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SOME INTERPOLATION FORMULAE FOR LIMITING APPROXIMATION SPACES

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ABSTRACT. We establish some interpolation formulae for limiting approximation spaces involving iterated logarithmic weights or, more generally, slowly varying weights. An application to the Lorentz-Karamata operator ideals is given.

1. Introduction

Let X be a quasi-normed space, and let $(G_n)_{n\geq 0}$ be a sequence of subsets of X satisfying the following conditions:

1. $G_0 = \{0\}$ and $G_n \subset G_{n+1}$ for all $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$;

2. $\lambda G_n \subset G_n$ for any scalar λ and $n \in \mathbb{N}_0$;

3. $G_n + G_m \subset G_{m+n}$ for all $m, n \in \mathbb{N}_0$.

Let $f \in X$, and put

$$E_n(f) = \inf\{\|f - g\|_X : g \in G_{n-1}\}, \quad n \in \mathbb{N}.$$

Let $\alpha \geq 0$, $0 < q \leq \infty$, and let b be a slowly varying function on $[1, \infty)$ (see Section 2). The approximation space $X_q^{(\alpha,b)}$ consists of all $f \in X$ for which the quasi-norm

$$\|f\|_{X_q^{(\alpha,b)}} = \left(\sum_{n=1}^{\infty} (n^\alpha b(n) E_n(f))^q \, n^{-1} \right)^{1/q}$$

is finite (with the usual modification when $q = \infty$). When $\alpha > 0$ and $b \equiv 1$, we get back the classical approximation spaces X_q^α (see, for instance, [15,25,26]). If $\alpha = 0$ and $b(t) = (1 + \ln t)^\gamma$, $t \geq 1$, $\gamma \in \mathbb{R}$, then we obtain the limiting approximation spaces $X_q^{(0,\gamma)}$ considered in [8,11,12,18].

Let $0 < p, q \leq \infty$, $\gamma > -1/p$ and $0 < \theta < 1$. Set $\delta = \theta(\gamma + 1/p) - 1/q$. The following interpolation formula was proved in [18, Theorem 4]:

$$(X, X_p^{(0,\gamma)})_{\theta,q} = X_q^{(0,\delta)}. \tag{1.1}$$

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Here $(\cdot,\cdot)_{\theta,q}$ denotes the classical real interpolation method (see Section 2).

In this paper, we extend the interpolation formula (1.1). More precisely, we will characterize the spaces $(X, X_p^{(0,b)})_{\theta,q}$ for an arbitrary slowly varying function b, in the case when $0 < p, q < \infty$ or $p = q = \infty$. For remaining values of parameters p and q, we will identify the spaces $(X, X_p^{(0,b)})_{\theta,q}$ when b is taken, in particular, to be an iterated logarithmic function (see Section 2). We also cover the limiting cases $\theta = 0$ (if $0 < q \leq \infty$) and $\theta = 1$ (if $q = \infty$) which were left open in [18, Theorem 4]. As an application, we derive an interpolation theorem for the Lorentz-Karamata operator ideals. In particular, we provide a generalization of the second interpolation formula in [14, Theorem 5.4].

The organization of the paper is as follows. All the necessary background, along with some auxiliary results, is given in Section 2. We derive discrete Hardy-type inequalities involving general weights in Section 3. Section 4 contains the main results. Finally, Section 5 provides an application to the interpolation of the Lorentz-Karamata operator ideals.

2. Preliminaries

2.1. **Some notations.** For two non-negative quantities a_1 and a_2, we write $a_1 \lesssim a_2$ to denote the inequality $a_1 \leq ca_2$ for some positive constant c, which is appropriately independent in a_1 and a_2. We write $a_1 \approx a_2$ if $a_1 \lesssim a_2$ and $a_2 \lesssim a_1$.

For two quasi-normed spaces X and Y, we write $X \hookrightarrow Y$ if X is continuously embedded in Y.

Let $0 < q \leq \infty$, and $\bar{\gamma} = (\gamma_1, \gamma_2, ..., \gamma_n) \in \mathbb{R}^n$. Following [16], we say $\bar{\gamma} \in \mathbb{M}_{q,m,S}$ (or $\bar{\gamma} \in \mathbb{M}_{q,m,G}$) for some $2 \leq m \leq n$ if $\gamma_1 = ... = \gamma_{m-1} = -1/q$ and $\gamma_m < -1/q$ (or $\gamma_m > -1/q$). Here, as usual, we interpret $1/\infty$ as 0. By $\bar{\gamma} \in \mathbb{M}_{q,1,S}$ (or $\bar{\gamma} \in \mathbb{M}_{q,1,G}$), we will mean $\gamma_1 < -1/q$ (or $\gamma_1 > -1/q$). Moreover, we will write $\bar{\gamma} =< \gamma >_n$ if $\gamma_1 = ... = \gamma_n = \gamma$.

2.2. **Slowly varying functions.** Let $b : [1, \infty) \to (0, \infty)$ be a Lebesgue measurable function. Following [17], we say b is slowly varying on $[1, \infty)$ if, for every $\varepsilon > 0$, the function $t^{\varepsilon} b(t)$ is equivalent to a non-decreasing function and the function $t^{-\varepsilon} b(t)$ is equivalent to a non-increasing function. Moreover, we say that $b : (0, 1] \to (0, \infty)$ is slowly varying on $(0, 1]$ if $b(1/t)$ is slowly varying on $[1, \infty)$. We refer to [17] for details on slowly varying functions.

The following result will be useful in Section 4.

Lemma 2.1. *Let $\beta > 0$, and assume that b is slowly varying on $[1, \infty)$. Then*

$$\sum_{k=n}^{\infty} k^{-\beta-1} b(k) \approx n^{-\beta} b(n),\ n \in \mathbb{N}.$$

Proof. In view of [17, Proposition 3.4.33], the proof simply follows from

$$\sum_{k=n}^{\infty} k^{-\beta-1} b(k) \approx \int_n^{\infty} x^{-\beta} b(x) \frac{dx}{x}.$$

□

2.3. **Iterated logarithmic functions.** Define positive functions $\lambda_1, \lambda_2, ..., \lambda_n$ on $(0,\infty)$ by

$$\lambda_1(t) = 1 + |\ln t|, \quad \lambda_k(t) = \lambda_1(\lambda_{k-1}(t)), \quad k = 2, 3, ..., n,$$

and put $\lambda^{\bar{\gamma}}(t)=\lambda_1^{\gamma_1}(t)\lambda_2^{\gamma_2}(t)...\lambda_n^{\gamma_n}(t)$, $t > 0$. It is easy to verify that the iterated logarithmic function $\lambda^{\bar{\gamma}}$ is slowly varying on $(0,1]$ and $[1,\infty)$.

The next three lemmas will be needed in what follows.

Lemma 2.2 ([16, Lemma 2])**.** *Let* $0 < q < \infty$.

(a) *If* $\bar{\gamma} \in \mathbb{M}_{q,m,S}$, *then*

$$\left(\int_0^t \lambda^{q\bar{\gamma}}(u)\frac{du}{u}\right)^{1/q} \approx \lambda^{\bar{\gamma}+<\frac{1}{q}>_m}(t), \quad 0 < t < 1.$$

(b) *If* $\bar{\gamma} \in \mathbb{M}_{q,m,G}$, *then*

$$\left(\int_t^1 \lambda^{q\bar{\gamma}}(u)\frac{du}{u}\right)^{1/q} \approx \lambda^{\bar{\gamma}+<\frac{1}{q}>_m}(t), \quad 0 < t < 1/2.$$

(c) *If* $\bar{\gamma} \in \mathbb{M}_{\infty,m,G}$, *then*

$$\sup_{t\le u<1} \lambda^{\bar{\gamma}}(u) \approx \lambda^{\bar{\gamma}}(t), \quad 0 < t < 1.$$

The second assertion of the previous lemma can be reformulated as follows.

Lemma 2.3. *Let* $0 < q < \infty$. *If* $\bar{\gamma} \in \mathbb{M}_{q,m,G}$, *then*

$$\left(1 + \int_t^1 \lambda^{q\bar{\gamma}}(u)\frac{du}{u}\right)^{1/q} \approx \lambda^{\bar{\gamma}+<\frac{1}{q}>_m}(t), \quad 0 < t < 1.$$

Remark 2.4. Let $0 < q < \infty$. The proof of Lemma 2 in [16] shows that

$$\int_0^1 \lambda^{q\bar{\gamma}}(u)\frac{du}{u} = \infty,$$

if $\bar{\gamma} \in \mathbb{M}_{q,m,G}$. Moreover, we have

$$\sup_{0<u\le 1} \lambda^{\bar{\gamma}}(u) = \infty,$$

if $\bar{\gamma} \in \mathbb{M}_{\infty,m,G}$.

The next technical result will be needed while making a change of variables in Section 4. In what follows, we say a positive function v on $(0,1)$ satisfies the condition (H_0) if v is increasing and differentiable such that $\lim_{t\to 0^+} v(t) = 0$, and $\lim_{t\to 1^-} v(t) = 1$.

Lemma 2.5. *[16, Lemma 3] Suppose that* $\bar{\gamma} \in \mathbb{M}_{\infty,m,S}$. *Then there is a function* v *on* $(0,1)$ *that satisfies the condition* (H_0) *with*

$$v \approx \lambda^{\bar{\gamma}} \quad \textit{and} \quad \frac{dv}{v} \approx \lambda^{-<1>_m}(t)\frac{dt}{t}.$$

2.4. **Real interpolation spaces.** Let A_0 and A_1 be two quasi-normed spaces. We say that (A_0, A_1) is a compatible couple if A_0 and A_1 are continuously embedded in the same Hausdorff topological vector space. For each $f \in A_0 + A_1$ and $t > 0$, the K-functional is defined by

$$\begin{aligned} K(t,f) &= K(t,f;A_0,A_1) \\ &= \inf\{\|f_0\|_{A_0} + t\|f_1\|_{A_1} : \ f_0 \in A_0, \ f_1 \in A_1, \ f = f_0 + f_1\}. \end{aligned}$$

In what follows, we always assume that the couple (A_0, A_1) is ordered in the sense that $A_1 \hookrightarrow A_0$.

Let $0 < q \le \infty$, $0 \le \theta \le 1$, and let b be slowly varying on $(0,1]$. The real interpolation space $\bar{A}_{\theta,q;b} = (A_0, A_1)_{\theta,q;b}$ is formed of those $f \in A_0$ for which the quasi-norm

$$\|f\|_{\bar{A}_{\theta,q;b}} = \left(\int_0^1 s^{-\theta q} b^q(s) K^q(s,f) \frac{ds}{s}\right)^{1/q}$$

is finite (with the usual modification when $q = \infty$); see [2, 22]. If $b \equiv 1$ and $0 < \theta < 1$, then we recover the classical real interpolation spaces $\bar{A}_{\theta,q}$ (see [6,7,27]).

In the sequel, we will work with the limiting spaces $\bar{A}_{0,q;b}$. It is not hard to verify that the limiting spaces $\bar{A}_{0,q;b}$ are intermediate, without any condition on b and q, for the couple (A_0, A_1), that is,

$$A_1 \hookrightarrow \bar{A}_{0,q;b} \hookrightarrow A_0.$$

However, in order to exclude the trivial case $\bar{A}_{0,q;b} = A_0$, we always assume that

$$\int_0^1 b^q(t) \frac{dt}{t} = \infty, \text{ if } \ 0 < q < \infty, \tag{2.1}$$

or,

$$\sup_{0<t\le 1} b(t) = \infty, \text{ if } \ q = \infty. \tag{2.2}$$

This observation immediately follows from the elementary fact that, as a function of t, $K(t,f)$ is non-decreasing. Put $\bar{A}_{0,q} = \bar{A}_{0,q;b}$ if $b \equiv 1$. Note that (2.1) is met if $b \equiv 1$. Therefore, the limiting spaces $\bar{A}_{0,q}$ also make sense for $q < \infty$, and these limiting spaces were considered in [23]. The reader is referred to [9] where the limiting spaces $\bar{A}_{1,q;K} = (A_0, A_1)_{1,q;K}$ have been defined for the ordered couple (A_0, A_1) with $A_0 \hookrightarrow A_1$. We should mention that definitions of the limiting spaces $\bar{A}_{0,q}$ and $\bar{A}_{1,q;K}$ can be extended, without using any auxiliary function b, from ordered couples to arbitrary general couples (see [1, 10]).

We put $\bar{A}_{0,q;\bar{\gamma}} = \bar{A}_{0,q;b}$ if b is the iterated logarithmic function $\lambda^{\bar{\gamma}}$ on $(0,1]$. In view of Remark 2.4, observe that the condition (2.1) or (2.2) is met if $\bar{\gamma} \in \mathbb{M}_{q,m,G}$.

2.5. **Limiting approximation spaces.** In this subsection, we make two comments about the limiting approximation spaces $X_q^{(0,b)}$ (which are defined in the Introduction). First, since the sequence $(E_n(f))$ is non-increasing with $E_1(f) = \|f\|_X$, we always assume that

$$\sum_{n=1}^{\infty} b^q(n) n^{-1} = \infty, \text{ if } \ 0 < q < \infty,$$

or,

$$\sup_{n\geq 1} b(n) = \infty, \text{ if } q = \infty,$$

so that the trivial case $X_q^{(0,b)} = X$ is excluded. Secondly, we put $X_q^{(0,\bar{\gamma})} = X_q^{(0,b)}$ if b is the iterated logarithmic function $\lambda^{\bar{\gamma}}$ on $[1,\infty)$.

3. Weighted Hardy-type inequalities

We derive discrete Hardy-type inequalities involving general weights. To this end, we need the following well-known power rule for series (see [5, Lemma 3]).

Lemma 3.1. *Let $s \in (0,\infty)$, and assume that (a_n) is a sequence of positive real numbers. Then*

$$\left(\sum_{k=1}^{n} a_k\right)^s \approx \sum_{k=1}^{n} a_k \left(\sum_{j=1}^{k} a_j\right)^{s-1}, \ n \in \mathbb{N}.$$

The next result is a discrete analogue of [2, Lemma 3.2].

Lemma 3.2. *Let $1 \leq s < \infty$, and suppose that (w_n) and (u_n) are sequences of positive real numbers. Put*

$$v_n = w_n^{1-s}\left(u_n \sum_{k=n}^{\infty} w_k\right)^s.$$

Then

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k a_k\right)^s w_n \lesssim \sum_{n=1}^{\infty} a_n^s v_n \tag{3.1}$$

holds for all sequences (a_n) of positive real numbers.

Proof. An application of Lemma 3.1 and Fubini's theorem yields

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k a_k\right)^s w_n \approx \sum_{k=1}^{\infty} u_k a_k \left(\sum_{j=1}^{k} u_j a_j\right)^{s-1} \sum_{n=k}^{\infty} w_n. \tag{3.2}$$

Now (3.1) trivially follows from (3.2) when $s = 1$. Let $s > 1$. Applying Hölder's inequality in (3.2), we obtain

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k a_k\right)^s w_n \lesssim \left(\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n} u_k a_k\right)^s w_n\right)^{1-\frac{1}{s}} \left(\sum_{n=1}^{\infty} a_n^s v_n\right)^{\frac{1}{s}},$$

which yields the desired estimate (3.1). The proof is complete. □

The next assertion is a discrete analogue of [2, Lemma 3.3].

Lemma 3.3. *Let $0 < s < 1$, and assume that (w_n) and (u_n) are sequences of positive real numbers. Put*

$$\tilde{v}_n = u_n \left(\sum_{k=1}^{n} u_k\right)^{s-1} \sum_{k=n}^{\infty} w_k.$$

Then

$$\sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}u_ka_k\right)^s w_n \lesssim \sum_{n=1}^{\infty}a_n^s\tilde{v}_n \tag{3.3}$$

holds for all non-increasing sequences (a_n) of positive real numbers.

Proof. Note that (3.2) also holds for $0<s<1$. Now (3.3) immediately follows from (3.2) in view of the trivial estimate $\sum\limits_{j=1}^{k}u_ja_j \geq a_k\sum\limits_{j=1}^{k}u_j$. □

We will also make use of the following weighted Hardy-type inequality for a supremum operator restricted to non-decreasing functions.

Lemma 3.4. *[24, Theorem 4.2] Let $0<s<\infty$, and assume that w, ϕ and v are non-negative functions on $(0,\infty)$. Then*

$$\int_0^{\infty}\left(\operatorname*{ess\,sup}_{t\leq u<\infty}\phi(u)h(u)\right)^s w(t)dt \lesssim \int_0^{\infty}h^s(t)v(t)dt \tag{3.4}$$

holds for all non-negative, non-decreasing functions h on $(0,\infty)$ if

$$\int_x^{\infty}\left(\operatorname*{ess\,sup}_{t\leq u<\infty}\phi(u)\right)^s w(t)dt \lesssim \int_x^{\infty}v(t)dt \tag{3.5}$$

and

$$\sup_{x\leq t<\infty}\left(\operatorname*{ess\,sup}_{t\leq u<\infty}\phi(u)\right)^s\left(\int_t^{\infty}v(u)du\right)^{-1} \lesssim \left(\int_0^{x}w(t)dt\right)^{-1} \tag{3.6}$$

hold for all $x>0$.

4. Interpolation formulae

First, we characterize the spaces $X_q^{(0,b)}$ as limiting real interpolation spaces between X and X_1^1. Here we note that the couple (X,X_1^1) is ordered with $X_1^1\hookrightarrow X$.

Lemma 4.1. *Let $0<q\leq\infty$, and assume that b is slowly varying on $[1,\infty)$. Then*

$$X_q^{(0,b)} = (X,X_1^1)_{0,q;b_0}$$

where $b_0(t)=b(1/t),\ 0<t\leq 1$.

Proof. Put $Y=(X,X_1^1)_{0,q;b_0}$, and take $f\in X$. Assume that $0<q<\infty$. By the usual discretization technique, it turns out that

$$\|f\|_Y \approx \left(\sum_{n=1}^{\infty}n^{-q-1}b^q(n)K^q(n,f;X_1^1,X)\right)^{1/q}.$$

Since (see [11] or [8, Lemma 2.6])

$$K(n,f;X_1^1,X)\approx\sum_{j=1}^{n}E_j(f),\ n\in\mathbb{N}, \tag{4.1}$$

we have

$$\|f\|_Y \approx \left(\sum_{n=1}^{\infty} n^{-q-1} b^q(n) \left(\sum_{j=1}^{n} E_j(f)\right)^q\right)^{1/q},$$

whence we get the estimate $\|f\|_Y \geq \|f\|_{X_q^{(0,b)}}$ immediately as the sequence $(E_n(f))$ is non-increasing. In order to derive $\|f\|_Y \lesssim \|f\|_{X_q^{(0,b)}}$, we apply Lemmas 3.2 and 3.3 with $w_n = n^{-q-1}b^q(n)$, $u_n = 1$ and $a_n = E_n(f)$. We can compute, with the aid of Lemma 2.1, that $v_n \approx \tilde{v}_n \approx n^{-1}b^q(n)$. Thus, it follows that $\|f\|_Y \lesssim \|f\|_{X_q^{(0,b)}}$ holds. The proof is complete in the case $0 < q < \infty$. Next we assume that $q = \infty$. This time, using (4.1), we get

$$\|f\|_Y \approx \sup_{n\geq 1} b(n) n^{-1} \sum_{j=1}^{n} E_j(f).$$

Once again the estimate $\|f\|_Y \geq \|f\|_{X_q^{(0,b)}}$ follows trivially, and in order to prove $\|f\|_Y \lesssim \|f\|_{X_q^{(0,b)}}$ we observe that

$$\|f\|_Y \lesssim c\|f\|_{X_q^{(0,b)}},$$

where

$$c = \sup_{n\geq 1} b(n) n^{-1} \sum_{j=1}^{n} b^{-1}(j).$$

We have to show that c is finite. We write

$$c = \sup_{n\geq 1} b(n) n^{-1} \sum_{j=1}^{n} j^{1/2} b^{-1}(j) j^{-1/2},$$

since b slowly varying, we get

$$c \lesssim \sup_{n\geq 1} n^{-1/2} \sum_{j=1}^{n} j^{-1/2}. \tag{4.2}$$

Now

$$\begin{aligned} \sum_{j=1}^{n} j^{-1/2} &= \sum_{j=0}^{n-1} (1+j)^{-1/2} \\ &\leq \sum_{j=0}^{n-1} \int_j^{j+1} x^{-1/2} dx \\ &= \int_0^n x^{-1/2} dx \\ &\approx n^{1/2}, \end{aligned}$$

which, in view of (4.2), establishes that c is finite. The proof is complete. □

The next result characterizes the limiting reiteration spaces $(A_0, (A_0, A_1)_{0,p;b})_{\theta,q}$. For other limiting reiteration formulae involving iterated logarithmic functions or slowly varying functions, the reader is referred to [2–4, 16, 19–22].

Lemma 4.2. (a) *Let $0 \le \theta < 1$, $0 < p, q < \infty$, and let b be slowly varying on $(0,1]$. Then*

$$(A_0, (A_0, A_1)_{0,p;b})_{\theta,q} = (A_0, A_1)_{0,q;\tilde{b}},$$

where

$$\tilde{b}(t) = \left(1 + \int_t^1 b^p(u)\frac{du}{u}\right)^{\theta/p-1/q} b^{p/q}(t),\ 0 < t \le 1.$$

(b) *Let $0 \le \theta \le 1$, and let b be slowly varying on $(0,1]$. Then*

$$(A_0, (A_0, A_1)_{0,\infty;b})_{\theta,\infty} = (A_0, A_1)_{0,\infty;\hat{b}},$$

where

$$\hat{b}(t) = b(t)(\sup_{t \le u < 1} b(u))^{\theta-1},\ 0 < t \le 1.$$

(c) *Let $0 < p < \infty$, and let b be slowly varying on $(0,1]$. Then*

$$(A_0, (A_0, A_1)_{0,p;b})_{1,\infty} = (A_0, A_1)_{0,p;b}.$$

(d) *Let $0 \le \theta < 1$, $0 < p < \infty$, and let $\bar{\gamma} \in \mathbb{M}_{p,m,G}$. Then*

$$(A_0, (A_0, A_1)_{0,p;\bar{\gamma}})_{\theta,\infty} = (A_0, A_1)_{0,\infty;\bar{\delta}},$$

where $\bar{\delta} = \theta(\bar{\gamma} + <\frac{1}{p}>_m)$.

(e) *Let $0 \le \theta < 1$, $0 < q < \infty$, and let $\bar{\gamma} \in \mathbb{M}_{\infty,m,G}$. Then*

$$(A_0, (A_0, A_1)_{0,\infty;\bar{\gamma}})_{\theta,q} = (A_0, A_1)_{0,q;\bar{\eta}},$$

where $\bar{\eta} = \theta\bar{\gamma} - <\frac{1}{q}>_m$.

Proof. (a) The assertion follows from [4, Theorem 11].

(b) Put $Y_1 = (A_0, (A_0, A_1)_{0,\infty;b})_{\theta,\infty}$ and $Z_1 = (A_0, A_1)_{0,\infty;\hat{b}}$. Let $f \in A_0$. According to Holmstedt-type estimate (2.19) in [2], we have

$$K(\rho(t), f; A_0, (A_0, A_1)_{0,\infty;b}) \approx \rho(t) \sup_{t \le u < 1} b(u)K(u, f),\ \ 0 < t < 1, \tag{4.3}$$

where

$$\rho(t) = \frac{1}{\sup\limits_{t \le u < 1} b(u)},\ \ 0 < t < 1.$$

Note that, in view of (2.2), we have $\lim\limits_{t \to 0^+} \rho(t) = 0$. Since ρ is also increasing, we obtain

$$\begin{aligned}
\|f\|_{Y_1} &\approx \sup_{0 < t \le 1} \rho^{1-\theta}(t) \sup_{t \le u < 1} b(u)K(u, f) \\
&= \sup_{0 < u \le 1} b(u)K(u, f) \sup_{0 \le t \le u} \rho^{1-\theta}(t) \\
&= \sup_{0 < u \le 1} \rho^{1-\theta}(u)b(u)K(u, f) \\
&= \|f\|_{Z_1},
\end{aligned}$$

as desired.

(c) Put $Y_2 = (A_0, (A_0, A_1)_{0,p;b})_{1,\infty}$ and $Z_2 = (A_0, A_1)_{0,p;b}$. Set

$$w(t) = \begin{cases} b(t), & 0 < t < 1, \\ \\ t^{-1}, & t \geq 1. \end{cases}$$

Then, according to Holmstedt-type estimate (2.19) in [2], we have

$$K(\sigma(t), f; A_0, (A_0, A_1)_{0,p;b}) \approx \sigma(t) \left(\int_t^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p}, \ 0 < t < 1, \tag{4.4}$$

where

$$\sigma(t) = \left(1 + \int_t^1 b^p(s) \frac{ds}{s} \right)^{-1/p}, \ 0 < t < 1.$$

Note that, in view of (2.1), we have $\lim_{t \to 0^+} \sigma(t) = 0$. Moreover, σ is increasing with $\lim_{t \to 1^-} \sigma(t) = 1$. Therefore,

$$\begin{aligned} \|f\|_{Y_2} &\approx \sup_{0<t<1} \left(\int_t^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p} \\ &= \left(\int_0^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p}, \end{aligned}$$

since $K(t,f) \approx \|f\|_{A_0}, \ t > 1$, we get

$$\left(\int_1^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p} \approx \|f\|_{A_0},$$

therefore,

$$\begin{aligned} \|f\|_{Y_2} &\approx \left(\int_0^1 b^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p} \\ &= \|f\|_{Z_2}, \end{aligned}$$

which completes the proof.

(d) Put $Y_3 = (A_0, (A_0, A_1)_{0,p;\bar{\gamma}})_{\theta,\infty}$ and $Z_3 = (A_0, A_1)_{0,\infty;\bar{\delta}}$. Using (4.4) with $b = \lambda^{\bar{\gamma}}$, we get

$$\|f\|_{Y_3} \approx \sup_{0<t<1} \sigma^{1-\theta}(t) \left(\int_t^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p}.$$

By Lemma 2.3, we get $\sigma(t) \approx \lambda^{-\bar{\gamma} - <\frac{1}{p}>_m}(t), \ 0 < t < 1$. Therefore,

$$\|f\|_{Y_3} \approx \sup_{0<t<1} \lambda^{\bar{\delta} - \bar{\gamma} - <\frac{1}{p}>_m}(t) \left(\int_t^\infty w^p(s) K^p(s,f) \frac{ds}{s} \right)^{1/p}.$$

Now the estimate $\|f\|_{Y_3} \gtrsim \|f\|_{Z_3}$ is immediate. For the converse estimate, we note that

$$\|f\|_{Y_3} \lesssim c \|f\|_{Z_3},$$

where

$$c = \sup_{0<t<1} \lambda^{\bar{\delta} - \bar{\gamma} - <\frac{1}{p}>_m}(t) \left(\int_t^\infty w^p(s) \lambda^{-p\bar{\delta}}(s) \frac{ds}{s} \right)^{1/p}.$$

Thus, it is enough to show that c is finite. Note that

$$\int_t^\infty w^p(s)\lambda^{-p\bar{\delta}}(s)\frac{ds}{s} = \int_t^1 \lambda^{p(\bar{\gamma}-\bar{\delta})}(s)\frac{ds}{s} + \int_1^\infty s^{-p}\lambda^{-p\bar{\delta}}(s)\frac{ds}{s},$$

by [17, Proposition 3.4.33], we have

$$\int_t^\infty w^p(s)\lambda^{-p\bar{\delta}}(s)\frac{ds}{s} = 1 + \int_t^1 \lambda^{p(\bar{\gamma}-\bar{\delta})}(s)\frac{ds}{s},$$

since $\bar{\gamma} \in \mathbb{M}_{p,m,G}$ implies that $\bar{\gamma} - \bar{\delta} \in \mathbb{M}_{p,m,G}$, we obtain, by Lemma 2.3, that

$$\int_t^\infty w^p(s)\lambda^{-p\bar{\delta}}(s)\frac{ds}{s} \approx \lambda^{p(\bar{\gamma}-\bar{\delta}+<\frac{1}{p}>_m)}(t),$$

which shows that c is finite. The proof is complete.

(e) Put $Y_4 = (A_0, (A_0, A_1)_{0,\infty;\bar{\gamma}})_{\theta,q}$ and $Z_4 = (A_0, A_1)_{0,q;\bar{\eta}}$. We use (4.3) with $b = \lambda^{\bar{\gamma}}$. Note that $\bar{\gamma} \in \mathbb{M}_{\infty,m,G}$ implies that $-\bar{\gamma} \in \mathbb{M}_{\infty,m,S}$. Thus an application of Lemma 2.2 (c) gives us $\rho(t) \approx \lambda^{-\bar{\gamma}}(t),\ \ 0 < t < 1$. Therefore,

$$K(\lambda^{-\bar{\gamma}}(t), f; A_0, (A_0, A_1)_{0,\infty;\bar{\gamma}}) \approx \lambda^{-\bar{\gamma}}(t) \sup_{t<s<1} \lambda^{\bar{\gamma}}(s)K(s, f),\ 0 < t < 1,$$

Therefore, making use of Lemma 2.5, we get at

$$\|f\|_{Y_4} \approx \left(\int_0^1 \lambda^{q(\bar{\eta}-\bar{\gamma})}(t)\left(\sup_{t\le s<1} \lambda^{\bar{\gamma}}(s)K(s, f)\right)^q \frac{dt}{t}\right)^{1/q}.$$

Observe that the estimate $\|f\|_{Y_4} \gtrsim \|f\|_{Z_4}$ is trivial. In order to obtain the converse estimate $\|f\|_{Y_4} \lesssim \|f\|_{Z_4}$, we apply Lemma 3.4 with $s = q$, $h(t) = K(t, f)$, $\phi(t) = \lambda^{\bar{\gamma}}(t)\chi_{(0,1)}(t)$, $w(t) = \lambda^{q(\bar{\eta}-\bar{\gamma})}(t)t^{-1}\chi_{(0,1)}(t)$ and $v(t) = \lambda^{q\bar{\eta}}(t)t^{-1}\chi_{(0,1)}(t)$. From Lemma 2.5, we can infer that ϕ is decreasing on $(0, 1)$. Thus, we see that (3.5) holds trivially, while (3.6) follows in view of Lemmas 2.2 (a) and 2.2 (b). Therefore, $\|f\|_{Y_4} \lesssim \|f\|_{Z_4}$ follows from Lemma 3.4. The proof is complete. □

The next result provides an extension of the interpolation formula (1.1).

Theorem 4.3. (a) *Let $0 \le \theta < 1$, $0 < p, q < \infty$, and let b be slowly varying on $[1, \infty)$. Then*

$$(X, X_p^{(0,b)})_{\theta,q} = X_q^{(0,\tilde{b})},$$

where

$$\tilde{b}(t) = \left(1 + \int_1^t b^p(u)\frac{du}{u}\right)^{\theta/p-1/q} b^{p/q}(t),\ t \ge 1. \tag{4.5}$$

(b) *Let $0 \le \theta \le 1$, and let b be slowly varying on $[1, \infty)$. Then*

$$(X, X_\infty^{(0,b)})_{\theta,\infty} = X_\infty^{(0,\hat{b})},$$

where

$$\hat{b}(t) = b(t)(\sup_{1<s\le t} b(s))^{\theta-1},\ t \ge 1. \tag{4.6}$$

(c) *Let $0 < p < \infty$, and let b be slowly varying on $[1, \infty)$. Then*

$$(X, X_p^{(0,b)})_{1,\infty} = X_p^{(0,b)}.$$

(d) *Let* $0 \leq \theta < 1$, $0 < p < \infty$, *and let* $\bar{\gamma} \in \mathbb{M}_{p,m,G}$. *Then*

$$(X, X_p^{(0,\bar{\gamma})})_{\theta,\infty} = X_\infty^{(0,\bar{\delta})},$$

where $\bar{\delta} = \theta(\bar{\gamma} + < \frac{1}{p} >_m)$.

(e) *Let* $0 \leq \theta < 1$, $0 < q < \infty$, *and let* $\bar{\gamma} \in \mathbb{M}_{\infty,m,G}$. *Then*

$$(X, X_\infty^{(0,\bar{\gamma})})_{\theta,q} = X_q^{(0,\bar{\delta})},$$

where $\bar{\delta} = \theta\bar{\gamma} - < \frac{1}{q} >_m$.

Proof. Put $b_0(t) = b(1/t),\ 0 < t \leq 1$. By Lemma 4.1, we get

$$(X, X_p^{(0,b)})_{\theta,q} = (X, (X, X_1^1)_{0,p;b_0})_{\theta,q},$$

now applying Lemma 4.2 (a) yields

$$(X, X_p^{(0,b)})_{\theta,q} = (X, X_1^1)_{0,q;\tilde{b}_0}$$

where $\tilde{b}_0(t) = \tilde{b}(1/t),\ 0 < t \leq 1$. Now apply Lemma 4.1 again to obtain $(X, X_1^1)_{0,q;\tilde{b}_0} = X_q^{(0,\tilde{b})}$ which completes the proof of the assertion (a). We omit the proofs of the remaining assertions since they can be done similarly using Lemmas 4.1 and 4.2. □

Corollary 4.4. *Let* $0 \leq \theta < 1$, $0 < p, q \leq \infty$, *and let* $\bar{\gamma} \in \mathbb{M}_{p,m,G}$. *Then*

$$(X, X_p^{(0,\bar{\gamma})})_{\theta,q} = X_q^{(0,\bar{\delta})},$$

where $\bar{\delta} = \theta(\bar{\gamma} + < \frac{1}{p} >_m) - < \frac{1}{q} >_m$.

Proof. Take $b(t) = \lambda^{\bar{\gamma}}(t),\ t \geq 1$. We just need to compute that

$$\tilde{b}(t) \approx \lambda^{\theta(\bar{\gamma} + < \frac{1}{p} >_m) - < \frac{1}{q} >_m}(t),\ t \geq 1, \tag{4.7}$$

if $0 < p, q < \infty$, and

$$\hat{b}(t) \approx \lambda^{\theta\bar{\gamma}},\ t \geq 1, \tag{4.8}$$

if $p = q = \infty$, where $\tilde{b}$ and $\hat{b}$ are defined by (4.5) and (4.6), respectively. Now (4.7) follows Lemma 2.3, and (4.8) follows from Lemma 2.2 (c). The proof is complete. □

Remark 4.5. *Let* $0 < \theta < 1$ *and* $\gamma > -1/p$. *Applying Corollary 4.4 to* $m = 1$ *with* $\bar{\gamma} = (\gamma)$, *we get back the interpolation formula (1.1).*

5. Application

Let E and F be Banach spaces, and let $\mathcal{L}(E, F)$ be the space of bounded linear operators acting from E to F. For each $T \in \mathcal{L}(E, F)$, put

$$a_n(T) = \inf\{\|T - R\|_{\mathcal{L}(E,F)} : R \in \mathcal{L}(E, F) \text{ with rank } R < n\},\ n \in \mathbb{N}.$$

Let $0 < q \leq \infty$ and let b be slowly varying on $[1, \infty)$. The Lorentz-Karamata operator ideal $\mathcal{L}_{\infty,q,b} = \mathcal{L}_{\infty,q,b}(E, F)$ is formed by all those $T \in \mathcal{L}(E, F)$ for which the quasi-norm

$$\|T\|_{\mathcal{L}_{\infty,q,b}} = \left(\sum_{n=1}^{\infty} (b(n) a_n(T))^q\, n^{-1} \right)^{1/q}$$

is finite. For $b(t) = (1 + \ln t)^{\gamma}$, $t \geq 1$, $\gamma \in \mathbb{R}$, the operator ideals $\mathcal{L}_{\infty,q,b}$ coincide with the Lorentz-Zygmund operator ideals $\mathcal{L}_{\infty,q,\gamma}$ (see, for example, [13, 18]).

Put

$$G_n = \{R \in \mathcal{L}(E,F) : \operatorname{rank} R \leq n\},\ n \in \mathbb{N}_0.$$

It is plain to check that the sequence $(G_n)_{n\geq 0}$ satisfies the conditions (1)-(3) (in the Introduction), and we have $E_n(T) = a_n(T)$. Therefore, in this case, the approximation spaces $X_q^{(0,b)}$ coincide with the operator ideals $\mathcal{L}_{\infty,q,b}$. Hence, writing down Theorem 4.3 (a) in this particular case $X = \mathcal{L}(E,F)$, we get the following interpolation theorem for the operator ideals $\mathcal{L}_{\infty,q,b}$.

Corollary 5.1. *Let $0 \leq \theta < 1$, $0 < p, q < \infty$, and let b be slowly varying on $[1,\infty)$. Then*

$$(\mathcal{L}(E,F), \mathcal{L}_{\infty,p,b})_{\theta,q} = \mathcal{L}_{\infty,q,\tilde{b}},$$

where

$$\tilde{b}(t) = \left(1 + \int_1^t b^p(u)\frac{du}{u}\right)^{\theta/p-1/q} b^{p/q}(t),\ t \geq 1.$$

Remark 5.2. If we take $\theta = 0$, $p = q$ and $b \equiv 1$ in Corollary 5.1, then we recover the second interpolation formula in [14, Theorem 5.4].

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