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A NONINTERSECTION PROPERTY FOR SOLUTIONS OF CONTINUOUS TIME OPTIMAL CONTROL PROBLEMS

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Abstract. In this work we study the structure of overtaking optimal programs of the continuous time Robinson-Solow-Srinivasan model and show that they have a nonintersection property.

1. Introduction

The study of the existence and the structure of solutions of optimal control problems defined on infinite intervals and on sufficiently large intervals has recently been a rapidly growing area of research. See, for example, [2, 4–14, 18, 19, 25, 28, 30, 33–35, 38, 49, 51] and the references mentioned therein. These problems arise in engineering [1, 26, 44], in models of economic growth [10, 15, 20–24, 29, 32, 39, 41–48, 53], in the game theory [16, 37, 50], in optimal control with PDE [17, 36, 40, 52] in infinite discrete models of solid-state physics related to dislocations in one-dimensional crystals [3] and in the theory of thermodynamical equilibrium for materials [27, 31]. In this paper we study the infinite horizon problem related to a continuous-time optimal control system describing the Robinson-Solow-Srinivasan model and establish a nonintersection property for their optimal solutions.

It should be mentioned that optimal control problems arising in economic dynamics usually are studied under the assumption that all their good programs converge to a turnpike which is an interior point of the set of admissible pairs [48, 50]. In this paper we study a large class of control systems for which the turnpike is not necessarily an interior point of the set of admissible pairs. This makes the situation more difficult and less understood.

One of the main topics in the infinite horizon optimal control theory is to study the existence and properties of solutions of problems over an infinite horizon using different optimality criteria. In the present paper, studying infinite horizon problems, we deal with the notion of good programs introduced by D. Gale in [15] which is of great usage in optimal control and economic dynamics (see, for example, [10, 44, 48] and the references mentioned therein) and with the notion of overtaking optimal program [10, 15, 41, 44, 48, 52, 53].

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2. The Robinson-Solow-Srinivasan model

Let R^1 (R^1_+) be the set of real (non-negative) numbers and let R^n be the n-dimensional Euclidean space with non-negative orthant

$$R^n_+ = \{x = (x_1, \dots, x_n) \in R^n : \ x_i \geq 0, \ i = 1, \dots, n\}.$$

For every pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, define their inner product by

$$xy = \sum_{i=1}^{n} x_i y_i$$

and let $x >> y$, $x > y$, $x \geq y$ have their usual meaning. Namely, for a given pair of vectors $x = (x_1, \dots, x_n), \ y = (y_1, \dots, y_n) \in R^n$, we say that $x \geq y$, if $x_i \geq y_i$ for all $i = 1, \dots, n$, $x > y$ if $x \geq y$ and $x \neq y$, and $x >> y$ if $x_i > y_i$ for all $i = 1, \dots, n$.

Let $e(i)$, $i = 1, \dots, n$, be the ith unit vector in R^n, and e be an element of R^n_+ all of whose coordinates are unity. For every $x \in R^n$, denote by $\|x\|$ its Euclidean norm in R^n.

Denote by mes(E) the Lebesgue measure of a Lebesgue measurable set $E \subset R^1$.

Let $a = (a_1, \dots, a_n) >> 0$, $b = (b_1, \dots, b_n) >> 0$, $d \in (0, 1)$, $c_i = b_i/(1+da_i)$, $i = 1, \dots, n$. We suppose:

$$\text{There exists } \sigma \in \{1, \dots, n\} \text{ such that for all}$$

$$i \in \{1, \dots, n\} \setminus \{\sigma\}, \ c_\sigma > c_i. \tag{2.1}$$

We now give a formal description of our technological structure.

Set

$$\Omega = \{(x, z) \in R^n_+ \times R^n : \ z + dx \geq 0 \text{ and } a(z + dx) \leq 1\}.$$

For every point $(x, z) \in \Omega$ set

$$\Lambda(x, z) = \{y \in R^n_+ : \ y \leq x \text{ and } ey \leq 1 - a(z + dx)\}.$$

Let I be either $[0, \infty)$ or $[0, T]$ with a positive number T. A pair of functions $(x(\cdot), y(\cdot))$ is called a program if $x : I \to R^n$ is an absolutely continuous (a.c.) function on any finite subinterval of I, $y : I \to R^n$ is a Lebesgue measurable function and if

$$(x(t), x'(t)) \in \Omega \text{ for almost every } t \in I,$$
$$y(t) \in \Lambda(x(t), x'(t)) \text{ for almost every } t \in I.$$

In the sequel if $I = [0, T]$, then the program $(x(\cdot), y(\cdot))$ is denoted by $(x(t), y(t))_{t=0}^T$ and if $I = [0, \infty)$, then the program $(x(\cdot), y(\cdot))$ is denoted by $(x(t), y(t))_{t=0}^\infty$.

Let $w : [0, \infty) \to [0, \infty)$ be a continuous strictly increasing concave and differentiable function which represents the preferences of the planner.

For every point $(x, z) \in \Omega$ set

$$u(x, z) = \max\{w(by) : \ y \in \Lambda(x, z)\}.$$

A golden-rule stock is a vector $\widehat{x} \in R^n_+$ such that a point $(\widehat{x}, 0)$ is a solution to the problem:

maximize $u(x, z)$ subject to

(i) $z \geq 0$; (ii) $(x, z) \in \Omega$.

It was shown in [20] that there exists a unique golden-rule stock

$$\widehat{x} = (1/(1 + da_\sigma))e(\sigma). \tag{2.2}$$

It is easy see that $\widehat{x}$ is a solution of the problem

$$w(by) \to \max,\ y \in \Lambda(\widehat{x}, 0).$$

Put

$$\widehat{y} = \widehat{x}. \tag{2.3}$$

For all integers $i = 1, \dots, n$ put

$$\widehat{q_i} = a_i b_i (1 + da_i)^{-1},\ \widehat{p_i} = w'(b\widehat{x})\widehat{q_i}. \tag{2.4}$$

Set

$$\xi_\sigma = 1 - d - 1/a_\sigma. \tag{2.5}$$

It was shown in [20] that

$$w(b\widehat{x}) \geq w(by) + \widehat{p}z \tag{2.6}$$

for every $(x, z) \in \Omega$ and for every $y \in \Lambda(x, z)$.

The following three propositions were obtained in [45].

Proposition 2.1. *Let m_0 be a positive number. Then there exists a positive number m_1 such that for every positive number T and every program $(x(t), y(t))_{t=0}^T$ which satisfies $x(0) \leq m_0 e$ the inequality $x(t) \leq m_1 e$ is valid for every number $t \in [0, T]$.*

We use the following notion of good programs.

A program $(x(t), y(t))_{t=0}^\infty$ is called good if there exist $M \in R^1$ such that

$$\int_0^T (w(by(t)) - w(b\widehat{y}))dt \geq M \text{ for all } T \geq 0.$$

A program is called bad if

$$\lim_{T\to\infty} \int_0^T (w(by(t)) - w(b\widehat{y}))dt = -\infty.$$

Proposition 2.2. *Any program $(x(t), y(t))_{t=0}^\infty$ that is not good is bad.*

Proposition 2.3. *For every point $x_0 \in R^n_+$ there exists a good program $(x(t), y(t))_{t=0}^\infty$ satisfying $x(0) = x_0$.*

In the sequel we use a notion of an overtaking optimal program.

A program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^\infty$ is overtaking optimal if for every program $(x(t), y(t))_{t=0}^\infty$ satisfying $x(0) = \tilde{x}(0)$ the inequality

$$\limsup_{T\to\infty} [\int_0^T w(by(t))dt - \int_0^T w(b\tilde{y}(t))dt] \leq 0$$

holds.

The following two theorems were obtained in [45].

Theorem 2.4. *Assume that a program $(x(t), y(t))_{t=0}^\infty$ is good. Then*

(i) $\lim_{t\to\infty} x(t) = \widehat{x}$.

(ii) *Let $\epsilon \in (0,1)$ and $L > 1$. Then there exists a positive number T_0 such that for every number $T \ge T_0$,*

$$mes(\{t \in [T, T+L] : \ \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon.$$

Theorem 2.5. *For every $x_0 \in R^n_+$ there exists an overtaking optimal program $(x(t), y(t))_{t=0}^{\infty}$ such that $x(0) = x$.*

3. A turnpike result

Let $z \in R^n_+$ and $T > 0$ be given. Define

$$U(z,T) = \sup\Big\{ \int_0^T w(by(t))dt : \ (x(t),y(t))_{t=0}^T \text{ is a program such that } x(0) = z \Big\}. \tag{3.1}$$

It is not difficult to see that $U(z,T)$ is a finite number.

Let $x_0, x_1 \in R^n_+$ and let $0 \le T_1 < T_2$. Define

$$U(x_0,x_1,T_1,T_2) = \sup\Big\{ \int_{T_1}^{T_2} w(by(t))dt : \ (x(t),y(t))_{t=T_1}^{T_2} \text{ is a program such that } x(T_1) = x_0, \ x(T_2) \ge x_1 \Big\}. \tag{3.2}$$

Here we assume that supremum over empty set is $-\infty$. In view of Proposition 2.1 and (5.2), $U(x_0,x_1,T_1,T_2) < \infty$. It is not difficult to see that for every point $z \in R^n_+$ and every positive number T, $U(z,T) = U(z,0,0,T)$.

The following theorem, obtained in [46], describes the structure of approximate optimal solutions of optimal control problems on sufficiently large intervals.

Theorem 3.1. *Let M, ϵ, L be positive numbers and let $\Gamma \in (0,1)$. Then there exist $T_* > 0$ and a positive number γ such that for each $T > 2T_*$, each $z_0, z_1 \in R^n_+$ satisfying $z_0 \le Me$ and $az_1 \le \Gamma d^{-1}$ and each program $(x(t), y(t))_{t=0}^T$ which satisfies*

$$x(0) = z_0, \ x(T) \ge z_1, \quad \int_0^T w(by(t))dt \ge U(z_0, z_1, 0, T) - \gamma$$

there are numbers τ_1, τ_2 such that $\tau_1 \in [0, T_*]$, $\tau_2 \in [T - T_*, T]$,

$$\|x(t) - \widehat{x}\| \le \epsilon \text{ for all } t \in [\tau_1, \tau_2]$$

and that for each number S satisfying $\tau_1 \le S \le \tau_2 - L$,

$$mes(\{t \in [S, S+L] : \|y(t) - \widehat{x}\| > \epsilon\}) \le \epsilon.$$

Moreover, if $\|x(0) - \widehat{x}\| \le \gamma$, then $\tau_1 = 0$ and if $\|x(T) - \widehat{x}\| \le \gamma$, then $\tau_2 = T$.

In our study we also use the following two auxiliary results. The first of them is obvious while the second one was obtained in [46].

Lemma 3.2. *Assume that nonnegative numbers T_1, T_2 satisfy $T_1 < T_2$,*

$$(x(t), y(t))_{t=T_1}^{T_2}$$

is a program and that $u \in R^n_+$. Then $(x(t) + e^{-d(t-T_1)}u, y(t))_{t=T_1}^{T_2})$ is also a program.

Lemma 3.3. *Let ϵ be a positive number. Then there exists a positive number δ such that for every pair of points $z, z' \in R^n_+$ satisfying*

$$\|z - \widehat{x}\|, \ \|z' - \widehat{x}\| \leq \delta$$

and every $T \in [2^{-1}, 2]$ there exists a program $(x(t), y(t))_{t=0}^{T}$ such that

$$x(0) = z, \ x(T) \geq z',$$

$$\|x(t) - \widehat{x}\|, \ |y(t) - \widehat{x}\| \leq \epsilon, \ t \in [0, T], \ \|x'(t)\| \leq \epsilon, \ t \in [0, T].$$

4. The main result

Theorem 4.1. *Assume that $(x(t), y(t))_{t=0}^{\infty}$ is an overtaking optimal program, $T_0 > 0$ and that*

$$x(0) = x(T_0). \tag{4.1}$$

Then for all integers $t \geq 0$,

$$x(t) = \widehat{x}$$

and for almost every $t \geq 0$,

$$y(t) = \widehat{x}.$$

Proof. In view of (4.1), there exists a program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^{\infty}$ such that

$$\tilde{x}(t) = x(t), \ \tilde{y}(t) = y(t), \ t \in [0, T_0] \tag{4.2}$$

and that for all $t \geq 0$,

$$\tilde{x}(t + T_0) = \tilde{x}(t), \ \tilde{y}(t + T_0) = \tilde{y}(t). \tag{4.3}$$

Proposition 2.2 implies that there are two cases:

(1) the program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^{\infty}$ is good;

(2) the program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^{\infty}$ is bad.

Assume that case (2) holds. By (4.2) and (4.3),

$$\begin{aligned} -\infty &= \lim_{k \to \infty} \int_0^{T_0 k} (w(b\tilde{y}(t)) - w(b\widehat{x}))dt \\ &= \lim_{k \to \infty} k \int_0^{T_0} (w(b\tilde{y}(t)) - w(b\widehat{x}))dt \\ &= \lim_{k \to \infty} k \int_0^{T_0} (w(by(t)) - w(b\widehat{x}))dt, \end{aligned}$$

where k is a natural number. Therefore

$$\int_0^{T_0} w(by(t))dt < T_0 w(b\widehat{x}). \tag{4.4}$$

In view of (4.4), there exists a positive number Δ such that

$$\Delta < T_0 w(b\widehat{x}) - \int_0^{T_0} w(by(t))dt. \tag{4.5}$$

There exists $\Delta_0 \in (0, \Delta)$ such that

$$2\Delta_0 e(\sigma) \leq \widehat{x}$$

$$w(b(\widehat{x} - \Delta_0 e(\sigma))) \geq w(b\widehat{x}) - \Delta/8. \tag{4.6}$$

Lemma 3.3 implies that there exists $\delta \in (0, \Delta_0)$ such that the following property holds:

(P) for each $z, z' \in R^n_+$ satisfying

$$\|z - \widehat{x}\|, \|z' - \widehat{x}\| \le \delta$$

and every $\tau \in [2^{-1}, 2]$ there exist a program $(u(t), v(t))_{t=0}^{\tau}$ such that

$$u(0) = z,\ u(\tau) \ge z',$$

$$\|u(t) - \widehat{x}\|,\ \|v(t) - \widehat{x}\| \le \Delta_0,\ t \in [0, \tau],$$

$$\|u'(t)\| \le \Delta_0,\ t \in [0, \tau]$$

and

$$w(b(\widehat{x} + \delta e)) \le w(\widehat{x}) + \Delta/8. \tag{4.7}$$

Theorem 2.4 implies that

$$\lim_{t \to \infty} x(t) = \widehat{x}.$$

and that there exists $T_1 > 0$ such that for all $t \ge T_1$,

$$\|x(t) - \widehat{x}\| \le \delta. \tag{4.8}$$

Property (P) and (4.8) imply that there exists a program

$$(x^{(1)}(t), y^{(1)}(t))_{t=0}^{T_0+T_1+1}$$

such that

$$x^{(1)}(t) = x(t + T_0), y^{(1)}(t) = y(t + T_0),\ t \in [0, T_0 + T_1], \tag{4.9}$$

$$x^{(1)}(T_0 + T_1 + 1) \ge \widehat{x}, \tag{4.10}$$

$$\|x^{(1)}(t) - \widehat{x}\|,\ \|y^{(1)}(t) - \widehat{x}\| \le \Delta_0,\ t \in [T_0 + T_1, T_0 + T_1 + 1], \tag{4.11}$$

$$\|(x^{(1)})'(t)\| \le \Delta_0,\ t \in [T_0 + T_1, T_0 + T_1 + 1]. \tag{4.12}$$

For all $t \in (T_0 + T_1 + 1, 2T_0 + T_1 + 1]$ set

$$x^{(1)}(t) = \widehat{x} + e^{-d(t - T_0 - T_1 - 1)}(x^{(1)}(T_0 + T_1 + 1) - \widehat{x}),\ y^{(1)}(t) = \widehat{y}. \tag{4.13}$$

Lemma 3.2, (4.10), (4.12) and (4.13) imply that $(x^{(1)}(t), y^{(1)}(t))_{t=0}^{2T_0+T_1+1}$ is a program. In view of (4.10) and (4.13),

$$x^{(1)}(2T_0 + T_1 + 1) \ge \widehat{x}. \tag{4.14}$$

Property (P) and (4.8) imply that there exists a program

$$(x^{(2)}(t), y^{(2)}(t))_{t=2T_0+T_1+1}^{2T_0+T_1+2}$$

such that

$$x^{(2)}(2T_0 + T_1 + 1) = \widehat{x},\ x^{(2)}(2T_0 + T_1 + 2) \ge x(2T_0 + T_1 + 2), \tag{4.15}$$

$$\|x^{(2)}(t) - \widehat{x}\|,\ \|y^{(2)}(t) - \widehat{x}\| \le \Delta_0,\ t \in [2T_0 + T_1 + 1, 2T_0 + T_1 + 2], \tag{4.16}$$

$$\|(x^{(2)})'(t)\| \le \Delta_0,\ t \in [2T_0 + T_1 + 1, 2T_0 + T_1 + 2]. \tag{4.17}$$

For all $t \in [2T_0 + T_1 + 1, 2T_0 + T_1 + 2]$ define

$$y^{(1)}(t) = y^{(2)}(t), \tag{4.18}$$

$$x^{(1)}(t) = x^{(2)}(t) + e^{-d(t-2T_0-T_1-1)}(x^{(1)}(2T_0 + T_1 + 1) - \widehat{x}). \tag{4.19}$$

Lemma 3.2, (4.14), (4.15) and (4.19) imply that $(x^{(1)}(t), y^{(1)}(t))_{t=0}^{2T_0+T_1+2}$ is a program. By (4.13) and (4.19), for all $t \in [2T_0 + T_1 + 1, 2T_0 + T_1 + 2]$,

$$\begin{aligned} x^{(1)}(t) &= x^{(2)}(t) + e^{-d(t-2T_0-T_1-1)}(\widehat{x} + e^{-dT_0}(x^{(1)}(T_0 + T_1 + 1) - \widehat{x}) - \widehat{x}) \\ &= x^{(2)}(t) + e^{-d(t-T_0-T_1-1)}(x^{(1)}(T_0 + T_1 + 1) - \widehat{x}). \end{aligned} \tag{4.20}$$

It follows from (4.9), (4.14), (4.15) and (4.19) that

$$x^{(1)}(0) = x(T_0) = x(0), \tag{4.21}$$

$$x^{(1)}(2T_0 + T_1 + 2) \geq x^{(2)}(2T_0 + T_1 + 2) \geq x(2T_0 + T_1 + 2). \tag{4.22}$$

Since the program $(x(t), y(t))_{t=0}^{\infty}$ is overtaking optimal Lemma 3.2, (4.5) and (4.22) imply that

$$\begin{aligned} 0 &\leq \int_0^{2T_0+T_1+2} w(by(t))dt - \int_0^{2T_0+T_1+2} w(by^{(1)}(t))dt \\ &= \int_0^{T_0} w(by(t))dt + \int_{2T_0+T_1}^{2T_0+T_1+2} w(by(t))dt - \int_{T_0+T_1}^{2T_0+T_1+2} w(by^{(1)}(t))dt \\ &< -\Delta + T_0 w(b\widehat{x}) + \int_{2T_0+T_1}^{2T_0+T_1+2} w(by(t))dt - \int_{T_0+T_1}^{2T_0+T_1+2} w(by^{(1)}(t))dt. \end{aligned} \tag{4.23}$$

In view of (4.8), for all $t \in [2T_0 + T_1, 2T_0 + T_1 + 2]$,

$$0 \leq y(t) \leq x(t) \leq \widehat{x} + \delta e. \tag{4.24}$$

By (4.7) and (4.24), for all $t \in [2T_0 + T_1, 2T_0 + T_1 + 2]$,

$$w(by(t)) \leq w(b(\widehat{x} + \delta e)) \leq w(b\widehat{x}) + \Delta/8. \tag{4.25}$$

It follows from (4.14), (4.18), (4.19), (4.23) and (4.25) that

$$\begin{aligned} 0 &< -\Delta + T_0 w(b\widehat{x}) + 2w(b\widehat{x}) + \Delta/4 - \int_{T_0+T_1}^{2T_0+T_1+2} w(by^{(1)}(t))dt \\ &\leq (T_0 + 2)w(b\widehat{x}) - 3\Delta/4 - \int_{T_0+T_1}^{T_0+T_1+1} w(by^{(1)}(t))dt \\ &\quad - \int_{T_0+T_1+1}^{2T_0+T_1+1} w(b\widehat{x})dt - \int_{2T_0+T_1+1}^{2T_0+T_1+2} w(by^{(1)}(t))dt \\ &\leq 2w(b\widehat{x}) - 3\Delta/4 - \int_{T_0+T_1}^{T_0+T_1+1} w(by^{(1)}(t))dt - \int_{2T_0+T_1+1}^{2T_0+T_1+2} w(by^{(1)}(t))dt. \end{aligned} \tag{4.26}$$

In view of (4.12), for all $t \in [T_0 + T_1, T_0 + T_1 + 1]$,

$$y^{(1)}(t) \geq \widehat{x} - \Delta_0 e(\sigma). \tag{4.27}$$

By (4.16) and (4.18), for all $t \in [2T_0 + T_1 + 1, 2T_0 + T_1 + 2]$,

$$y^{(1)}(t) = y^{(2)}(t) \geq \widehat{x} - \Delta_0 e(\sigma). \tag{4.28}$$

It follows from (4.6), (4.27) and (4.28) that for all $t \in [T_0+T_1, T_0+T_1+1] \cup [2T_0+T_1+1, 2T_0+T_1+2]$,

$$w(by^{(1)}(t)) \geq w(b\widehat{x}) - \Delta/8. \tag{4.29}$$

By (4.26) and (4.29),

$$0 \leq 2w(b\widehat{x}) - 3\Delta/4 - 2(w(b\widehat{x}) - \Delta/8) \leq 3\Delta/4 + \Delta/4,$$

a contradiction. The contradiction we have reached proves that case (2) does not hold. Thus the program $(\tilde{x}(t), \tilde{y}(t))_{t=0}^{\infty}$ is good. Together with Theorem 2.4, (4.2) and (4.3) this implies that

$$x(t) = \widehat{x},\ t \in [0, T_0]$$
$$y(t) = \widehat{x},\ t \in [0, T_0] \text{ (a. e.) }.$$

Combined with Theorem 3.1 and the inequality $a\widehat{x} < d^{-1}$ this completes the proof of Theorem 4.1. □

5. The one-dimensional case

Assume that $n = 1$ and that $(x(t), y(t))_{t=0}^{\infty}$ is an overtaking optimal program. There are three cases:

$$x(0) = \widehat{x};\ x(0) > \widehat{x};\ x(0) < \widehat{x}.$$

If $x(0) = \widehat{x}$, then in view of Theorem 3.1, for all $t \geq 0$, $x(t) = \widehat{x}$ and for a. e. $t \geq 0$, $y(t) = \widehat{x}$. Assume that

$$x(0) > \widehat{x}. \tag{5.1}$$

Theorem 2.4 implies that

$$\lim_{t\to\infty} x(t) = \widehat{x}. \tag{5.2}$$

It follows from Theorems 3.1 and 4.1, (5.1) and (5.2) that

$$x(t) \geq \widehat{x} \text{ for all } t \geq 0, \tag{5.3}$$

$$x(t) < x(0) \text{ for all } t > 0. \tag{5.4}$$

If $x(\tau) = \widehat{x}$ for some $\tau > 0$, then there exists $\tau_0 > 0$ such that

$$x(t) = \widehat{x},\ y(t) = \widehat{x} \text{ for all } t \geq \tau_0,$$
$$x(t) > \widehat{x} \text{ for all } t \in [0, \tau_0),$$

the function x is strictly decreasing on the interval $[0, \tau_0]$.

Assume that

$$x(t) \neq \widehat{x} \text{ for all } t \geq 0.$$

It is not difficult to see that

$$x(t) > \widehat{x} \text{ for all } t \geq 0$$

and the function x is strictly decreasing on $[0, \infty)$.

Assume that

$$x(0) < \widehat{x}. \tag{5.5}$$

Theorem 2.4 implies (5.2). It follows from Theorem 3.1, (5.2) and (5.5) that

$$x(t) \leq \widehat{x} \text{ for all } t \geq 0,$$

$$x(t) > x(0) \text{ for all } t > 0.$$

It is not difficult to see that if $x(\tau) = \widehat{x}$ for some $\tau > 0$, then there exists $\tau_0 > 0$ such that

$$x(t) = \widehat{x},\ y(t) = \widehat{x} \text{ for all } t \geq \tau_0,$$

$$x(t) < \widehat{x} \text{ for all } t \in [0, \tau_0),$$

the function x is strictly increasing on the interval $[0, \tau_0]$.

Assume that

$$x(t) \neq \widehat{x} \text{ for all } t \geq 0.$$

It is not difficult to see that

$$x(t) < \widehat{x} \text{ for all } t \geq 0$$

and the function x is strictly increasing on $[0, \infty)$.

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