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STRUCTURAL COMPACTNESS AND STABILITY OF DOUBLY NONLINEAR FLOWS

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Abstract. This work deals with the Cauchy problem for quasilinear parabolic PDEs of the form $D_t\partial\gamma(u)+\alpha(u)\ni h$, with γ convex and lower semi-continuous and α maximal monotone. A qualified variational formulation of this problem is given by means of Fitzpatrick's variational representation of maximal monotone operators, extending a method which was pioneered by Brezis, Ekeland and Nayroles for gradient flows. On this basis, *structural compactness* and *structural stability* of this problem are proved, by using De Giorgi's Γ-convergence and a nonlinear topology of weak type. This result can be applied to several quasilinear PDEs of mathematical physics.

Foreword. Professor Felix Earl Browder was one of the pioneers in the study of nonlinear operators in Banach spaces and of related partial differential equations.

I am indebted to professors Simeon Reich and Alexander J. Zaslavski who invited me to contribute to the present volume in His memory.

1. Introduction

In this work we deal with doubly-nonlinear parabolic equations of the form

$$D_t\partial\gamma(u)+\alpha(u)\ni h \qquad (D_t:=\partial/\partial t), \tag{1.1}$$

with γ convex and lower semi-continuous and α maximal monotone. We formulate the corresponding flow as a qualified minimization principle, via Fitzpatrick's variational formulation of maximal monotone operators of [11], and ideas of Brezis and Ekeland [5] and Nayroles [15]. Our main purpose is to prove the stability w.r.t. arbitrary perturbations of data and operators. This rests upon a nonlinear topology of weak type, and on De Giorgi's notion of Γ-convergence (see [8] and e.g. the monographs [2], [3], [7]).

This class of linear equations was also studied e.g. in [1], [6], [10], [12], [19].

Variational representation of flows. Let V be a reflexive real Banach space, H be a real Hilbert space, and $V\subset H=H'\subset V'$ with dense injections. Let

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$\alpha : V \to \mathcal{P}(V')$ be a maximal monotone operator. The present analysis is based on that of the (simply-nonlinear) maximal monotone flow

$$D_t u + \alpha(u) \ni h \qquad \text{in } V', \text{ a.e. in }]0,T[\ (D_t := \partial/\partial t). \tag{1.2}$$

In [11] Fitzpatrick introduced a *representation* of maximal monotone operators $\alpha : V \to \mathcal{P}(V')$ via a qualified minimization principle, see Section 2.

Brezis and Ekeland [5] and Nayroles [15] assumed that $\alpha = \partial g$ for some lower semi-continuous convex function $g : V \to \mathbf{R} \cup \{+\infty\}$, set

$$\Phi(v, v^*) = \int_0^T [g(v) + g^*(v^* - D_t v) - \langle v^*, v\rangle]\, dt + \frac{1}{2}\|v(T)\|_H^2 - \frac{1}{2}\|u(0)\|_H^2 \qquad \forall (v, v^*) \in L^2(0,T; V \times V'), \tag{1.3}$$

and reformulated the inclusion (1.2) as

$$u \in L^2(0,T;V) \cap H^1(0,T;V') \qquad \Phi(u,h) = \inf \Phi(\cdot, h) = 0. \tag{1.4}$$

We shall refer to this result as the *B.E.N. principle*, Here we extend it to the doubly-nonlinear inclusion (1.1) without assuming α to be a subdifferential, and extend the compactness and stability analysis of (1.2), along the lines of [21], [23], [24].

Structural compactness and structural stability. Besides stability w.r.t. perturbations of the data (e.g., source terms, initial- and boundary-values), one may consider robustness w.r.t. perturbations of the structure of the problem (e.g. linear and nonlinear operators in differential equations). We shall call this *structural stability*, This is a prerequisites for the numerical treatment and efficient control of systems, and has an obvious applicative motivation, since data and operators are accessible just with some approximation. Structural stability is here associated with *structural compactness,* namely the existence of a convergent subsequence of the solutions of the perturbed problems.

For variational principles, these structural properties are fulfilled by De Giorgi's theory of Γ-convergence. This typically applies to stationary models, and the present article extends that approach to flows.

Here the choice of the topology for data and operators plays a key role. The topology must be so weak that, under minimal assumptions of boundedness and coerciveness, there exists a sequence of solutions of the perturbed problems that converges with respect to that topology. In order to fulfill a compactness theorem, this topology is necessarily of weak type. However, the weak topology is not appropriate, since it would not provide the existence of a recovery sequences: to this purpose a slightly stronger topology is needed. An answer to this question is provided by what we name the *nonlinear weak topology* of $V \times V'$, which is intermediate between the strong and the weak topology of $V \times V'$, see [21].

Plan of work. In Section 2 we outline the variational formulation of the flow (1.2), which is the point of departure for the study of (1.1). In Section 3 we introduce the notions of structural compactness and structural stability, and in Sections 4 we prove these properties for the Cauchy problem for (1.1). Finally, in Sections 5 we show that this result can be applied to several quasilinear PDEs issued from mathematical physics.

2. Fitzpatrick's theory and extended B.E.N. principle

In this section we review a result of Fitzpatrick [11] for monotone operators and the related extension of the B.E.N. principle.

The Fitzpatrick theory. Let V be a real Banach space with norm $\|\cdot\|$, dual norm $\|\cdot\|_{V'}$, and duality pairing $\langle\cdot,\cdot\rangle$.

Let an operator $\alpha : V \to \mathcal{P}(V')$ be proper and measurable (this includes maximal monotone operators). In [11] Fitzpatrick defined what is now called the *Fitzpatrick function* of α:

$$
\begin{aligned}
f_\alpha(v,v^*) &:= \langle v^*, v\rangle + \sup\big\{\langle v^* - \widetilde{v}^*, \widetilde{v} - v\rangle : \widetilde{v} \in V, \widetilde{v}^* \in \alpha(\widetilde{v})\big\} \\
&= \sup\big\{\langle v^*, \widetilde{v}\rangle - \langle \widetilde{v}^*, \widetilde{v} - v\rangle : \widetilde{v} \in V, \widetilde{v}^* \in \alpha(\widetilde{v})\big\} \quad \forall (v,v^*) \in V\times V',
\end{aligned}
\tag{2.1}
$$

and proved the following result.

Theorem 2.1 ([11]). *$\alpha : V \to \mathcal{P}(V')$ is maximal monotone if and only if*

$$f_\alpha(v,v^*) \geq \langle v^*, v\rangle \qquad \forall (v,v^*) \in V\times V', \tag{2.2}$$

$$f_\alpha(v,v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.3}$$

More generally, nowadays one says that a function f (variationally) *represents* a proper measurable operator $\alpha : V \to \mathcal{P}(V')$ whenever

$$f : V\times V' \to \mathbf{R} \cup \{+\infty\} \ \text{ is convex and lower semi-continuous,} \tag{2.4}$$

$$f(v,v^*) \geq \langle v^*, v\rangle \qquad \forall (v,v^*) \in V\times V', \tag{2.5}$$

$$f(v,v^*) = \langle v^*, v\rangle \quad \Leftrightarrow \quad v^* \in \alpha(v). \tag{2.6}$$

One accordingly says that α is *representable,* that f is a (convex) *representative function,* that f *represents* α, and so on. The Fitzpatrick function f_α thus represents α. We shall denote by $\mathcal{F}(V)$ the class of the functions that fulfill (2.4) and (2.5). Representable operators are monotone, see [11], but need not be either cyclically monotone or maximal monotone. However, by Theorem 2.1 any operator that is represented by its Fitzpatrick function is maximal monotone. The notion of representative function can be extended dropping the requirement of convexity, but leaving the lower semi-continuity.

Examples. (i) For any proper convex and lower-semi-continuous function $g : V \to \mathbf{R} \cup \{+\infty\}$, the Fenchel function $f : (v,v^*) \mapsto g(v) + g^*(v^*)$ represents the operator ∂g.

(ii) Let $A : V \to V'$ be a linear, bounded and invertible monotone operator, and define the convex and continuous mapping

$$F_b : V \times V' \to \mathbf{R} : (v,v^*) \mapsto b[\langle Av, v\rangle + \langle v^*, A^{-1}v^*\rangle] \qquad \forall b > 0. \tag{2.7}$$

$F_{1/2}$ is the Fenchel function of the operator A. For any $b > 1/2$, F_b represents the nonmaximal monotone operator $\alpha(0) = \{0\}$, $\alpha(v) = \emptyset$ for any $v \neq 0$. For $0 < b < 1/2$, F_b does not represents any operator, as (2.5) fails.

(iii) Let us denote by $\Delta^{-1} : H^{-1}(\Omega) \to H^1_0(\Omega)$ the inverse of the Laplace operator with homogeneous Dirichlet boundary condition, and by $\langle \cdot, \cdot \rangle$ the duality pairing between $H^{-1}(\Omega)$ and $H^1_0(\Omega)$ such that $\langle u, v \rangle = \int_\Omega u(x)\, v(x)\, dx$ for any $u, v \in H^1_0(\Omega)$. If $\varphi : \mathbf{R}^n \times \mathbf{R}^N \to \mathbf{R}$ represents the mapping γ in $\mathbf{R}^N$ in the sense of (2.4)–(2.6), then the function

$$\psi(v, v^*) = \int_\Omega [\varphi(\nabla v, \nabla \Delta^{-1} v^*) - \langle v^*, v \rangle]\, dx \qquad \forall (v, v^*) \in H^1_0(\Omega) \times H^{-1}(\Omega) \tag{2.8}$$

represents the maximal monotone operator

$$\alpha : H^1_0(\Omega) \to \mathcal{P}(H^{-1}(\Omega)) : v \mapsto -\nabla \cdot \gamma(\nabla v). \tag{2.9}$$

This is proved e.g. in Example 3.2 of [21].

Further examples of representative functions are provided e.g. in [20], [21].

The extended B.E.N. principle. Let us assume that we are given a triplet of real Hilbert spaces

$$V \subset H = H' \subset V' \ \text{ with continuous and dense injections,} \tag{2.10}$$

and assume that

$$\begin{aligned} &\alpha : V \to \mathcal{P}(V') \text{ is maximal monotone,} \\ &\exists A, B \in \mathbf{R} : \forall v \in V, \quad \|\alpha(v)\|_{V'} \le A \|v\|_V + B. \end{aligned} \tag{2.11}$$

Let us fix any $u^* \in L^2(0, T; V')$, any $u^0 \in H$, and consider the Cauchy problem

$$\begin{cases} u \in \mathcal{V} := L^2(0, T; V) \cap H^1(0, T; V'), \\ D_t u + \alpha(u) \ni u^* \qquad \text{in } V', \text{ a.e. in }]0, T[, \\ u(0) = u^0. \end{cases} \tag{2.12}$$

Let us set $\mathcal{V}_{u^0} = \{v \in \mathcal{V} : v(0) = u^0\}$ and

$$\begin{aligned} &\Psi(v, v^*) := \int_0^T [\langle D_t v - v^*, v \rangle + \varphi(v, v^* - D_t v)]\, dt \\ &\qquad\qquad\qquad\qquad\qquad \forall (v, v^*) \in \mathcal{V}_{u^0} \times L^2(0, T; V'), \\ &\Psi(v, v^*) := +\infty \qquad \text{for any other } (v, v^*) \in L^2(0, T; V \times V'). \end{aligned} \tag{2.13}$$

Theorem 2.2 (Extended B.E.N. principle)**.** *Let* (2.10) *and* (2.11) *be fulfilled, and let φ represent the operator α in V. The functional Ψ then represents the operator $D_t + \alpha$ in $L^2(0, T; V)$, and the Cauchy problem* (2.12) *is equivalent to*

$$u \in \mathcal{V}_{u^0}, \qquad \Psi(u, u^*) = 0. \tag{2.14}$$

Proof. It is clear that Ψ is convex and lower semicontinuous. As φ fulfills (2.4) and (2.5), these properties also hold for Ψ in $L^p(0, T; V) \times L^{p'}(0, T; V')$.

As φ is representative, $\Psi \ge 0$; hence $\Psi(u, u^*) = 0$ only if this integrand vanishes a.e. in $]0, T[$. As φ represents α, (2.12) follows. Ψ thus represents the operator $D_t + \alpha$ in $L^2(0, T; V)$, and (2.14) entails (2.12). Conversely, if (2.12) is fulfilled then the integrand of (2.13) vanishes, and (2.14) follows. □

Time-integrated extended B.E.N. principle. Let us first define the measure $d\mu(t) = (T - t)dt$. For any Hilbert space X, let us introduce the Hilbert space

$$L^2_\mu(0,T;X) := \Big\{\mu\text{-measurable } v :]0,T[\to X : \int_0^T \|v(t)\|^2_X \, d\mu(t) < +\infty\Big\}, \tag{2.15}$$

and note that

$$L^2_\mu(0,T;V) \subset L^2_\mu(0,T;H) = L^2_\mu(0,T;H)' \subset L^2_\mu(0,T;V)' = L^2_\mu(0,T;V'), \tag{2.16}$$

with continuous and dense injections. Let us set

$$\begin{aligned} \mathcal{V}_\mu &:= \big\{v \in L^2_\mu(0,T;V) : D_t v \in L^2_\mu(0,T;V')\big\}, \\ \mathcal{V}_{\mu,u^0} &:= \big\{v \in \mathcal{V}_\mu : v(0) = u^0\big\}. \end{aligned} \tag{2.17}$$

Note that $\mathcal{V}_\mu$ equipped with the graph norm is a Hilbert space, and the injection $\mathcal{V}_\mu \to L^2_\mu(0,T;H)$ is compact.

Next we apply a further time integration to the potential Ψ. Recalling the elementary identity $\int_0^T d\tau \int_0^\tau f(t)\,dt = \int_0^T f(t)\,d\mu(t)$ for any $f \in L^1(0,T)$, we set

$$\begin{aligned} \widetilde{\Psi}(v,v^*) &:= \int_0^T \big[\langle D_t v - v^*, v\rangle + \varphi(v, v^* - D_t v)\big]\,d\mu(t) \\ &\qquad\qquad (v,v^*) \in \mathcal{V}_{\mu,u^0} \times L^2_\mu(0,T;V'), \\ \widetilde{\Psi}(v,v^*) &:= +\infty \qquad \text{for any other } (v,v^*) \in L^2_\mu(0,T;V \times V'). \end{aligned} \tag{2.18}$$

Theorem 2.3 (Time-integrated extended B.E.N. principle)**.** *Assume that* (2.10) *and* (2.11) *are fulfilled, and that φ represents α in V. Then $\widetilde{\Psi}$ represents the operator $D_t + \alpha$ in $L^2_\mu(0,T;V)$, and the problem*

$$u \in \mathcal{V}_{\mu,u^0}, \qquad D_t u + \alpha(u) \ni u^* \qquad \text{in } L^2_\mu(0,T;V') \tag{2.19}$$

is equivalent to

$$u \in \mathcal{V}_{\mu,u^0}, \qquad \widetilde{\Psi}(u,u^*) = 0. \tag{2.20}$$

Proof. The argument of Proposition 2.2 can be repeated almost verbatim, since the further time integration preserves the conditions (2.4)–(2.6). □

3. Structural compactness, structural stability and nonlinear weak topology

In this section we illustrate the notions of structural compactness and structural stability, and apply them to maximal monotone flows, here reformulated as *null-minimization principles*, We also introduce what we shall refer to as the *nonlinear weak topology* of $V \times V'$, which will be used in the next section.

Structural compactness and structural stability. We illustrate these notions in an abstract set-up. Let X be a topological space and $\mathcal{G}$ be a family of functionals $X \to \mathbf{R} \cup \{+\infty\}$, equipped with Γ-convergence. We shall use the following terminology:

(i) the problem of minimizing these functionals will be called *structurally compact* if the family $\mathcal{G}$ is sequentially compact, and the corresponding minimizers range in a sequentially relatively compact subset of X. This definition is instrumental to the next one.

(ii) the minimization problem will be said *structurally stable* if [1]

$$\begin{cases} u_n \to u & \text{in } X \\ \Phi_n \xrightarrow{\Gamma} \Phi & \text{in } \mathcal{G} \\ \Phi_n(u_n) - \inf \Phi_n \to 0 \end{cases} \quad \Rightarrow \quad \Phi(u) = \inf \Phi. \tag{3.1}$$

The selection of the notion of convergence in X is crucial. Structural compactness and structural stability are in competition: the convergence must be sufficiently weak in order to allow for sequential compactness, and at the same time it must be sufficiently strong to provide structural stability. The (typically stationary) Γ-convergence is especially appropriate for this problem.

The nonlinear weak topology, Dealing with the structural stability of the null-minimization of (2.20), one must pass to the limit in the term $\int_0^T \langle v_n^* - D_t v_n, v_n\rangle \, d\mu(t)$. This induces us to complement the weak topology of $L^2_\mu(0,T;V\times V')$ with the convergence

$$\int_0^T \langle v_n^*, v_n\rangle \, d\mu(t) \to \int_0^T \langle v^*, v\rangle \, d\mu(t).$$

In view of the use of Γ-convergence, this will provide the existence of a so-called recovery sequence. [2]

More specifically, let us still denote by π the duality pairing of $V\times V'$: $\pi(v,v^*) = \langle v^*, v\rangle$. Along the lines of [21], we shall name *nonlinear weak topology* of $V\times V'$, and denote by $\widetilde{\pi}$, the coarsest among the topologies of this space that are finer than the weak topology, and for which the mapping π is continuous. For any sequence $\{(v_n, v_n^*)\}$ in $V\times V'$, thus [3]

$$\begin{aligned} &(v_n, v_n^*) \underset{\widetilde{\pi}}{\to} (v, v^*) \quad \text{in } V\times V' \quad \Leftrightarrow \\ &v_n \rightharpoonup v \quad \text{in } V, \quad v_n^* \rightharpoonup v^* \quad \text{in } V', \quad \langle v_n^*, v_n\rangle \to \langle v^*, v\rangle, \end{aligned} \tag{3.2}$$

and similarly for nets. This construction is extended to the space $L^2_\mu(0,T;V\times V')$ in an obvious way: in this case the duality product reads $(v,v^*) \mapsto \int_0^T \langle v^*, v\rangle \, d\mu(t)$, and we set

$$\begin{aligned} &(v_n, v_n^*) \underset{\widetilde{\pi}}{\to} (v, v^*) \quad \text{in } L^2_\mu(0,T;V\times V') \quad \Leftrightarrow \\ &v_n \rightharpoonup v \ \text{ in } L^2_\mu(0,T;V), \ v_n^* \rightharpoonup v^* \ \text{ in } L^2_\mu(0,T;V') \ \text{ and} \\ &\int_0^T \langle v_n^*, v_n\rangle \, d\mu(t) \to \int_0^T \langle v^*, v\rangle \, d\mu(t), \end{aligned} \tag{3.3}$$

and similarly for nets.

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[1] By $\Phi_n \xrightarrow{\Gamma} \Phi$ we mean that the sequence $\{\Phi_n\}$ Γ-converges to Φ in the sense of De Giorgi, see e.g. [7].

[2] Here it does not seem appropriate to use the product of the weak topology of $L^2_\mu(0,T;V)$ by the strong topology of $L^2_\mu(0,T;V')$. Indeed, dealing with parabolic problems, v_n^* is replaced by $v_n^* - D_t v$, and the sequence $\{D_t v_n\}$ typically converges just weakly in $L^2_\mu(0,T;V')$.

[3] We denote the strong, weak, and weak star convergence respectively by $\to$, $\rightharpoonup$, $\overset{*}{\rightharpoonup}$.

$\Gamma\widetilde{\pi}$-compactness and $\Gamma\widetilde{\pi}$-stability of $\mathcal{F}(L^2_\mu(0,T;V))$. As the weak topology and the nonlinear weak topology $\widetilde{\pi}$ are nonmetrizable, some caution is needed in dealing with *sequential* Γ-convergence with respect to either topology; for the former topology see e.g. [2], [7]. For functions defined on a topological space, the definition of Γ-convergence involves the filter of the neighborhoods of each point. If the space is metrizable, that notion can equivalently be formulated in terms of the family of converging sequences; but this does not hold in general. We shall refer to these two notions as *topological* and *sequential* Γ-convergence, respectively. Hereafter reference to the topological notion should be understood, if not otherwise stated.

The similarity between the nonlinear weak topology of $V\times V'$ and that of $L^2_\mu(0,T;V\times V')$ is obvious. Dealing with flows, here we are mainly concerned with the latter; for the sake of simplicity, we shall however develop our discussion for the former, and leave the obvious reformulation of the other one to the reader.

It is known that bounded subsets of a separable and reflexive space equipped with the weak topology are metrizable. The same holds for the nonlinear weak topology $\widetilde{\pi}$ of $V\times V'$, as it was proved in [21]. This property is at the basis of the next statement, where we define $\mathcal{F}(V)$ and $\mathcal{F}(L^2_\mu(0,T;V))$ as above.

Theorem 3.1 (Γ-compactness and Γ-stability)**.** *Let V be a separable real Banach space, and $\{\gamma_n\}$ be an equi-coercive sequence in $\mathcal{F}(V)$, in the sense that*

$$\sup_{n\in\mathbf{N}}\big\{\|v\|_V+\|v^*\|_{V'}:(v,v^*)\in V\times V',\gamma_n(v,v^*)\le C\big\}<+\infty\quad\forall C\in\mathbf{R}. \tag{3.4}$$

Then: (i) *There exists $\gamma: V\times V'\to\mathbf{R}\cup\{+\infty\}$ such that, possibly extracting a subsequence, γ_n $\Gamma\widetilde{\pi}$-converges to γ both topologically and sequentially.*

(ii) *This convergence entails that $\gamma\in\mathcal{F}(V)$.*

(iii) *If α_n (α, resp.) is the operator that is represented by γ_n (γ, resp.) for any n, then*

$$\forall\ \textit{sequence}\ \{(v_n,v_n^*)\in\operatorname{graph}(\alpha_n)\},\quad (v_n,v_n^*)\underset{\widetilde{\pi}}{\rightrightarrows}(v,v^*)\quad\Rightarrow\quad v^*\in\alpha(v). \tag{3.5}$$

That is, the superior limit in the sense of Kuratowski of the graph of the α_ns is included in the graph of α.

Proof. Part (i) is Theorem 4.4 of [21].

As the functional π is obviously $\widetilde{\pi}$-continuous, the sequential $\widetilde{\pi}$-lower semicontinuity of the γ_ns and the property "$\gamma_n\ge\pi$" are preserved by $\Gamma\widetilde{\pi}$-convergence; thus $\gamma\in\mathcal{F}(V)$ (whereas in general $\gamma\notin\mathcal{F}(V)$). On the other hand, it is known that passage to the Γ-limit preserves the convexity. If $\gamma_n\in\mathcal{F}(V)$ for any n, namely if the γ_ns are also convex, then the same holds for γ. Part (ii) is thus established.

Next let the operators $\{\alpha_n\}$ and α be as prescribed in part (iii), $(v_n,v_n^*)\in\operatorname{graph}(\alpha_n)$ for any n, and $(v_n,v_n^*)\underset{\widetilde{\pi}}{\rightrightarrows}(v,v^*)$. As γ_n represents α_n, we have $\gamma_n(v_n,v_n^*)=\langle v_n^*,v_n\rangle$ for any n. By the definition of $\Gamma\widetilde{\pi}$-convergence, if $(v_n,v_n^*)\to_{\widetilde{\pi}}(v,v^*)$ we then get

$$\gamma(v,v^*)\le\liminf_{n\to\infty}\gamma_n(v_n,v_n^*)=\liminf_{n\to\infty}\langle v_n^*,v_n\rangle=\langle v^*,v\rangle. \tag{3.6}$$

Thus $v^*\in\alpha(v)$, as γ represents α. The implication (3.5) is thus established. □

Remarks 3.2. (i) By the same argument, Theorem 3.1 holds also if the space V is replaced by $L^2_\mu(0,T;V)$, which is the case of interest for the next two sections.

(ii) In general the sequence of operators $\{\alpha_n\}$ does not converge in the sense of Kuratowski.

(iii) A counterexample shows that the limit of subdifferential operators is maximal monotone but need not be a subdifferential, see Section 5 of [21]. Thus the class of gradient flows is not structurally stable, at variance with that of maximal monotone flows.

Let $\{\varphi_n\}$ be a sequence of representative functions of $\mathcal{F}(V)$, and define the superposition (i.e., Nemytskiĭ-type) operators [4]

$$\psi_n : L^2_\mu(0,T;V\times V') \to L^1(0,T) : w \mapsto \varphi_n(w(\cdot)) \quad \forall n. \tag{3.7}$$

The following question arises:

$$\begin{aligned}&\text{if } \psi_n\ \Gamma\widetilde{\pi}\text{-converges to some operator } \psi,\\ &\text{is then } \psi \text{ necessarily a superposition operator, too?}\end{aligned} \tag{3.8}$$

In other terms, does a mapping $\varphi : V\times V' \to \mathbf{R}\cup\{+\infty\}$ exist such that $\psi_w = \varphi(w(\cdot))$ for any $w \in L^2_\mu(0,T;V\times V')$? Denoting by α the function that is represented by φ, in the limit we would then get the equation $D_t\partial\gamma(u) + \alpha(u) \ni h$, so with an instantaneous (i.e., memoryless) relation between u and $h - D_t\partial\gamma(u)$; see e.g. [18].

The next statement provides a positive answer to the question (3.8).

Theorem 3.3 ([24]). *Let V be a real separable Hilbert space, and let $\{\varphi_n\}$ be a sequence of representative functions $V\times V' \to \mathbf{R}$ such that*

$$\begin{aligned}&\exists C_1,...,C_4 > 0 : \forall n, \forall (v,v^*) \in V\times V',\\ &C_1\|(v,v^*)\|^2_{V\times V'} - C_2 \le \varphi_n(v,v^*) \le C_3\|(v,v^*)\|^2_{V\times V'} + C_4,\end{aligned} \tag{3.9}$$

$$\varphi_n(0,0) = 0. \tag{3.10}$$

Let us define the functionals

$$\Phi_n(v,v^*) = \int_0^T \varphi_n(v(t),v^*(t))\,d\mu(t) \qquad \forall (v,v^*) \in L^2_\mu(0,T;V\times V'), \forall n. \tag{3.11}$$

Then: (i) There exists a functional $\Phi : L^2_\mu(0,T;V\times V') \to \mathbf{R}$ such that, possibly extracting a subsequence,

$$\Phi_n \overset{\Gamma\widetilde{\pi}}{\to} \Phi \qquad \text{sequentially in } L^2_\mu(0,T;V\times V'). \tag{3.12}$$

(ii) This entails that there exists a representative function $\varphi : V\times V' \to \mathbf{R}$ such that

$$\Phi(v,v^*) = \int_0^T \varphi(v(t),v^*(t))\,d\mu(t) \qquad \forall (v,v^*) \in L^2_\mu(0,T;V\times V'). \tag{3.13}$$

[4] These operators should not be confused with the functions $\gamma_n \in \mathcal{F}(L^2_\mu(0,T;V))$ of the Remark 3.2 (i).

4. Structural properties of doubly-nonlinear flows

In this section we study the structural compactness and structural stability of doubly-nonlinear flows of the form

$$D_t w + \alpha(u) \ni h, \qquad w \in \partial\gamma(u), \tag{4.1}$$

with α maximal monotone operator, and γ lower semicontinuous and convex.

Let V, H be real separable Hilbert spaces, let the sequences $\{\alpha_n\}, \{\varphi_n\}$ and $\{\gamma_n\}$ be given such that

$$V \subset H \text{ with compact and dense injection,} \tag{4.2}$$

$$\alpha_n : V \to \mathcal{P}(V') \text{ is maximal monotone, } \forall n, \tag{4.3}$$

$$\varphi_n \text{ is the Fitzpatrick function of } \alpha_n, \forall n, \tag{4.4}$$

$$\gamma_n : H \to \mathbf{R} \text{ is convex and lower semicontinuous, } \forall n, \tag{4.5}$$

and assume that

$$\exists C_1, C_2 > 0 : \forall n, \forall v \in V, \quad \langle \alpha_n(v), v\rangle \geq C_1 |v\|_V^2 - C_2, \tag{4.6}$$

$$\exists C_3, C_4 > 0 : \forall n, \forall v \in V, \quad \|\alpha_n(v)\|_{V'} \leq C_3\|v\|_V + C_4, \tag{4.7}$$

$$\exists \bar{C}_1, ..., \bar{C}_4 > 0 : \forall n, \forall v \in H, \quad \bar{C}_1\|v\|_H^2 - \bar{C}_2 \leq \gamma_n(v) \leq \bar{C}_3\|v\|_H^2 + \bar{C}_4, \tag{4.8}$$

$$0 \in \alpha_n(0) \qquad \forall n. \tag{4.9}$$

(The latter condition is not really restrictive: if it is not satisfied, it can be recovered by selecting any $b_n \in \alpha_n(0)$ and then replacing α_n by $\bar{\alpha}_n = \alpha_n - b_n$.) Let two sequences $\{w_n^0\}$ and $\{h_n\}$ be also given such that

$$w_n^0 \to w^0 \quad \text{in } H, \tag{4.10}$$

$$h_n \to h \quad \text{in } L^2(0,T;V'). \tag{4.11}$$

For any n, we are now able to formulate the following initial-value problem:

$$\begin{cases} u_n \in L^2(0,T;V), \quad w_n \in L^2(0,T;H) \cap H^1(0,T;V'), \\ D_t w_n + \alpha(u_n) \ni h \quad \text{in } V', \text{ a.e. in }]0,T[, \\ w_n \in \partial\gamma(u_n) \quad \text{in } H, \text{ a.e. in }]0,T[, \\ w_n(0) = w_n^0. \end{cases} \tag{4.12}$$

Lemma 4.1 ([1, 6, 10, 12]). *Under the above hypotheses on the sequences $\{\alpha_n\}, \{h_n\}, \{\gamma_n\}, \{w_n^0\}$, for any n the initial-value problem (4.12) has at least one solution. Moreover, the sequence $\{(u_n, w_n)\}$ is bounded in $L^2(0,T;V) \times \big(L^2(0,T;H) \cap H^1(0,T;V')\big)$.*

Null-Minimization. Next we reformulate the problem (4.12) as a null-minimization principle. Let us first define the measure $d\mu(t) = (T-t)dt$, the Hilbert space

$$\mathcal{W} := \big\{v \in L^2_\mu(0,T;H) : D_t v \in L^2_\mu(0,T;V')\big\}, \tag{4.13}$$

and the affine subspace $\mathcal{W}_{\mu,w_n^0} := \{v \in \mathcal{W} : v(0) = w_n^0\}$. Notice that

$$\begin{aligned}
&\int_0^T \langle D_t w, v\rangle \, d\mu(t) = \int_0^T d\tau \int_0^\tau \langle D_t w, v\rangle \, dt \\
&= \int_0^T d\tau \int_0^\tau D_t\gamma_n^*(w) dt = \int_0^T \gamma_n^*(w)\, dt - T\gamma_n^*(w(0)) \\
&\forall v \in L^2_\mu(0,T;V), \forall w \in \mathcal{W}, \text{ with } w \in \partial\gamma_n(v) \text{ a.e. in }]0,T[.
\end{aligned} \tag{4.14}$$

For any n, we introduce a representative function φ_n of the operator α and the twice-time-integrated functional

$$\begin{aligned}
\Xi_n(u,w,h) := &\int_0^T d\tau \int_0^\tau \big[\gamma_n(u) + \gamma_n^*(w) - (w,u)_H\big]\, dt \\
&+ \left(\int_0^T d\tau \int_0^\tau \big[\varphi_n(u, h - D_t w) - \langle h - D_t w, u\rangle\big]\, dt\right)^+ \\
&\qquad \forall (u,w,h) \in L^2_\mu(0,T;V) \times \mathcal{W}_{\mu,w_n^0} \times L^2_\mu(0,T;V'), \\
\Xi_n(u,w,h) := +\infty \qquad &\text{for any other } (u,w,h) \in L^2_\mu(0,T;V) \times \mathcal{W} \times L^2_\mu(0,T;V').
\end{aligned} \tag{4.15}$$

Note that the first integrand is nonnegative, because of the Fenchel inequality; hence $\Xi_n \geq 0$. Moreover, by (4.14),

$$\begin{aligned}
&\Xi_n(u,w,h) = \int_0^T \big[\gamma_n(u) + \gamma_n^*(w) - (w,u)_H\big]\, d\mu(t) \\
&+ \left(\int_0^T \big[\varphi_n(u, h - D_t w) - \langle h, u\rangle\big]\, d\mu(t) + \int_0^T \gamma_n^*(w)\, dt - T\gamma_n^*(w_n^0)\right)^+ \\
&\qquad \forall (u,w,h) \in L^2_\mu(0,T;V) \times \mathcal{W}_{\mu,w_n^0} \times L^2_\mu(0,T;V').
\end{aligned} \tag{4.16}$$

Proposition 4.2 ([17]). *For any n, the pair (u_n, w_n) solves the initial-value problem* (4.12) *if and only if*

$$\begin{cases}
u_n \in L^2_\mu(0,T;V), \quad w_n \in \mathcal{W}, \\
\Xi_n(u_n, w_n, h_n) = 0 \ \Big(= \inf\limits_{L^2_\mu(0,T;V)\times\mathcal{W}} \Xi_n(\cdot,\cdot,h_n)\Big).
\end{cases} \tag{4.17}$$

Moreover, this null-minimization problem is equivalent to

$$\begin{cases}
\displaystyle\int_0^T \big[\gamma_n(u_n) + \gamma_n^*(w_n) - (w_n,u_n)_H\big]\, d\mu(t) = 0, \\
\displaystyle\int_0^T \big[\varphi_n(u_n, h_n - D_t w_n) - \langle h_n, u_n\rangle\big]\, d\mu(t) + \int_0^T \gamma_n^*(w_n)\, dt - T\gamma_n^*(w_n^0) \leq 0.
\end{cases} \tag{4.18}$$

Proof. Let us assume that $\Xi_n(u,w,h) = 0$. By $(4.12)_3$, the first integrand of (4.15) is nonnegative. Hence

$$\begin{cases}
\gamma_n(u) + \gamma_n^*(w) - (w,u)_H = 0 & \text{a.e. in }]0,T[, \\
\varphi_n(u, h - D_t w) - \langle h - D_t w, u\rangle \leq 0 & \text{a.e. in }]0,T[.
\end{cases} \tag{4.19}$$

$(4.19)_1$ entails that $w \in \gamma_n(u)$ a.e. in $Q := \Omega\times]0,T[$, whence $\langle D_t w, u\rangle = D_t\gamma_n^*(w)$ a.e. in $]0,T[$. Therefore, by $(4.19)_2$,

$$\varphi_n(u, h - D_t w) - \langle h, u\rangle + D_t\gamma_n^*(w(\tau)) = \varphi_n(u, h - D_t w) - \langle h - D_t w, u\rangle \le 0, \tag{4.20}$$

a.e. in $]0,T[$. As φ_n is a representative function the latter term is nonnegative, hence it vanishes whenever $\Xi_n(u,w,h) = 0$. We then conclude that $D_t w_n + \alpha_n(u_n) \ni h_n$ in V', a.e. in $]0,T[$.

The null-minimization problem (4.17) is thus equivalent to the system (4.18), and in turn this entails the Cauchy problem. The converse of the latter implication is straightforward. □

Lemma 4.3 ([24]). *Let* (4.2)–(4.9) *be fulfilled, and set*

$$\Phi_n(v, v^*) = \int_0^T \varphi_n(v, v^*)\, d\mu(t) \qquad \forall (v, v^*) \in L^2_\mu(0,T; V\times V'), \forall n. \tag{4.21}$$

Then: (i) *There exists a functional* Φ *such that, up to extracting a subsequence,*

$$\Phi_n \overset{\Gamma\widetilde{\pi}}{\to} \Phi \qquad \textit{sequentially in } L^2_\mu(0,T; V\times V'). \tag{4.22}$$

(ii) *There exists a representative function* $\varphi : V\times V' \to \mathbf{R}$ *such that, setting*

$$\Phi(v, v^*) = \int_0^T \varphi(v, v^*)\, d\mu(t) \qquad \forall (v, v^*) \in L^2_\mu(0,T; V\times V'), \forall n. \tag{4.23}$$

The first part of this lemma stems from Theorem 3.1. The second part is proved in Section 4 of [24].

Theorem 4.4 (Structural compactness and structural stability). *Let* (4.2)–(4.11) *be fulfilled, and for any* n *let* (u_n, w_n) *be a solution of the Cauchy problem* (4.12). *Then:*

(i) *There exist* $u \in L^2(0,T;V)$ *and* $w \in \mathcal{W}$ *such that, possibly extracting a subsequence,*

$$u_n \rightharpoonup u \qquad \textit{in } L^2(0,T;V), \tag{4.24}$$

$$w_n \overset{*}{\rightharpoonup} w \qquad \textit{in } \mathcal{W}. \tag{4.25}$$

(ii) *There exists a convex and lower semicontinuous function* $\gamma : H \to \mathbf{R}$ *such that*

$$\bar{C}_1\|v\|_H^2 - \bar{C}_2 \le \gamma(v) \le \bar{C}_3\|v\|_H^2 + \bar{C}_4 \qquad \forall v \in H, \tag{4.26}$$

$$\gamma_n \overset{\Gamma}{\to} \gamma \qquad \textit{weakly in } L^2(0,T;H). \tag{4.27}$$

(iii) *Let* φ *be as in Lemma 4.3, and let* $\alpha : V \to \mathcal{P}(V')$ *be the operator that is represented by this function. If*

$$\gamma_n^*(w_n^0) \to \gamma^*(w^0), \tag{4.28}$$

then

$$\begin{cases} u \in L^2(0,T;V), \quad w \in L^2(0,T;H) \cap H^1(0,T;V'), \\ D_t w + \alpha(u) \ni h \quad \textit{in } V', \textit{ a.e. in }]0,T[, \\ w \in \partial\gamma(u) \quad \textit{in } H, \textit{ a.e. in }]0,T[, \\ w(0) = w^0. \end{cases} \tag{4.29}$$

Proof. We split this argument into several steps.

(1) Lemma 4.1 yields (4.24) and (4.25), up to extracting subsequences. As the injection $\mathcal{W} \to L^2(0,T;H)$ is compact, it follows that

$$\int_0^T (u_n, w_n)\, d\mu(t) \to \int_0^T (u, w)\, d\mu(t). \tag{4.30}$$

(2) A result of Γ-compactness similar to Theorem 3.1 holds for the weak topology, see e.g. [7] (p. 95). By (4.5) and (4.8), then there exists a convex and lower semicontinuous function $\gamma : H \to \mathbf{R}$ such that, possibly extracting a subsequence, (4.26) and (4.27) hold. After e.g. [2] (pp. 282-283), this entails that

$$\gamma_n^* \xrightarrow{\Gamma} \gamma^* \qquad \text{strongly in } L^2(0,T;H). \tag{4.31}$$

(3) Hence

$$\liminf_{n\to\infty} \int_0^T \big[\gamma_n(u_n) + \gamma_n^*(w_n)\big]\, d\mu(t) \geq \int_0^T \big[\gamma(u) + \gamma^*(w)\big]\, d\mu(t). \tag{4.32}$$

As $w_n \in \partial\gamma_n(u_n)$ for any n, by the Fenchel equality we have

$$\int_0^T \big[\gamma_n(u_n) + \gamma_n^*(w_n) - (u_n, w_n)\big]\, d\mu(t) = 0 \qquad \forall n. \tag{4.33}$$

On the other hand by the Fenchel inequality

$$\int_0^T \big[\gamma(u) + \gamma^*(w) - (u, w)\big]\, d\mu(t) \geq 0. \tag{4.34}$$

By (4.31)–(4.34) the equality holds in (4.34), and, possibly extracting further sequences,

$$w \in \partial\gamma(u) \quad \text{in } H, \text{ a.e. in }]0,T[, \tag{4.35}$$

$$\int_0^T \gamma_n(u_n)\, d\mu(t) \to \int_0^T \gamma(u)\, d\mu(t), \tag{4.36}$$

$$\int_0^T \gamma_n^*(w_n)\, d\mu(t) \to \int_0^T \gamma^*(w)\, d\mu(t). \tag{4.37}$$

(4) By (4.14), (4.24), (4.25), (4.28), (4.35) and (4.37) we have

$$\begin{aligned} &\int_0^T \langle D_t w_n, u_n\rangle\, d\mu(t) = \int_0^T \gamma_n^*(w_n)\, d\tau - T\gamma_n^*(w_n(0)) \\ &\to \int_0^T \gamma^*(w)\, d\tau - T\gamma^*(w^0) = \int_0^T \langle D_t w, u\rangle\, d\mu(t). \end{aligned} \tag{4.38}$$

Thus

$$(u_n, h_n - D_t w_n) \underset{\pi}{\rightrightarrows} (u, h - D_t w) \qquad \text{in } L^2(0,T;V \times V'). \tag{4.39}$$

Therefore

(4.40)

$$
\begin{aligned}
&\int_0^T \varphi(u, h - D_t w)\, d\mu(t) \overset{(3.12),(4.22)}{\le} \liminf_{n\to\infty} \int_0^T \varphi_n(u_n, h_n - D_t w_n)\, d\mu(t) \\
&\overset{(4.18)_2}{\le} \liminf_{n\to\infty} \left\{ \int_0^T \langle u_n, h_n\rangle\, d\mu(t) - \int_0^T \gamma_n^*(w_n)\, dt + T\gamma_n^*(w_n^0) \right\} \\
&\overset{(4.28),(4.37)}{\le} \int_0^T \langle u, h\rangle\, d\mu(t) - \int_0^T \gamma^*(w)\, dt + T\gamma^*(w^0) \overset{(4.35)}{=} \int_0^T \langle u, h - D_t w\rangle\, d\mu(t).
\end{aligned}
$$

By (4.28), (4.30), (4.32), (4.37) and (4.40), we get

$$
\Xi(u, w, h) \le \liminf_{n\to\infty} \Xi_n(u_n, w_n, h_n). \tag{4.41}
$$

As $0 \le \Xi(u, w, h)$ and $\Xi_n(u_n, w_n, h_n) = 0$ for any n, we infer that $\Xi(u, w, h) = 0$.

In conclusion, u fulfills the twice-time-integrated BEN principle, which by Proposition 4.2 is equivalent to the problem (4.29). □

Remarks 4.5. (i) A similar procedure could be used to prove the structural compactness and structural stability of doubly-nonlinear equations of the form

$$
\alpha(D_t u) + \partial\gamma(u) \ni h, \tag{4.42}
$$

if $\alpha : H \to \mathcal{P}(H)$ is maximal monotone, and $\gamma : V \to \mathbf{R} \cup \{+\infty\}$ is convex and lower semicontinuous; see [22]. This class of doubly-nonlinear equations was also studied e.g. in [6], [13], [16], [17].

(ii) It would be certainly desirable to prove that in this case the limit operator is representable. However, this is not always needed for representable operators, as we pointed out in Remark 3.2 (iii).

5. Application to PDEs of mathematical physics

Theorem 4.4 provides the structural compactness and structural stability of the Cauchy problem associated with several quasilinear PDEs issued from mathematical physics. In this section we just illustrate some examples.

Example 1. The doubly-nonlinear parabolic system

$$
\begin{cases} D_t w - \nabla\cdot g(x, \nabla u) \ni h \\ w \in \partial\gamma(x, u) \end{cases} \tag{5.1}
$$

can represent nonlinear heat conduction coupled with a nonlinear relation between the energy density, w, and the temperature, u. If the mapping $\partial\gamma(x, \cdot)$ is multivalued, then this equation can also account for phase transitions with nonlinear diffusion. (This might be labelled a *doubly-nonlinear Stefan problem.)* Existence of a solution of the associated flow is known, see e.g. [1], [6], [10], [12].

Let Ω be a bounded Lipschitz domain of $\mathbf{R}^N$ ($N \geq 1$), and assume that $g(x, v, \xi) = g_1(\xi) + g_2(x, v, \xi)$, with g_1 and g_2 such that

$$
\begin{aligned}
& g_1 : \mathbf{R}^N \to \mathcal{P}(\mathbf{R}^N) \qquad g_2 : \Omega \times \mathbf{R}^N \to \mathbf{R}^N, \\
& g_1 \ \text{ is maximal monotone,} \\
& g_2 \ \text{ is measurable, and } g_2(x, \cdot) \text{ is maximal monotone for a.e. } x, \\
& \exists c_1, c_2 > 0 : \text{for a.e. } x, \forall \xi, \quad |g_1(\xi)| + |g_2(x, \xi)| \leq c_1 |\xi| + c_2.
\end{aligned}
\tag{5.2}
$$

Moreover, assume that

$$
\begin{aligned}
& \gamma : \Omega \times \mathbf{R}^M \to \mathbf{R} \cup \{+\infty\} \ \ (M \geq 1), \\
& x \mapsto \gamma(x, v) \ \text{ is Lebesgue-measurable, } \forall v, \\
& v \mapsto \gamma(x, v) \ \text{ is convex, for a.e. } x.
\end{aligned}
\tag{5.3}
$$

Under these regularity assumptions, Theorem 4.4 provides the structural compactness and the structural stability of the Cauchy problem associated to this model.

Example 2. For $N = 3$ one can also deal with the parabolic vector equation

$$
D_t \partial \gamma(x, H) + \nabla \times g(x, \nabla \times H) \ni 0 \qquad (\nabla \times := \text{curl}).
\tag{5.4}
$$

This equation arises by coupling
(i) the Faraday law of magnetic induction, $D_t B + \nabla \times E = 0$,
(ii) the Ampère law $J = \nabla \times H$,
(iii) a magnetic constitutive relation of the form $B \in \partial \gamma(x, H)$,
(iv) a nonlinear Ohm law $E \in g(x, J)$.

The Ampère law is here written neglecting the displacement current, by the so-called *eddy current approximation*,

Denoting the outward-oriented unit normal vector-field on $\partial\Omega$ by ν, we assume properties analogous to (5.2), with $g_1 : \mathbf{R}^3 \to \mathcal{P}(\mathbf{R}^3)$ and $g_2 : \Omega \times \mathbf{R}^3 \to \mathbf{R}^3$. We also use the function spaces

$$
\begin{aligned}
& H = \big\{ v \in L^2(\Omega)^3 : \nabla \cdot v = 0 \text{ in } \mathcal{D}'(\Omega) \big\}, \\
& V = \big\{ v \in H : \nabla \times v \in L^2(\Omega)^3, \ \nu \times v = 0 \text{ in } H^{-1/2}(\partial\Omega)^3 \big\},
\end{aligned}
\tag{5.5}
$$

Notice that $V \subset H$ with dense and compact injection. The operator

$$
V \to \mathcal{P}(V') : v \mapsto \nabla \times g_1(\nabla \times v) + \nabla \times g_2(x, \nabla \times v)
$$

is maximal monotone, and Theorem 4.4 then provides the structural compactness and the structural stability of the Cauchy problem that is associated to (5.4).

Example 3. One can also deal with the following system of $2M$ scalar equations

$$
\begin{cases}
D_t w - \nabla \cdot g(x, \nabla u) \ni h \\
w \in \partial \gamma(x, u),
\end{cases}
\tag{5.6}
$$

with $u, w : \Omega \times]0, T[\to \mathbf{R}^M$, assuming that the functions

$$
\gamma : \Omega \times \mathbf{R}^M \to \mathbf{R} \cup \{+\infty\} \ (M \geq 1), \quad g : \Omega \times \mathbf{R}^{M \times N} \to \mathcal{P}(\mathbf{R}^{M \times N})
\tag{5.7}
$$

fulfill analogous regularity conditions to those of the Example 1.

Equations of this form arise in the *thermodynamics of irreversible processes,* and apply to a large class of phenomena, see e.g. [9], Chap. 8 of [14] and Chap. V of [19]. In this case the potential $\gamma(x,u)$ represents the negative of the entropy density, u is the vector of M scalar state variables (density of internal energy, concentration, ecc.), $g(x,\nabla u)$ is the vector of M generalized fluxes (each an element of $\mathbf{R}^N$).

If the mapping $\xi \mapsto g(x,\xi)$ is maximal monotone for a.e. x, then Theorem 4.4 provides the structural compactness and the structural stability of the corresponding Cauchy problem.

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