2} w(s) \, \frac{ds}{s} + \int_{t^2}^{\infty} \frac{t^2}{s} w(s) \, \frac{ds}{s},$$

we estimate the integrals from the right-hand side separately. Since (6.1) holds for w, we have

$$\int_1^{t^2} w(s) \, \frac{ds}{s} = 2 \int_1^{t} w(u^2) \, \frac{du}{u} \leq 2M \int_1^{t} w(u) \, \frac{du}{u} \leq 2M \tilde{w}(t)$$

and

$$\begin{aligned} \int_{t^2}^{\infty} \frac{t^2}{s} w(s) \, \frac{ds}{s} &= \int_t^{\infty} \frac{t}{u} w(tu) \, \frac{du}{u} \leq \int_t^{\infty} \frac{t}{u} w(u^2) \, \frac{du}{u} \\ &\leq M \int_t^{\infty} \frac{t}{u} w(u) \, \frac{du}{u} \leq M \tilde{w}(t). \end{aligned}$$

Thus, the function $\tilde{w}$ satisfies condition (6.1) with the constant $3M$.

Finally, observe that from the quasi-concavity of w it follows

$$\tilde{w}(2^k) = \sum_{i=1}^{\infty} \int_{2^{i-1}}^{2^i} \min(1, \frac{2^k}{s}) w(s) \, \frac{ds}{s} \asymp \sum_{i=1}^{\infty} \min(1, 2^{k-i}) w(2^i).$$

Therefore, applying Corollary 6.1 and Lemma 6.2, we get the first assertion. The second one follows now from the fact that in the case when $w(t) = \log^{\alpha}(ct)$ we have $\tilde{w}(t) \asymp \log^{\alpha+1}(ct)$. $\square$

Let us note that the spaces $\vec{A}^K_{\ell_1(\alpha)}$ can be defined also for negative values of α. Indeed, though the space $\ell_1(\alpha) := \ell_1(k^\alpha)$, $\alpha < 0$, is not intermediate with respect to the Banach couple $(\ell_\infty, \ell_\infty(2^k))$, the norms of nonincreasing nonnegative sequences in the spaces $\ell_1(\alpha)$ and $\ell_1(\alpha) \cap \ell_\infty$ coincide, and hence we have $\vec{A}^K_{\ell_1(\alpha)} = \vec{A}^K_{\ell_1(\alpha)\cap\ell_\infty}$. Then, an immediate inspection of the proof of Corollary 6.3 shows that the equation

$$\vec{A}^K_{\ell_1(\alpha)} = \vec{A}^J_{\ell_1(\alpha+1)}$$

can be extended to the case of $\alpha > -1$.

Therefore, if $\beta > 0$ and $\vec{A} = (L_1[0,1], L_\infty[0,1])$, then

$$\begin{aligned}
\|a\|_{\vec{A}^J_{\ell_1(\beta)}} &\asymp \sum_{k=1}^{\infty} K(2^{-k}, a; L_1, L_\infty) k^{\beta-1} = \sum_{k=1}^{\infty} \int_0^{2^{-k}} a^*(t) \, dt \; k^{\beta-1} \\
&\asymp \int_0^1 \int_0^s a^*(t) \, dt \, \log^{\beta-1}(e/s) \, d\log(e/s) \\
&= \int_0^1 a^*(t) \int_t^1 \log^{\beta-1}(e/s) \, d\log(e/s) \, dt \\
&\asymp \int_0^1 a^*(t) \log^{\beta}(e/t) \, dt = \|a\|_{L(\log L)^\beta}.
\end{aligned}$$

Thus, for every $\beta > 0$ we have

$$(L_1, L_\infty)^J_{\ell_1(\beta)} = L(\log L)^\beta.$$

Combining this together with (5.6) and (5.7), we obtain the following discrete version of (1.2): for arbitrary $k_0 \geq 1$

$$\Sigma_{k \geq k_0}(L_{1+1/k}) = L_1 \quad \text{and} \quad \Sigma_{k \geq k_0}(k^\beta L_{1+1/k}) = L(\log L)^\beta.$$

Acknowledgment

The first author was supported by the Ministry of Science and Higher Education of the Russian Federation, project 1.470.2016/1.4, and by the RFBR grant 18-01-00414. The second author was partially supported by the Ministry of Science and Higher Education of the Russian Federation within a state contract with the Federal Scientific Research Centre "Crystallography and Photonics" of Russian Academy of Sciences under agreement 007-GZ/Part3363/26 and by the RFBR grant 17-01-00138.

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Manuscript received February 22 2019
revised May 15 2019

S. V. Astashkin
Faculty of Mathematics, Samara National Research University, 443086 Samara, Russia
E-mail address: `astash56@mail.ru`

K. V. Lykov
Image Processing Systems Institute — Branch of the Federal Scientific Research Centre "Crystallography and Photonics" of Russian Academy of Sciences, Molodogvardejskaya st. 151, 443001 Samara, Russia and Faculty of Information Technology, Samara National Research University, 443086 Samara, Russia
E-mail address: `alkv@list.ru`