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NONLINEAR SPECTRAL RESOLUTION

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Abstract. A quasi-product on a normed space is defined. In addition, the notions of the eigenvectors of linear operators can be extended for nonlinear operators. Based on the quasi-product and the generalized eigenvectors, spectral theorems for certain possibly nonlinear operators which include bounded symmetric linear operators as special cases can be proved. Operational calculus of the class of possibly nonlinear operators is developed and nonlinear spectral operators are given. Two classes of operators, nonlinear generalizations of linear operators on finite dimensional vector spaces and compact linear operators on normed spaces, and two operators associated with the ones playing a role in quantum mechanics are given for illustrations.

1. Introduction

Operator theory has been at the heart of research in analysis (see [1]; [8], Chapter 4). Moreover, as implied by [7], considering nonlinear case should be essential. Developing useful results for the operators holds the promise for wide applications of nonlinear functional analysis to a variety of scientific areas.

In classical functional analysis, the space of bounded linear operators is a normed space endowed with a sensible norm. Further, by defining the composition of two bounded linear operators as the operation of multiplication, the space of the bounded linear operators is also a normed algebra. In Section 2, a normed function is defined and some set of possibly nonlinear operators from a normed space into a normed space turns out to be a normed space. Further, if the domain and the range of the possibly nonlinear operators are the same normed algebra, the normed space of the possibly nonlinear operators can be a normed algebra by defining an operation of multiplication for two operators.

Spectral theory is one of main topics of modern functional analysis and its applications (see [5]; [10]). Spectral theory for certain classes of linear operators has been well developed (see [3]), particularly symmetric linear operators in a Hilbert space. Spectral theory for the nonlinear operators is an emerging field in functional analysis (see [2]). However, relatively little has been done for the spectral resolution of the possibly nonlinear operator of interest and associated extensions which are main objectives of this article. The inner product is important for the development of spectral theory in Hilbert spaces. Therefore, in order to develop the spectral

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decomposition of the possibly nonlinear operator on a normed space, a "product" which is referred to as the quasi-product and plays the role analogous to the inner product in the inner product space is proposed in next section. Section 3 defines a generalized real definite operator which can be considered as the generalization of the symmetric linear operator on the Hilbert space. Furthermore, the generalized eigenvalue which can be considered as the generalized version of the eigenvalue in the linear case is also defined. Spectral theorems of the bounded generalized real definite operator are given in the section. Extensions of the results in Section 3, including the operational calculus of the generalized real definite operator and nonlinear spectral operators (see [4]), are given in Section 4. Finally, several nonlinear operators along with their applications are given for illustrations in Section 5. For readability, only proofs of main results are given in these sections. Proofs of other results are delegated to Section 6. Note that the results in this article can be used to prove spectral theorems for a more general class of possibly nonlinear operators which includes bounded normal linear operators as special cases. In addition, the spectral representations for unbounded nonlinear operators can be also proved based on these results.

Hereafter $D(F)$ and $R(F)$ are denoted as the domain and the range of an operator F, respectively, and the notation $||\cdot||_Z$ is denoted as the norm of the normed space Z. The spaces of interest are normed spaces implicitly. On the other hand, Banach spaces or normed algebras will be indicated explicitly. Note that the vector spaces and the normed spaces of interest in this article are not trivial, i.e., not only including the zero element.

2. Nonlinear functional spaces

2.1. **Basics.** Let X and Y be normed spaces over a field K with some sensible norms, where K is either the real field R or the complex field C. Let $V(X,Y)$ be the set of all operators from X into Y, i.e., the set of arbitrary maps of X into Y. Let the algebraic operations of $F_1, F_2 \in V(X,Y)$ be the operators with $(F_1+F_2)(x) = F_1(x)+F_2(x)$ and $(\alpha F_1)(x) = \alpha F_1(x)$ for $x \in X$, where $\alpha \in K$ is a scalar. Also let the zero element in $V(X,Y)$ be the operator with the image equal to the zero element in Y.

In this subsection, basic properties of the nonlinear functional spaces are given. The proofs of several theorems and corollaries, including Theorem 2.2, Corollary 2.3, Theorem 2.4, Corollary 2.5, and Theorem 2.7, are quite routine and are delegated to Section 6.

Theorem 2.1. *$V(X,Y)$ is a vector space over the field K.*

The routine proof of the above theorem is not presented. Define a non-negative extended real-valued function p, i.e., the range of p including ∞, on $V(X,Y)$ by

$$p(F) = \max\left(\sup_{x\neq 0, x\in X} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y\right)$$

for $F \in V(X,Y)$. The non-negative extended real-valued function p is a generalization of the norm for linear operators. Let $B(X,Y)$, the subset of $V(X,Y)$, consist of all operators with $p(F)$ being finite.

Theorem 2.2. *p is a norm on $B(X,Y)$ and $[B(X,Y),p]$ is a normed space.*

Hereafter the norm p is used, i.e., $||F||_{B(X,Y)} = p(F)$ for $F \in B(X,Y)$. Note that bounded linear operators lie in $B(X,Y)$. As $X = Y$, the notation $B(X) = B(X,X)$ is used.

Let the notation of the composition of two operators be $\circ$ hereafter.

Corollary 2.3. *Let $F_1 \in B(X,Y)$ and $F_2 \in B(Y,Z)$, where Z is a normed space. Then*

$$\|F_1(x)\|_Y \le \|F_1\|_{B(X,Y)}\|x\|_X, x \neq 0.$$

If $F_1(0) = F_2(0) = 0$, then

$$\|F_2 \circ F_1\|_{B(X,Z)} \le \|F_1\|_{B(X,Y)}\|F_2\|_{B(Y,Z)}.$$

Let $BC(X,Y)$, the subspace of $B(X,Y)$, consist of bounded continuous operators. It is well known that the space of all bounded linear operators from a normed space X to a Banach space Y is complete. The analogous results also hold for the spaces $B(X,Y)$ and $BC(X,Y)$.

Theorem 2.4. *If Y is a Banach space, then $B(X,Y)$ and $BC(X,Y)$ are Banach spaces.*

Unlike a linear operator, a continuous nonlinear operator F might not be bounded. The following corollary gives sufficient conditions for the boundedness of a continuous operator.

Corollary 2.5. *Let $\mathcal{K}$ be a compact subset of X. Let $F : X \to Y$ be continuous on $\mathcal{K}$. $F \in B(X,Y)$ if the following conditions hold:*

(a) $\|F(\mathcal{K}^c)\|_{B(X,Y)} \le M$, *where $\mathcal{K}^c$ is the complement of the set $\mathcal{K}$ and M is a positive number.*
(b) $\lim_{x\to 0} \|F(x)\|_Y/\|x\|_X$ *exists and is finite.*

The linear functionals in the dual space of a normed space can distinguish the points of the normed space. Since the dual space of the normed space X is a subspace of $B(X,K)$, the results hold for the nonlinear functionals in $B(X,K)$. Further, the following theorem gives the counterpart of the one for the bounded linear functionals in the second dual space.

Theorem 2.6. *Let $x \neq 0$ and $g_x : B(X,K) \to K$ be defined by $g_x(F) = F(x)$ for $F \in B(X,K)$. Then $g_x \in B[B(X,K),K]$ and*

$$||x||_X = \sup_{F\neq 0, F\in B(X,K)} \frac{|F(x)|}{\|F\|_{B(X,K)}} = ||g_x||_{B[B(X,K),K]}\,.$$

Proof. For any $x \neq 0$, there exists a bounded linear functional $F_x \in B(X,K)$ such that $F_x(x) = ||x||_X$ and $||F_x||_{B(X,K)} = 1$. Thus,

$$\sup_{F\neq 0, F\in B(X,K)} \frac{|F(x)|}{\|F\|_{B(X,K)}} \ge \frac{|F_x(x)|}{\|F_x\|_{B(X,K)}} = \|x\|_X.$$

On the other hand, $||x||_X \geq \sup_{F\neq 0, F\in B(X,K)} |F(x)| / \|F\|_{B(X,K)}$ because by Corollary 2.3, $|F(x)| \leq ||F||_{B(X,K)} ||x||_X$. Finally,

$$||g_x||_{B[B(X,K),K]} = \sup_{F\neq 0, F\in B(X,K)} \frac{|F(x)|}{\|F\|_{B(X,K)}} = ||x||_X$$

because $g_x(0) = 0$. □

The Banach space of all bounded linear operators on a Banach space X is a Banach algebra with the multiplication being the composition of the operators. The Banach space $B(X)$ is not a Banach algebra with the multiplication being the composition of the operators and X being a Banach space. However, $B(X)$ can be a normed algebra or a Banach algebra depending on X being a normed algebra or a Banach algebra (see Theorem 2.4) as the multiplication of the operators is defined properly.

Theorem 2.7. *Let X be a normed (Banach) algebra. Define the multiplication of F_1 and F_2 in $V(X, X)$ by*

$$(F_1 * F_2)(x) = \frac{F_1(x)F_2(x)}{\|x\|_X}, x \neq 0, x \in X,$$

and

$$(F_1 * F_2)(0) = F_1(0)F_2(0).$$

Then $B(X)$ is a normed (Banach) algebra. If X has a unit, $B(X)$ is a normed (Banach) algebra with a unit e.

2.2. **Quasi-product.**

Definition 2.8. Let X be a normed space and S be a subset of X. A quasi-product $[\cdot,\cdot]_S$ on S is a mapping (or a map) of $S \times S$ into the scalar field K with the following properties:

(a)

$$[x, x]_S \geq 0$$

for $x \in S$.

(b)

$$|[x, y]_S| \leq \bar{c}\, ||x||_X\, ||y||_X$$

for $x, y \in S$, where $\bar{c}$ is a positive number.

(c)

$$\left[\sum_{i=1}^{n} \alpha_i x_i, y\right]_S = c(y) \sum_{i=1}^{n} \alpha_i \left[x_i, y\right]_S$$

for any $n \geq 1$, $x_1, \ldots, x_n, y, \sum_{i=1}^{n} \alpha_i x_i \in S$ and $\alpha_1, \ldots, \alpha_n \in K$, where $c : S \to R$ is a positive bounded function and is bounded away from 0.

A quasi-product is quasi-symmetric if and only if $[x,y]_S = q(x,y)\overline{[y,x]_S}$, where $\overline{z}$ is the conjugate of the complex number z and $q : S \times S \to R$ is a positive bounded function and is bounded away from 0. A quasi-symmetric quasi-product is symmetric if and only if $[x,y]_S$ is equal to the conjugate of $[x,y]_S$, i.e., $[x,y]_S = \overline{[y,x]_S}$.

Remark 2.9. If $S = X$, $c(y) = \overline{c} = 1$, $[x,x]_X = 0$ implies $x = 0$, and the quasi-product is symmetric, X is an inner product space with the inner product being the quasi-product. On the other hand, an inner product $< \cdot, \cdot >_X$ on the inner product space X can be a symmetric quasi-product by setting $< x, y >_X = [x,y]_X$.

The following are examples of the quasi-products on some normed spaces or the subset of some normed space.

Example 2.10. Let X be an inner product space with an inner product $< \cdot, \cdot >_X$ and the norm induced by the inner product. Then

$$[x,y]_X = c(y) < x, y >_X$$

for $x, y \in X$, where c is a positive bounded function on X and is bounded away from 0, for example,

$$c(y) = \frac{||y||_X}{||y||_X + 1} + k$$

and $k > 0$.

Example 2.11. Let X be the real normed space of real-valued Lebesgue integrable functions on the measurable set $\Omega \subset R$ with the norm $||x||_X = \int_\Omega |x| d\mu$ for $x \in X$, where μ is the Lebesgue measure. Let the symmetric quasi-product defined by

$$[x,y]_X = \int_\Omega x d\mu \int_\Omega y d\mu$$

for $x, y \in X$.

Example 2.12. Let X be the real normed space of real-valued bounded functions on the compact domain $\Omega \in R$ with the supremum norm and the subset S of X consist of bounded Lebesgue integrable functions with positive integrals. Define the non-symmetric quasi-product on S by

$$[x,y]_S = \int_\Omega x d\mu \sup_{t \in \Omega} y(t),$$

where μ is the Lebesgue measure.

Note that the quasi-product function $[x,y]_X = f_y(x)$, considered as a function of x, is continuous on X. In addition, if the quasi-product is symmetric, it is jointly continuous.

The bilinear form on a Hilbert space has a representation associated with a unique continuous linear operator (see [5], Theorem 3.8-4; [9], Chapter VI, Theorem 1.2). The following theorem can be considered as the nonlinear counterpart of the linear case.

Theorem 2.13. *Let $F \in B(X,Y)$ with $F(0)=0$ and there exists a positive function d defined on Y and being bounded away from 0 such that $[y,y]_Y = d(y)||y||_Y^2$ for $y \in Y$. Then $h : X \times Y \to K$ has a unique representation*

$$h(x,y) = [F(x),y]_Y$$

if and only if the function h has the following properties:

(a) $$|h(x,y)| \leq \overline{c}\,||x||_X\,||y||_Y$$

for $x \in X$ and $y \in Y$, where $\overline{c}$ is a positive number.

(b) *For $y \in Y$, there exists $z \in Y$ such that*

$$h(x,y) = [z,y]_Y$$

for any $x \in X$.

Proof. The "only if" part is obvious. To prove "if" part, define $F(x) = z$ with $F(0) = 0$. F is well-defined since for $x_1 = x_2$,

$$h(x_1,y) = [z,y]_Y = h(x_2,y)$$

and thus $F(x_1) = F(x_2) = z$. Further, for $F \neq 0$, $F \in B(X,Y)$ because there exists $c_1 > 0$ such that

$$\begin{aligned}
&||F||_{B(X,Y)} \\
= &\sup_{x\neq 0, x\in X, F(x)\neq 0} \frac{||F(x)||_Y^2}{||x||_X\,||F(x)||_Y} \\
\leq &\ c_1 \sup_{x\neq 0, x\in X, F(x)\neq 0} \frac{|[F(x),F(x)]_Y|}{||x||_X\,||F(x)||_Y} \\
\leq &\ c_1 \sup_{x\neq 0, x\in X, y\neq 0, y\in Y} \frac{|[F(x),y]_Y|}{||x||_X\,||y||_Y} \\
\leq &\ c_1\overline{c}.
\end{aligned}$$

To prove the uniqueness of F, let $h(x,y) = [F(x),y]_Y = [G(x),y]_Y$ for $x \in X$ and $y \in Y$. Then by the properties of the quasi-product,

$$|[F(x),y]_Y - [G(x),y]_Y| \geq c_2\,|[F(x)-G(x),y]_Y|$$

implies $[F(x)-G(x),y]_Y = 0$ for any $y \in Y$ and thus

$$c_3\,||F(x)-G(x)||_Y^2 \leq [F(x)-G(x),F(x)-G(x)]_Y = 0,$$

where c_2 depending on y and c_3 depending on x are both positive numbers. Therefore, $F(x) = G(x)$. □

3. Generalized real definite operators

The goal of this section is to formulate the spectral resolutions of some class of possibly nonlinear operators. In this section and the following subsection, i.e., Section 3 and Section 4.1, suppose that the operator $F : X \to X$ of interest satisfies $F(0) = 0$ and the quasi-products on X exist. For $\tilde{F} : X \to X$ with $\tilde{F}(0) \neq 0$, the corresponding spectral resolution can be obtained by the shift $\tilde{F}(x) = F(x) +$

$s(x), x \in X$, where $s : X \to X$ satisfies $s(0) = \tilde{F}(0)$, for example, $s(x) = \tilde{F}(0)$ (also see Remark 4.10 in Section 4.1).

The spectral resolution of bounded symmetric linear operators, due to Hilbert, is a limit of Riemann-Stieltjes sums in the sense of operator convergence (also see [6], Chapter 31), i.e., having the form of Riemann-Stieltjes integral. The spectral resolutions of the generalized real definite operators defined in this section also have the form of Riemann-Stieltjes integral in terms of the quasi-product or in the sense of operator convergence as certain classes of quasi-products are employed. For the bounded symmetric linear operators, the spectral resolution involves both linear projection operators and positive linear operators. The nonlinear counterparts of the positive linear operators and the linear projection operators are defined and their basic properties are given in the first two subsections. Then, the spectral resolutions are given in the last subsection.

3.1. **G-positive operators.** Linear spectral theory for the symmetric operators involves the positive operators. Similarly, for the nonlinear operators with real spectrums, the spectral theorems also involve the positive operators. The main theorem in this subsection, Theorem 3.7, indicates that the multiplication of two positive operators remains positive given some sufficient conditions imposed on the quasi-product. Based on the theorem, two corollaries give the existence and the uniqueness of the square root of the positive operator. Furthermore, the spectral theorem in Section 3.3 can be proved by using the theorem. Note that the proofs of Theorem 3.7, Corollary 3.8, and Corollary 3.9 in this subsection are delegated to Section 6.

The positive operator, possibly nonlinear, is defined as follows.

Definition 3.1. Let X be a normed space. An operator $F : X \to X$ is generalized real definite if and only if there exist a quasi-product $[\cdot, \cdot]_X$ and an operator $g : X \to X$ satisfying $g(x) \neq 0$ for $x \neq 0$ and

$$[F(x), g(x)]_X \in R$$

for $x \in X$. Furthermore, F is g-positive, denoted by $F \geq 0$, if and only if

$$[F(x), g(x)]_X \geq 0$$

for $x \in X$. For two operators $F_1 : X \to X$ and $F_2 : X \to X$,

$$F_1 \geq F_2$$

if and only if

$$F_1 - F_2 \geq 0.$$

If X is a normed algebra, a g-positive operator F has a square root if and only if there exists an operator $G : X \to X$ such that $[G(x)]^2 = G(x)G(x) = F(x)$ for $x \in X$. G is then called a square root of F.

If $F = 0$, F is g-positive. $G = 0$ is the square root of $F = 0$ defined on the normed algebra X.

Remark 3.2. The generalized real definiteness of the operator F relies on both the operator g and the quasi-product. If F is a symmetric linear operator on a Hilbert space, the operator g is the identity map, and the quasi-product is the inner product on the Hilbert space, F is generalized real definite and F being positive implies F being g-positive. Therefore, the g-positivity extends the notion of the positivity.

If $F : D(F) \to Y$ and $g : D(F) \to Y$ satisfies $g(x) \neq 0$ for $x \neq 0$ and $x \in D(F)$, the above definition can be modified and the associated expressions are $[F(x), g(x)]_Y \in R$ and $[F(x), g(x)]_Y \geq 0$ for $x \in D(F)$, where $D(F)$ is a subset of X.

Some basic properties of the positive operators are given by the following lemma. The routine proofs are not presented.

Lemma 3.3. *Let X be a normed space and F_1, F_2, G_1, G_2 be the operators from X into X.*

(a) *$F_1 \geq F_2$ if and only if $[F_1(x), g(x)]_X \geq [F_2(x), g(x)]_X$ for $x \in X$.*
(b) *If $\alpha \in R$, $\alpha \geq 0$, and $F_1 \geq 0$, then $\alpha F_1 \geq 0$.*
(c) *If $F_1 \geq 0$ and $F_2 \geq 0$, then $F_1 + F_2 \geq 0$.*
(d) *If $F_1 \leq G_1$ and $F_2 \leq G_2$, then $F_1 + F_2 \leq G_1 + G_2$.*

For a symmetric linear operator F on a Hilbert space X, the normed value of $F(x)$ and the associated inner product satisfy the equation $< (F \circ F)(x), x >_X = ||F(x)||_X^2$. It turns out that the analogous equation (inequality) defined below plays a key role in the development of the spectral resolutions of the possibly nonlinear operators.

Definition 3.4. Let X be a normed space and $g : X \to X$ satisfy $g(x) \neq 0$ for $x \neq 0$. For $x, y \in X$, a quasi-product has a left integral domain if and only if for $g(x) \neq 0$, $[y, g(x)]_X = 0$ implies $y = 0$. Further, if X is a normed algebra with a unit 1, for $x, y_1, y_2 \in X$, a quasi-product preserves the positivity if and only if $[y_1 y_2, g(x)]_X \geq 0$ as $[y_1, g(x)]_X \geq 0$ and $[y_2, g(x)]_X \geq 0$. In addition, a quasi-product is square bounded below if and only if for $x, y \in X$, there exists a positive number $\underline{k}$ such that $[y^2, g(x)]_X \geq \underline{k}||y||_X^2||g(x)||_X$ as $y = 1$ or $[y, g(x)]_X \in R$.

The properties of the quasi-product in Definition 3.4 rely on the operator g and g is assumed to be the same as the one corresponding to the generalized real definite operators of interest (see Definition 3.1) hereafter. Note that the quasi-product has these properties on the set $X \times R(g)$ by the above definition and thus another way is to define these properties on the subsets of $X \times X$. However, only the set $X \times R(g)$ is of interest in this article and hence Definition 3.4 serves the purpose.

In the following, the operators involved in the spectral theorems of the generalized real definite operators, the positive and negative parts of the operator, are defined.

Definition 3.5. Let X be a normed space and F be a generalized real definite operator. The operator $|F|$ is defined by

$$|F|(x) = F(x)$$

if $[F(x), g(x)]_X \geq 0$ and

$$|F|(x) = -F(x)$$

if $[F(x), g(x)]_X < 0$ for $x \in X$. The positive part of F is

$$F^+ = \frac{|F| + F}{2}$$

and the negative part of F is

$$F^- = \frac{|F| - F}{2}.$$

A direct check gives the following lemma.

Lemma 3.6. *Let F be generalized real definite.*

(a) *Let X be a normed space.*
Then $|F| \geq 0$.

(b) *Let X be a unital normed algebra. Then $|F|^2 = F^2$, where for $x \in X$, $|F|^2$ is defined by $|F|^2(x) = |F|(x)|F|(x)$ and F^2 is defined by $F^2(x) = F(x)F(x)$. Furthermore, if the quasi-product is square bounded below or preserves the positivity, then $|F|$ is a square root of F^2.*

The above lemma implies that the quasi-product being square bounded below or preserving the positivity is a sensible condition. Since otherwise, the square of a generalized real definite operator or even a g-positive operator might not be g-positive.

It is natural to ask when the g-positive operator has a square root. It turns out that the quasi-product having a left integral domain or preserving the positivity is a key sufficient condition. The following theorem indicates that the pointwise multiplication of two g-positive operators is g-positive given some sufficient condition.

Theorem 3.7. *Let X be a unital Banach algebra, $F \geq 0$ and $H \geq 0$. If there exists a quasi-product (not necessarily square bounded below) preserving the positivity with X being not necessarily commutative or a square bounded below quasi-product having a left integral domain with X being commutative, then $FH \geq 0$, where $(FH)(x) = F(x)H(x)$ for $x \in X$.*

For a positive symmetric linear operator, the positive square root of the operator is uniquely determined. In the following, the sufficient conditions for the existence and the uniqueness of the positive square root of the g-positive operator are given.

Corollary 3.8. *Let X be a unital Banach algebra.*

(a) *If G, the square root of the g-positive operator F, exists and $G \geq 0$, the commutative unital Banach algebra X is an integral domain, and the quasi-product has a left integral domain, then G is unique and is denoted by $G = F^{1/2}$.*

(b) *If there exists a square bounded below quasi-product preserving the positivity on X (not necessarily commutative) or a square bounded below quasi-product having a left integral domain with X being commutative, $0 \leq F \leq 1_X$, then F has a square root $G \geq 0$ and G commutes with any operator $W \in V(X, X)$ which commutes with F, where $1_X(x) = 1$ for $x \in X$.*

Corollary 3.9. *Let X be a unital Banach algebra and $B(X)$ be the Banach algebra with the multiplication operation given in Theorem 2.7.*

(a) *If $G \in B(X)$ satisfying $G^{*2} = G * G = F$ exists and $F, G \geq 0$, the commutative unital Banach algebra X is an integral domain, and the quasi-product has a left integral domain, then G is unique.*
(b) *Suppose that there exists a square bounded below quasi-product preserving the positivity on X (not necessarily commutative) or a square bounded below quasi-product having a left integral domain with X being commutative. For a g-positive operator $F \in B(X)$, there exists a g-positive operator $G \in B(X)$ such that $G^{*2} = G*G = F$ and G commutes with any operator $W \in V(X, X)$ which commutes with F.*

Remark 3.10. Let $\tilde{F} \geq 0$ and $\tilde{H} \geq 0$ with $\tilde{F}(0) \neq 0$ or $\tilde{H}(0) \neq 0$, and $g(0) \neq 0$. Then given the sufficient conditions in Theorem 3.7 and Corollary 3.8, the results also hold for $\tilde{F}$ and $\tilde{H}$. Furthermore, given the sufficient conditions in Corollary 3.9 and $\tilde{F} \in B(X)$, the g-positive operator $G \in B(X)$ defined by $G(x) = (||\tilde{F}||_{B(X)}/k)^{1/2}||x||_X \hat{G}(x)$ for $x \neq 0$ and $G(0) = k^{-1/2}\hat{G}(0)$ satisfies $G * G = \tilde{F}$, where $\hat{G}^2 = \hat{F}$ and $\hat{F}(x) = k\tilde{F}(x)/(||\tilde{F}||_{B(X)}||x||_X)$ for $x \neq 0$, $\hat{F}(0) = k\tilde{F}(0)$, and where k is some number.

3.2. **Projection operators.** The spectral resolution of a bounded symmetric linear operator involves both the projection operator and the spectrum of the operator. The counterparts of these quantities for a generalized real definite operator are defined and some basic facts about these quantities are given in this subsection. The first two lemmas, Lemma 3.12 and Lemma 3.13, can be used to prove Lemma 3.18, the main result of this subsection. Lemma 3.18 can be applied to prove the spectral theorem in next subsection. Note that the proofs of Lemma 3.12, Lemma 3.13, Lemma 3.16, and Lemma 3.17 in this subsection are delegated to Section 6.

The nonlinear projection operator involved in the spectral resolution of the generalized real definite operator is defined first.

Definition 3.11. For a subset S containing 0 of a normed space X, the corresponding projection operator $E_S : X \to X$ is defined by $E_S(x) = x$ if $x \in S$ and $E_S(x) = 0$ otherwise. If X is a unital normed algebra, the projection indicator $1_S : X \to X$ is defined by $1_S(x) = 1$ if $x \in S$ and $1_S(x) = 0$ otherwise.

Denote $N(F)$ as the null space (set) of an operator F, i.e., $N(F) = \{x : F(x) = 0, x \in X\}$. The following two lemmas give the basic properties of the projection operator (indicator) and the positive and negative parts of F. Let $S_2 \backslash S_1$ denote the intersection of the set S_2 and the complement of the set S_1.

Lemma 3.12. *Let S_1 and S_2 both containing 0 be the subsets of a normed space X.*

(a) $||E_{S_1}||_{B(X)} \leq 1$.
(b) *The following are equivalent.*
 (i)
$$E_{S_1} \circ E_{S_2} = E_{S_2} \circ E_{S_1} = E_{S_1}.$$
 (ii) $S_1 \subset S_2$.
 (iii) $N(E_{S_2}) \subset N(E_{S_1})$.

(iv) $||E_{S_1}(x)||_X \leq ||E_{S_2}(x)||_X$ *for* $x \in X$.

If X *is a unital normed algebra, the following are equivalent.*

(i)*

$$1_{S_1}1_{S_2} = 1_{S_2}1_{S_1} = 1_{S_1}.$$

(ii)* $S_1 \subset S_2$.

(iii)* $N(1_{S_2}) \subset N(1_{S_1})$.

(iv)* $||1_{S_1}(x)||_X \leq ||1_{S_2}(x)||_X$ *for* $x \in X$.

(c) *Let* $S_1 \subset S_2$.

Then $E_{S_2-S_1} = E_{S_2} - E_{S_1}$ *is idempotent, i.e.,* $E_{S_2-S_1} \circ E_{S_2-S_1} = E_{S_2-S_1}$, *and the range of* $E_{S_2-S_1}$ *is* $S_2 - S_1 = (S_2 \backslash S_1) \cup \{0\}$. *If* X *is a unital normed algebra,* $1_{S_2-S_1} = 1_{S_2} - 1_{S_1}$ *satisfies* $1_{S_2-S_1}1_{S_2-S_1} = 1_{S_2-S_1}$.

Lemma 3.13. *Let* X *be a unital normed algebra,* F *be generalized real definite, and* $S = N(F^+)$.

(a) $|F|$, F^+, *and* F^- *commute with every operator* $W \in V(X,X)$ *which commutes with* F.

(b) $F^+F^- = F^-F^+ = 0$.

(c) $F^+1_S = 1_SF^+ = 0$ *and* $F^-1_S = 1_SF^- = F^-$.

(d) $F1_S = 1_SF = -F^-$ *and* $F(1_X - 1_S) = (1_X - 1_S)F = F^+$.

(e) $F^+ \geq 0$ *and* $F^- \geq 0$.

The spectrum corresponding to the quasi-product and the g-positivity is defined as follows.

Definition 3.14. Let $F : D(F) \to X$, where $D(F) \subset X$ and X is a normed space. Let $\gamma : D(F) \to X$ be a g-positive operator (see Remark 3.2) satisfying $[\gamma(x), g(x)]_X = k_1(x)||x||_X||g(x)||_X$ and $||\gamma(x)||_X = k_2(x)||x||_X$ for $x \in D(F)$, where both k_1 and k_2 are positive bounded functions and are bounded away from 0. The g-resolvent set of F, denoted by $\rho(F)$ and $\rho(F) \subset C$, consists of the scalars λ such that $R_\lambda = (F - \lambda\gamma)^{-1}$ exists (see [5], A1.2), is bounded, and $D(R_\lambda)$ is a dense set of X. The set $\sigma(F) = C\backslash\rho(F)$ is referred to as the g-spectrum of F. If $F(x) = \lambda\gamma(x)$ for some $x \neq 0$, x is referred to as the g-eigenvector of F corresponding to the g-eigenvalue λ.

In this article, the generalized real definite operators of interest and the g-positive operator γ are in relation to the same quasi-product and the same operator g.

Remark 3.15. The above definition can be also used for the operator $\tilde{F}$ with $\tilde{F}(0) \neq 0$. Moreover, if X and Y are Banach spaces and G, J are in the space of continuous operators from X into Y, the classical eigenvalue λ of the pair (G, J) corresponding to the eigenvector $x \neq 0$ satisfies the equation $G(x) = \lambda J(x)$ (see [2], Chapter 9.5, Chapter 10). If X and Y are normed spaces, the g-eigenvalue λ and the g-eigenvector $x \neq 0$ can be defined similarly for the operators $F : D(F) \to Y$ and $\gamma : D(F) \to Y$ (also see Remark 3.2), i.e., $F(x) = \lambda\gamma(x)$, where $D(F)$ is a subset of X.

The following two lemmas, Lemma 3.16 and Lemma 3.17, give the results for values of the quasi-product of the bounded operators in $B(X)$ and the g-eigenvalues of the bounded generalized real definite operators, respectively.

Lemma 3.16. *Let $F \in B(X)$, where X is a normed space. There exists a positive number $\overline{k}$ such that*

$$|[F(x), g(x)]_X| \leq \overline{k}\, ||x||_X\, ||g(x)||_X$$

for $x \in X$.

Lemma 3.17. *Let F be generalized real definite defined on X, where X is a normed space. Then all g-eigenvalues of F, if exist, are real. Further, if $F \in B(X)$, all g-eigenvalues, if exist, lie in some bounded interval of R.*

Note that the results in Lemma 3.17 also hold for the generalized real definite operator $\tilde{F}$ with $\tilde{F}(0) \neq 0$, i.e., all g-eigenvalues of $\tilde{F}$, if exist, are real and lie in some bounded interval of R if $\tilde{F} \in B(X)$.

Hereafter denote $F_\lambda = F - \lambda\gamma$ and let 1_λ and E_λ be the projection indicator and the projection operator corresponding to $N(F_\lambda^+)$, respectively, where $\lambda \in R$. In addition, let $\Delta = \mu - \lambda$, $E_\Delta = E_\mu - E_\lambda$, and $1_\Delta = 1_\mu - 1_\lambda$, where $\lambda < \mu$.

Lemma 3.18. *Let X be a unital normed algebra. Suppose that $F \in B(X)$ is generalized real definite, $\mu, \lambda \in R$, and $\lambda < \mu$.*

(a) $||1_\lambda(x)||_X \leq ||1_\mu(x)||_X$ *for $x \in X$.*

(b) $\lim_{\lambda \to -\infty} \sup_{x \neq 0, x \in X} 1_\lambda(x) = 0$. *Further,* $\lim_{\lambda \to -\infty} E_\lambda = 0$, $\lim_{\lambda \to \infty} 1_\lambda = 1_X$, *and* $\lim_{\lambda \to \infty} E_\lambda = I$ *with respect to the norm topology* $||\cdot||_{B(X)}$, *where I is the identity operator defined on X, i.e., $I(x) = x$ for $x \in X$.*

(c) $\lambda\gamma \circ E_\Delta \leq F1_\Delta \leq \mu\gamma \circ E_\Delta$.

Proof. (a): If $x \neq 0$ and $1_\lambda(x) = 1$, i.e., $x \in N(F_\lambda^+)$, $[F_\lambda(x), g(x)]_X \leq 0$ then and thus $[F(x), g(x)]_X \leq [\lambda\gamma(x), g(x)]_X$. Since $[\lambda\gamma(x), g(x)]_X < [\mu\gamma(x), g(x)]_X$, $[F(x), g(x)]_X < [\mu\gamma(x), g(x)]_X$ and $[F_\mu(x), g(x)]_X = [F(x) - \mu\gamma(x), g(x)]_X < 0$ thus. The last inequality implies $F_\mu^+(x) = 0$ and $1_\mu(x) = 1$ then. Therefore, $||1_\lambda(x)||_X \leq ||1_\mu(x)||_X$ for $x \in X$.

(b): By Lemma 3.16, there exist positive numbers $\overline{k}$, k_1, and k_2 such that

$$\begin{aligned} & [F_\lambda(x), g(x)]_X \\ \geq\ & k_1 \{[F(x), g(x)]_X - [\lambda\gamma(x), g(x)]_X\} \\ \geq\ & k_1(-\overline{k} - k_2\lambda)\, ||x||_X\, ||g(x)||_X \\ >\ & 0 \end{aligned}$$

for $\lambda < -\overline{k}/k_2$ and $x \neq 0$. This gives that $1_\lambda(x) = 0$ for $\lambda < -\overline{k}/k_2$. Similarly, there exist positive numbers k_1^* and k_2^* such that

$$\begin{aligned} & [F_\lambda(x), g(x)]_X \\ \leq\ & k_1^* \{[F(x), g(x)]_X - [\lambda\gamma(x), g(x)]_X\} \\ \leq\ & k_1^*(\overline{k} - k_2^*\lambda)\, ||x||_X\, ||g(x)||_X \\ <\ & 0 \end{aligned}$$

for $\lambda > \overline{k}/k_2^*$ and $x \neq 0$. This gives that $1_\lambda = 1_X$ for $\lambda > \overline{k}/k_2^*$. The results for E_λ follow because $E_\lambda(x) = 1_\lambda(x)x$ for $x \in X$.

(c): Since $F_\mu 1_\Delta = F1_\Delta - \mu\gamma \circ E_\Delta$ and $-F_\mu 1_\Delta = -F_\mu 1_\mu(1_\mu - 1_\lambda) = F_\mu^-(1_\mu - 1_\lambda) \geq 0$ by Lemma 3.12 (b), Lemma 3.13 (d), and Lemma 3.13 (e), hence $F1_\Delta \leq \mu\gamma \circ E_\Delta$.

Similarly, because $F_\lambda 1_\Delta = F 1_\Delta - \lambda\gamma \circ E_\Delta$ and $F_\lambda 1_\Delta = F_\lambda(1_X - 1_\lambda)(1_\mu - 1_\lambda) = F^+_\lambda(1_\mu - 1_\lambda) \geq 0$ by Lemma 3.12 (b), Lemma 3.13 (d), and Lemma 3.13 (e), hence $F1_\Delta \geq \lambda\gamma \circ E_\Delta$ holds. □

3.3. **Spectral theorems.** In this subsection, the spectral resolutions of the generalized real definite operators in terms of the quasi-product and with respect to some topology are stated in Theorem 3.21 and Theorem 3.23, respectively. Let X be a unital Banach algebra in the subsection.

The spectral resolution of interest can be defined based on the lemma below.

Lemma 3.19. *Let $F \in B(X)$ be generalized real definite. There exists a bounded interval $[m, M]$ with any partition $\{s_j\}$ satisfying $m = s_0 < s_1 < \cdots < s_n = M$ and $\Delta_j = s_j - s_{j-1} < \epsilon_n$ such that $F_n = \sum_{j=1}^{n} \lambda_j(\gamma \circ E_{\Delta_j})$ converges to an operator in $B(X)$ with respect to the norm topology $||\cdot||_{B(X)}$ and the convergence is independent of the choice of $\lambda_j \in (s_{j-1}, s_j]$ as $n \to \infty$, where $1_\lambda(x) = 0$ for $x \neq 0$ as $\lambda = m$, $1_\lambda = 1_X$ as $\lambda = M$, and $0 < \epsilon_n \underset{n\to\infty}{\longrightarrow} 0$.*

Proof. By Lemma 3.12 (b) and Lemma 3.18 (b), there exist m and M such that $1_\lambda(x) = 0$ for $x \neq 0$ as $\lambda \leq m$ and $1_\lambda = 1_X$ as $\lambda \geq M$. Since for fixed $x \neq 0$, $1_m(x) = 0$, $1_M(x) = 1$, and $1_\lambda(x)$, considered as a function of λ, is right-continuous, there exists $\lambda_x \in [m, M]$ such that $1_{\lambda_x}(x) = 1$ and $1_\lambda(x) = 0$ for $\lambda < \lambda_x$. Define the operator $\tilde{F}(x) = \lambda_x\gamma(x)$ as $x \neq 0$ and $\tilde{F}(0) = 0$. Then $\tilde{F} \in B(X)$. Because $\lambda_x \in (s_{j-1}, s_j]$ for some j and $\Delta_j < \epsilon_n$, hence

$$\begin{aligned}
& \left|\left|F_n(x) - \tilde{F}(x)\right|\right|_X \\
= \ & ||(\lambda_j - \lambda_x)\gamma(x)||_X \\
\leq \ & \overline{k}\epsilon_n \, ||x||_X
\end{aligned}$$

for $x \in X$, where $\overline{k}$ is some positive number. □

The following definition gives the spectral resolution in terms of operator convergence.

Definition 3.20. Let X be a unital Banach algebra, $F \in B(X)$ be generalized real definite, and $\{s_j\}$ be any partition of a bounded interval $[m, M]$ with $m = s_0 < s_1 < \cdots < s_n = M$ and $\Delta_j = s_j - s_{j-1} < \epsilon_n$, where $0 < \epsilon_n \underset{n\to\infty}{\longrightarrow} 0$. If $\sum_{j=1}^{n} \lambda_j(\gamma \circ E_{\Delta_j})$ converges to an operator in the sense of operator convergence, i.e., with respect to the norm topology $||\cdot||_{B(X)}$, and the convergence is independent of the choice of λ_j for $\lambda_j \in (s_{j-1}, s_j]$ as $n \to \infty$, the limit operator is denoted as $\int_m^M \lambda d(\gamma \circ E_\lambda)$. Furthermore, if $f(\lambda_1)1_{s_1} + \sum_{j=2}^{n} f(\lambda_j)1_{\Delta_j}$ converges to an operator in the sense of operator convergence and the convergence is independent of the choice of λ_j as $n \to \infty$, the limit operator is denoted as $\int_m^M f(\lambda)d1_\lambda$, where $f : R \to V(X, X)$ is a mapping from R into $V(X, X)$.

The following two theorems give the spectral representations of the generalized real definite operators in terms of the quasi-product and with respect to the norm topology $||\cdot||_{B(X)}$, respectively. Moreover, $[F(x), g(x)]_X, x \in X$, can be expressed as an ordinary Riemann-Stieltjes integral in terms of a certain equivalence relation.

The involved equivalence relation $\equiv$ for two functionals G_1 and G_2 from X into K is denoted as $G_1 \equiv G_2$ (or $G_1(x) \equiv G_2(x)$ for convenience) if and only if there exist positive numbers $\underline{k}$ and $\overline{k}$ such that

$$\underline{k}\,|G_2(x)| \le |G_1(x)| \le \overline{k}\,|G_2(x)|$$

for all $x \in X$.

Theorem 3.21. *Let $F \in B(X)$ be generalized real definite. Then*

$$[F(x), g(x)]_X = \left[\left[\int_m^M \lambda d(\gamma \circ E_\lambda)\right](x), g(x)\right]_X$$

and

$$[F(x), g(x)]_X \equiv \int_m^M \lambda dw_x(\lambda)$$

for $x \in X$, where $[m, M]$ is some bounded interval depending on F, $w_x(\lambda) = [(\gamma \circ E_\lambda)(x), g(x)]_X$, and the second integral is the ordinary Riemann-Stieltjes integral.

Proof. $\int_m^M \lambda d(\gamma \circ E_\lambda)$ exists by Lemma 3.19, where $1_m(x) = 0$ for $x \neq 0$ and $1_M = 1_X$. Note that $\sum_{j=1}^n \Delta_j = M - m$ and $\sum_{j=1}^n 1_{\Delta_j}(x) = 1$ for $x \neq 0$. Then $F = F\sum_{j=1}^n 1_{\Delta_j} = \sum_{j=1}^n F1_{\Delta_j}$. By Lemma 3.18 (c) and Lemma 3.3 (b), (c), and (d),

$$\sum_{j=0}^{n-1} s_j(\gamma \circ E_{\Delta_{j+1}}) \le F \le \sum_{j=1}^{n} s_j(\gamma \circ E_{\Delta_j})$$

and hence

$$0 \le \sum_{j=1}^n s_j(\gamma \circ E_{\Delta_j}) - F \le \sum_{j=1}^n \Delta_j(\gamma \circ E_{\Delta_j}) \le \epsilon_n \sum_{j=1}^n \gamma \circ E_{\Delta_j}.$$

Then by Lemma 3.3 (a),

$$\begin{aligned}
&\left[\sum_{j=1}^n s_j(\gamma \circ E_{\Delta_j})(x) - F(x), g(x)\right]_X \\
\le\ &\left[\epsilon_n \sum_{j=1}^n (\gamma \circ E_{\Delta_j})(x), g(x)\right]_X \\
=\ &[\epsilon_n \gamma(x), g(x)]_X \\
\le\ &\overline{k}\epsilon_n\,||x||_X\,||g(x)||_X
\end{aligned}$$

for $x \in X$, where $\overline{k}$ is some positive number. $[[\int_m^M \lambda d(\gamma \circ E_\lambda) - F](x), g(x)]_X = 0$ by the continuity of the quasi-product and $[[\int_m^M \lambda d(\gamma \circ E_\lambda)](x), g(x)]_X = [F(x), g(x)]_X$ thus. By property (c) of the quasi-product,

$$\left[\sum_{j=1}^n \lambda_j(\gamma \circ E_{\Delta_j})(x), g(x)\right]_X \equiv \sum_{j=1}^n \lambda_j\left[(\gamma \circ E_{\Delta_j})(x), g(x)\right]_X$$

for $x \in X$ and

$$\begin{aligned}
&\int_m^M \lambda dw_x(\lambda) \\
\equiv\ & \lim_{n\to\infty} \sum_{j=1}^n \lambda_j \left[(\gamma \circ E_{\Delta_j})(x), g(x)\right]_X \\
\equiv\ & \lim_{n\to\infty} \left[\sum_{j=1}^n \lambda_j (\gamma \circ E_{\Delta_j})(x), g(x)\right]_X \\
=\ & [F(x), g(x)]_X
\end{aligned}$$

thus. □

In the following, the spectral resolution of the generalized real definite operator in terms of operator convergence is given. The key sufficient condition for the operator convergence in this subsection and in Section 4.1 is defined below.

Definition 3.22. The uniform spectral representation condition on a unital Banach algebra X is as follows: There exists a quasi-product which

(a) is square bounded below and preserves the positivity
or
(b) has the left integral domain.

Theorem 3.23. *Let $F \in B(X)$ be generalized real definite. If the uniform spectral representation condition on X holds, then F has the spectral representation*

$$F = \int_m^M \lambda d(\gamma \circ E_\lambda),$$

where $[m, M]$ is some bounded interval depending on F.

Proof. $\int_m^M \lambda d(\gamma \circ E_\lambda)$ exists by Lemma 3.19. If the quasi-product has the left integral domain, i.e., the uniform spectral representation condition (b) satisfied, the equation $[[\int_m^M \lambda d(\gamma \circ E_\lambda) - F](x), g(x)]_X = 0$ implies that $F(x) = [\int_m^M \lambda d(\gamma \circ E_\lambda)](x)$ for every $x \in X$. Then by Lemma 3.19, $F = \int_m^M \lambda d(\gamma \circ E_\lambda)$.

Next, assume that the uniform spectral representation condition (a) holds. Let $F_n = \sum_{j=1}^n s_j(\gamma \circ E_{\Delta_j})$. Then

$$0 \le F_n - F \le \epsilon_n \sum_{j=1}^n \gamma \circ E_{\Delta_j}$$

and thus

$$0 \le (F_n - F)^2 \le \epsilon_n^2 \left(\sum_{j=1}^n \gamma \circ E_{\Delta_j}\right)^2$$

by Theorem 3.7. Because the quasi-product is square bounded below, there exist positive numbers k_1 and k_2 such that for $x \in X$,

$$\begin{aligned} & k_1 \left|\left|F_n(x) - F(x)\right|\right|_X^2 \left|\left|g(x)\right|\right|_X \\ \leq\ & \left[(F_n - F)^2 (x), g(x)\right]_X \\ \leq\ & \left[\epsilon_n^2 \left(\sum_{j=1}^{n} \gamma \circ E_{\Delta_j}\right)^2 (x), g(x)\right]_X \\ \leq\ & k_2 \epsilon_n^2 \left|\left|x\right|\right|_X^2 \left|\left|g(x)\right|\right|_X . \end{aligned}$$

Hence $||F_n - F||_{B(X)} \underset{n\to\infty}{\longrightarrow} 0$ holds. □

Remark 3.24. Theorem 3.21 and Theorem 3.23 under only the uniform spectral representation condition (b) also hold as X is a Banach space, i.e., the condition for X being relaxed. The key is to use $F \circ E_S$ in place of $F 1_S$ in the associated results, where S containing 0 is a subset of X.

4. Extensions

In this section, the operational calculus of the bounded generalized real definite operators is given in the first subsection. Furthermore, the bounded generalized real definite operators with the spectral representation in the sense of operator convergence turns out to be associated with a class of operators defined in the second subsection.

4.1. **Operational calculus.** In this subsection, the extensions of the spectral theorem in the previous subsection to polynomial functions and continuous functions are stated in two theorems, Theorem 4.3 and Theorem 4.9, respectively. Let X be a unital Banach algebra as in Section 3.3. The polynomial of the generalized real definite operator is defined below. Note that the proofs of the lemmas in this subsection are delegated to Section 6.

Definition 4.1. Let $B(X)$ be a unital Banach algebra with the multiplication operation $*$ given in Theorem 2.7. Let $p(\lambda) = \sum_{i=0}^{n} a_i \lambda^i$ be a polynomial in λ with real coefficients a_i, i.e., p being over the real field. Then $p(F) = \sum_{i=0}^{n} a_i F^{*i}$, where $F \in B(X)$ and $F^{*0} = e$.

Lemma 4.2. *Let $F_{1n}, F_1, F_{2n}, F_2 \in B(X)$. If $F_{1n} \underset{n\to\infty}{\longrightarrow} F_1$ and $F_{2n} \underset{n\to\infty}{\longrightarrow} F_2$ in the sense of operator convergence, then $F_{1n} * F_{2n} \underset{n\to\infty}{\longrightarrow} F_1 * F_2$ in the sense of operator convergence.*

Theorem 4.3. *Let $F \in B(X)$ be generalized real definite. Then $p(F)$ has the spectral representation*

$$p(F) = \int_m^M p(\lambda\gamma) d1_\lambda$$

if the uniform spectral representation condition on X holds.

Proof. Because for $k > j$,

$$1_{\Delta_j} 1_{\Delta_k} = \left(1_{s_j} - 1_{s_{j-1}}\right)\left(1_{s_k} - 1_{s_{k-1}}\right) = 1_{s_j} - 1_{s_j} - 1_{s_{j-1}} + 1_{s_{j-1}} = 0$$

by Lemma 3.12 (b) and thus $(\gamma \circ E_{\Delta_j})(\gamma \circ E_{\Delta_k}) = \gamma^2 1_{\Delta_j} 1_{\Delta_k} = 0$. Therefore,

$$\left[\sum_{j=1}^{n} \lambda_j (\gamma \circ E_{\Delta_j})\right]^{*k} = \sum_{j=1}^{n} \lambda_j^k (\gamma \circ E_{\Delta_j})^{*k} = \sum_{j=1}^{n} (\lambda_j \gamma)^{*k} 1_{\Delta_j},$$

where k is a nonnegative integer. By the condition imposed on the quasi-product and then using Theorem 3.23, $\sum_{j=1}^{n} \lambda_j (\gamma \circ E_{\Delta_j}) \underset{n\to\infty}{\longrightarrow} F$ in the sense of operator convergence. By Lemma 4.2,

$$\sum_{j=1}^{n} (\lambda_j \gamma)^{*k} 1_{\Delta_j} = \left[\sum_{j=1}^{n} \lambda_j (\gamma \circ E_{\Delta_j})\right]^{*k} \underset{n\to\infty}{\longrightarrow} F^{*k}$$

and hence

$$p(\lambda_1 \gamma) 1_{s_1} + \sum_{j=2}^{n} p(\lambda_j \gamma) 1_{\Delta_j} = p\left[\sum_{j=1}^{n} \lambda_j (\gamma \circ E_{\Delta_j})\right] \underset{n\to\infty}{\longrightarrow} p(F),$$

i.e., $p(F) = \int_m^M p(\lambda\gamma) d1_\lambda$ in the sense of operator convergence. □

For a bounded self-adjoint linear operator T, the normed value of $p(T)$ is not greater than the normed value of $p(\lambda)$, where $p(\lambda)$ is considered as an element in the space of all continuous functions defined on some compact interval. The following lemma can be considered as the counterpart of the one for the bounded self-adjoint linear operator. Let $C([m, M]$ be the Banach space of all real-valued continuous functions defined on $[m, M]$ with the supremum norm.

Lemma 4.4. *Let $F \in B(X)$ be generalized real definite. If the uniform spectral representation condition* (a) *holds or the uniform spectral representation condition* (b) *holds with X being commutative and the quasi-product being square bounded below, then there exist a positive number $\overline{k}$ and a bounded interval $[m, M]$ depending on F such that for any polynomial function p with real coefficients,*

$$||p(F)||_{B(X)} \le \overline{k} \max_{\lambda \in [m,M]} |p(\lambda)| = \overline{k}\, ||p||_{C([m,M])}\,.$$

For any $f \in C([m, M])$, the operator $f(F)$ and its spectral theorem can be defined and established based on Lemma 4.4. The following theorem gives the existence of the limit operator of a sequence of polynomials of the generalized real definite operator.

Theorem 4.5. *Let $F \in B(X)$ be generalized real definite. If the uniform spectral representation condition* (a) *holds or the uniform spectral representation condition* (b) *holds with X being commutative and the quasi-product being square bounded below, then there exists a bounded interval $[m, M]$ depending on F such that $\{p_n(F)\}$ converges to an operator in $B(X)$ in the norm $||\cdot||_{B(X)}$, where $\{p_n\}$ is any convergent sequence of polynomial functions with real coefficients in the space $C([m, M])$.*

Proof. By Lemma 4.4, for any $\epsilon > 0$, there exists a positive number N such that for $n, l > N$,

$$||p_n(F) - p_l(F)||_{B(X)} \leq \overline{k}\, ||p_n - p_l||_{C([m,M])} < \epsilon$$

and thus $\{p_n(F)\}$ is a Cauchy sequence in the Banach space $B(X)$, where $\overline{k}$ is some positive number. $\square$

According to the above theorem, the operator corresponding to any $f \in C([m, M])$ and the generalized real definite operator $F \in B(X)$ can be defined as follows.

Definition 4.6. Let $B(X)$ be a unital Banach algebra. If there exists a sequence of polynomial functions $\{p_n\}$ defined on $[m, M]$ over the real field converging uniformly to f in $C([m, M])$, i.e., the convergence being in the norm $|| \cdot ||_{C([m,M])}$, and the limit of the sequence of operators $\{p_n(F)\}$ in the norm $|| \cdot ||_{B(X)}$ corresponding to the generalized real definite operator $F \in B(X)$ exists, then the limit in $B(X)$ is denoted by $f(F)$.

As the above condition on X holds, $f(F)$ exists and is unique, i.e., $f(F)$ being the limit corresponding to any sequence of polynomial functions converging to f in $C([m, M])$, as indicated by the corollary below.

Corollary 4.7. *Let $F \in B(X)$ be generalized real definite. If the uniform spectral representation condition* (a) *holds or the uniform spectral representation condition* (b) *holds with X being commutative and the quasi-product being square bounded below, then $f(F)$ exists for any $f \in C([m, M])$ and is independent of the choice of the sequence of polynomial functions in $C([m, M])$, i.e., $f(F)$ being the limit of any sequence of operators $\{p_n(F)\}$ satisfying that $\{p_n\}$ defined on $[m, M]$ over the real field converges uniformly to f in $C([m, M])$, where $[m, M]$ is some bounded interval depending on F.*

Proof. Because there exists a sequence of polynomial functions converging uniformly to f in $C([m, M])$ by Weierstrass theorem, $f(F)$ exists by Theorem 4.5 and Definition 4.6. Next, let $\{p_n\}$ and $\{p_n^*\}$ be the sequences of polynomial functions both converging uniformly to f in $C([m, M])$. Then $p_n(F) \underset{n\to\infty}{\longrightarrow} f(F)$ and $p_n^*(F) \underset{n\to\infty}{\longrightarrow} f^*(F)$ by Theorem 4.5. Furthermore, because

$$\begin{aligned} & ||f(F) - f^*(F)||_{B(X)} \\ \leq \;& ||f(F) - p_n(F)||_{B(X)} + ||p_n(F) - p_n^*(F)||_{B(X)} + ||p_n^*(F) - f^*(F)||_{B(X)} \end{aligned}$$

and

$$||p_n(F) - p_n^*(F)||_{B(X)} \leq \overline{k}\, ||p_n - p_n^*||_{C([m,M])} \,,$$

letting $n \to \infty$ gives $f(F) = f^*(F)$. $\square$

The following lemma is an extension of Lemma 4.4 to continuous functions and can be used to prove the spectral theorem corresponding to the continuous functions.

Lemma 4.8. *Let $F \in B(X)$ be generalized real definite. If the uniform spectral representation condition* (a) *holds or the uniform spectral representation condition* (b) *holds with X being commutative and the quasi-product being square bounded* *below, then there exist a positive number $\overline{k}$ and a bounded interval $[m, M]$ depending on F such that for any $f \in C([m, M])$,*

$$||f(F)||_{B(X)} \leq \overline{k} \max_{\lambda \in [m,M]} |f(\lambda)| = \overline{k}\, ||f||_{C([m,M])}\,.$$

Theorem 4.9. *If the uniform spectral representation condition* (a) *holds or the uniform spectral representation condition* (b) *holds with X being commutative and the quasi-product being square bounded below, then for a generalized real definite operator $F \in B(X)$ and any $f \in C([m, M])$, $f(F)$ has the spectral representation*

$$f(F) = \int_m^M f(\lambda\gamma) d1_\lambda,$$

where $[m, M]$ is some bounded interval depending on F.

Proof. There exist a bounded interval $[m, M]$ and a positive number $\bar{k}$ such that $||f(\lambda\gamma)||_{B(X)} \leq \overline{k}||f||_{C([m,M])}$ for any $f \in C([m, M])$ and $\lambda \in [m^*, M^*]$ by Lemma 4.8 with $1_{m^*}(x) = 0$ for $x \neq 0$, $1_{M^*} = 1$, and $[m^*, M^*] \subset [m, M]$. Let $\{p_l\}$ be a sequence of polynomial functions defined on $[m, M]$ over the real field converging uniformly to f in $C([m, M])$. By Definition 4.6, Theorem 4.3, Corollary 2.3, and Lemma 4.8, for every $\epsilon > 0$, there exist positive numbers N and N_l such that for $n > N_l$ and $l > N$, $||p_l(F) - f(F)||_{B(X)} \leq \epsilon/3$, $||p_l(\lambda_1\gamma)1_{s_1} + \sum_{j=2}^n p_l(\lambda_j\gamma)1_{\Delta_j} - p_l(F)||_{B(X)} \leq \epsilon/3$, and

$$\begin{aligned}
&\left|\left|f(\lambda_1\gamma)1_{s_1} + \sum_{j=2}^n f(\lambda_j\gamma)1_{\Delta_j} - p_l(\lambda_1\gamma)1_{s_1} - \sum_{j=2}^n p_l(\lambda_j\gamma)1_{\Delta_j}\right|\right|_{B(X)} \\
\leq\ & \max_{m^* \leq \lambda \leq M^*} ||f(\lambda\gamma) - p_l(\lambda\gamma)||_{B(X)} \\
\leq\ & \overline{k}\, ||f - p_l||_{C([m,M])} \\
\leq\ & \epsilon/3.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\left|\left|f(\lambda_1\gamma)1_{s_1} + \sum_{j=2}^n f(\lambda_j\gamma)1_{\Delta_j} - f(F)\right|\right|_{B(X)} \\
\leq\ & \left|\left|f(\lambda_1\gamma)1_{s_1} + \sum_{j=2}^n f(\lambda_j\gamma)1_{\Delta_j} - p_l(\lambda_1\gamma)1_{s_1} - \sum_{j=2}^n p_l(\lambda_j\gamma)1_{\Delta_j}\right|\right|_{B(X)} \\
+\ & \left|\left|p_l(\lambda_1\gamma)1_{s_1} + \sum_{j=2}^n p_l(\lambda_j\gamma)1_{\Delta_j} - p_l(F)\right|\right|_{B(X)} + ||p_l(F) - f(F)||_{B(X)} \\
\leq\ & \epsilon
\end{aligned}$$

and the result holds. □

Remark 4.10. For the generalized real definite operator $\tilde{F} \in B(X)$ with $\tilde{F}(0) \neq 0$, i.e., $\tilde{F} = F + \tilde{F}(0)e$ and the projection indicator 1_λ corresponding to the null space of F_λ^+, the spectral resolution of the polynomial $p(\tilde{F})$ given the condition in Theorem 4.3 is

$$p(\tilde{F}) = \int_m^M p\left[\lambda\gamma + \tilde{F}(0)e\right] d1_\lambda.$$

Similarly, the spectral integral of the operator $f(\tilde{F})$ given the condition in Theorem 4.9 is

$$f(\tilde{F}) = \int_m^M f\left[\lambda\gamma + \tilde{F}(0)e\right] d1_\lambda.$$

4.2. **Nonlinear spectral operators.** In [4], the theory of the linear spectral operators has been discussed thoroughly. In this subsection, the nonlinear spectral operators based on the projection operator given in Definition 3.11 are defined and a basic result, Theorem 4.14, is given.

Definition 4.11. Let X be a normed space. A spectral projection E on $(m, M]$ is an operator-valued function from the subsets $\cup_{i=1}^n (a_i, b_i]$ of $(m, M]$ into $B(X)$, $m, M \in R, (a_i, b_i] \subset (m, M]$, with the following properties.

(a) $E\{(a_i, b_i]\}$ is a projection operator, i.e., $E\{(a_i, b_i]\}(x) = x$ if $x \in S$ and $E\{(a_i, b_i]\}(x) = 0$ otherwise, where S containing 0 is some subset of X.
(b) $E(\phi) = 0$ and $E\{(m, M]\} = I$.
(c)

$$E\{(a_1, b_1] \cap (a_2, b_2]\} = E\{(a_1, b_1]\} \circ E\{(a_2, b_2]\} = E\{(a_2, b_2]\} \circ E\{(a_1, b_1]\}.$$

In addition, if $(a_1, b_1] \cap (a_2, b_2] = \phi$, then

$$E\{(a_1, b_1] \cup (a_2, b_2]\} = E\{(a_1, b_1]\} + E\{(a_2, b_2]\}.$$

The existence of the spectral operators of interest is due to the following lemma. The proof of this lemma is analogous to Lemma 3.19 and is not presented.

Lemma 4.12. *Let X be a Banach space and $\{s_j\}$ be a partition of a bounded interval $[m, M]$ with $m = s_0 < s_1 < \cdots < s_n = M$, $s_j - s_{j-1} < \epsilon_n$ and $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Then, $\sum_{j=1}^n f(\lambda_j) E\{(s_{j-1}, s_j]\}$ converges to an operator in $B(X)$ with respect to the norm topology $||\cdot||_{B(X)}$ and the convergence is independent of the choice of the points $\lambda_j \in (s_{j-1}, s_j]$ as $n \to \infty$, where $f \in C([m, M])$.*

Based on the above lemma, the resulting limit operator can be defined.

Definition 4.13. Let X be a Banach space and $\{s_j\}$ be a partition of a bounded interval $[m, M]$ with $m = s_0 < s_1 < \cdots < s_n = M$, $s_j - s_{j-1} < \epsilon_n$ and $\epsilon_n \underset{n\to\infty}{\longrightarrow} 0$. Then, the limit operator F of $\sum_{j=1}^n f(\lambda_j) E\{(s_{j-1}, s_j]\}$ as $n \to \infty$ with respect to the norm topology $||\cdot||_{B(X)}$ is denoted as

$$F = \int_m^M f(\lambda) dE,$$

where $\lambda_j \in (s_{j-1}, s_j]$ and $f \in C([m, M])$. The operator F is referred to as the nonlinear spectral operator with respect to the spectral projection E on $(m, M]$ and the function f. The class of the nonlinear spectral operators with respect to the spectral projection E on $(m, M]$ and any function $f \in C([m, M])$ is denoted as $S_{E,C([m,M])}(X)$.

Theorem 4.14. *$S_{E,C([m,M])}(X)$ is a subspace of $B(X)$, where X is a Banach space. Further, If $E\{(m, \bar{\lambda}]\} - E\{(m, \underline{\lambda}]\} \neq 0$ for any $\underline{\lambda}, \bar{\lambda} \in (m, M]$ and $\underline{\lambda} < \bar{\lambda}$, then $S_{E,C([m,M])}(X)$ is a Banach space.*

Proof. $S_{E,C([m,M])}(X)$ is a subspace of $B(X)$ because for two operators $F_1, F_2 \in S_{E,C([m,M])}(X)$ corresponding to f_1 and f_2 in $C([m, M]$, respectively, and $\alpha \in K$,

$$\begin{aligned} & (\alpha F_1 + F_2) \\ = & \lim_{n\to\infty} \sum_{j=1}^{n} \alpha f_1(\lambda_j) E\{(s_{j-1}, s_j]\} \\ & + \lim_{n\to\infty} \sum_{j=1}^{n} f_2(\lambda_j) E\{(s_{j-1}, s_j]\} \\ = & \lim_{n\to\infty} \sum_{j=1}^{n} [\alpha f_1(\lambda_j) + f_2(\lambda_j)] E\{(s_{j-1}, s_j]\} \\ = & \int_m^M [\alpha f_1(\lambda) + f_1(\lambda)] \, dE, \end{aligned}$$

and $\alpha f_1 + f_2 \in C([m, M])$.

To prove the completeness of $S_{E,C([m,M])}(X)$, let $F_n \underset{n\to\infty}{\longrightarrow} F$, where $\{F_n\} \subset S_{E,C([m,M])}(X)$ and $F \in B(X)$. Then $\{F_n\}$ is Cauchy. Further, if the required condition holds, there exists x_λ depending on λ such that for any $\lambda \in (m, M]$,

$$\begin{aligned} & \left|\left| \left\{ \sum_{j=1}^{n} [f_l(\lambda_j) - f_m(\lambda_j)] E\{(s_{j-1}, s_j]\} \right\} (x_\lambda) \right|\right|_X \\ = & \, ||[f_l(\lambda) - f_m(\lambda)] \, x_\lambda||_X \\ = & \, |f_l(\lambda) - f_m(\lambda)| \, ||x_\lambda||_X \, , \end{aligned}$$

where f_l and f_m lie in $C([m, M])$ corresponding to the spectral operators F_l and F_m, respectively. Then by letting $n \to \infty$ in the above equation, for any $\epsilon > 0$, there exists a positive integer N such that for $l, m > N$,

$$||f_l - f_m||_{C([m,M])} \leq \epsilon.$$

Hence $\{f_n\}$ is Cauchy and there exists a function $f \in C([m, M])$ such that $f_n \underset{n\to\infty}{\longrightarrow} f$ with respect to $|| \cdot ||_{C([m,M])}$ owing to the completeness of $C([m, M])$. Then,

$F_n \underset{n\to\infty}{\longrightarrow} \int_m^M f dE$ because for $x \neq 0$,

$$
\begin{aligned}
& \left\| \left[F_n - \int_m^M f(\lambda) dE \right] (x) \right\|_X \\
= \quad & \left\| \left\{ \int_m^M \left[f_n(\lambda) - f(\lambda) \right] dE \right\} (x) \right\|_X \\
\leq \quad & ||f_n - f||_{C([m,M])} \, ||x||_X \, .
\end{aligned}
$$

Therefore, $F = \int_m^M f dE$ and $F \in S_{E,C([m,M])}(X)$, i.e., $S_{E,C([m,M])}(X)$ being closed. □

Remark 4.15. If $\gamma = I$, the bounded generalized real definite operators with the spectral representation lie in $S_{E,C([m,M])}(X)$. Note that a more general class of nonlinear spectral operators can be defined as the limiting operators of the operators $\sum_{j=1}^{n} f(\lambda_j)(\gamma \circ E\{(s_{j-1}, s_j]\})$ as $n \to \infty$ with respect to the norm topology $||\cdot||_{B(X)}$ and can be denoted as

$$F = \int_m^M f(\lambda) d(\gamma \circ E).$$

Moreover, Lemma 4.12 and Theorem 4.14 can be generalized for the class of the nonlinear spectral operators.

If X is a Hilbert space, a class of projection operators other than the ones give in Definition 3.11 and in Definition 4.11 (a) can be defined as follows. For a subset S containing 0 of a closed subspace Y of a Hilbert space X, the corresponding projection operator $E_S : X \to X$ is defined by $E_S(x) = y$ if $y \in S$ and $E_S(x) = 0$ otherwise, where $x = y + y^{\perp}$, $y \in Y$, $y^{\perp} \in Y^{\perp}$, and where $Y^{\perp}$ is the orthogonal complement of Y. Then the spectral representation of a bounded symmetric linear operator defined on X is a special case of the corresponding nonlinear spectral operator given in the previous paragraph.

5. Examples

In this section, the nonlinear generalizations of two classes of linear operators, including linear operators on finite dimensional spaces and compact linear operators, and nonlinear counterparts of the linear multiplication operator and the linear differentiation operator are given via examples. In addition, some associated applications are given.

5.1. **Operators on finite dimensional spaces.** In this subsection, let X be an m dimensional vector space. Note that any linear operator T on X is bounded, i.e., $T \in B(X)$. Further, let $\{T_i : i = 1, \ldots, m^2\}$ be a basis of the space of all linear operators on X. Then the linear operator T has the form $T = \sum_{i=1}^{m^2} a_i T_i$, where $a_i \in K$. The following examples can be considered as different nonlinear generalizations of T.

Example 5.1. Let $F = \sum_{i=1}^{n} a_i F_i$, where $F_i \in V(X, X)$. If $F_i \in B(X)$, then $F \in B(X)$. F is generalized real definite if F_i are generalized real definite and $a_i \in R$. F_i can be nonlinear, for example, $F_i = x/\|x\|, x \neq 0$ or $F_i = x^2$ for X

being a Banach algebra. If F_i is the projection operators given in Definition 3.11, $\cup_{i=1}^n R(F_i) = X$, $R(F_i) \cap R(F_j) = \{0\}, i \neq j$, and γ is the identity operator, then a_i are the g-eigenvalues of F corresponding to the g-eigenvectors lying in $R(F_i)\backslash\{0\}$, the g-resolvent set of F is $\rho(F) = C\backslash\{a_i : i = 1, \ldots, n\}$, i.e., the set of g-eigenvalues being the g-spectrum, and F is a nonlinear spectral operator for $a_i \in R$, i.e., $F \in S_{E,C([m,M])}(X)$.

The possibly nonlinear operator equation $F(x) = y$ for $x, y \in X$, can be solved by the nonlinear spectral resolution. If F has the nonlinear spectral representation $F = \sum_{i=1}^n a_i(\gamma \circ E_{\Delta_i})$, the nontrivial solution $x_h \neq 0$ for the homogeneous equation $F(x) = 0$ is any vector in the union of the projected sets corresponding to $a_j = 0$, i.e., x_h lying in the union of the sets $R(E_{\Delta_j})$ corresponding to $a_j = 0$. On the other hand, for the nonhomogeneous equation, i.e., $y \neq 0$, the solution exists if $y \in \cup_{i=1}^n R[a_i(\gamma \circ E_{\Delta_i})]$ and the nonhomogeneous equation has no solution if $y \notin \cup_{i=1}^n R[a_i(\gamma \circ E_{\Delta_i})]$. Unlike a solution in a linear system, $x_h + x_p$ might not be the solution for $F(x) = y$. Note that the operator γ can be the ones other than the identity operator, for example, for $x \neq 0$, $\gamma(x) = x/\|x\|_X$ and $\gamma(x) = x^2/\|x\|_X, x^2 \neq 0$, along with X being a Banach space or a Banach algebra, respectively, and thus more operator equations corresponding to the nonlinear spectral operators can be solved. If γ is the identity operator and for $y \in R(a_j E_{\Delta_j})\backslash\{0\}, a_j \neq 0$, then $x_p = a_j^{-1} y$ is a solution of the nonhomogeneous equation $F(x) = y$, i.e., the equation being solvable for any $y \in [\cup_{a_i \neq 0} R(a_i E_{\Delta_i})]\backslash\{0\}$.

Example 5.2. Let F defined by $F(x) = \sum_{i=1}^n A_i(x)T_i(x), x \in X$, where $A_i \in V(X, K)$. If $R(A_i)$ is bounded, i.e., $|A_i(x)| \leq c_i, x \in X$, for example, for $x \neq 0$, $A_i(x) = c_i \min\{\|x\|_X, 1/\|x\|_X\}$ or $A_i(x) = c_i\|x^2\|_X/\|x\|_X^2$ with X being a Banach algebra, and c_i are some positive numbers, then $F \in B(X)$. F is generalized real definite if T_i are generalized real definite and A_i are real valued functionals, i.e., $K = R$. If T_i are linear projection operators, and $R(T_i) \cap R(T_j) = \{0\}, i \neq j$, then $A_i(x)$ is the g-eigenvalues of F corresponding to the g-eigenvectors $x \in R(T_i)\backslash\{0\}$ and the identity operator γ.

The possibly nonlinear equation of interest is $F(x) = \sum_{i=1}^n A_i(x)T_i(x) = y$ for $x, y \in X$. If y does not lie in the space spanned by $\cup_{i=1}^n R(T_i)$, then the operator equation has no solution. On the other hand, any nonzero vector lies in the set $\{(\cap_{i=1}^n \{x : A_i(x) = 0\}) \cup [\cap_{i=1}^n N(T_i)]\}$ is a nontrivial solution of the homogeneous equation $F(x) = 0$. In addition, if T_i are the linear projection operators with $R(T_i)$ orthogonal to $R(T_i)$, $i \neq j$, $A_i = A_i \circ T_i$, A_i is homogeneous of degree 1 for $i = 1, \ldots, n$, i.e., $A_i(\alpha x) = \alpha A_i(x), \alpha \in C$, $A_i(x) \neq 0$ for $x \in R(T_i)\backslash\{0\}$, and $K = C$, then the solutions exist for the nonhomogeneous equation $F(x) = y$ with any nonzero vector y lying in the space spanned by $\cup_{i=1}^n R(T_i)$ and the solutions are $\sum_{i=1}^n c_i y_i$, where $y = \sum_{i=1}^n y_i$, $y_i \in R(T_i)$, and c_i satisfy that $c_i^2 A_i(y_i) = 1$ for $y_i \neq 0$ and $c_i = 0$ for $y_i = 0$.

The other operator equation $F(x) = \sum_{i=1}^n A_i(x)(\gamma \circ E_{S_i})(x) = y$ for $x, y \in X$, can be solved under some sufficient conditions, where E_{S_i} is the projection operators given in Definition 3.11, S_i contain not only the zero vector, and $S_i \cap S_j = \{0\}, i \neq j$. If $y \in \cup_{i=1}^n R[A_i(\gamma \circ E_{S_i})]$, then at least one solution exists and no solution exists otherwise. Specifically, if γ is the identity operator, then the results are given as follows. Any nonzero vector lies in the set $\{(\cap_{i=1}^{n}\{x : A_i(x) = 0\}) \cup (\cup_{i=1}^{n} S_i)^c\}$ is a nontrivial solution of the homogeneous equation $F(x) = 0$, while for $y \in [\cup_{i=1}^{n} R(A_i E_{S_i})]\backslash\{0\}$, there exists at least one solution with the form $cy, c \in K$, for the nonhomogeneous equation $F(x) = y$.

5.2. **Compact operators.** A compact operator, possibly nonlinear, maps a bounded set in a normed space X into a precompact (or relatively compact) set in X. The following are two classes of compact operators on normed spaces and Hilbert spaces, respectively.

Example 5.3. Suppose the operator γ give in Definition 3.14 is the identity operator. Let F defined by $F(x) = \sum_{i=1}^{n} A_i(x)T_i(x), x \in X$, where T_i are compact linear operators on X, $A_i \in V(X,K)$, and $R(A_i)$ are bounded. $F \in B(X)$ and F is a compact operator. In addition, F is generalized real definite if T_i are all generalized real definite and A_i are all real valued functionals. If $n = 1$, i.e., $F = A_1T_1$, and λ is the eigenvalue of T_1 corresponding the eigenvector x_λ in X, then $\lambda A_1(x)$ is the g-eigenvalue of F corresponding to the g-eigenvector x_λ, but $\lambda A_1(x)$ might not be the g-eigenvalue for the vector cx_λ, where $c \in K$.

Example 5.4. If X is a Hilbert space with the inner product $< \cdot,\cdot >_X$, any compact linear operator T has the representation $T = \sum_{i=1}^{\infty} s_iT_i$, where the singular values $\{s_i\}$ converging to 0 is a decreasing sequence, $s_i \geq 0$, and the linear operators $T_i(x) =< x, e_i >_X \ f_i$, and where both $\{e_i\}$ and $\{f_i\}$ are orthonormal sequences in X. Therefore, a possibly nonlinear operator F can be defined by $F(x) = \sum_{i=1}^{\infty} A_i(x)T_i(x)$, where $A_i \in V(X,K)$ and $\sum_{i=1}^{\infty} \|A_i\|_{sup} < \infty$, i.e., $\{\|A_i\|_{sup}\}$ lying in l^1 space (see [6], Chapter 5), and where $\|A_i\|_{sup} = sup_{x\in X}|A_i(x)|$. $F \in B(X)$ and F is a compact operator. Furthermore, F is generalized real definite if T_i are all generalized real definite and A_i are all real valued functionals. For $1 \leq i \leq n$, $A_i(x)$ are the g-eigenvalues of F corresponding to the g-eigenvectors $x \neq 0$ lying in the one dimensional space spanned by e_i and $\gamma = \sum_{i=1}^{n} T_i + P$, where P is the linear projection operator with $R(P)$ being the orthogonal complement of the space spanned by $\{e_1, \ldots, e_n\}$. On the other hand, any nonzero vector lying in the orthogonal complement of the closure of the space spanned by $\{e_i\}$ is the g-eigenvector corresponding to the g-eigenvalue 0.

By replacing the compact linear operators T_i in Example 5.3 and Example 5.4 with some operators depending on the projection operator E_S and imposing some condition on S, two classes of compact operators are given as follows.

Example 5.5. Let $F_1 = \sum_{i=1}^{n} A_{1i}(F \circ E_{S_{1i}})$ and $F_2 = \sum_{i=1}^{\infty} A_{2i}(F \circ E_{S_{2i}})$, where $F \in BC(X,X)$, S_{1i} and S_{2i} all containing 0 are precompact sets in the normed space X, A_{1i} and A_{2i} are the functionals A_i given in Example 5.3 and Example 5.4, respectively. $F_1, F_2 \in B(X)$ and both F_1 and F_1 are compact operators.

If the operator γ give in Definition 3.14 is equal to F, $R(S_{1i}) \cap R(S_{1j}) = \{0\}$ and $R(S_{2i}) \cap R(S_{2j}) = \{0\}$ for $i \neq j$, $A_{1i}(x)$ are the g-eigenvalues of F_1 corresponding to the g-eigenvectors x lying in $S_{1i}\backslash\{0\}$, while $A_{2i}(x)$ are the g-eigenvalues of F_2 corresponding to the g-eigenvectors x lying in $S_{2i}\backslash\{0\}$.

5.3. **Multiplication operator and differentiation operator.** In this subsection, two examples of nonlinear operators associated with the linear multiplication operator and the linear differentiation operator (see [5], Chapter 10.7) which are related to the position operator and the momentum operator (see [5], Chapter 11), respectively, in quantum mechanics are given. Some facts about the spectrums of these operators are proved in Theorem 5.9.

Let $L_p(-\infty,\infty), 1 \leq p < \infty$, be the spaces of all complex-valued functions x defined on $(-\infty,\infty)$ satisfying that $|x|^p$ is integrable with respect to the Lebesgue measure.

Example 5.6. Let the linear operator $T_m : S_{T_m} \to L_2(-\infty,\infty)$ defined by $T_m(x) = x_0 x$ for $x \in S_{T_m}$, where $x_0(t) = t$ for $t \in (-\infty,\infty)$ and S_{T_m}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x_0 x \in L_2(-\infty,\infty)$. The relevant nonlinear operator $F_m : S_{F_m} \to L_1(-\infty,\infty)$ defined by $F_m(x) = x_0|x|^2$ for $x \in S_{F_m}$, where S_{F_m}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x_0|x|^2 \in L_1(-\infty,\infty)$. Note that $S_{T_m} \subset S_{F_m}$.

Example 5.7. Let the linear operator $T_d : S_{T_d} \to L_2(-\infty,\infty)$ defined by $T_d(x) = ix^{'}$ for $x \in S_{T_d}$, where $x^{'}$ is the derivative of x and S_{T_d}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x^{'} \in L_2(-\infty,\infty)$. The relevant nonlinear operator $F_d : S_{F_d} \to L_1(-\infty,\infty)$ defined by $F_d(x) = ix^{'}\bar{x}$ for $x \in S_{F_d}$, where S_{F_d}, a subset of $L_2(-\infty,\infty)$, consists of all functions satisfying $x^{'}\bar{x} \in L_1(-\infty,\infty)$, and where $\bar{x}$ is the conjugate function of x. Note that $S_{T_d} \subset S_{F_d}$.

Remark 5.8. In the above examples,

$$< T_m(e), e >_{L_2(-\infty,\infty)} = \left[F_m(e), |e|^2\right]_{L_1(-\infty,\infty)}$$

for any unit vector $e \in S_{T_m}$ and

$$< T_d(e), e >_{L_2(-\infty,\infty)} = \left[F_d(e), |e|^2\right]_{L_1(-\infty,\infty)}$$

for any unit vector $e \in S_{T_d}$, where $< \cdot,\cdot >_{L_2(-\infty,\infty)}$ is the inner product on $L_2(-\infty,\infty)$ and for $x_1, x_2 \in L_1(-\infty,\infty)$ the quasi-product is defined by

$$[x_1, x_2]_{L_1(-\infty,\infty)} = \int_{-\infty}^{\infty} x_1(t)dt \overline{\int_{-\infty}^{\infty} x_2(t)dt}.$$

$e \in S_{T_m}$ or $e \in S_{T_d}$ is referred to as the state function (or the wave function) and T_m and T_d are the operators corresponding to the observables (see [5], Chapter 11.1) in quantum mechanics.

The linear operator T_m is self-adjoint with the spectrum being all of R and has no eigenvalue (see [5], Chapter 10.7). Similarly, the properties of the nonlinear operators given in above examples are as follows.

Theorem 5.9. *If $S_{F_m} = S_{T_m} \cap L_1(-\infty,\infty)$ and $S_{F_d} = S_{T_d} \cap L_1(-\infty,\infty)$, i.e., both $D(F_m) \subset L_1(-\infty,\infty)$ and $D(F_d) \subset L_1(-\infty,\infty)$, the quasi-product is*

$$[x_1, x_2]_{L_1(-\infty,\infty)} = \int_{-\infty}^{\infty} x_1(t)dt \overline{\int_{-\infty}^{\infty} x_2(t)dt}$$

for $x_1, x_2 \in L_1(-\infty,\infty)$, $g(x) = |x|^2$, $\gamma(x) = ||x||_{L_1(-\infty,\infty)} ||x^2||^{-1}_{L_1(-\infty,\infty)} |x|^2$ for $x \neq 0$ and $x \in S_{F_m}$ or $x \in S_{F_d}$, then F_m is generalized real definite but F_d is not *generalized real definite, and both have C as their g-spectrums. However, F_m has no g-eigenvalue but any pure imaginary number is a g-eigenvalue of F_d.*

Proof. The proof for F_m is given first. Let e_x be any unit vector in S_{F_m}. Then

$$\left[F_m(e_x), |e_x|^2\right]_{L_1(-\infty,\infty)} =< T_m(e_x), e_x >_{L_2(-\infty,\infty)} \in R$$

since T_m is self-adjoint. For any $x \in S_{F_m}$, $x \neq 0$,

$$x = ||x||_{L_1(-\infty,\infty)} \left(x/\,||x||_{L_1(-\infty,\infty)}\right) = ||x||_{L_1(-\infty,\infty)}\, e_x$$

and thus

$$\left[F_m(x), |x|^2\right]_{L_1(-\infty,\infty)} = ||x||^4_{L_1(-\infty,\infty)} \left[F_m(e_x), |e_x|^2\right]_{L_1(-\infty,\infty)} \in R,$$

i.e., F_m being generalized real definite. Next is to prove that F_m has no g-eigenvalue with respect to the operator γ. For any $\lambda \in C$, there does not exist $x \neq 0$ on a set of measure greater than 0 satisfying

$$||F_{m\lambda}(x)||_{L_1(-\infty,\infty)} = \int_{-\infty}^{\infty} \left| t - \lambda ||x||_{L_1(-\infty,\infty)} ||x^2||^{-1}_{L_1(-\infty,\infty)} \right| |x(t)|^2\, dt = 0,$$

i.e., F_m having no g-eigenvalue, where $F_{m\lambda} = F_m - \lambda\gamma$. Because $F_{m\lambda}(x) = F_{m\lambda}(\bar{x})$ for any $x \in S_{F_m}$ satisfying $x \neq \bar{x}$, $F_{m\lambda}$ is not injective and $F^{-1}_{m\lambda}$ does not exist for any $\lambda \in C$, i.e., $\rho(F_m)$ being an empty set.
Next, F_d is not generalized real definite because

$$[F_d(x_\mu), g(x_\mu)]_{L_1(-\infty,\infty)} = \frac{-i\mu}{2} \notin R,$$

where $\mu > 0$ and $x_\mu(t) = \mu^{1/2} \exp[(-\mu t)/2]$ for $t \geq 0$ and $x_\mu(t) = 0$ elsewhere. Furthermore,

$$F_d(x_\mu) = \left[\frac{(-i)\mu^{3/2}}{4}\right] \gamma(x_\mu)$$

and

$$F_d(x^*_\mu) = \left[\frac{i\mu^{3/2}}{4}\right] \gamma(x^*_\mu),$$

i.e., any pure imaginary number being a g-eigenvalue, where the functions $x^*_\mu(t) = \mu^{1/2} \exp[(\mu t)/2]$ for $t \leq 0$ and $x^*_\mu(t) = 0$ elsewhere. Finally, because $F_{d\lambda}$ is not injective for any $\lambda \in C$, $\rho(F)$ is an empty set, where $F_{d\lambda} = F_d - \lambda\gamma$.

□

Remark 5.10. If S_{T_d} given in Theorem 5.9 consists of all functions $x \in L_2(-\infty,\infty)$ satisfying $x' \in L_2(-\infty,\infty)$ and being absolutely continuous on every compact interval of R, then T_d and F_d have the following properties.

- T_d is self-adjoint and F_d is generalized real definite on S_{F_d}.
- The spectrum of T_d is all of R but F_d has C as its g-spectrum.
- T_d has no eigenvalue and F_d has no g-eigenvalue.

The above results imply that the self-adjointness or the generalized real definiteness relies on the domain of the operator of interest.

6. Proofs of ancillary results

The proofs of some lemmas, theorems, and corollaries in Section 2.1, Section 3.1, Section 3.2, and Section 4.1, are given in this section.

6.1. **Proofs: Section 2.1.**

The proofs of Theorem 2.2, Corollary 2.3, Theorem 2.4, Corollary 2.5, and Theorem 2.7 are given in the subsection.

6.1.1. *Proof of Theorem 2.2.* First, for $F \in B(X,Y)$, $p(F) \geq 0$. Secondly $p(F) = 0$ gives $\max(\sup_{x\neq 0, x\in X} \|F(x)\|_Y/\|x\|_X, \|F(0)\|_Y) = 0$. Hence $F(x) = 0$ for $x \neq 0$ and $F(0) = 0$, i.e., $F = 0$. Thirdly, for $\alpha \in K$,

$$\begin{aligned}
p(\alpha F) &= \max\left(\sup_{x\neq 0, x\in X} \frac{\|\alpha F(x)\|_Y}{\|x\|_X}, \|\alpha F(0)\|_Y\right) \\
&= |\alpha| \max\left(\sup_{x\neq 0, x\in X} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y\right) \\
&= |\alpha|\, p(F).
\end{aligned}$$

Finally, for $F_1, F_2 \in B(X,Y)$,

$$\begin{aligned}
p(F_1+F_2) &= \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)+F_2(x)\|_Y}{\|x\|_X}, \|F_1(0)+F_2(0)\|_Y\right) \\
&\leq \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_Y}{\|x\|_X} + \sup_{x\neq 0, x\in X} \frac{\|F_2(x)\|_Y}{\|x\|_X},\right. \\
&\qquad \left.\|F_1(0)\|_Y + \|F_2(0)\|_Y\right) \\
&\leq \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_Y}{\|x\|_X}, \|F_1(0)\|_Y\right) \\
&\qquad + \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_2(x)\|_Y}{\|x\|_X}, \|F_2(0)\|_Y\right) \\
&= p(F_1) + p(F_2).
\end{aligned}$$

6.1.2. *Proof of Corollary 2.3.* If $x \neq 0$,

$$\frac{\|F_1(x)\|_Y}{\|x\|_X} \leq \sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_Y}{\|x\|_X} \leq \|F_1\|_{B(X,Y)}$$

and hence $||F_1(x)||_Y \leq ||F_1||_{B(X,Y)}||x||_X$. In addition, if $F_1(0) = F_2(0) = 0$,

$$\begin{aligned}
\|F_2 \circ F_1\|_{B(X,Z)} &= \sup_{x\neq 0, x\in X} \frac{\|(F_2\circ F_1)(x)\|_Z}{\|x\|_X} \\
&\leq \sup_{x\neq 0, x\in X} \frac{\|F_2\|_{B(Y,Z)}\|F_1(x)\|_Y}{\|x\|_X} \\
&= \|F_2\|_{B(Y,Z)} \left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_Y}{\|x\|_X}\right) \\
&= \|F_2\|_{B(Y,Z)}\|F_1\|_{B(X,Y)}.
\end{aligned}$$

6.1.3. *Proof of Theorem 2.4.* Let $\{F_n\}$ be a Cauchy sequence in the space $B(X,Y)$. Then for every positive ϵ, there exists a positive integer N such that for $m,n > N$, $||F_n - F_m||_{B(X,Y)} < \epsilon$. Then if $x \neq 0$,

$$\|F_n(x) - F_m(x)\|_Y \leq \|F_n - F_m\|_{B(X,Y)}\|x\|_X < \epsilon\|x\|_X$$

by Corollary 2.3 and

$$\|F_n(0) - F_m(0)\|_Y \leq \|F_n - F_m\|_{B(X,Y)} < \epsilon.$$

Thus, $\{F_n(x)\}$ is Cauchy in Y for $x \in X$ and $F_n(x) \underset{n\to\infty}{\longrightarrow} y, y \in Y$, owing to the completeness of Y. Define an operator $F : X \to Y$ by $F(x) = y$. For $x \neq 0$, by the continuity of the norm,

$$\begin{aligned}\|F_n(x) - F(x)\|_Y &= \|F_n(x) - \lim_{m\to\infty} F_m(x)\|_Y = \lim_{m\to\infty} \|F_n(x) - F_m(x)\|_Y \\ &\leq \epsilon\|x\|_X.\end{aligned}$$

In addition,

$$\|F_n(0) - F(0)\|_Y = \lim_{m\to\infty} \|F_n(0) - F_m(0)\|_Y \leq \epsilon.$$

Thus, $F_n - F \in B(X,Y)$ and $F \in B(X,Y)$. Finally, because $||F_n - F||_{B(X,Y)} \leq \epsilon$, $\{F_n\}$ converges to F.

Let $\{F_n\}$ be a Cauchy sequence in the space $BC(X,Y)$. There exists an operator $F \in B(X,Y)$ such that $F_n \underset{n\to\infty}{\longrightarrow} F$ by the completeness of $B(X,Y)$. Then as $x_m \underset{m\to\infty}{\longrightarrow} x$,

$$\begin{aligned}\|F(x_m) - F(x)\|_Y &\leq \|F_n(x_m) - F(x_m)\|_Y + \|F_n(x_m) - F_n(x)\|_Y \\ &\quad +\|F_n(x) - F(x)\|_Y\end{aligned}$$

and hence $F(x_m) \underset{m\to\infty}{\longrightarrow} F(x)$, i.e., $F \in BC(X,Y)$, by Corollary 2.3 and the continuity of F_n.

6.1.4. *Proof of Corollary 2.5.* By condition (b), there exist positive numbers δ and M_1 such that $\|F(x)\|_Y/\|x\|_X \leq M_1$ for $||x||_X < \delta$. By condition (a),

$$\sup_{\{x:\|x\|_X \geq \delta, x\in X\}\cap\mathcal{K}^c} \|F(x)\|_Y/\|x\|_X \leq M/\delta.$$

Let $\delta^* = \max(\delta, \sup_{x\in\mathcal{K}} \|x\|_X)$. Thus,

$$\sup_{\{x:\delta\leq\|x\|_X\leq\delta^*, x\in X\}\cap\mathcal{K}} \|F(x)\|_Y/\|x\|_X \leq M_2$$

since the continuous function $\|F(\cdot)\|_Y/\|\cdot\|_X$ is bounded over the compact set $\{x : \delta \leq \|x\|_X \leq \delta^*, x \in X\} \cap \mathcal{K}$, where $M_2 > 0$. Finally,

$$\begin{aligned}\|F\|_{B(X,Y)} &= \max\Bigg(\sup_{\|x\|_X<\delta, x\in X} \frac{\|F(x)\|_Y}{\|x\|_X}, \sup_{\{x:\delta\leq\|x\|_X\leq\delta^*, x\in X\}\cap\mathcal{K}} \frac{\|F(x)\|_Y}{\|x\|_X}, \\ &\qquad \sup_{\{x:\|x\|_X\geq\delta, x\in X\}\cap\mathcal{K}^c} \frac{\|F(x)\|_Y}{\|x\|_X}, \|F(0)\|_Y\Bigg) \\ &\leq \max\left(M_1, M_2, \frac{M}{\delta}, \|F(0)\|_Y\right)\end{aligned}$$

and hence $F \in B(X,Y)$.

6.1.5. *Proof of Theorem 2.7.* Let $F_1, F_2, F_3 \in B(X)$. First, if $x \neq 0$,

$$\begin{aligned}
[(F_1 * F_2) * F_3](x) &= \frac{(F_1 * F_2)(x)F_3(x)}{\|x\|_X} \\
&= \frac{F_1(x)F_2(x)F_3(x)}{\|x\|_X^2} \\
&= \frac{F_1(x)(F_2 * F_3)(x)}{\|x\|_X} \\
&= [F_1 * (F_2 * F_3)](x).
\end{aligned}$$

Besides,

$$\begin{aligned}
[(F_1 * F_2) * F_3](0) &= (F_1 * F_2)(0)F_3(0) \\
&= F_1(0)F_2(0)F_3(0) \\
&= F_1(0)(F_2 * F_3)(0) \\
&= [F_1 * (F_2 * F_3)](0).
\end{aligned}$$

Secondly, if $x \neq 0$,

$$\begin{aligned}
[F_1 * (F_2 + F_3)](x) &= \frac{F_1(x)(F_2 + F_3)(x)}{\|x\|_X} \\
&= \frac{F_1(x)F_2(x) + F_1(x)F_3(x)}{\|x\|_X} \\
&= (F_1 * F_2)(x) + (F_1 * F_3)(x).
\end{aligned}$$

In addition,

$$\begin{aligned}
[F_1 * (F_2 + F_3)](0) &= F_1(0)(F_2 + F_3)(0) \\
&= F_1(0)F_2(0) + F_1(0)F_3(0) \\
&= (F_1 * F_2)(0) + (F_1 * F_3)(0).
\end{aligned}$$

$[(F_1+F_2)*F_3](x) = (F_1*F_3)(x)+(F_2*F_3)(x)$ and $[\alpha(F_1*F_2)](x) = [(\alpha F_1)*F_2](x) = [F_1 * (\alpha F_2)](x)$ for $x \in X$ and $\alpha \in K$, can be proved analogously. Further,

$$\begin{aligned}
&\|F_1 * F_2\|_{B(X)} \\
&= \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)F_2(x)\|_X}{\|x\|_X^2}, \|F_1(0)F_2(0)\|_X\right) \\
&\leq \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_X}{\|x\|_X} \sup_{x\neq 0, x\in X} \frac{\|F_2(x)\|_X}{\|x\|_X}, \|F_1(0)\|_X \|F_2(0)\|_X\right) \\
&\leq \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_1(x)\|_X}{\|x\|_X}, \|F_1(0)\|_X\right) \\
&\quad \max\left(\sup_{x\neq 0, x\in X} \frac{\|F_2(x)\|_X}{\|x\|_X}, \|F_2(0)\|_X\right) \\
&= \|F_1\|_{B(X)}\|F_2\|_{B(X)}.
\end{aligned}$$

If 1 is the unit element in X, the unit element e in $B(X)$ is given by $e(x) = \|x\|_X 1$ for $x \neq 0$ and $e(0) = 1$. Then

$$\begin{aligned}(F * e)(x) &= \frac{F(x)e(x)}{\|x\|_X} = F(x)1 = F(x) \\ &= 1F(x) = \frac{e(x)F(x)}{\|x\|_X} = (e * F)(x)\end{aligned}$$

for $x \neq 0$,

$$\begin{aligned}(F * e)(0) &= F(0)e(0) = F(0)1 = F(0) \\ &= 1F(0) = e(0)F(0) = (e * F)(0),\end{aligned}$$

and

$$\begin{aligned}\|e\|_{B(X)} &= \max\left(\sup_{x\neq 0, x\in X} \frac{\|e(x)\|_X}{\|x\|_X}, \|e(0)\|_X\right) \\ &= \|1\|_X \\ &= 1,\end{aligned}$$

where the last 1 is the unit element in the scalar field.

6.2. **Proofs: Section 3.1.** In this subsection, the proofs Theorem 3.7, Corollary 3.8, and Corollary 3.9 in Section 3.1 are given.

6.2.1. *Proof of Theorem 3.7.* If the quasi-product preserves the positivity, $[(FH)(x), g(x)]_X \geq 0$ for $x \in X$ given that both $[F(x), g(x)]_X \geq 0$ and $[H(x), g(x)]_X \geq 0$. On the other hand, consider that the square bounded below quasi-product has a left integral domain on the commutative unital Banach algebra X. For $F = 0$ or $H = 0$, the result holds. If $F(x) \neq 0$ and $x \neq 0$, there exists a positive function $\alpha_1 : X \to R$ such that $[F(x), g(x)]_X \geq [\alpha_1(x)F^2(x), g(x)]_X$ by the properties of the quasi-product, i.e., $F - \alpha_1 F^2 \geq 0$. Define a sequence of g-positive operators $\{F_n\}$ by $F_{n+1} = F_n - \alpha_n F_n^2$ and $F_1 = F$, where $\alpha_n : X \to R$ are positive functions. Thus, $F_1 = \sum_{i=1}^n \alpha_i F_i^2 + F_{n+1}$. Similarly, define a sequence of g-positive operators $\{H_n\}$ by $H_{n+1} = H_n - \beta_n H_n^2$ and $H_1 = H$, where $\beta_n : X \to R$ are positive functions. Because the quasi-product is square bounded below, by property (c) of the quasi-product, there exists a positive number $\underline{c}$ such that

$$\underline{c}\sum_{i=1}^{n} \left|\left|(\alpha_i^{1/2}F_i)(x)\right|\right|_X^2 ||g(x)||_X \leq \left[\sum_{i=1}^{n}(\alpha_i F_i^2)(x), g(x)\right]_X \leq [F(x), g(x)]_X$$

and thus $||(\alpha_n^{1/2}F_n)(x)||_X \underset{n\to\infty}{\longrightarrow} 0$. This gives $F_n(x) \underset{n\to\infty}{\longrightarrow} 0$ since otherwise

$$\alpha_n(x) = \frac{[F_n(x), g(x)]_X}{\overline{k}\,[F_n^2(x), g(x)]_X}$$

converges to a nonzero positive number by the left integral domain property of the quasi-product and $\left|\left|(\alpha_n^{1/2}F_n)(x)\right|\right|_X$ does not converge to 0, i.e., a contradiction, where $\overline{k}$ is some positive number. Then $\sum_{i=1}^n (\alpha_i F_i^2)(x) \underset{n\to\infty}{\longrightarrow} F(x)$ and

$\sum_{i=1}^{n}(\beta_i H_i^2)(x) \underset{n\to\infty}{\longrightarrow} H(x)$. Finally,

$$\begin{aligned}
& [(FH)(x), g(x)]_X \\
= & \lim_{n\to\infty}\left[\sum_{i=1}^{n}(\alpha_i F_i^2)(x)\sum_{j=1}^{n}(\beta_j H_j^2)(x), g(x)\right]_X \\
= & \lim_{n\to\infty}\left[\sum_{i=1}^{n}\sum_{j=1}^{n}(\alpha_i\beta_j F_i^2 H_j^2)(x), g(x)\right]_X \\
\geq & \underline{k}\lim_{n\to\infty}\sum_{i=1}^{n}\sum_{j=1}^{n}\left[(\alpha_i\beta_j F_i^2 H_j^2)(x), g(x)\right]_X \\
\geq & 0
\end{aligned}$$

by the properties of the quasi-product and the commutativity of X, where $\underline{k}$ is some positive number.

6.2.2. *Proof of Corollary 3.8.* (a): Let $\hat{G}^2 = G^2 = F$. Then $G^2 - \hat{G}^2 = (G+\hat{G})(G-\hat{G}) = 0$ by the commutativity of X. Thus, $\hat{G}(x) = G(x)$ or $\hat{G}(x) = -G(x)$ for $x \in X$ because X is an integral domain. If $\hat{G}(x) = -G(x)$ holds, $0 \leq [\hat{G}(x), g(x)]_X = [-G(x), g(x)]_X \leq 0$ and thus $\hat{G}(x) = G(x) = 0$ by the left integral domain property of the quasi-product.
(b): Define a sequence of operators $\{S_n\}$ by $S_{n+1} = S_1 + S_n^2/2$ and let $R_n = 1_X - S_n$, where $S_1 = (1_X - F)/2$. $0 \leq S_1 \leq 1_X$, $R_1 \geq 0$, and $S_{n+1} = (2S_1 + S_n^2)/2 \geq 0$ for every n by Lemma 3.3 (b), (c) and by induction. Also, by induction, $S_n \geq S_m$ for $n > m$. Further, $R_{n+1} = 1_X - S_{n+1} = F/2 + R_n(1_X + S_n)/2 \geq 0$ for every n by Theorem 3.7 and by induction. Then $R_m^2 - R_n^2 = (S_n - S_m)(R_m + R_n) \geq 0$ by Theorem 3.7 and hence $[R_m^2(x), g(x)]_X \geq [R_n^2(x), g(x)]_X$ for $x \in X$ by Lemma 3.3 (a). Note that $\{[R_n^2(x), g(x)]\}$ is a decreasing convergent sequence. Since

$$\begin{aligned}
& (R_m - R_n)^2 \\
= & R_m^2 + R_n^2 - 2R_m R_n \\
\leq & R_m^2 - R_n^2
\end{aligned}$$

and hence

$$\begin{aligned}
& ||R_m(x) - R_n(x)||_X^2 \, ||g(x)||_X \\
\leq & \overline{k}_1[(R_m - R_n)^2(x), g(x)]_X \\
\leq & \overline{k}_1[R_m^2(x) - R_n^2(x), g(x)]_X \\
\leq & \overline{k}_1\overline{k}_2\left\{[R_m^2(x), g(x)]_X - [R_n^2(x), g(x)]_X\right\},
\end{aligned}$$

both $\{R_n(x)\}$ and $\{S_n(x)\}$ are Cauchy sequences, where $\overline{k}_1$ and $\overline{k}_2$ are some positive numbers. Define the operator S by $S(x) = \lim_{n\to\infty} S_n(x)$. Because

$$0 \leq [S_n(x), g(x)]_X \leq [1_X(x), g(x)]_X,$$

$0 \leq S \leq 1_X$ by the continuity of the quasi-product. Then $\lim_{n\to\infty} S_{n+1}(x) = S_1(x) + \lim_{n\to\infty} S_n^2(x)/2$ gives $(1_X - S)^2(x) = F(x)$, i.e., $G = 1_X - S$.

Because S_n is a polynomial of F, $WS_n = S_nW$ and

$$\lim_{n\to\infty} W(x)S_n(x) = W(x)S(x) = S(x)W(x) = \lim_{n\to\infty} S_n(x)W(x).$$

Hence,

$$W(x)G(x) = W(x) - W(x)S(x) = W(x) - S(x)W(x) = G(x)W(x).$$

6.2.3. *Proof of Corollary 3.9.* (a): The proof follows the lines given in the one of Corollary 3.8 (a).
(b): For $F \in B(X)$ and $F \neq 0$, there exists a positive number k such that the operator $\hat{F}$ defined by $\hat{F}(x) = kF(x)/(||F||_{B(X)}||x||_X)$ for $x \neq 0$ and $\hat{F}(0) = 0$ satisfying $0 \leq \hat{F} \leq 1_X$. By Corollary 3.8, there exists a positive operator $\hat{G}$ satisfying $\hat{G}^2 = \hat{F}$. Then the operator G defined by $G(x) = (||F||_{B(X)}/k)^{1/2}||x||_X\hat{G}(x)$ for $x \in X$ is the g-positive operator in $B(X)$ satisfying $G * G = F$. Finally, the community of G follows directly by the community of $\hat{G}$.

6.3. **Proofs: Section 3.2.** In this subsection, the proofs of Lemma 3.12, Lemma 3.13, Lemma 3.16, and Lemma 3.17 in Section 3.2 are given.

6.3.1. *Proof of Lemma 3.12.* Because $||E_{S_1}(x)||_X \leq ||x||_X$, $||E_{S_1}||_{B(X)} \leq 1$. Because the proof of (b) for the projection operators is very similar to the one for the projection indicators, only the one for the projection indicators is given.
(i)* implies (iv)*: If $x \in N(1_{S_1})$, $||1_{S_1}(x)||_X = 0 \leq ||1_{S_2}(x)||_X$. On the other hand, if $x \notin N(1_{S_1})$, $1_{S_1}(x) = 1$ and thus $1 = 1_{S_1}(x) = 1_{S_1}(x)1_{S_2}(x) = 1_{S_2}(x)$, i.e., $||1_{S_1}(x)||_X = ||1_{S_2}(x)||_X$.
(iv)* implies (iii)*: If $x \in N(1_{S_2})$, then $||1_{S_1}(x)||_X \leq ||1_{S_2}(x)||_X = 0$ and hence $1_{S_1}(x) = 0$, i.e., $x \in N(1_{S_1})$.
(iii)* implies (ii)*: If $x \in S_1$ and $x \neq 0$, $1_{S_1}(x) = 1$ and $x \notin N(1_{S_1})$. Thus, $x \notin N(1_{S_2})$, $1_{S_2}(x) = 1$, and $x \in S_2$.
(ii)* implies (i)*: If $x \notin S_1$, $0 = 1_{S_1}(x) = 1_{S_1}(x)1_{S_2}(x) = 1_{S_2}(x)1_{S_1}(x)$. On the other hand, if $x \in S_1$, then $x \in S_2$ and thus $1 = 1_{S_1}(x) = 1_{S_2}(x) = 1_{S_1}(x)1_{S_2}(x) = 1_{S_2}(x)1_{S_1}(x)$.
To prove (c), there are 3 cases: $x \in S_1$, $x \in S_2\backslash S_1$, or $x \in X\backslash S_2$. If $x \in S_1$, $E_{S_2-S_1}(x) = x - x = 0$ and $E_{S_2-S_1} \circ E_{S_2-S_1}(x) = E_{S_2-S_1}(0) = 0 = E_{S_2-S_1}(x)$. If $x \in S_2\backslash S_1$, $E_{S_2-S_1}(x) = x - 0 = x$ and $E_{S_2-S_1} \circ E_{S_2-S_1}(x) = E_{S_2-S_1}(x)$. If $x \in X\backslash S_2$, $E_{S_2-S_1}(x) = 0 - 0 = 0$ and $E_{S_2-S_1} \circ E_{S_2-S_1}(x) = E_{S_2-S_1}(0) = 0 = E_{S_2-S_1}(x)$. Finally, by (b), (i)*,

$$\begin{aligned}
& 1_{S_2-S_1}1_{S_2-S_1} \\
= \ & 1_{S_2}^2 + 1_{S_1}^2 - 1_{S_1}1_{S_2} - 1_{S_2}1_{S_1} \\
= \ & 1_{S_2} + 1_{S_1} - 1_{S_1} - 1_{S_1} \\
= \ & 1_{S_2} - 1_{S_1} \\
= \ & 1_{S_2-S_1}.
\end{aligned}$$

6.3.2. *Proof of Lemma 3.13.* (a): Because W commutes with F and $-F$, W commutes with $|F|$. Further,

$$F^+W = \frac{FW + |F|\,W}{2} = \frac{WF + W\,|F|}{2} = WF^+$$

and

$$F^-W = \frac{|F|\,W - FW}{2} = \frac{W\,|F| - WF}{2} = WF^-.$$

(b): By (a) and Lemma 3.6 (b),

$$F^-F^+ = F^+F^- = \frac{(|F| + F)\,(|F| - F)}{4} = \frac{|F|^2 - F^2 + F\,|F| - |F|\,F}{4} = 0.$$

(c): If $x \in S$, $F^+(x) = 0$ and $1_S(x) = 1$. Then $F^+(x)1_S(x) = 1_S(x)F^+(x) = 0$ and $F^-(x)1_S(x) = 1_S(x)F^-(x) = F^-(x)$. On the other hand, if $x \notin S$, then $F^+(x) = F(x) \neq 0$, i.e., $|F|(x) = F(x)$, $F^-(x) = [|F|(x) - F(x)]/2 = 0$, and $1_S(x) = 0$. Hence $F^+(x)1_S(x) = 1_S(x)F^+(x) = 0$ and $F^-(x)1_S(x) = 1_S(x)F^-(x) = 0 = F^-(x)$.
(d): $1_SF = F1_S = (F^+ - F^-)1_S = F^+1_S - F^-1_S = 0 - F^- = -F^-$ by (c) and thus $(1_X - 1_S)F = F(1_X - 1_S) = F1_X - F1_S = F + F^- = F^+$.
(e): $F^- = (F^- + F^+)1_S = |F|1_S \geq 0$ and $0 \leq |F|(1_X - 1_S) = |F| - F^- = F^+$ by (c).

6.3.3. *Proof of Lemma 3.16.* By property (b) of the quasi-product and Corollary 2.3,

$$|[F(x), g(x)]_X| \leq \overline{c}\,||F||_{B(X)}\,||x||_X\,||g(x)||_X$$

for $x \in X$ and $\overline{k}$ can be $\overline{c}||F||_{B(X)}$ thus, where $\overline{c}$ is some positive number.

6.3.4. *Proof of Lemma 3.17.* For any g-eigenvalue λ corresponding to the g-eigenvector $x \neq 0$, $[F(x), g(x)]_X = [\lambda\gamma(x), g(x)]_X = k(x)\lambda||x||_X||g(x)||_X \in R$ by property (c) of the quasi-product and thus $\lambda \in R$, where k is a positive bounded function defined on X and is bounded away from 0. Further, if $F \in B(X)$, there exists a positive number $\overline{k}$ such that

$$|k(x)|\,|\lambda|\,||x||_X\,||g(x)||_X = |[\lambda\gamma(x), g(x)]_X| \leq \overline{k}\,||x||_X\,|\,||g(x)||_X$$

by Lemma 3.16 and thus $|\lambda| \leq \overline{k}/|k(x)|$, i.e., $|\lambda|$ lying in a bounded interval of R due to k being bounded away from 0.

6.4. **Proofs: Section 4.1.** In this subsection, the proofs of three lemmas, Lemma 4.2, Lemma 4.4, and Lemma 4.8 in Section 4.1, are given.

6.4.1. *Proof of Lemma 4.2.*

$$\begin{aligned}
& ||F_{1n} * F_{2n} - F_1 * F_2||_{B(X)} \\
\leq\ & ||(F_{1n} - F_1) * F_{2n}||_{B(X)} + ||F_1 * (F_{2n} - F_2)||_{B(X)} \\
\leq\ & ||F_{1n} - F_1||_{B(X)} \left(||F_{2n} - F_2||_{B(X)} + ||F_2||_{B(X)}\right) \\
& + ||F_1||_{B(X)}\,||F_{2n} - F_2||_{B(X)}\,,
\end{aligned}$$

then $||(F_{1n} * F_{2n}) - (F_1 * F_2)||_{B(X)} \underset{n\to\infty}{\longrightarrow} 0$.

6.4.2. *Proof of Lemma 4.4.* $p(\lambda) \geq 0$ for $\lambda \in [m, M]$ implying $p(F) \geq 0$ is proved first. If p is over the real field and $p(\lambda) \geq 0$ for $\lambda \in [m, M]$, then p has the form

$$p(\lambda) = a \prod_{i=1}^{n_1} (\lambda - \alpha_i) \prod_{j=1}^{n_2} (\beta_j - \lambda) \prod_{k=1}^{n_3} \left[(\lambda - \gamma_k)^2 + \kappa_k^2 \right],$$

where n_1, n_2, and n_3 are nonnegative integers, the associated product is equal to 1 as $n_l = 0, l = 1, 2, 3$, $a \geq 0$, $\gamma_k, \kappa_k \in R$, $\alpha_i \leq m$, and $\beta_j \geq M$. Then by Lemma 3.16 and the properties of the quasi-product, there exist positive numbers k_1, k_2, and k_3 such that for $m \leq -k_2/k_3$ and $x \in X$

$$\begin{aligned} & [(F - \alpha_i e)(x), g(x)]_X \\ \geq\ & k_1 \left([F(x), g(x)]_X - [\alpha_i e(x), g(x)]_X \right) \\ \geq\ & k_1 \left(-k_2 \|x\|_X \|g(x)\|_X - \alpha_i k_3 \|x\|_X \|g(x)\|_X \right) \\ \geq\ & 0. \end{aligned}$$

Similarly, there exists a number $M > 0$ such that $[(\beta_j e - F)(x), g(x)]_X \geq 0$ for $\beta_j \geq M$ and $x \in X$. Because $F - \alpha_i e \geq 0$, $\beta_j e - F \geq 0$, and $(F - \gamma_k e)^{*2} + (\kappa_k e)^{*2} \geq 0$ by the properties of the quasi-product, thus $p(F) \geq 0$ by Theorem 3.7.

Let $\overline{c} = \max_{\lambda \in [m,M]} |p(\lambda)|$. Because $\overline{c} - p(\lambda) \geq 0$ and $\overline{c} + p(\lambda) \geq 0$ for $\lambda \in [m, M]$, $\overline{c}e - p(F) \geq 0$ and $\overline{c}e + p(F) \geq 0$. Therefore, $\overline{c}^2 e^2 \geq p^2(F)$ by Theorem 3.7. By the square bounded below property of the quasi-product,

$$\underline{k} \|p(F)(x)\|_X^2 \|g(x)\|_X \leq \left[p^2(F)(x), g(x) \right]_X \leq k\overline{c}^2 \|e(x)\|_X^2 \|g(x)\|_X$$

for $x \in X$ and hence $\|p(F)\|_{B(X)} \leq (k/\underline{k})^{1/2} \overline{c}$, where $\underline{k}$ and k are some positive numbers.

6.4.3. *Proof of Lemma 4.8.* Because

$$\begin{aligned} & \|f(F)\|_{B(X)} \\ \leq\ & \|f(F) - p_n(F)\|_{B(X)} + \|p_n(F)\|_{B(X)} \\ \leq\ & \|f(F) - p_n(F)\|_{B(X)} + \overline{k} \left(\|p_n - f\|_{C([m,M])} + \|f\|_{C([m,M])} \right) \end{aligned}$$

by Lemma 4.4, the result holds by letting $n \to \infty$, where p_n is a sequence of polynomial functions defined on $[m, M]$ over the real field converging uniformly to f in $C([m, M])$.

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