Pure and Applied Functional Analysis
Volume 6, Number 2, 2021, 399–415

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

A STRONG CONVERGENCE THEOREM UNDER A NEW SHRINKING PROJECTION METHOD FOR TWO DEMIGENERALIZED MAPPINGS IN A BANACH SPACE

WATARU TAKAHASHI

Abstract. In this paper, we establish a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the common fixed point set of two demigeneralized mappings in a Banach space by using a new shrinking projection method. Moreover we apply our result to obtain well-known and new strong convergence theorems in a Hilbert space and in a Banach space.

1. Introduction

Let H be a real Hilbert space and let C be a nonempty, closed and convex subset of H. For a mapping $U : C \to H$, we denote by $F(U)$ the set of fixed points of U. Let k be a real number with $0 \le k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseud-contraction and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\|Ux - q\|^2 \le \|x - q\|^2 + k\|x - Ux\|^2.$$

From this, we have that

$$\|Ux - x\|^2 + \|x - q\|^2 + 2\langle Ux - x, x - q\rangle \le \|x - q\|^2 + k\|x - Ux\|^2.$$

Therefore, we have that

$$2\langle x - Ux, x - q\rangle \ge (1 - k)\|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. A mapping $U : C \to H$ is called generalized hybrid [10] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \le \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$, where $\mathbb{R}$ is the set of real numbers. Such a mapping U is called (α, β)-generalized hybrid. If U is (α, β)-generalized hybrid and $F(U) \neq \emptyset$, then we have that, for $x \in C$ and $q \in F(U)$,

$$\alpha\|q - Ux\|^2 + (1 - \alpha)\|q - Ux\|^2 \le \beta\|q - x\|^2 + (1 - \beta)\|q - x\|^2$$

2020 *Mathematics Subject Classification.* 47H05, 47H09.
Key words and phrases. Fixed point, generalized projection, generalized resolvent, shrinking projection method, duality mappinng.

and hence $\|Ux - q\|^2 \le \|x - q\|^2$. From this, we have that

$$2\langle x - q, x - Ux\rangle \ge \|x - Ux\|^2. \tag{1.2}$$

On the other hand, such a mapping exists in a Banach space. Let E be a smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \ne \emptyset$. Then, for the generalized resolvent J_λ of B for $\lambda > 0$, we have from [3, 23] that, for any $x \in E$ and $q \in B^{-1}0$,

$$\langle J_\lambda x - q, Jx - JJ_\lambda x\rangle \ge 0.$$

Then we get $\langle J_\lambda x - x + x - q, Jx - JJ_\lambda x\rangle \ge 0$ and hence

$$\begin{aligned} 2\langle x - q, Jx - JJ_\lambda x)\rangle &\ge 2\langle x - J_\lambda x, Jx - JJ_\lambda x\rangle \\ &= \phi(x, J_\lambda x) + \phi(J_\lambda x, x) \\ &\ge \phi(x, J_\lambda x), \end{aligned} \tag{1.3}$$

where J is the duality mapping on E and

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2, \quad \forall x, y \in E.$$

Motivated by (1.1), (1.2) and (1.3), Takahashi, Wen and Yao [27] defined a nonlinear mapping as follows: Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \ne \emptyset$ is called η-demigeneralized if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, Jx - JUx\rangle \ge (1 - \eta)\phi(x, Ux),$$

where J is the duality mapping on E. According to this definition, we have that a k-strict pseud-contraction U with $F(U) \ne \emptyset$ is k-demigeneralized, a generalized hybrid mapping U with $F(U) \ne \emptyset$ is 0-demigeneralized and the metric resolvent J_λ with $B^{-1}0 \ne \emptyset$ is 0-demigeneralized. On the other hand, we know a strong convergence theorem under the shrinking projection method which was proved by Takahashi, Takeuchi and Kubota [26] for finding a fixed point of a nonexpansive mapping in a Hilbert space.

Theorem 1.1 ([26])**.** *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let T be a nonexpansive mapping of C into H. Assume that $F(T) \ne \emptyset$. Let $x_1 \in C$ and $C_1 = C$. Let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} y_n = (1 - \lambda_n)x_n + \lambda_n T x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|x_n - z\|\}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a \in \mathbb{R}$ and $\{\lambda_n\} \subset (0, \infty)$ satisfy the following:

$$0 < a \le \lambda_n \le 1, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in F(T)$, where $z_0 = P_{F(T)} x_1$.

In this paper, using a new shrinking projection method, we establish a strong convergence theorem for finding a common element of the set of zero points of a maximal monotone operator and the set of common fixed points of two demigeneralized mappings in a Banach space. Moreover we apply our result to obtain well-known and new strong convergence theorems in a Hilbert space and in a Banach space.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf \left\{ 1 - \frac{\|x+y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x-y\| \ge \epsilon \right\}$$

for all ϵ with $0 \le \epsilon \le 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [6, 16].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x+ty\| - \|x\|}{t} \tag{2.1}$$

exists. In the case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [22] and [23]. In this connection, see also the paper by Reich [15]. We know the following result.

Lemma 2.1 ([22])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x-y, Jx-Jy \rangle \ge 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x-y, Jx-Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x-y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \le \phi(x, y) \le (\|x\| + \|y\|)^2; \tag{2.3}$$

$$\phi(x,y) = \phi(x,z) + \phi(z,y) + 2\langle x - z, Jz - Jy\rangle; \tag{2.4}$$

$$2\langle x - y, Jz - Jw\rangle = \phi(x,w) + \phi(y,z) - \phi(x,z) - \phi(y,w). \tag{2.5}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x,y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.6}$$

The following lemma was proved by Kamimura and Takahashi [9].

Lemma 2.2 ([9])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z,x) = \min_{y \in C} \phi(y,x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. For example, see [1,2,9].

Lemma 2.3 ([1,2,9])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x_1 \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x_1$;
(2) $\langle z - y, Jx_1 - Jz\rangle \geq 0, \quad \forall y \in C$.

Lemma 2.4 ([1,9])**.** *Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space. Then*

$$\phi(y, \Pi_C x_1) + \phi(\Pi_C x_1, x_1) \leq \phi(y, x_1), \quad \forall y \in C,\ x_1 \in E.$$

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^*\rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to [4,18]; see also [23, Theorem 3.5.4].

Theorem 2.5 ([4,18])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r. In fact, let $x \in E$. There exists $x_r \in D(B)$ from $Jx \in E^* = R(J + rB)$ such that

$$Jx \in Jx_r + rBx_r.$$

We show that such a solution x_r is unique. Take $z_1, z_2 \in D(B)$ such that

$$Jx \in Jz_1 + rBz_1 \text{ and } Jx \in Jz_2 + rBz_2.$$

We have $\frac{1}{r}(Jx - Jz_1) \in Bz_1$ and $\frac{1}{r}(Jx - Jz_2) \in Bz_2$. Since B and J are monotone, we have

$$\begin{aligned}
0 &\le \left\langle z_1 - z_2, \frac{1}{r}(Jx - Jz_1) - \frac{1}{r}(Jx - Jz_2) \right\rangle \\
&= \frac{1}{r}\langle z_1 - z_2, Jx - Jz_1 - (Jx - Jz_2)\rangle \\
&= -\frac{1}{r}\langle z_1 - z_2, Jz_1 - Jz_2\rangle \le 0
\end{aligned}$$

and hence

$$\langle z_1 - z_2, Jz_1 - Jz_2)\rangle = 0.$$

Since E is strictly convex, we have from Lemma 2.1 that $z_1 = z_2$. We define J_r by $x_r = J_r x$. Such $J_r, r > 0$ are called the generalized resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [23].

3. Main Result

In this section, using the new shrinking projection method we introduced, we prove a strong convergence theorem for finding a common element of the zero point set of a maximal monotone operator and the common fixed point set of two demigeneralized mappings in a Banach space. The ideas of the proof are due to [19–21, 25]. Let E be a smooth and strictly convex Banach space and let J be the duality mapping on E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called (η, s)-demigeneralized [14, 27] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, Jx - JUx\rangle \ge (1 - \eta)\phi(x, Ux) + s\phi(Ux, x), \tag{3.1}$$

where $F(U)$ is the set of fixed points of U. In particular, if $s = 0$ in (3.1), then the mapping U is as follows:

$$2\langle x - q, Jx - JUx\rangle \ge (1 - \eta)\phi(x, Ux)$$

for all $x \in C$ and $q \in F(U)$. Especially, such $(\eta, 0)$-demigeneralized mappings in the class of demigeneralized mappings are important and called η-demigeneralized.

Examples.

(1) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \le k < 1$. If U is a k-strict pseudo-contraction [5] and $F(U) \neq \emptyset$, then U is $(k, 0)$-demigeneralized [27].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. If U is $(\alpha,\,\beta)$-generalized hybrid and $F(U) \neq \emptyset$, then U is $(0,0)$-demigeneralized [27], i.e.,

$$2\langle x-u, x-Ux\rangle \geq \|x-Ux\|^2, \quad \forall x \in C,\ u \in F(U).$$

Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [11,12] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx-Ty\|^2 \leq \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x,y \in C.$$

It is also hybrid [24] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx-Ty\|^2 \leq \|x-y\|^2 + \|Tx-y\|^2 + \|Ty-x\|^2, \quad \forall x,y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [8].

(3) Let E be a strictly convex, reflexive and smooth Banach space and let C be a nonempty, closed and convex subset of E. Let Π_C be the generalized projection of E onto C. Then Π_C is $(0,1)$-demigeneralized. In fact, since Π_C is the generalized projection of E onto C, we have that, for any $x \in E$ and $q \in C$,

$$2\langle \Pi_C x - q, Jx - J\Pi_C x\rangle \geq 0.$$

Then we get

$$2\langle \Pi_C x - x + x - q, Jx - J\Pi_C x\rangle \geq 0$$

and hence

$$\begin{aligned}2\langle x-q, Jx - J\Pi_C x\rangle &\geq 2\langle x - \Pi_C x, Jx - J\Pi_C x\rangle \\ &= \phi(x, \Pi_C x) + \phi(\Pi_C x, x).\end{aligned}$$

This means that Π_C is $(0,1)$-demigeneralized. Furthermore, since

$$\phi(x, \Pi_C x) + \phi(\Pi_C x, x). \geq \phi(x, \Pi_C x),$$

Π_C is also $(0,0)$-demigeneralized, i.e., 0-demigeneralized.

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the generalized resolvent Q_λ is $(0,1)$-demigeneralized, i.e.,

$$\begin{aligned}2\langle x-q, Jx - JQ_\lambda x\rangle &\geq 2\langle x - Q_\lambda x, Jx - JQ_\lambda x\rangle \\ &= \phi(x, Q_\lambda x) + \phi(Q_\lambda x, x).\end{aligned}$$

Furthermore, since

$$\phi(x, Q_\lambda x) + \phi(Q_\lambda x, x). \geq \phi(x, Q_\lambda x),$$

Q_λ is also $(0,0)$-demigeneralized, i.e., 0-demigeneralized.

The following lemma is important and crucial in the proof of our main result which was proved in [27]. For the sake of completeness, we give the proof.

Lemma 3.1 ([27])**.** *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0,\infty)$, respectively. Let U be an $(\eta, 0)$-demigeneralized* *mapping of C into E. Then $F(U)$ is closed and convex. In particular, if U is (η, s)-demigeneralized, then $F(U)$ is closed and convex.*

Proof. Assume that U is an $(\eta, 0)$-demigeneralized. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$\langle q - q_n, Jq - JUq \rangle \geq \frac{1-\eta}{2}\phi(q, Uq).$$

From $q_n \to q$, we have $0 \geq \frac{1-\eta}{2}\phi(q, Uq)$. From $1 - \eta > 0$, we get $0 \geq \phi(q, Uq)$ and hence $q = Uq$. This implies that $F(U)$ is closed. Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $z = \alpha p + (1-\alpha)q$, where $\alpha \in [0, 1]$. From the definition of U, we have that, for $x \in C$ and $u \in F(U)$,

$$\langle x - u, Jx - JUx \rangle \geq \frac{1-\eta}{2}\phi(x, Ux).$$

This implies from (2.5) that

$$\phi(x, Ux) + \phi(u, x) - \phi(u, Ux) \geq (1-\eta)\phi(x, Ux)$$

and hence

$$\phi(u, x) + \eta\phi(x, Ux) \geq \phi(u, Ux).$$

Using this, we have that for $z = \alpha p + (1-\alpha)q$ and $p, q \in F(U)$,

$$\begin{aligned}
\phi(z, Uz) &= \|z\|^2 - 2\langle z, JUz \rangle + \|Uz\|^2 \\
&= \|z\|^2 - 2\langle \alpha p + (1-\alpha)q, JUz \rangle + \|Uz\|^2 \\
&= \|z\|^2 - 2\alpha\langle p, JUz \rangle - 2(1-\alpha)\langle q, JUz \rangle + \|Uz\|^2 \\
&= \|z\|^2 + \alpha\phi(p, Uz) + (1-\alpha)\phi(q, Uz) - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&\leq \|z\|^2 + \alpha(\phi(p, z) + \eta\phi(z, Uz)) \\
&\qquad + (1-\alpha)(\phi(q, z) + \eta\phi(z, Uz)) - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&= \|z\|^2 + \alpha(\|p\|^2 - 2\langle p, Jz \rangle + \|z\|^2 + \eta\phi(z, Uz)) \\
&\qquad + (1-\alpha)(\|q\|^2 - 2\langle q, Jz \rangle + \|z\|^2 + \eta\phi(z, Uz)) \\
&\qquad - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&= 2\|z\|^2 - 2\langle \alpha p + (1-\alpha)q, Jz \rangle + \eta\phi(z, Uz) \\
&= 2\|z\|^2 - 2\langle z, Jz \rangle + \eta\phi(z, Uz) \\
&= \eta\phi(z, Uz)
\end{aligned}$$

and hence $0 \leq (\eta - 1)\phi(z, Uz)$. We have from $0 > \eta - 1$ that $\phi(z, Uz) = 0$. Since E is strictly convex, we have $z = Uz$. This means that $F(U)$ is convex. If U is (η, s)-demigeneralized, then U is $(\eta, 0)$-demigeneralized and hence $F(U)$ is closed and convex. $\square$

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.2. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty, closed and convex subset of E such that JC is closed and convex. Let $A \subset E \times E^*$ be a maximal monotone operator and let $Q_r = (J + rA)^{-1}J$ be the generalized resolvent of A for all $r > 0$. Let $\eta, \tau \in (-\infty, 1)$ and let S and T be η and τ-demigeneralized mappings from C into itself, respectively, such that they are demiclosed and $\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} u_n = Q_{r_n} z_n, \\ z_n = J^{-1}(\beta_n J v_n + (1 - \beta_n) J T v_n,) \\ v_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) J S x_n), \\ C_{n+1} = \Big\{ z \in C_n : 2\langle z_n - z, J z_n - J u_n \rangle \geq \phi(z_n, u_n) + \phi(u_n, z_n), \\ \qquad\qquad 2\langle v_n - z, J v_n - J z_n \rangle \geq (1 - \tau)\phi(v_n, z_n) \\ \qquad\qquad \text{and} \quad 2\langle x_n - z, J x_n - J v_n \rangle \geq (1 - \eta)\phi(x_n, v_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where J is the duality mapping on E, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $1 - \alpha_n \geq b > 0$ and $1 - \beta_n \geq c > 0$ for some $b, c \in (0, 1)$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is the generalized projection of E onto Ω.

Proof. It follows that C_n is closed and convex for all $n \in \mathbb{N}$. We show that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. It is obvious that $\Omega \subset C_1 = C$. Suppose that $\Omega \subset C_k$ for some $k \in \mathbb{N}$. To show $\Omega \subset C_{k+1}$, let us show that

$$2\langle z_k - z, J z_k - J u_k \rangle \geq \phi(z_k, u_k) + \phi(u_k, z_k),$$

$2\langle v_k - z, J v_k - J z_k \rangle \geq (1 - \tau)\phi(v_k, z_k)$ and

$$2\langle x_k - z, J x_k - J v_k \rangle \geq (1 - \eta)\phi(x_k, v_k)$$

for all $z \in \Omega$. Let $z \in \Omega$. Since Q_{r_k} is the generalized resolvent, we have that

$$\langle Q_{r_k} z_k - z, J z_k - J Q_{r_k} z_k \rangle \geq 0$$

for all $z \in \Omega \subset A^{-1}0$. Thus, we get that

$$\langle Q_{r_k} z_k - z_k + z_k - z, J z_k - J Q_{r_k} z_k \rangle \geq 0$$

and hence

$$2\langle z_k - z, J z_k - J Q_{r_k} z_k \rangle \geq 2\langle z_k - Q_{r_k} z_k, J z_k - J Q_{r_k} z_k \rangle.$$

We have from (2.5) that

$$2\langle z_k - z, J z_k - J Q_{r_k} z_k \rangle \geq \phi(z_k, Q_{r_k} z_k) + \phi(Q_{r_k} z_k, z_k).$$

This implies that

$$2\langle z_k - z, J z_k - J u_k \rangle \geq \phi(z_k, u_k) + \phi(u_k, z_k).$$

Since $u \in \Omega$ and T is τ-demigeneralized, we have that

$$\begin{aligned} \phi(v_k, z_k) &= \phi\big(v_k, J^{-1}(\beta_k J v_k + (1 - \beta_k) J T v_k)\big) \\ &= \|v_k\|^2 - 2\langle v_k, \beta_k J v_k + (1 - \beta_k) J T v_k \rangle \\ &\quad + \|\beta_k J v_k + (1 - \beta_k) J T v_k\|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \|v_k\|^2 - 2\beta_k\|v_k\|^2 - 2(1-\beta_k)\langle v_k, JTv_k\rangle \\
&\quad + \beta_k\|v_k\|^2 + (1-\beta_k)\|Tv_k\|^2 \\
&= (1-\beta_k)\|v_k\|^2 - 2(1-\beta_k)\langle v_k, JTv_k\rangle + (1-\beta_k)\|Tv_k\|^2 \\
&= (1-\beta_k)\phi(v_k, Tv_k)
\end{aligned}
$$

and hence

$$
\begin{aligned}
2\langle v_k - z, Jv_k - Jz_k\rangle &= 2\langle v_k - z, Jv_k - (\beta_k Jv_k + (1-\beta_k)JTv_k)\rangle \\
&= 2(1-\beta_k)\langle v_k - z, Jv_k - JTv_k\rangle \\
&\geq (1-\beta_k)(1-\tau)\phi(v_k, Tv_k) \qquad (3.2)\\
&\geq (1-\tau)\phi\big(v_k, J^{-1}(\beta_k Jv_k + (1-\beta_k)JTv_k)\big) \\
&= (1-\tau))\phi(v_k, z_k).
\end{aligned}
$$

Similarly, we have that

$$2\langle x_k - z, J(x_k - v_k)\rangle \geq (1-\eta)\phi(x_k, v_k)..$$

Then $\Omega \subset C_{k+1}$. We have by mathematical induction that $\Omega \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

We have that $F(S)$ and $F(T)$ are closed and convex from Lemma 3.1 . We also have that $A^{-1}0$ is closed and convex. Thus Ω is nonempty, closed and convex. Then there exists $w_1 \in \Omega$ such that $w_1 = \Pi_\Omega x_1$. From $x_n = \Pi_{C_n} x_1$, we have that

$$\phi(x_n, x_1) \leq \phi(y, x_1)$$

for all $y \in C_n$. Since $w_1 \in \Omega \subset C_n$, we have that

$$\phi(x_n, x_1) \leq \phi(w_1, x_1). \tag{3.3}$$

From $x_n = \Pi_{C_n} x_1$ and $x_{n+1} \in C_{n+1} \subset C_n$, we have that

$$\phi(x_n, x_1) \leq \phi(x_{n+1}, x_1).$$

Thus $\{\phi(x_n, x_1)\}$ is bounded and nondecreasing. Then there exists the limit of $\{\phi(x_n, x_1)\}$. Put $\lim_{n\to\infty} \phi(x_n, x_1) = c$. For any $m, n \in \mathbb{N}$ with $m \geq n$, we have $C_m \subset C_n$. From $x_m = \Pi_{C_m} x_1 \in C_m \subset C_n$ and Lemma 2.4, we have that

$$\phi(\Pi_{C_n} x_1, x_1) + \phi(x_m, \Pi_{C_n} x_1) \leq \phi(x_m, x_1).$$

This implies that

$$\phi(x_m, x_n) \leq \phi(x_m, x_1) - \phi(x_n, x_1) \leq c - \phi(x_n, x_1). \tag{3.4}$$

Since $c - \phi(x_n, x_1) \to 0$ as $n \to \infty$, we have from Lemma 2.2 that $\{x_n\}$ is a Caushy sequence. By the completeness of C, there exists a point $w_0 \in C$ such that

$$x_n \to w_0. \tag{3.5}$$

To complete the proof, it is sufficient to show that $w_1 = \Pi_\Omega x_1 = w_0$. From (3.5), we have that

$$\|x_n - x_{n+1}\| \to 0. \tag{3.6}$$

From $x_{n+1} = \Pi_{C_{n+1}} x_1$, we have $x_{n+1} \in C_{n+1}$. This implies that

$$2\langle x_n - x_{n+1}, Jx_n - Jv_n\rangle \geq (1-\eta)\phi(x_n, v_n). \tag{3.7}$$

Furthermore, we claim that $\{Jx_n - Jv_n\}$ is bounded. That $\{Jx_n - Jv_n\}$ is bounded is proved as follows. For proving this, from

$$\|Jx_n - Jv_n\| = \|(1-\beta_n)(Jx_n - JSx_n)\|,$$

we may prove that $\{Sx_n\}$ is bounded. Since

$$2\langle x_n - z, Jx_n - JSx_n\rangle \geq (1-\eta)(x_n, Sx_n)$$

for $z \in F(S)$, we have from (2.5) that

$$\phi(x_n, Sx_n) + \phi(z, x_n) - \phi(z, Sx_n) \geq (1-\eta)\phi(x_n, Sx_n)$$

and hence

$$\eta\phi(x_n, Sx_n) + \phi(z, x_n) \geq \phi(z, Sx_n).$$

In the case of $\eta \leq 0$, we have $\phi(z, x_n) \geq \phi(z, Sx_n)$. Thus, we have that, for $u \in F(S)$,

$$\begin{aligned}(\|z\| - \|Sx_n\|)^2 &\leq \phi(z, Sx_n)\\ &\leq \phi(z, x_n) \leq (\|z\| + \|x_n\|)^2.\end{aligned}$$

Using this, we have that

$$\|Sx_n\| \leq (\|z\| + \|x_n\|) + \|z\|.$$

This implies that $\{Sx_n\}$ is bounded. In the case of η withh $0 < \eta < 1$, we have

$$\eta\phi(x_n, Sx_n) + \phi(z, x_n) \geq \phi(z, Sx_n).$$

Thus, we have that, for $z \in F(S)$,

$$\begin{aligned}(\|z\| - \|Sx_n\|)^2 &\leq \phi(z, Sx_n)\\ &\leq \phi(z, x_n) + \eta\phi(x_n, Sx_n)\\ &\leq (\|z\| + \|x_n\|)^2 + \eta(\|x_n\| + \|Sx_n\|)^2\\ &\leq \big(\|z\| + \|x_n\| + \sqrt{\eta}(\|x_n\| + \|Sx_n\|)\big)^2.\end{aligned}$$

From this, we have that

$$|\|z\| - \|Sx_n\|| \leq \|z\| + \|x_n\| + \sqrt{\eta}(\|x_n\| + \|Sx_n\|)$$

and hence

$$(1-\sqrt{\eta})\|Sx_n\| \leq (1+\sqrt{\eta})\|x_n\| + 2\|z\|.$$

Then, we have that

$$\|Sx_n\| \leq \Big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\Big).$$

This implies that $\{Sx_n\}$ is bounded. We have from (3.7) that $\phi(x_n, v_n) \to 0$. Then we have from Lemma 2.2 that

$$\lim_{n\to\infty} \|x_n - v_n\| = 0. \tag{3.8}$$

Since E is uniformly smooth, we have that $Jx_n - Jv_n \to 0$. From $1 - \alpha_n \geq b > 0$ and

$$\|Jx_n - Jv_n\| = \|(1-\alpha_n)(Jx_n - JSx_n)\| \geq b\|Jx_n - JSx_n\|,$$

we have that $Jx_n - JSx_n \to 0$. Since E^* is uniformly smooth, we have that

$$\lim_{n\to\infty} \|x_n - Sx_n\| = 0. \tag{3.9}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$2\langle v_n - x_{n+1}, Jv_n - Jz_n \rangle \geq (1-\tau)\phi(v_n, z_n)$$

and hence

$$2\langle v_n - x_n + x_n - x_{n+1}, Jv_n - Jz_n \rangle \geq (1-\beta_n)(1-\tau)\phi(v_n, z_n).$$

As in the proof of boundedness of $\{Jx_n - Jv_n\}$, we have that $\{Jv_n - Jz_n\}$ is bounded. From $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - v_n\| \to 0$, we have that $\lim_{n\to\infty} \phi(v_n, z_n) = 0$. Using Lemma 2.2, we have that $v_n - z_n \to 0$. As in the proof of $x_n - Sx_n \to 0$, we have that

$$\lim_{n\to\infty} \|v_n - Tv_n\| = 0. \tag{3.10}$$

We also have from $x_{n+1} \in C_{n+1}$ that

$$2\langle z_n - x_{n+1}, Jz_n - Ju_n \rangle \geq \phi(z_n, u_n) + \phi(u_n, z_n).$$

From $\|z_n - x_{n+1}\| \leq \|z_n - v_n\| + \|v_n - x_n\| + \|x_n - x_{n+1}\|$, $z_n - v_n \to 0$, $v_n - x_n \to 0$ and $x_n - x_{n+1} \to 0$, we have $\|z_n - x_{n+1}\| \to 0$. Then we get that

$$\lim_{n\to\infty} \phi(z_n, u_n) = 0$$

and hence

$$\lim_{n\to\infty} \|z_n - Q_{r_n} z_n\| = 0. \tag{3.11}$$

Since $x_n \rightharpoonup w_0$ and S is demiclosed, we have from (3.9) that $w_0 \in F(S)$. Similarly, since $v_n \rightharpoonup w_0$ and T is demiclosed, we have from (3.10) that $w_0 \in F(T)$. We show $w_0 \in A^{-1}0$. Since E is uniformly smooth, from $u_n = Q_{r_n} z_n$ and (3.11) we have that

$$\lim_{n\to\infty} \|Jz_n - Ju_n\| = 0.$$

From $r_n \geq a$, we have

$$\lim_{n\to\infty} \frac{1}{r_n} \|Jz_n - Ju_n\| = 0.$$

Therefore, we have

$$\lim_{n\to\infty} \|A_{r_n} z_n\| = \lim_{n\to\infty} \frac{1}{r_n} \|Jz_n - Ju_n\| = 0.$$

For $(p, p^*) \in A$, from the monotonicity of A, we have $\langle p - u_n, p^* - A_{r_n} z_n \rangle \geq 0$ for all $n \geq 0$. From $u_n \rightharpoonup w_0$, we get $\langle p - w_0, p^* \rangle \geq 0$. From the maximallity of A, we have $w_0 \in A^{-1}0$. Therefore, we have $w_0 \in \Omega$.

From $w_1 = \Pi_\Omega x_1$, $w_0 \in \Omega$ and (3.3), we have that

$$\phi(w_1, x_1) \leq \phi(w_0, x_1) = \lim_{n\to\infty} \phi(x_n, x_1) \leq \phi(w_1, x_1).$$

Then we get that $\phi(w_1, x_1) = \phi(w_0, x_1)$ and hence $w_0 = w_1$. Therefore, we have $x_n \to w_0 = w_1$. This completes the proof. $\square$

4. Applications

In this section, using Theorem 3.2, we prove strong convergence theorems under a new shrinking projection method in a Hilbert space and in a Banach space. We know the following result obtained by Marino and Xu [13]; see also [28].

Lemma 4.1 ([13, 28]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [10]; see also [29].

Lemma 4.2 ([10, 29]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

As a direct consequence of Theorem 3.2, we obtain the following result.

Theorem 4.3. *Let E be a uniformly convex and uniformly smooth Banach space. Let $A \subset E \times E^*$ be a maximal monotone operator satisfying $A^{-1}0 \neq \emptyset$ and let $Q_r = (J + rA)^{-1}J$ be the generalized resolvent of A for all $r > 0$. Let S and T be relatively nonexpansive mappings from E into itself such that*

$$\Omega = F(S) \cap F(T) \cap A^{-1}0 \neq \emptyset.$$

For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by

$$\begin{cases} u_n = Q_{r_n} z_n, \\ z_n = J^{-1}(\beta_n J v_n + (1 - \beta_n) JT v_n), \\ v_n = J^{-1}(\alpha_n J x_n + (1 - \alpha_n) JS x_n), \\ C_{n+1} = \Big\{ z \in C_n : 2\langle z_n - z, Jz_n - Ju_n \rangle \geq \phi(z_n, u_n) + \phi(u_n, z_n), \\ \qquad\qquad\qquad \phi(z, z_n) \leq \phi(z, v_n) \text{ and } \phi(z, v_n) \leq \phi(z, x_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where J is the duality mapping on E, $\{\alpha_n\}, \{\beta_n\} \subset [0, 1]$ and $\{r_n\} \subset [a, \infty)$ for some $a > 0$. If $1 - \alpha_n \geq b > 0$ and $1 - \beta_n \geq c > 0$ for some $b, c \in (0, 1)$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is the generalized projection of E onto Ω.

Proof. Since S and T are relatively nonexpansive, S and T are 0-demigeneralized mappings such that they are demiclosed. We also have that $\phi(z, z_n) \leq \phi(z, v_n)$ is equivalent to

$$2\langle v_n - z, Jv_n - Jz_n \rangle \geq \phi(v_n, z_n).$$

Similarly, $\phi(z, v_n) \leq \phi(z, x_n)$ is equivalent to

$$2\langle x_n - z, Jx_n - Jv_n \rangle \geq \phi(x_n, v_n).$$

Therefore, we obtain Theorem 4.3 from Theorem 3.2. □

Let E be a Banach space and let $f : E \to (-\infty, \infty]$ be a proper, lower semicontinuous and convex function. Define the subdifferential of f as follows:

$$\partial f(x) = \{x^* \in E^* : f(y) \geq \langle y - x, x^* \rangle + f(x), \ \forall y \in E\}$$

for all $x \in E$. Then, we know that ∂f is a maximal monotone operator; see [17] for more details.

Theorem 4.4. *Let E be a uniformly convex and uniformly smooth Banach space and let C be a nonempty, closed and convex subset of E such that JC is closed and convex. Let S be a relatively nonexpansive mapping from C into itself. Let $\tau \in (-\infty, 1)$ and let T be a τ-demigeneralized mapping from C into itself such that it is demiclosed and $F(T) \neq \emptyset$. Suppose that $\Omega = F(S) \cap F(T) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J^{-1}\big(\alpha_n J z_n + (1-\alpha_n) JTv_n\big), \\ v_n = J^{-1}\big(\beta_n J x_n + (1-\beta_n) JSx_n\big), \\ C_{n+1} = \Big\{ z \in C_n : 2\langle v_n - z, Jv_n - Jz_n\rangle \geq (1-\tau)\phi(v_n, z_n) \\ \qquad\qquad\qquad\qquad\qquad\qquad \text{and } \phi(z, v_n) \leq \phi(z, x_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If $1-\alpha_n \geq b > 0$ and $1-\beta_n \geq c > 0$ for some $b, c \in (0,1)$, then $\{x_n\}$ converges strongly to $\Pi_\Omega x_1$, where Π_Ω is the generalized projection of E onto Ω.

Proof. Set $A = \partial i_C$ in Theorem 3.2, where i_C is the indicator function, that is,

$$i_C = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then, we have that ∂i_C is a maximal monotone operator and $Q_r = \Pi_C$ for all $r > 0$. In fact, for any $x \in E$ and $r > 0$, we have from Lemma 2.3 that

$$\begin{aligned} z = Q_r x &\Leftrightarrow Jz + r\partial i_C(z) \ni Jx \\ &\Leftrightarrow Jx - Jz \in r\partial i_C(z) \\ &\Leftrightarrow i_C(y) \geq \Big\langle y - z, \frac{Jx - Jz}{r} \Big\rangle + i_C(z), \ \forall y \in E \\ &\Leftrightarrow 0 \geq \langle y - z, Jx - Jz\rangle, \ \forall y \in C \\ &\Leftrightarrow z = \arg\min_{y \in C} \phi(y, x) \\ &\Leftrightarrow z = \Pi_C \end{aligned}$$

and $u_n = z_n$ in Theorem 3.2. Therefore, from Theorem 3.2, we obtain Theorem 4.4. $\square$

The following is a strong convergence theorem for nonexpansive mappings and k-strict pseudo-contractions in a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $k \in [0,1)$. Let $T : C \to C$ be a nonexpansive mapping and let $U : C \to C$ be a k-strict pseudo-contraction such that $F(U) \neq \emptyset$. Suppose that $\Omega = F(T) \cap F(U) \neq \emptyset$. Let For $x_1 \in C$ and $C_1 = C$, let*

$\{x_n\}$ be a sequence generated by

$$\begin{cases} z_n = \beta_n v_n + (1-\beta_n)Uv_n, \\ v_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle v_n - z, v_n - z_n\rangle \geq (1-k)\|v_n - Uz_n\|^2 \\ \qquad\qquad\qquad \text{and } 2\langle x_n - z, x_n - v_n\rangle \geq \|x_n - v_n\|^2 \ \Big\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If $1-\alpha_n \geq b > 0$ and $1-\beta_n \geq c > 0$ for some $b, c \in (0,1)$, then $\{x_n\}$ converges strongly to a point $P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. Since U be a k-strict pseudo-contraction of C into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is k-demigeneralized. From Lemma 4.1, U is demiclosed. We also have that a nonexpansive maping T is 0-demigeneralized and demiclosed. Furthermore, putting $A = 0$ in Theorem 3.2, we have that $Q_r = I$ for all $r > 0$. Therefore, we have the desired result from Theorem 3.2. □

The following is a strong convergence theorem for nonexpansive mappings and generalized hybrid mappings in a Hilbert space.

Theorem 4.6. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $T : C \to C$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $U : C \to C$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Suppose that $\Omega = F(T) \cap F(U) \neq \emptyset$. For $x_1 \in C$ and $C_1 = C$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = \beta_n v_n + (1-\beta_n)Uv_n, \\ v_n = \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle v_n - z, v_n - z_n\rangle \geq \|v_n - Uz_n\|^2 \\ \qquad\qquad\qquad \text{and } 2\langle x_n - z, x_n - v_n\rangle \geq \|x_n - v_n\|^2 \Big\}, \\ x_{n+1} = P_{C_{n+1}}x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$. If $1-\alpha_n \geq b > 0$ and $1-\beta_n \geq c > 0$ for some $b, c \in (0,1)$, then $\{x_n\}$ converges strongly to a point $P_\Omega x_1$, where P_Ω is the metric projection of H onto Ω.

Proof. Since U be a generalized hybrid mapping of C into itself such that $F(U) \neq \emptyset$, from (2) in Examples, U is 0-demigeneralized. From Lemma 4.2, U is demiclosed. We also have that a nonexpansive maping T is 0-demigeneralized and demiclosed. Furthermore, putting $A = 0$ in Theorem 3.2, we have that $Q_r = I$ for all $r > 0$. Therefore, we have the desired result from Theorem 3.2. □

The following is a strong convergence theorem for two generalized projections in a Banach space.

Theorem 4.7. *Let E be a uniformly convex and uniformly smooth Banach space. Let C and D be nonempty, closed and convex subsets of E and let Π_C and Π_D be the* *generalized projections of E onto C and D, respectively. Suppose that $C \cap D \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J^{-1}(\beta_n Jv_n + (1-\beta_n)J\Pi_D v_n), \\ v_n = \Pi_C x_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle v_n - z, Jv_n - Jz_n\rangle \geq \phi(v_n, z_n) \\ \qquad\qquad\qquad \text{and} \quad 2\langle x_n - z, Jx_n - Jv_n\rangle \geq \phi(x_n, v_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset [0,1]$. If $1 - \beta_n \geq c > 0$ for some $c \in (0,1)$, then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap D$, where $z_0 = \Pi_{C\cap D} x_1$.

Proof. Since Π_C is the generalized projection of E onto C, Π_C is 0-demigeneralized from (3) in Examples. Since Π_D is the generalized projection of E onto D, from (3) in Examples, Π_D is 0-demigeneralized. We also have that if $\{u_n\}$ is a sequence in E such that $u_n \rightharpoonup p$ and $u_n - \Pi_D u_n \to 0$, then $p = \Pi_D p$. In fact, assume that $u_n \rightharpoonup p$ and $u_n - \Pi_D u_n \to 0$. It is clear that $\Pi_D u_n \rightharpoonup p$. Furthermore, since E is uniformly smooth, we have that $\|Ju_n - J\Pi_D u_n\| \to 0$. Since Π_D is the generalized projection of E onto D, we have that

$$\langle \Pi_D u_n - \Pi_D p, Ju_n - J\Pi_D u_n - (Jp - J\Pi_D p)\rangle \geq 0.$$

Therefore, $\langle p - \Pi_D p, -(Jp - J\Pi_D p)\rangle \geq 0$. This implies that

$$\phi(p, \Pi_D p) + \phi(\Pi_D p, p) \leq 0$$

and hence $p = \Pi_D p$. Therefore, Π_D is demiclosed. Similarly, Π_C is demiclosed. Furthermore, putting $A = 0$ in Theorem 3.2, we have that $Q_r = I$ for all $r > 0$. Therefore, we have the desired result from Theorem 3.2. □

The following is a strong convergence theorem for two generalized resolvents in a Banach space.

Theorem 4.8. *Let E be a uniformly convex and uniformly smooth Banach space. Let G and B be maximal monotone operators of E into E^*. Let J_λ be the generalized resolvent of G for $\lambda > 0$ and let R_μ be the generalized resolvent of B for $\mu > 0$. Suppose that $G^{-1}0 \cap B^{-1}0 \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J^{-1}(\beta_n Jv_n + (1-\beta_n)JR_\mu v_n), \\ v_n = J_\lambda x_n, \\ C_{n+1} = \Big\{ z \in C_n : 2\langle v_n - z, Jv_n - Jz_n\rangle \geq \phi(v_n, z_n) \\ \qquad\qquad\qquad \text{and} \quad 2\langle x_n - z, Jx_n - Jv_n\rangle \geq \phi(x_n, v_n) \Big\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{\beta_n\} \subset [0,1]$. If $1 - \beta_n \geq c > 0$ for some $c \in (0,1)$, then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap B^{-1}0$, where $z_0 = \Pi_{G^{-1}0\cap B^{-1}0} x_1$.

Proof. Since R_μ is the generalized resolvent of B on E, from (4) in Examples, R_μ is 0-demigeneralized. We also have that if $\{u_n\}$ is a sequence in E such that $u_n \rightharpoonup p$ and $u_n - R_\mu u_n \to 0$, then $p = R_\mu p$. In fact, assume that $u_n \rightharpoonup p$ and $u_n - R_\mu u_n \to 0$. It is clear that $R_\mu u_n \rightharpoonup p$. Furthermore, since E is uniformly smooth, we have that $\|Ju_n - JR_\mu u_n\| \to 0$. Since R_μ is the generalized resolvent of B, we have from [3] that

$$\langle R_\mu u_n - R_\mu p, Ju_n - JR_\mu u_n - (Jp - JR_\mu p)\rangle \geq 0.$$

Therefore, $\langle p - R_\mu p, -(Jp - JR_\mu p)\rangle \geq 0$. This implies that

$$\phi(p, R_\mu p) + \phi(R_\mu p, p) \leq 0$$

and hence $p = R_\mu p$. Therefore, R_μ is demiclosed. Similarly, J_λ is 0-demigeneralized and demiclosed. Furthermore, putting $A = 0$ in Theorem 3.2, we have that $Q_r = I$ for all $r > 0$. Therefore, we have the desired result from Theorem 3.2. □

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in: Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50,

[2] Y. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator in Banach spaces*, Panamer. Math. J. **4** (1994), 39–54.

[3] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[4] F. E. Browder, *Nonlinear maximal monotone operators in Banach spaces*, Math. Ann. **175** (1968), 89–113.

[5] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[6] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[7] T. Ibaraki, Y. Kimura and W. Takahashi, *Convegence theorems for generalized projections and maximal monotone operators in Banach spaces*, Abstr. Appl. Anal. **2003** (2003), 621–629.

[8] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in: Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85,

[9] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM. J. Optim. **13** (2002), 938–945.

[10] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert space*, Taiwanese J. Math. **14** (2010), 2497–2511.

[11] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM. J. Optim. **19** (2008), 824–835.

[12] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces*, Arch. Math. (Basel) **91** (2008), 166–177.

[13] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[14] S. Matsushita, K. Nakajo and W. Takahashi, *Strong convergence theorems obtained by a generalized projections hybrid method for families of mappings in Banach spaces*, Nonlinear Appl. **73** (2010), 1466–1480.

[15] S. Reich, *On the asymptotic behavior of nonlinear semigroups and the range of accretive operators*, J. Math. Anal. Appl. **79** (1981), 113–126.

[16] S. Reich, *Book Review: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Bull. Amer. Math. Soc. **26** (1992), 367–370.

[17] R. T. Rockafellar, *On the maximal monotonicity of subdifferential mappings*, Pacific J. Math. **33** (1970), 209–216.
[18] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
[19] S. Takahashi and W. Takahashi, *The split common null point problem and the shrinking projection method in two Banach spaces*, Linear Nonlinear Anal. **1** (2015), 297–304.
[20] S. Takahashi and W. Takahashi, *The split common null point problem and the shrinking projection method in Banach spaces*, Optimization, **65** (2016), 281–287.
[21] S. Takahashi and W. Takahashi, *Split common null point problem and shrinking projection method for generalized resolvents in two Banach spaces*, J. Nonlinear Convex Anal. **17** (2016), 2171–2182.
[22] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[23] W. Takahashi, *Convex Analysis and Approximation of Fixed Points*, Yokohama Publishers, Yokohama, 2000 (Japanese).
[24] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[25] W. Takahashi, *The split feasibility problem and the shrinking projection method in Banach spaces*, J. Nonlinear Convex Anal. **16** (2015), 1449–1459.
[26] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.
[27] W. Takahashi, C.-F. Wen and J.-C. Yao, *Strong convergence theorem by shrinking projection method for new nonlinear mappings in Banach spaces and applications*, Optimization **66** (2017), 609–621.
[28] W. Takahashi, N.-C. Wong, and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.
[29] W. Takahashi, J.-C. Yao and K. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.

Manuscript received December 24 2019
revised February 4 2020

Wataru Takahashi
Research Center for Interneural Computing, China Medical University Hospital,China Medical University, Taichung 40447, Taiwan; Keio Research and Education Center for Natural Sciences, Keio University, Kouhoku-ku, Yokohama 223-8521, Japan; and Department of Mathematical and Computing Sciences, Tokyo Institute of Technology, Ookayama, Meguro-ku, Tokyo 152-8552, Japan
E-mail address: `wataru@is.titech.ac.jp; wataru@a00.itscom.net`