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AN ALTERNATIVE THEOREM FOR GRADIENT SYSTEMS

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Abstract. In this paper, given two Banach spaces X, Y and a C^1 functional $\Phi : X \times Y \to \mathbf{R}$, under general assumptions, we show that either Φ has a saddle-point in $X \times Y$ or, for each convex and dense set $S \subseteq Y$, there is some $\tilde{y} \in S$ such that $\Phi(\cdot, \tilde{y})$ has at least three critical points in X, two of which are global minima. Also, an application to non-cooperative elliptic systems is presented.

The present paper is part of the extensive program of studying consequences and applications of certain general minimax theorems ([9, 10, 12–25]) which cannot be directly deduced by the classical Fan-Sion theorem ([5, 26]).

Here, we are interested in gradient systems. Precisely, given two Banach spaces X, Y and a C^1 functional $\Phi : X \times Y \to \mathbf{R}$, we are interested in the existence of critical points for Φ, that is in the solvability of the system

$$\begin{cases} \Phi'_x(x, y) = 0 \\ \\ \Phi'_y(x, y) = 0, \end{cases}$$

where Φ'_x (resp. Φ'_y) is the derivative of Φ with respect to x (resp. y).

Let $I : X \to \mathbf{R}$. As usual, I is said to be coercive if $\lim_{\|x\| \to +\infty} I(x) = +\infty$. I is said to be quasi-concave (resp. quasi-convex) if the set $I^{-1}([r, +\infty[)$ (resp. $I^{-1}(]-\infty, r]))$ is convex for all $r \in \mathbf{R}$. When I is C^1, it is said to satisfy the Palais-Smale condition if each sequence $\{x_n\}$ in X such that $\sup_{n \in \mathbf{N}} |I(x_n)| < +\infty$ and $\lim_{n \to \infty} \|I'(x_n)\|_{X^*} = 0$ admits a strongly convergent subsequence.

Here is our main abstract theorem:

Theorem 1. *Let X, Y be two real reflexive Banach spaces and let $\Phi : X \times Y \to \mathbf{R}$ be a C^1 functional satisfying the following conditions:*
(a) the functional $\Phi(x, \cdot)$ is quasi-concave for all $x \in X$ and the functional $-\Phi(x_0, \cdot)$ is coercive for some $x_0 \in X$;
(b) there exists a convex set $S \subseteq Y$ dense in Y, such that, for each $y \in S$, the functional $\Phi(\cdot, y)$ is weakly lower semicontinuous, coercive and satisfies the Palais-Smale condition.

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Then, either the system

$$\begin{cases} \Phi'_x(x,y) = 0 \\ \\ \Phi'_y(x,y) = 0 \end{cases}$$

has a solution (x^*, y^*) *such that*

$$\Phi(x^*, y^*) = \inf_{x\in X} \Phi(x, y^*) = \sup_{y\in Y} \Phi(x^*, y),$$

or, for every convex set $T \subseteq S$ *dense in* Y*, there exists* $\tilde{y} \in T$ *such that equation*

$$\Phi'_x(x, \tilde{y}) = 0$$

has at least three solutions, two of which are global minima in X *of the functional* $\Phi(\cdot, \tilde{y})$.

Proof. Assume that there is no solution (x^*, y^*) of the system

$$\begin{cases} \Phi'_x(x,y) = 0 \\ \\ \Phi'_y(x,y) = 0 \end{cases}$$

such that

$$\Phi(x^*, y^*) = \inf_{x\in X} \Phi(x, y^*) = \sup_{y\in Y} \Phi(x^*, y).$$

We consider both X, Y endowed with the weak topology. Notice that, by (a), $\Phi(x, \cdot)$ is weakly upper semicontinuous in Y for all $x \in X$ and weakly sup-compact for $x = x_0$. As a consequence, the functional $y \to \inf_{x\in X} \Phi(x, y)$ is weakly sup-compact and so it attains its supremum. Likewise, by (b), $\Phi(\cdot, y)$ is weakly inf-compact for all $y \in S$. By continuity and density, we have

$$\sup_{y\in Y} \Phi(x,y) = \sup_{y\in S} \Phi(x,y) \tag{1}$$

for all $x \in X$. As a consequence, the functional $x \to \sup_{y\in Y} \Phi(x, y)$ is weakly inf-compact and so it attains its infimum. Therefore, the occurrence of the equality

$$\sup_Y \inf_X \Phi = \inf_X \sup_Y \Phi$$

is equivalent to the existence of a point $(\hat{x}, \hat{y}) \in X \times Y$ such that

$$\sup_{y\in Y} \Phi(\hat{x}, y) = \Phi(\hat{x}, \hat{y}) = \inf_{x\in X} \Phi(x, \hat{y}).$$

But, for what we are assuming, no such a point can exist and hence we have

$$\sup_Y \inf_X \Phi < \inf_X \sup_Y \Phi. \tag{2}$$

So, in view of (1) and (2), we also have

$$\sup_S \inf_X \Phi < \inf_X \sup_S \Phi.$$

At this point, we are allowed to apply Theorem 1.1 of [20]. Therefore, there exists $\tilde{y} \in S$ such that the functional $\Phi(\cdot, \tilde{y})$ has at least two global minima in X and so, thanks to Corollary 1 of [8], the same functional has at least three critical points. □

The next result is a consequence of Theorem 1.

Theorem 2. *Let X, Y be two real Hilbert spaces and let $J : X \times Y \to \mathbf{R}$ be a C^1 functional satisfying the following conditions:*
(a_1) the functional $y \to \frac{1}{2}\|y\|_Y^2 + J(x, y)$ is quasi-convex for all $x \in X$ and coercive for some $x \in X$;
(b_1) there exists a convex set $S \subseteq Y$ dense in Y such that, for each $y \in S$, the operator $J'_x(\cdot, y)$ is compact and

$$\limsup_{\|x\|_X \to +\infty} \frac{J(x,y)}{\|x\|_X^2} < \frac{1}{2} \; ; \tag{3}$$

Then, either the system

$$\begin{cases} x = J'_x(x, y) \\ \\ y = -J'_y(x, y) \end{cases}$$

has a solution (x^*, y^*) such that

$$\begin{aligned} \frac{1}{2}(\|x^*\|_X^2 - \|y^*\|_Y^2) - J(x^*, y^*) &= \inf_{x\in X} \left(\frac{1}{2}(\|x\|_X^2 - \|y^*\|_Y^2) - J(x, y^*) \right) \\ &= \sup_{y \in Y} \left(\frac{1}{2}(\|x^*\|_X^2 - \|y\|_Y^2) - J(x^*, y) \right), \end{aligned}$$

or, for every convex set $T \subseteq S$ dense in Y, there exists $\tilde{y} \in T$ such that the equation

$$x = J'_x(x, \tilde{y})$$

has at least three solutions, two of which are global minima in X of the functional $x \to \frac{1}{2}\|x\|_X^2 - J(x, \tilde{y})$.

Proof. PROOF. Consider the function $\Phi : X \times Y \to \mathbf{R}$ defined by

$$\Phi(x, y) = \frac{1}{2}(\|x\|_X^2 - \|y\|_Y^2) - J(x, y)$$

for all $(x, y) \in X \times Y$. Clearly, Φ is C^1 and one has

$$\Phi'_x(x, y) = x - J'_x(x, y),$$
$$\Phi'_y(x, y) = -y - J'_y(x, y)$$

for all $(x, y) \in X \times Y$. We want to apply Theorem 1 such a Φ. Of course, Φ satisfies (a) in view of (a_1). Concerning (b), notice that, for each $y \in S$, the functional $J(\cdot, y)$ is sequentially weakly continuous since $J'_x(\cdot, y)$ is compact ([27], Corollary 41.9). Moreover, from (3) it immediately follows that $\Phi(\cdot, y)$ is coercive and so, by the Eberlein-Smulyan theorem, it is weakly lower semicontinuous. Finally, $\Phi(\cdot, y)$ satisfies the Palais-Smale condition in view of Example 38.25 of [27]. Now, the conclusion follows directly from Theorem 1. □

We now present an application of Theorem 2 to non-cooperative elliptic systems.

In what follows, $\Omega \subset \mathbf{R}^n$ $(n \geq 2)$ is a bounded smooth domain. We consider $H_0^1(\Omega)$ equipped with the scalar product

$$\langle u, v \rangle = \int_\Omega \nabla u(x) \nabla v(x) dx.$$

We denote by $\mathcal{A}$ the class of all functions $H : \Omega \times \mathbf{R}^2 \to \mathbf{R}$, with $H(x, 0, 0) = 0$ for all $x \in \Omega$, which are measurable in Ω, C^1 in $\mathbf{R}^2$ and satisfy

$$\sup_{(x,u,v)\in\Omega\times\mathbf{R}^2} \frac{|H_u(x, u, v)| + |H_v(x, u, v)|}{1 + |u|^p + |v|^q} < +\infty$$

where $p, q > 0$, with $p < \frac{n+2}{n-2}$ and $q \leq \frac{n+2}{n-2}$ when $n > 2$. Given $H \in \mathcal{A}$, we are interested in the problem

$$(P_H) \qquad \begin{cases} -\Delta u = H_u(x, u, v) & \text{in } \Omega \\ -\Delta v = -H_v(x, u, v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega, \end{cases}$$

H_u (resp. H_v) denoting the derivative of H with respect to u (resp. v).

As usual, a weak solution of (P_H) is any $(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)$ such that

$$\int_\Omega \nabla u(x)\nabla\varphi(x)dx = \int_\Omega H_u(x, u(x), v(x))\varphi(x)dx,$$

$$\int_\Omega \nabla v(x)\nabla\psi(x)dx = -\int_\Omega H_v(x, u(x), v(x))\psi(x)dx$$

for all $\varphi, \psi \in H^1_0(\Omega)$.

Define the functional $I_H : H^1_0(\Omega) \times H^1_0(\Omega) \to \mathbf{R}$ by

$$I_H(u, v) = \frac{1}{2}\left(\int_\Omega |\nabla u(x)|^2dx - \int_\Omega |\nabla v(x)|^2dx\right) - \int_\Omega H(x, u(x), v(x))dx$$

for all $(u, v) \in H^1_0(\Omega) \times H^1_0(\Omega)$.

Since $H \in \mathcal{A}$, the functional I_H is C^1 in $H^1_0(\Omega) \times H^1_0(\Omega)$ and its critical points are precisely the weak solutions of (P_H). Also, we denote by λ_1 the first eigenvalue of the Dirichlet problem

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Our result on (P_H) is a follows:

Theorem 3. *Let $H \in \mathcal{A}$ be such that*

$$(4) \qquad \limsup_{|u|\to+\infty} \frac{\sup_{x\in\Omega}\sup_{|v|\leq r} H(x, u, v)}{u^2} \leq 0$$

for all $r > 0$, and

$$(5) \qquad \text{meas}\left(\left\{x \in \Omega : \sup_{u\in\mathbf{R}} H(x, u, 0) > 0\right\}\right) > 0.$$

Moreover, assume that either $H(x,u,\cdot)$ is convex for all $(x,u)\in\Omega\times\mathbf{R}$, or

$$L := \sup_{(v,\omega)\in\mathbf{R}^2, v\neq\omega} \frac{\sup_{(x,u)\in\Omega\times\mathbf{R}} |H_v(x,u,v)-H_v(x,u,\omega)|}{|v-\omega|} < +\infty. \tag{6}$$

Set

$$\lambda^* = \frac{1}{2}\inf\left\{\frac{\int_\Omega |\nabla w(x)|^2dx}{\int_\Omega H(x,w(x),0)dx} : w\in H_0^1(\Omega), \int_\Omega H(x,w(x),0)dx>0\right\}$$

and assume that $\lambda^* < \frac{\lambda_1}{L}$ when (6) holds.

Then, for each $\lambda > \lambda^*$, with $\lambda < \frac{\lambda_1}{L}$ when (6) holds, either the problem

$$\begin{cases} -\Delta u = \lambda H_u(x,u,v) & \text{in } \Omega \\ -\Delta v = -\lambda H_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-zero weak solution belonging to $L^\infty(\Omega)\times L^\infty(\Omega)$, or, for each convex set $T\subseteq H_0^1(\Omega)\cap L^\infty(\Omega)$ dense in $H_0^1(\Omega)$, there exists $\tilde{v}\in T$ such that the problem

$$\begin{cases} -\Delta u = \lambda H_u(x,u,\tilde{v}(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least three weak solutions, two of which are global minima in $H_0^1(\Omega)$ of the functional $I_{\lambda H}(\cdot,\tilde{v})$.

Proof. Define the functional $J : H_0^1(\Omega)\times H_0^1(\Omega)\to\mathbf{R}$ by

$$J(u,v) = \int_\Omega H(x,u(x),v(x))dx$$

for all $(u,v)\in H_0^1(\Omega)\times H_0^1(\Omega)$. Notice that (5) implies $\sup_{u\in H_0^1(\Omega)} J(u,0) > 0$ ([16], pp. 135-136). Consequently, $\lambda^* < +\infty$. Fix $\lambda > \lambda^*$, with $\lambda < \frac{\lambda_1}{L}$ when (6) holds. We want to apply Theorem 2 to λJ. Concerning (a_1), notice that, for each $u\in H_0^1(\Omega)$, the functional $v\to\frac{1}{2}\int_\Omega|\nabla v(x)|^2dx+\lambda J(u,v)$ is strictly convex and coercive. This is clear when $H(x,\xi,\cdot)$ is convex for all $(x,\xi)\in\mathbf{R}^2$. When (6) holds, the operator $J'_v(u,\cdot)$ turns out to be Lipschitzian in $H_0^1(\Omega)$ with Lipschitz constant $\frac{L}{\lambda_1}$ ([11], p. 165). So, the operator $v\to v-\lambda J'_v(u,v)$ is uniformly monotone and then the claim follows from a classical result ([27], pp. 247-249). Concerning (b_1), fix $v\in H_0^1(\Omega)\cap L^\infty(\Omega)$. Notice that $J'_u(\cdot,v)$ is compact due to restriction on p (recall that $H\in\mathcal{A}$). Moreover, in view of (4), for each $\epsilon>0$, there exists $\delta>0$ such that

$$H(x,t,s)\le\epsilon t^2$$

for all $x\in\Omega$, $s\in\left[-\|v\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)}\right]$ and $t\in\mathbf{R}\setminus[-\delta,\delta]$. But H is bounded on each bounded subset of $\Omega\times\mathbf{R}^2$, and so, for a suitable constant $c>0$, we have

$$H(x,t,s)\le\epsilon t^2+c \tag{7}$$

for all $(x,t,s)\in\Omega\times\mathbf{R}\times\left[-\|v\|_{L^\infty(\Omega)},\|v\|_{L^\infty(\Omega)}\right]$. Of course, from (7) it follows that

$$\limsup_{\|u\|\to+\infty}\frac{J(u,v)}{\|u\|^2}\le\epsilon$$

and so

$$\limsup_{\|u\|\to+\infty}\frac{J(u,v)}{\|u\|^2}\le 0$$

since $\epsilon>0$ is arbitrary. Hence, λJ satisfies (3). Now suppose that there exists a convex set $T\subseteq H^1_0(\Omega)\cap L^\infty(\Omega)$ dense in $H^1_0(\Omega)$ such that, for each $v\in T$, the problem

$$\begin{cases}-\Delta u=\lambda H_u(x,u,v(x)) & \text{in } \Omega\\ \\ u=0 & \text{on } \partial\Omega\end{cases}$$

has at most two weak solutions. Then, Theorem 2 ensures the existence of a weak solution (u^*,v^*) of the problem

$$\begin{cases}-\Delta u=\lambda H_u(x,u,v) & \text{in } \Omega\\ \\ -\Delta v=-\lambda H_v(x,u,v) & \text{in } \Omega\\ \\ u=v=0 & \text{on } \partial\Omega\end{cases}$$

such that

$$I_{\lambda H}(u^*,v^*)=\inf_{u\in H^1_0(\Omega)}I_{\lambda H}(u,v^*)=\sup_{v\in H^1_0(\Omega)}I_{\lambda H}(u^*,v). \tag{8}$$

From (8), in view of Theorem 1 of [3] (see Remark 5, p. 1631), it follows that $u^*,v^*\in L^\infty(\Omega)$. We show that $(u^*,v^*)\neq(0,0)$. If $v^*\neq 0$, we are done. So, assume $v^*=0$. Since $\lambda>\lambda^*$, we have

$$\inf_{u\in H^1_0(\Omega)}\left(\frac{1}{2}\int_\Omega|\nabla u(x)|^2dx-\lambda\int_\Omega H(x,u(x),0)dx\right)<0. \tag{9}$$

But then, since $\int_\Omega H(x,0,0)dx=0$, from (9) and the first equality in (8), it follows that $u^*\neq 0$, and the proof is complete. □

For previous results on problem (P_H) (markedly different from Theorem 3) we refer to [1,4,6,7].

A joint application of Theorem 3 with the main result in [2] gives the following:

Theorem 4. *Let $H\in\mathcal{A}$ satisfy the assumptions of Theorem 3. Moreover, suppose that $\inf_{\Omega\times\mathbf{R}^2}H_u\ge 0$ and that, for each $(x,v)\in\Omega\times\mathbf{R}$, the function $u\to\frac{H_u(x,u,v)}{u}$ is strictly decreasing in $]0,+\infty[$.*

Then, for every $\lambda > \lambda^*$, with $\lambda < \frac{\lambda_1}{L}$ when (6) *holds, the problem*

$$\begin{cases} -\Delta u = \lambda H_u(x,u,v) & \text{in } \Omega \\ -\Delta v = -\lambda H_v(x,u,v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-zero weak solution belonging to $L^\infty(\Omega) \times L^\infty(\Omega)$.

Proof. Fix $\lambda > \lambda^*$, with $\lambda < \frac{\lambda_1}{L}$ when (6) holds. Fix also $v \in C_0^\infty(\Omega)$. Since $\inf_{\Omega\times\mathbf{R}^2} H_u \geq 0$, the bounded weak solutions of the problem

$$\begin{cases} -\Delta u = \lambda H_u(x,u,v(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are continuous and non-negative in $\overline{\Omega}$. As a consequence, in view of Theorem 1 of [2], the problem

$$\begin{cases} -\Delta u = \lambda H_u(x,u,v(x)) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at most one non-zero bounded weak solution. Now, the conclusion follows directly from Theorem 3. □

Finally, notice the following corollary of Theorem 4:

Theorem 5. *Let $F, G : \mathbf{R} \to \mathbf{R}$ be two C^1 functions, with $FG - F(0)G(0) \in \mathcal{A}$, satisfying the following conditions:*
(a_2) F is non-negative, increasing, $\lim_{u\to+\infty} \frac{F(u)}{u^2} = 0$ and the function $u \to \frac{F'(u)}{u}$ is strictly decreasing in $]0,+\infty[$;
(b_2) G is positive and convex.
Finally, let $\alpha \in L^\infty(\Omega)$, with $\alpha > 0$. Set

$$\lambda_\alpha^* = \frac{1}{2G(0)} \inf\left\{ \frac{\int_\Omega |\nabla w(x)|^2 dx}{\int_\Omega \alpha(x)(F(w(x)) - F(0))dx} : w \in H_0^1(\Omega), \int_\Omega \alpha(x)(F(w(x)) - F(0))dx > 0 \right\}.$$

Then, for every $\lambda > \lambda_\alpha^*$, the problem

$$\begin{cases} -\Delta u = \lambda\alpha(x)G(v(x))F'(u) & \text{in } \Omega \\ -\Delta v = -\lambda\alpha(x)F(u(x))G'(v) & \text{in } \Omega \\ u = v = 0 & \text{on } \partial\Omega \end{cases}$$

has a non-zero weak solution belonging to $L^\infty(\Omega) \times L^\infty(\Omega)$.

Proof. Apply Theorem 4 to the function $H : \Omega \times \mathbf{R}^2 \to \mathbf{R}$ defined by

$$H(x, u, v) = \alpha(x)(F(u)G(v) - F(0)G(0))$$

for all $(x, u, v) \in \Omega \times \mathbf{R}^2$. Checking that H satisfies the assumptions of Theorem 4 is an easy task. □

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