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A MULTIDIMENSIONAL APPROACH TO TRAFFIC ANALYSIS

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Abstract. We first propose a multidimensional variational inequality problem and then reformulate the equilibria of a traffic network model as solutions to this problem. Furthermore, we establish existence and uniqueness results for the variational inequality we have proposed. We also provide a method for finding the equilibria of the traffic network model and numerically illustrate our results.

1. Introduction

The concept of a variational inequality problem originated in the Calculus of Variations. The Signorini problem was the first problem involving a variational inequality [17]. In the early 1960s, the Italian mathematician Guido Stampacchia [29] initiated a systematic study of this subject. He studied infinite dimensional variational inequality problems as an analytic tool for tackling free boundary problems governed by nonlinear partial differential equations of elliptic or parabolic type. The book by Kinderlehrer and Stampacchia [20] provides an extensive exposition of variational inequality problems and a detailed account of their applications to solving infinite dimensional problems. Some years later, the study of variational inequality problems in finite dimensional spaces was initiated by Smith [28] and Dafermos [10]. They set up the traffic assignment problem in terms of a finite dimensional variational inequality. For more information on finite dimensional variational inequalities and traffic network problems we refer the interested reader to the comprehensive two-volume treatise by Facchinei and Pang [14], and to the recent paper [18].

In 1967 Lions and Stampacchia [21], and Brezis [4] introduced the time-dependent (evolutionary) variational inequality problem, and developed an existence and uniqueness theory regarding it. Daniele et al. [12] set forth a dynamic traffic network equilibrium problem in terms of a time-dependent variational inequality.

Since then, many other economics related problems such as spatial price equilibrium problems [11], internet problems with multiple classes of traffic [24], Nash equilibrium problems [2], pollution control problems [26] and dynamic financial equilibrium problems [7] have been studied via time-dependent variational inequalities.

During the last decades, the multitime concept has been applied to optimization theory in the framework of multitime control problems. However, it is worth mentioning that this concept was first introduced in physics. More precisely, several

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science and engineering problems can be reformulated in terms of optimization problems which are governed by m-flow type PDEs (multitime evolution systems) and by cost functionals expressed as curvilinear integrals or multiple integrals. Udriste and Tevy [34] were the first to present the basic optimization problems involving path-independent curvilinear integrals and multiple integrals. Mathematically, these integrals are equivalent, but their meanings are completely different in real life problems. Subsequently, Jayswal et al. [19] introduced a variational inequality in terms of path-independent curvilinear integrals and demonstrated the relation between this variational inequality problem and a multitime variational problem. Recently, Singh et al. [27] have formulated a variational inequality problem in terms of multiple integrals and also have established its equivalence with certain multiple integral type multitime variational problems, which were investigated in [23, 32, 33] in many ways. For more contributions, see [31].

Motivated by the above studies, we introduce in the present paper a multidimensional (multitime-dependent) variational inequality problem involving multiple integrals and establish existence and uniqueness results for it. In order to illustrate the utilization of this multidimensional variational inequality in economics, we interpret the equilibria of a multidimensional traffic network model in terms of such a variational inequality and also discuss these equilibria using the Wardrop condition. Finally, we propose a method for finding the equilibria of our multidimensional traffic network model, and numerically demonstrate the validity and applicability of our results. The key difference between our results and the relevant results obtained in [3, 8, 12] is that the latter are investigated with respect to a linear dimension of time (in a fixed time interval) while our results pertain to multidimensional time, which varies between the opposite diagonal points of a hyperparallelepiped.

Our paper is organized as follows: preliminaries and our formulation of the problem are presented in Section 2. A user-oriented form of the multidimensional traffic network equilibria is developed in Section 3 and an existence result for these equilibria is established in Section 4. Numerical illustrations and the procedure for finding the equilibria of our multidimensional traffic network model are provided in Section 5. Finally, we present our conclusions in Section 6.

2. Preliminaries and problem formulation

In order to formulate our multidimensional traffic network model, we start by introducing important notations and mathematical tools. Our traffic network is made of the set of nodes N, which represent airports, railway stations, crossings, etc., and are connected by the set of directed links L. Furthermore, W represents the set of origin-destination pairs and V represents the entire set of routes. We assume that each route $r \in V$ links exactly one origin-destination pair. The set of all $r \in V$ which link a given $w \in W$ is denoted by $V(w)$. Let $f(t) \in \mathbb{R}^V$ be the multitime-dependent flow trajectory and for each $r \in V$, let $f_r(t)$ represent the flow trajectory over the multitime t in the route r. Here, t is defined as $t = (t^\alpha) \in \Omega_{l_\circ, l_1} \subset \mathbb{R}^m$, where $\alpha = (1, 2, \ldots, m)$ denotes a multitime parameter of evolution or a multitime, for short. Geometrically, $\Omega_{l_\circ, l_1}$ is a hyperparallelepiped in $\mathbb{R}^m$ with the opposite diagonal points $l_\circ = (l_\circ^1, l_\circ^2, \ldots, l_\circ^m)$ and $l_1 = (l_1^1, l_1^2, \ldots, l_1^m)$, and by using the product order relation on $\mathbb{R}^m$, it can be written as an interval $[l_\circ, l_1]^m$. Our functional setting for the flow trajectories is the reflexive Banach function space $L^p(\Omega_{l_\circ,l_1},\mathbb{R}^V)$, where $p>1$, the dual space of which is the Banach function space $L^q(\Omega_{l_\circ,l_1},\mathbb{R}^V)$, $\frac{1}{p}+\frac{1}{q}=1$. We use the following notation to denote the value of the functional represented by $h(t)$ at the point $f(t)$:

$$\langle\langle f(t),h(t)\rangle\rangle=\int_{\Omega_{l_\circ,l_1}}\langle f(t),h(t)\rangle dt,\ f(t)\in L^p(\Omega_{l_\circ,l_1},\mathbb{R}^V),$$

and $h(t)\in L^q(\Omega_{l_\circ,l_1},\mathbb{R}^V)$, where $\langle .,.\rangle$ denotes the Euclidean inner product and $dt=dt^1\dots dt^m$ denotes the volume element of $\Omega_{l_\circ,l_1}$.

We impose the restrictions that every feasible flow has to satisfy the multitime-dependent capacity constraints

$$\lambda(t)\le f(t)\le\mu(t),\ \text{a.e. on } \Omega_{l_\circ,l_1},$$

and the traffic conservation law/demand requirements

$$\phi f(t)=\rho(t),\ \text{a.e. on } \Omega_{l_\circ,l_1},$$

where $\lambda(t),\ \mu(t)\in L^p(\Omega_{l_\circ,l_1},\mathbb{R}^V)$ are given bounds with $\lambda(t)\le\mu(t)$ and the function $\rho(t)\in L^p(\Omega_{l_\circ,l_1},\mathbb{R}^W)$ is the given demand. Here $\rho(t)\ge 0$ and $\phi=\phi_{r,w}$ is the pair link incidence matrix, the entries of which are equal to 1 if route r links the pair w and 0 otherwise. The abbreviation "a.e." stands for "almost everywhere" throughout the paper. We also assume that

$$\phi\lambda(t)\le\rho(t)\le\phi\mu(t),\ \text{a.e. on } \Omega_{l_\circ,l_1},$$

an assumption which implies the non-emptiness of the set of feasible flows

$$K:=\{f(t)\in L^p(\Omega_{l_\circ,l_1},\mathbb{R}^V):\lambda(t)\le f(t)\le\mu(t)\text{ and }\phi f(t)=\rho(t),$$

a.e. on $\Omega_{l_\circ,l_1}\}$.

Remark 2.1. It can easily be proven that the feasible set K is convex, closed and bounded. Consequently, it is weakly compact.

From now on, for notational simplicity, the multitime-dependent flow trajectories are written without explicitly mentioning t. For example, we denote the multitime-dependent flow trajectory by f. Furthermore, for each $f\in K$, the cost trajectory is determined by a mapping $F:K\to L^q(\Omega_{l_\circ,l_1},\mathbb{R}^V)$ and denoted by $F(f)$.

Now we formulate our multidimensional variational inequality problem as follows: **(MDVIP)** to find $f\in K$ such that

$$\int_{\Omega_{l_\circ,l_1}}\langle F(f),h-f\rangle dt\ge 0\ \ \forall\ h\in K.$$

In the sequel we denote the solution set of (MDVIP) by A.

Special Case. If $m=1$, then $\Omega_{l_\circ,l_1}$ is simply the closed real interval $[l_\circ,l_1]$. Furthermore, for more convenience, we put $l_\circ=0$ and $l_1=T$ (which denotes an arbitrary time) and then $\Omega_{l_\circ,l_1}=[0,T]$ (a fixed time interval). Consequently, in this case (MDVIP) reduces to the following problem:
Find $f\in K$ such that

$$\int_0^T\langle F(f),h-f\rangle dt\ge 0\ \forall\ h\in K,$$

where the feasible set $K = \{f(t) \in L^p([0,T], \mathbb{R}^V) : \lambda(t) \leq f(t) \leq \mu(t)$ and $\phi f(t) = \rho(t)$ a.e. on $[0,T]\}$. This is an evolutionary (time-dependent) variational inequality problem of the kind introduced by Lions and Stampacchia [21], and Brezis [4]. This problem has been explored by Daniele et al. [12], Cojocaru et al. [8], and by Barbagallo [3] in the context of traffic network equilibrium problems with a linear dimension of time.

In view of the definition of an equilibrium flow for a dynamic traffic network problem as defined by Daniele et al. [12], we set forth the following definition for a multidimensional traffic network model in terms of our (MDVIP).

Definition 2.2. $H \in K$ is an equilibrium flow if and only if $H \in A$.

Equilibrium flows of traffic network problems have been studied in terms of the Wardrop condition. An equivalence between the Wardrop condition and classical variational inequality problems was established by Daniele et al. [12], and the vector form of the Wardrop condition was explored by Raciti [25]. In a similar manner, we consider the following multitime-dependent Wardrop condition.

(MWC) For an arbitrary $f \in K$ and a.e. on $\Omega_{l_\circ,l_1}$, the multitime-dependent Wardrop condition is defined as follows:

$$F_u(f) < F_s(f) \Rightarrow f_u = \mu_u \text{ or } f_s = \lambda_s \ \forall\ w \in W,\ \forall\ u, s \in V(w).$$

3. User-oriented multidimensional traffic network equilibria

In this section we study a more amenable form of our multidimensional traffic network model. Concretely, we provide an equivalent form of the equilibrium flows of our multidimensional traffic network model by means of the multitime-dependent Wardrop condition. As a matter of fact, because of its form, the multitime-dependent Wardrop condition is more responsive to the user than the definition of equilibrium flows given in Definition 2.2.

The following theorem is the main result of this section.

Theorem 3.1. *Let $f \in K$ be an arbitrary flow. Then f is an equilibrium flow if and only if it satisfies the (MWC).*

Proof. First we assume that $f \in K$ satisfies the (MWC). Given an origin-destination pair $w \in W$, we define the following two sets:

$$R := \{u \in V(w) : f_u < \mu_u\},\ S := \{s \in V(w) : f_s > \lambda_s\}.$$

It follows from (MWC) that

$$F_u(f) \geq F_s(f)\ \forall\ u \in R,\ \forall\ s \in S \text{ and a.e. on } \Omega_{l_\circ,l_1}. \tag{3.1}$$

Inequality (3.1) implies that there exists a real number $v \in \mathbb{R}$ such that

$$\sup_{s\in S} F_s(f) \leq v \leq \inf_{u\in R} F_u(f),\ \text{a.e. on } \Omega_{l_\circ,l_1}.$$

Suppose that $h \in K$ is an arbitrary flow. Then, for a.e. on $\Omega_{l_\circ,l_1}$, we have

$$\forall\ r \in V(w),\ F_r(f) < v \Rightarrow r \notin R.$$

Next, we note that if $r \notin R$, then $f_r = \mu_r$ and $(h_r - f_r) \leq 0$. Consequently, we have $(F_r(f) - v)(h_r - f_r) \geq 0$ a.e. on $\Omega_{l_\circ,l_1}$. In the same way, we obtain that for a.e. on

$\Omega_{l_\circ,l_1}$ and for all $r \in V(w)$ such that $F_r(f) > v$, we also have $(F_r(f)-v)(h_r-f_r) \geq 0$ a.e. on $\Omega_{l_\circ,l_1}$. Thus we have

$$\begin{aligned}\langle F(f), h-f\rangle &= \sum_{w\in W}\sum_{r\in V(w)} F_r(f)(h_r-f_r) \\ &= \sum_{w\in W}\sum_{r\in V(w)} (F_r(f)-v)(h_r-f_r) + v\sum_{w\in W}\sum_{r\in V(w)}(h_r-f_r) \qquad (3.2)\\ &\geq 0, \text{ a.e. on } \Omega_{l_\circ,l_1}.\end{aligned}$$

Note that in the above inequality, the value of the term $\sum\limits_{w\in W}\sum\limits_{r\in V(w)}(h_r-f_r)$ is zero because of another form of the traffic conservation law/demand requirements, that is, $\sum\limits_{r\in V(w)} f_r(t) = \rho_w(t)$ for all $f\in K$ and $w \in W$, a.e. on $\Omega_{l_\circ,l_1}$.

Since $h \in K$ is arbitrary, (3.2) implies that

$$\int_{\Omega_{l_\circ,l_1}} \langle F(f), h-f\rangle dt \geq 0 \;\forall\; h \in K.$$

Therefore, f is indeed an equilibrium flow, as claimed.

To prove the converse assertion, we suppose to the contrary that f is an equilibrium flow, but that it does not satisfy the (MWC). It then follows that there exist an origin-destination pair $w \in W$ and routes $u, s \in V(w)$ together with a set $G_{l_\circ,\frac{l_\circ+l_1}{2}} \subset \Omega_{l_\circ,l_1}$ of positive measure such that

$$F_u(f) < F_s(f),\; f_u < \mu_u,\; f_s > \lambda_s,\text{ a.e. on } G_{l_\circ,\frac{l_\circ+l_1}{2}},$$

where $G_{l_\circ,\frac{l_\circ+l_1}{2}}$ is a hyperparallelepiped in $\mathbb{R}^m$ with the opposite diagonal points $l_\circ = (l_\circ^1, l_\circ^2, \ldots, l_\circ^m)$ and $\frac{l_\circ+l_1}{2} = (\frac{l_\circ^1+l_1^1}{2}, \frac{l_\circ^2+l_1^2}{2}, \ldots, \frac{l_\circ^m+l_1^m}{2})$. By using the product order relation on $\mathbb{R}^m$, this set can be written as an interval $\left[l_\circ, \frac{l_\circ+l_1}{2}\right]^m$. Next, we consider the function

$$\nu(t) := \min\{\mu_u(t) - f_u(t), f_s(t) - \lambda_s(t)\},\; t \in G_{l_\circ,\frac{l_\circ+l_1}{2}}.$$

Then $\nu(t) > 0$ a.e. on $G_{l_\circ,\frac{l_\circ+l_1}{2}}$. We now define a multitime-dependent flow trajectory $h \in L^p(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$ by

$$h_u := f_u + \nu(t),\; h_s := f_s - \nu(t),\; h_r := f_r \text{ for } r \neq u, s,\text{ a.e. on } G_{l_\circ,\frac{l_\circ+l_1}{2}}.$$

Then, $h \in K$ clearly satisfies $h = f$ outside $G_{l_\circ,\frac{l_\circ+l_1}{2}}$. Now we have

$$\begin{aligned}\int_{\Omega_{l_\circ,l_1}} \langle F(f), h-f\rangle dt &= \int_{G_{l_\circ,\frac{l_\circ+l_1}{2}}} \langle F(f), h-f\rangle dt \\ &= \int_{G_{l_\circ,\frac{l_\circ+l_1}{2}}} \nu(t)(F_u(f) - F_s(f))dt \\ &< 0,\end{aligned}$$

an inequality which shows that f is not an equilibrium flow after all. The contradiction we have reached enables us to conclude that f does satisfy the (MWC). This completes the proof of the theorem. $\square$

4. Existence and uniqueness of multidimensional traffic network equilibria

In this section we establish the existence and uniqueness of equilibria of our multidimensional traffic network model, which we have formulated in terms of a multidimensional variational inequality problem.

In this connection we recall that the first existence and uniqueness theorem for solutions of the classical variational inequality problem was established by Stampacchia [29] in 1964. Thereafter, many existence results were established by Browder [5] using monotonicity methods. Several recent existence results for vector variational inequality problems can be found in [1,6,30]. We need the following definitions and lemma, which are influenced by [6,15,16].

Let M be a nonempty, closed and convex subset of the set of feasible flows K.

Definition 4.1. The cost function F is said to be strictly monotone if

$$\langle\langle F(x) - F(y), x - y\rangle\rangle > 0 \ \ \forall\, x, y \in K \text{ and } x \neq y.$$

Definition 4.2. The cost function F is said to be demi-continuous at the point $a \in K$ if it is strongly–weakly sequentially continuous at this point, that is, if the sequence $\{F(x_n)\}$ weakly converges to $F(a)$ for all sequences $\{x_n\} \subset K,\ x_n \to a$. Here the symbol " $\to$ " stands for strong convergence.

Definition 4.3. The convex hull of a finite subset $\{x_1, x_2, \ldots, x_n\}$ of M is defined as follows:

$$co\{x_1, x_2, \ldots, x_n\} := \left\{ \sum_{i=1}^{n} \beta_i x_i, : \sum_{i=1}^{n} \beta_i = 1, \text{ for some } \beta_i \in [0,1] \right\}.$$

Definition 4.4. A (set-valued) mapping $\Gamma : M \to 2^K$ is said to be a KKM mapping if for any finite subset $\{x_1, x_2, \ldots, x_n\}$ of M, we have

$$\mathrm{co}\{x_1, x_2, \ldots, x_n\} \subset \bigcup_{i=1}^{n} \Gamma(x_i).$$

The following lemma is a special case of Ky Fan's infinite-dimensional version of the classical Knaster-Kuratowski-Mazurkiewicz theorem.

Lemma 4.5 ([15] KKM-Fan Theorem))**.** *Let $\Gamma : M \to 2^K$ be a KKM mapping such that $\Gamma(x)$ is a closed subset of K for each $x \in M$. If $\Gamma(x)$ is compact for at least one $x \in M$, then*

$$\bigcap_{x \in M} \Gamma(x) \neq \emptyset.$$

We are now ready to state and prove our existence result.

Theorem 4.6. *Assume that the cost function F is demi-continuous, and that there exist a nonempty compact set $C \subset M$ and a point $y \in C$ such that for all $\overline{x} \in M \backslash C$, we have*

$$\int_{\Omega_{l_o,l_1}} \langle F(\overline{x}), y - \overline{x} \rangle dt < 0.$$

Then (MDVIP) has a solution.

Proof. First, we construct two set-valued mappings as follows: $\Gamma_1 : M \to 2^K$ is defined by

$$\Gamma_1(x^*) := \left\{ x \in M : \int_{\Omega_{l_o,l_1}} \langle F(x^*), x - x^* \rangle dt < 0 \right\}$$

for each $x^* \in M$ and $\Gamma_2 : M \to 2^K$ is defined by

$$\Gamma_2(x) := \left\{ x^* \in M : \int_{\Omega_{l_o,l_1}} \langle F(x^*), x - x^* \rangle dt \geq 0 \right\}$$

for each $x \in M$. It is clear that $x \in \Gamma_2(x)$. Therefore, $\Gamma_2(x)$ is nonempty for all $x \in M$. At the next stage, we are going to prove that Γ_2 is a KKM mapping. To this end, suppose to the contrary that Γ_2 is not a KKM mapping. Then there exists a finite subset $\{x_1, x_2, \dots, x_n\}$ of M such that

$$\mathrm{co}\{x_1, x_2, \dots, x_n\} \not\subset \bigcup_{i=1}^{n} \Gamma_2(x_i). \tag{4.1}$$

The definition of the convex hull yields that there exists a point $\hat{y} \in \mathrm{co}\{x_1, x_2, \dots, x_n\}$, that is,

$$\hat{y} = \sum_{i=1}^{n} \beta_i x_i,$$

where $\sum_{i=1}^{n} \beta_i = 1$ for some $\beta_i \in [0,1]$, such that

$$\hat{y} \notin \bigcup_{i=1}^{n} \Gamma_2(x_i).$$

Thus, for any $i = \{1, 2, \dots, n\}$, we have

$$\int_{\Omega_{l_o,l_1}} \langle F(\hat{y}), x_i - \hat{y} \rangle dt < 0,$$

which implies that $\{x_1, x_2, \dots, x_n\} \subset \Gamma_1(\hat{y})$. Further, it can easily be seen that the set-valued mapping Γ_1 has convex point images, that is, $\Gamma_1(x^*)$ is convex for each $x^* \in M$. It follows that $\mathrm{co}\{x_1, x_2, \dots, x_n\} \subset \Gamma_1(\hat{y})$ and hence $\hat{y} \in \Gamma_1(\hat{y})$ too. Therefore, we have

$$\int_{\Omega_{l_o,l_1}} \langle F(\hat{y}), \hat{y} - \hat{y} \rangle dt < 0,$$

which is impossible. Consequently, Γ_2 is, in fact, a KKM mapping, as claimed. Now suppose that $x \in M$ is arbitrary and $\{x_n\}_{n=0}^{\infty}$ is a sequence in $\Gamma_2(x)$ which converges strongly to y. Since $x_n \in \Gamma_2(x)$, we have

$$\int_{\Omega_{l_\circ,l_1}} \langle F(x_n), x - x_n \rangle dt \geq 0$$

for each natural number n. As the cost function F is strongly-weakly sequentially continuous, by taking the limit as $n \to \infty$ in the above inequalities, we obtain

$$\int_{\Omega_{l_\circ,l_1}} \langle F(y), x - y \rangle dt \geq 0.$$

Therefore, $y \in \Gamma_2(x)$. Hence the point images of the set-valued mapping $\Gamma_2(x)$ are closed (in the strong topology) for all $x \in M$. Since $\Gamma_2(y) \subset C$ for $y \in C \subset M$, it follows that $\Gamma_2(y)$ is compact (in the strong topology) for each $y \in C$. It now follows from the KKM-Fan Theorem that

$$\bigcap_{x \in M} \Gamma_2(x) \neq \emptyset,$$

that is, there exists a point $x^* \in M$ such that $x^* \in \Gamma_2(x)$ for all $x \in M$. In other words, there exists a point $x^* \in M$ such that

$$\int_{\Omega_{l_\circ,l_1}} \langle F(x^*), x - x^* \rangle dt \geq 0 \ \forall \ x \in M. \tag{4.2}$$

Therefore (MDVIP) has indeed a solution x^* in the set $M \subset K$, as asserted. □

The following corollary provides the uniqueness of the solution of (MDVIP).

Corollary 4.7. *If the cost function F is strictly monotone on K, then (MDVIP) has a unique solution.*

Proof. Suppose to the contrary that (MDVIP) does not have a unique solution. Let $x_1 \in K$ be a solution of (MDVIP). Then we have

$$\int_{\Omega_{l_\circ,l_1}} \langle F(x_1), x - x_1 \rangle dt \geq 0 \ \forall \ x \in K. \tag{4.3}$$

If $x_2 \in K$ is another solution of (MDVIP) and $x_1 \neq x_2$, then we get

$$\int_{\Omega_{l_\circ,l_1}} \langle F(x_2), \hat{x} - x_2 \rangle dt \geq 0 \ \ \forall \ \hat{x} \in K. \tag{4.4}$$

Inequality (4.3) can be rewritten as

$$\langle\langle F(x_1), x_2 - x_1 \rangle\rangle \geq 0 \tag{4.5}$$

and the strict monotonicity of the function F yields

$$\langle\langle F(x_1) - F(x_2), x_1 - x_2 \rangle\rangle > 0. \tag{4.6}$$

Adding inequalities (4.5) and (4.6), we obtain

$$\langle\langle F(x_2), x_1 - x_2 \rangle\rangle < 0,$$

which, however, contradicts inequality (4.4). Hence x_2 is not a solution of (MDVIP) after all. Therefore (MDVIP) does have a unique solution, as asserted. □

5. Experimental study of the multidimensional traffic network model

In this section we study our multidimensional traffic network model by using the theory of projected dynamical systems (PDS). We are motivated by the work of Cojocaru et al. [8]. These authors were also concerned with the historical and practical aspects of projected dynamical systems, and explored them very well. At this point it is worthwhile recalling that Dupuis and Nagurney [13] had already studied projected dynamical systems and established their connections with the classical variational inequality problem. Influenced by these works, we consider the following projected dynamical system on the feasible set K for $p = 2$:

$$\textbf{(PDS)}\ x(\cdot,\tau) \in K \text{ such that } \frac{dx(\cdot,\tau)}{d\tau} = \Pi_K(x(\cdot,\tau), -F(x(\cdot,\tau))),$$

$x(\cdot,0) = x_\circ(\cdot) \in K$, where $F : K \to L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$ is a Lipschitz continuous vector field and the operator $\Pi_K : K \times L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V) \to L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$ is defined by

$$\Pi_K(x(\cdot), v(\cdot)) := \lim_{\delta \to 0^+} \frac{\operatorname{proj}_K(x(\cdot) + \delta v(\cdot)) - x(\cdot)}{\delta}\ \forall\ x(\cdot) \in K$$

and $v(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$. Here $\operatorname{proj}_K(\cdot)$ is the nearest point projection of a given vector onto the set K (we recall its definition below). Furthermore, in order to avoid confusion between the multitime t and the time τ, we represent elements of the space $L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$ at fixed moments $t \in \Omega_{l_\circ,l_1}$ by $x(\cdot)$. Indeed, in the formulation of (PDS), the time τ is different from the multitime t. For all $t \in \Omega_{l_\circ,l_1}$, a solution of (MDVIP) represents a static state of the underlying system and the static states define one or more equilibrium curves when t varies over the set $\Omega_{l_\circ,l_1}$. On the other hand, τ defines the dynamics of the system over the interval $[0,\infty)$ until it reaches one of the equilibria on the curves. It is clear that the solutions to (PDS) lie in the class of absolutely continuous functions with respect to τ, taking $[0,\infty)$ to K. In order to describe the procedure for solving the (MDVIP), we need to recall the following definitions (compare [9, 22]).

Definition 5.1. $x^*(\cdot) \in K$ is called a critical point of (PDS) if

$$\Pi_K(x^*(\cdot), -F(x^*(\cdot))) = 0.$$

Definition 5.2. The nearest point projection of a point $x(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$ onto the set K is defined by

$$\operatorname{proj}_K\big(x(\cdot)\big) := \arg\min_{y(\cdot)\in K} \big\|x(\cdot) - y(\cdot)\big\|.$$

Remark 5.3. For each $x(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$, $\operatorname{proj}_K(x(\cdot))$ enjoys the following property:

$$\langle\langle x(\cdot) - \operatorname{proj}_K(x(\cdot)), y(\cdot) - \operatorname{proj}_K(x(\cdot))\rangle\rangle \le 0\ \forall\ y(\cdot) \in K.$$

Definition 5.4. The polar set K° associated to K is defined by

$$K^\circ := \big\{y(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V) : \langle\langle y(\cdot), x(\cdot)\rangle\rangle \le 0\ \forall\ x(\cdot) \in K\big\}.$$

Definition 5.5. The tangent cone to the set K at a point $x(\cdot) \in K$ is defined by

$$T_K(x(\cdot)) = \operatorname{cl}\left(\bigcup_{\lambda>0} \frac{K - x(\cdot)}{\lambda}\right),$$

where cl denotes the closure operation.

Definition 5.6. The normal cone of K at a point $x(\cdot) \in K$ is defined by

$$N_K\big(x(\cdot)\big) := \{y(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V) : \langle\langle y(\cdot), z(\cdot) - x(\cdot)\rangle\rangle \leq 0 \ \forall \ z(\cdot) \in K\}.$$

This can also be written as $T_K\big(x(\cdot)\big) = \big[N_K\big(x(\cdot)\big)\big]^\circ$.

The following propositions are direct consequences of Proposition 2.1 and 2.2 in [9].

Proposition 5.7. *For all* $x(\cdot) \in K$ *and* $v(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$, $\Pi_K(x(\cdot), v(\cdot))$ *exists and* $\Pi_K(x(\cdot), v(\cdot)) = proj_{T_K(x(\cdot))}(v(\cdot))$.

Proposition 5.8. *For all* $x(\cdot) \in K$, *there exists* $n(\cdot) \in N_K(x(\cdot))$ *such that* $\Pi_K(x(\cdot), v(\cdot)) = v(\cdot) - n(\cdot) \ \forall \ v(\cdot) \in L^2(\Omega_{l_\circ,l_1}, \mathbb{R}^V)$.

The following theorem establishes a relationship between solutions to (MDVIP) and the critical points of (PDS).

Theorem 5.9. $x^*(\cdot) \in K$ *is a solution of (MDVIP) if and only if it is a critical point of (PDS).*

Proof. Suppose first that $x^*(\cdot) \in K$ is a solution to (MDVIP), that is,

$$\int_{\Omega_{l_\circ,l_1}} \langle F(x^*(\cdot)), x(\cdot) - x^*(\cdot)\rangle dt \geq 0 \ \forall \ x(\cdot) \in K.$$

In other words,

$$\langle\langle F(x^*(\cdot)), x(\cdot) - x^*(\cdot)\rangle\rangle \geq 0 \ \forall \ x(\cdot) \in K,$$

so that

$$-F(x^*(\cdot)) \in N_K(x^*(\cdot)).$$

Proposition 5.8 now implies that

$$\Pi_K(x^*(\cdot), -F(x^*(\cdot))) = 0. \tag{5.1}$$

Thus $x^*(\cdot)$ is a critical point of (PDS), as claimed.

Conversely, assume that $x^*(\cdot)$ is a critical point of (PDS). Then (5.1) holds. It follows from Proposition 5.7 that

$$\operatorname{proj}_{T_K(x^*(\cdot))}(-F(x^*(\cdot))) = 0.$$

Using Remark 5.3, we get

$$\langle\langle -F(x^*(\cdot)), z\rangle\rangle \leq 0 \ \forall \ z \in T_K(x^*(\cdot)),$$

which yields

$$-F(x^*(\cdot)) \in N_K(x^*(\cdot)).$$

In other words, $x^*(\cdot)$ is indeed a solution of (MDVIP), as asserted. □

Now we are in a position to present our method for solving (MDVIP). Theorem 5.9 means that any point on a curve of equilibria in the set $\Omega_{l_\circ,l_1}$ is a critical point of (PDS) and *vice versa*. Also, the existence and uniqueness of equilibria have already been established in the previous section. Keeping this information in mind, we employ the following discretization of $\Omega_{l_\circ,l_1}$: $(l_\circ^1, l_\circ^2, \dots, l_\circ^m) = (t_0^1, t_0^2, \dots, t_0^m) < (t_1^1, t_1^2, \dots, t_1^m) < \dots < (t_i^1, t_i^2, \dots, t_i^m) < \dots < (t_n^1, t_n^2, \dots, t_n^m) = (l_1^1, l_1^2, \dots, l_1^m)$. Then for each $t_i = (t_i^1, t_i^2, \dots, t_i^m\}$, $i = 0, 1, \dots, n$, we obtain a sequence of (PDS) on the distinct, finite-dimensional, closed and convex sets K_{t_i}. After computing all the critical points of each (PDS), we get a sequence of critical points which by interpolation yields the curves of equilibria. In order to illustrate how to apply this procedure in practice, we consider a transportation network pattern of a city called X, which is shown in Figure 1. This transportation network is made of twelve nodes and thirteen links. We assume that two persons called Y and Z live at nodes "a" and "l", respectively, and City X has two railway stations at nodes "f" and "h". Furthermore, we assume that persons Y and Z have to catch a train from the railway stations "f" and "h", respectively. Thus, the origin-destination pairs are $w_1 = (a, f)$ and $w_2 = (l, h)$, which are respectively connected by the following routes:

$$w_1 : \begin{cases} V_1 = & (a,b) \cup (b,c) \cup (c,d) \cup (d,e) \cup (e,f) \\ V_2 = & (a,b) \cup (b,h) \cup (h,g) \cup (g,f), \end{cases}$$

$$w_2 : \begin{cases} V_3 = & (l,k) \cup (k,j) \cup (j,i) \cup (i,h) \\ V_4 = & (l,a) \cup (a,b) \cup (b,h). \end{cases}$$

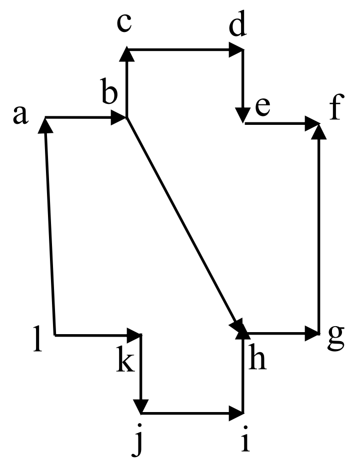

FIGURE 1. A Traffic Network Model of the City X

Let $p = 2$, $m = 2$ and $\Omega_{l_\circ,l_1} = \Omega_{0,3} = [0,3]^2$. The set of feasible flows is given by

$$K = \{f(t) \in L^2(\Omega_{0,3}, \mathbb{R}^4) : (0,0,0,0) \le (f_1(t), f_2(t), f_3(t), f_4(t)) \\ \le (t^1 + t^2 + 1, t^1 + t^2 + 2,\ 2t^1 + 2t^2 + 2, t^1 + t^2 + 5) \\ \text{and } f_1(t) + f_2(t) = 2t^1 + 2t^2 + 2, f_3(t) + f_4(t) = 3t^1 + 3t^2 + 3, \text{ a.e. on } \Omega_{0,3}\},$$

and the cost function $F : K \to L^2(\Omega_{0,3}, \mathbb{R}^4)$ is defined by

$$F(x(t)) := (x_1(t) + x_1^2(t), x_2(t) + x_2^2(t), x_3(t) + x_3^2(t), x_4(t) + x_4^2(t)),$$

where $x(t) = (x_1(t), x_2(t), x_3(t), x_4(t))$. It can easily be proven that the hypotheses of Theorem 4.6 are satisfied and the cost function F is strictly monotone on the set K. Therefore, (MDVIP) has a unique solution on K. We select $t_i \in \big\{\big(\frac{k}{6}, \frac{k}{6}\big) : k \in \{0, 1, 2, \ldots, 18\}\big\}$. Then we get a sequence of (PDS) defined on the feasible set

$$K_{t_i} = \{f \in L^2(\Omega_{0,3}, \mathbb{R}^4) : (0, 0, 0, 0) \le (f_1(t_i), f_2(t_i), f_3(t_i), f_4(t_i))$$

$$\le (t_i^1 + t_i^2 + 1, t_i^1 + t_i^2 + 2,\ 2t_i^1 + 2t_i^2 + 2, t_i^1 + t_i^2 + 5)$$

$$\text{and } f_1(t_i) + f_2(t_i) = 2t_i^1 + 2t_i^2 + 2, f_3(t_i) + f_4(t_i) = 3t_i^1 + 3t_i^2 + 3, \text{ a.e. on } \Omega_{0,3}\}.$$

For calculating the unique equilibrium, we have the following system at t_i: to find the point $x^*(t_i) = (x_1^*(t_i), x_2^*(t_i), x_3^*(t_i), x_4^*(t_i)) \in K_{t_i}$ such that

$$-F(x_1^*(t_i), x_2^*(t_i), x_3^*(t_i), x_4^*(t_i)) \in N_{K_{t_i}}(x_1^*(t_i), x_2^*(t_i), x_3^*(t_i), x_4^*(t_i)).$$

After a simple computation, we indeed find the equilibrium points. They are listed in Table 1. Interpolating the points of Table 1, we finally get the curves of equilibria. They are displayed in Figure 2.

Table 1. Numerical Results

$t_i = \{t_i^1, t_i^2\}$	$x_1^*(t_i)$	$x_2^*(t_i)$	$x_3^*(t_i)$	$x_4^*(t_i)$
$\{0, 0\}$	1	1	1.5	1.5
$\{\frac{1}{6}, \frac{1}{6}\}$	1.332	1.332	1.998	1.998
$\{\frac{1}{3}, \frac{1}{3}\}$	1.666	1.666	2.499	2.499
$\{\frac{1}{2}, \frac{1}{2}\}$	2	2	3	3
$\{\frac{2}{3}, \frac{2}{3}\}$	2.332	2.332	3.498	3.498
$\{\frac{5}{6}, \frac{5}{6}\}$	2.666	2.666	3.999	3.999
$\{1, 1\}$	3	3	4.5	4.5
$\{\frac{7}{6}, \frac{7}{6}\}$	3.332	3.332	4.998	4.998
$\{\frac{4}{3}, \frac{4}{3}\}$	3.666	3.666	5.499	5.499
$\{\frac{3}{2}, \frac{3}{2}\}$	4	4	6	6
$\{\frac{5}{3}, \frac{5}{3}\}$	4.332	4.332	6.498	6.498
$\{\frac{11}{6}, \frac{11}{6}\}$	4.666	4.666	6.999	6.999
$\{2, 2\}$	5	5	7.5	7.5
$\{\frac{13}{6}, \frac{13}{6}\}$	5.332	5.332	7.998	7.998
$\{\frac{7}{3}, \frac{7}{3}\}$	5.666	5.666	8.499	8.499
$\{\frac{5}{2}, \frac{5}{2}\}$	6	6	9	9
$\{\frac{8}{3}, \frac{8}{3}\}$	6.332	6.332	9.498	9.498
$\{\frac{17}{6}, \frac{17}{6}\}$	6.666	6.666	9.999	9.999
$\{3, 3\}$	7	7	10.5	10.5

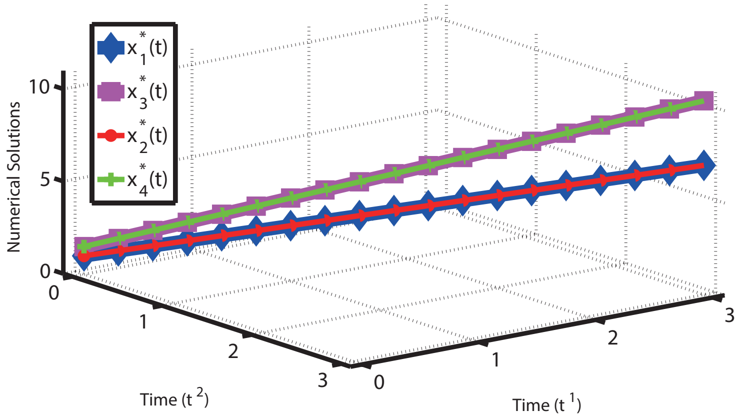

FIGURE 2. The Traffic Network Pattern of City X

6. Conclusions

In this paper we have made a contribution towards a multidimensional study of traffic network equilibrium problems. We have formulated a multidimensional variational inequality problem and discussed the equilibria of a multidimensional traffic network model of a city in terms of this problem. We have also established existence and uniqueness results for these equilibria and provided a method for finding them.

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