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MAPS WITH WEAKLY SEQUENTIALLY CLOSED GRAPHS SATISFYING COMPACTNESS CONDITIONS ON COUNTABLE SETS

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Abstract. We present general fixed point results for maps with weakly sequentially closed graphs satisfying certain weak compactness type conditions on countable sets.

1. Introduction

In this paper motivated by Mönch's fixed point theorem [3–5] we present general fixed point results for maps with weakly sequentially closed graphs (satisfying some weak compactness type condition on countable sets) defined on Banach spaces. Our theory is based on fixed point results in the literature for self maps with weakly sequentially closed graphs defined on convex weakly compact sets. This paper is a companion to our paper [4] where another approach is presented.

Now we recall the following result from the literature [4].

Theorem 1.1. *Let Q be a nonempty, convex, weakly compact subset of a metrizable locally convex linear topological space E. Suppose $F : Q \to K(Q)$ has weakly sequentially closed graph; here $K(Q)$ denotes the family of nonempty, convex, weakly compact subsets of Q. Then F has a fixed point in Q.*

2. Fixed Point Theory

We begin immediately with our first main result.

Theorem 2.1. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following condition holds:*

$$(2.1)\qquad \begin{cases} A \subseteq Q, \ \ A = \overline{co}\,(\{x_0\} \cup F(A)) \ \ \textit{with } \ C \subseteq A \\ \textit{countable and } \ C \subseteq \overline{co}\,(\{x_0\} \cup F(C)), \\ \textit{implies } \ \overline{C^w} \ \ \textit{is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Remark 2.2. (a). In the proof below E being a Banach space can be replaced by any Hausdorff locally convex linear topological space with the following properties (provided there is an analogue of Theorem 1.1 in this space): (i). E is such that

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the weak closure of a subset Ω of E is weakly compact if and only if Ω is weakly sequentially compact (we say E is a Eberlein–Šmulian space if it satisfies this property; see [2], pg. 549–551), and (ii). for any convex set $D \subseteq E$ if $x \in \overline{D}$ then there exists a sequence x_1, $x_2, ...$ in D with x_n converging to x.

(b). Note by mistake we unfortunately left out the condition that E has to be a Eberlein–Šmulian space only in Section 3 in [4].

Remark 2.3. (a). In (2.1) in fact $\overline{C^w}$ is weakly compact implies $A\,(= \overline{A^w} = \overline{A})$ is weakly compact (see the proof below).

(b). From the proof below we see that we can remove the assumption that Q is closed and convex in the statement of Theorem 2.1 provided we assume $\overline{co}\,(\{x_0\} \cup F(Q)) \subseteq Q$.

Proof. Let $\mathbf{H}$ be the family of all subsets D of Q with $\overline{co}\,(\{x_0\} \cup F(D)) \subseteq D$. Note $\mathbf{H} \neq \emptyset$ since $Q \in \mathbf{H}$ (recall $F(Q) \subseteq Q$, $x_0 \in Q$ and Q is closed and convex). Let

$$D_0 = \cap_{D \in \mathbf{H}} D \quad \text{and} \quad D_1 = \overline{co}\,(\{x_0\} \cup F(D_0)).$$

We now show $D_1 = D_0$. Now for <u>any</u> $D \in \mathbf{H}$ we have since $D_0 \subseteq D$ that

$$D_1 = \overline{co}\,(\{x_0\} \cup F(D_0)) \subseteq \overline{co}\,(\{x_0\} \cup F(D)) \subseteq D,$$

so as a result $D_1 \subseteq D_0$. Also since $D_1 \subseteq D_0$ we have $F(D_1) \subseteq F(D_0)$ so

$$\overline{co}\,(\{x_0\} \cup F(D_1)) \subseteq \overline{co}\,(\{x_0\} \cup F(D_0)) = D_1,$$

and as a result $D_1 \in \mathbf{H}$, so $D_0 \subseteq D_1$. Consequently

$$D_0 = \overline{co}\,(\{x_0\} \cup F(D_0)). \tag{2.2}$$

We now claim

$$D_0\,(= \overline{D_0} = \overline{D_0}^w) \quad \text{is weakly compact.} \tag{2.3}$$

Suppose the claim is false. Then from the Eberlein–Šmulian theorem [1], pp. 430, there exists a sequence y_1, y_2, ... in D_0 without a weakly convergent subsequence. Let $C_1 = \{y_1, y_2, ...\}$ and note $C_1 \subseteq D_0$. Next we construct a countable set $C_2 \subseteq D_0$ with $C_1 \subseteq C_2$ and

$$C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(C_2)).$$

To see this first note $D_0 = \overline{co}\,(\{x_0\} \cup F(D_0))$ and $C_1 \subseteq D_0$ so each y_n (i.e. each element of the countable set C_1) is the limit of a sequence of finite convex combination of points from $\{x_0\} \cup F(D_0)$ so there exists a countable set $Q_0 \subseteq \{x_0\} \cup F(D_0)$ with $y_n \in \overline{co}\,(Q_0)$ for each n i.e. $C_1 \subseteq \overline{co}\,(Q_0)$. In particular there exists a countable set $A_2 \subseteq D_0$ with

$$C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(A_2)). \tag{2.4}$$

Let $C_2 = C_1 \cup A_2$. Note $C_1 \subseteq C_2$, $C_2 \subseteq D_0$ (since $A_2 \subseteq D_0$ and $C_1 \subseteq D_0$) and since $A_2 \subseteq C_2$ we have from (2.4) that

$$C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(C_2)).$$

Proceed (as above) and we obtain countable sets $C_3, C_4, ...$ with $C_n \subseteq D_0$ for $n \in \{1, 2, ...\}$, $C_n \subseteq C_{n+1}$ for $n \in \{1, 2, ...\}$ and

$$C_n \subseteq \overline{co}\,(\{x_0\} \cup F(C_{n+1})) \quad \text{for} \quad n \in \{1, 2...\}.$$

Let $C = \cup_{n=1}^{\infty} C_n$. For each $x \in C$ we have $x \in C_n$ for some $n \in \{1, 2, ...\}$ so

$$x \in \overline{co}\,(\{x_0\} \cup F(C_{n+1})) \subseteq \overline{co}\,(\{x_0\} \cup F(C)).$$

Thus $C \subseteq \overline{co}\,(\{x_0\} \cup F(C))$. Now (2.1) guarantees that $\overline{C}^w$ is weakly compact. This is a contradiction since $\overline{C}^w$ contains the sequence $\{y_n\}$ (note $C_1 \subseteq C$ and $\overline{C}^w \subseteq \overline{D_0}^w = D_0$) which has no weakly convergent subsequence.

Thus (2.3) holds i.e D_0 is weakly compact. Now $F : D_0 \to K(D_0)$ (see (2.2)) has weakly sequentially closed graph. Now apply Theorem 1.1. □

Remark 2.4. Of course if instead of (2.1) in Theorem 2.1 we assumed

$$A \subseteq Q,\;\; A = \overline{co}\,(\{x_0\} \cup F(A)) \;\text{ implies }\; A \;\text{ is weakly compact}$$

then immediately F has a fixed point in Q (the proof is easier since it follows from (2.2) and Theorem 1.1); note E being a Banach space can be replaced by any space where there is an analogue of Theorem 1.1 in this space (for example E could be a metrizable locally convex linear topological space).

We now present a Mönch type result [3–5] provided an extra assumption is added (see (2.5) below).

Theorem 2.5. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following conditions hold:*

$$\text{(2.5)}\qquad \begin{cases} A \subseteq Q,\;\; A = \overline{co}\,(\{x_0\} \cup F(A)), \;\; \textit{for any} \\ \textit{countable set } N \subseteq A \;\; \textit{there exists a countable set} \\ P \subseteq A \;\; \textit{with } \; \overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{P}^w \end{cases}$$

and

$$\text{(2.6)}\qquad \begin{cases} A \subseteq Q,\;\; A = \overline{co}\,(\{x_0\} \cup F(A)) \;\; \textit{with } \; C \subseteq A \\ \textit{countable and } \; \overline{C}^w = \overline{co}\,(\{x_0\} \cup F(C)), \\ \textit{implies } \; \overline{C}^w \;\; \textit{is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Proof. Let $\mathbf{H}$ and D_0 be as in Theorem 2.1 and note

$$\text{(2.7)}\qquad D_0 = \overline{co}\,(\{x_0\} \cup F(D_0)).$$

We claim

$$\text{(2.8)}\qquad D_0 \;\text{ is weakly compact.}$$

Suppose the claim is false. Then there exists a sequence y_1, y_2, ... in D_0 without a weakly convergent subsequence. Let $C_1 = \{y_1, y_2, ...\}$. Next we construct a countable set $C_2 \subseteq D_0$ with $C_1 \subseteq C_2$ and

$$C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(C_2)) \;\text{ and }\; \overline{co}\,(\{x_0\} \cup F(C_1)) \subseteq \overline{C_2}^w.$$

To see this notice as in Theorem 2.1 there exists a countable set $A_2 \subseteq D_0$ with

$$\text{(2.9)}\qquad C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(A_2)).$$

From (2.5) there exists a countable set $P_2 \subseteq D_0$ with

$$\text{(2.10)}\qquad \overline{co}\,(\{x_0\} \cup F(C_1)) \subseteq \overline{P_2}^w.$$

Let $C_2 = C_1 \cup A_2 \cup P_2$. Note $C_1 \subseteq C_2$, $C_2 \subseteq D_0$ (since $A_2 \subseteq D_0$, $C_1 \subseteq D_0$ and $P_2 \subseteq D_0$) and since $A_2 \subseteq C_2$ and $P_2 \subseteq C_2$ we have from (2.9) and (2.10) that

$$C_1 \subseteq \overline{co}\,(\{x_0\} \cup F(C_1)) \ \text{ and } \ \overline{co}\,(\{x_0\} \cup F(C_1)) \subseteq \overline{C_2}^w.$$

Proceed (as above) and we obtain countable sets $C_3, C_4, ...$ with $C_n \subseteq D_0$ for $n \in \{1, 2, ...\}$, $C_n \subseteq C_{n+1}$ for $n \in \{1, 2, ...\}$,

$$C_n \subseteq \overline{co}\,(\{x_0\} \cup F(C_{n+1})) \quad \text{and} \quad \overline{co}\,(\{x_0\} \cup F(C_n)) \subseteq \overline{C_{n+1}}^w \quad \text{for} \quad n \in \{1, 2...\}.$$

Let $C = \cup_{n=1}^{\infty} C_n$. Now as in Theorem 2.1 we have

$$C \subseteq \overline{co}\,(\{x_0\} \cup F(C)). \tag{2.11}$$

Also since $C_1 \subseteq C_2 \subseteq .....$ (so $F(C_1) \subseteq F(C_2) \subseteq ...$) we have

$$\begin{aligned} co\,(\{x_0\} \cup F(C)) &= co\,(\{x_0\} \cup F\,(\cup_{n=1}^{\infty} C_n)) = co\,(\{x_0\} \cup [\cup_{n=1}^{\infty} F(C_n)]) \\ &\subseteq \cup_{n=1}^{\infty} co\,(\{x_0\} \cup F(C_n)) \subseteq \cup_{n=1}^{\infty} \overline{C_{n+1}}^w \subseteq \overline{C^w} \end{aligned}$$

since $C_n \subseteq C$ for $n \in \{1, 2, ...\}$. Thus

$$\overline{co}\,(\{x_0\} \cup F(C)) \subseteq \overline{C^w}$$

and this together with (2.11) yields

$$\overline{C^w} = \overline{co}\,(\{x_0\} \cup F(C)).$$

Now (2.6) guarantees that $\overline{C^w}$ is weakly compact. This is a contradiction since $\overline{C^w}$ contains the sequence $\{y_n\}$ which has no weakly convergent subsequence. Thus (2.8) holds and $F : D_0 \to K(D_0)$ has weakly sequentially closed graph. Now apply Theorem 1.1. □

Remark 2.6. (a). If for example F is single valued then (2.5) holds. To see this first note since $N \subseteq A$ that $\overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{co}\,(\{x_0\} \cup F(A)) = A$ i.e. $\overline{co}\,(\{x_0\} \cup F(N)) \subseteq A$. Next note $co\,(\{x_0\} \cup F(N))$ is weakly separable; to see this recall F is single valued and the convex hull of a countable set is separable (see [1], pp. 51, and recall subsets of separable sets in pseudometric spaces are separable or alternatively adjust slightly the argument in [1], pp. 51) so weakly separable (see (b) below). Thus there exists a countable set $P \subseteq E$ with $P \subseteq co\,(\{x_0\} \cup F(N)) \subseteq \overline{P^w}$. Now since $\overline{co}\,(\{x_0\} \cup F(N)) \subseteq A$ we have $P \subseteq co\,(\{x_0\} \cup F(N)) \subseteq A$. Note also that $\overline{co}\,(\{x_0\} \cup F(N)) = \overline{P^w}$.

Of course in the argument above we could replace F single valued with maps F which map countably sets to countable (or separable) sets and then again (2.5) holds.

(b). If a Hausdorff locally convex space X is separable then it is weakly separable. To see this let M be countable and dense in X so $\overline{A} = X$. Then immediately we have $X = \overline{A} \subseteq \overline{A^w}$, so X is weakly separable.

In our final two results we will replace $A = \overline{co}\,(\{x_0\} \cup F(A))$ with $A = co\,(\{x_0\} \cup F(A))$ (this condition is in the spirit of [3–5]) in Theorem 2.1 and Theorem 2.5.

Theorem 2.7. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following condition holds:*

$$(2.12) \qquad \begin{cases} A \subseteq Q, \;\; A = co\,(\{x_0\} \cup F(A)) \;\; \text{with} \;\; C \subseteq A \\ \text{countable and} \;\; C \subseteq \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies} \;\; \overline{C^w} \;\; \text{is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Remark 2.8. From the proof below we see that we can remove the assumption that Q is convex in the statement of Theorem 2.7 provided we assume $co\,(\{x_0\} \cup F(Q)) \subseteq Q$.

Proof. Let $\mathbf{H}$ be the family of all subsets D of Q with $co\,(\{x_0\} \cup F(D)) \subseteq D$ and let $D_0 = \cap_{D \in \mathbf{H}} D$. Essentially the same argument as in Theorem 2.1 yields $D_0 = co\,(\{x_0\} \cup F(D_0))$. A slight adjustment of the argument in Theorem 2.1 guarantees that $\overline{D_0}^w$ is weakly compact. Consider the map $F^\star : \overline{D_0}^w \to K(\overline{D_0}^w)$ given by $F^\star(x) = F(x) \cap \overline{D_0}^w$; this is clear once we show the map is well defined i.e. once we show $F^\star(x) \neq \emptyset$ for each $x \in \overline{D_0}^w$. To see this first note since $D_0 = co\,(\{x_0\} \cup F(D_0))$ that $F(D_0) \subseteq D_0 \subseteq \overline{D_0}^w$ so $D_0 \subseteq F^{-1}(\overline{D_0}^w)$; here $F^{-1}(\overline{D_0}^w) = F^l(\overline{D_0}^w) = \{z : F(z) \cap \overline{D_0}^w \neq \emptyset\}$. Now let $x \in \overline{D_0}^w$. Now since $\overline{D_0}^w$ is weakly compact the Eberlein–Šmulian theorem [2], pg. 549, guarantees that there is a sequence (x_n) in D_0 with $x_n \rightharpoonup x$ (here $\rightharpoonup$ denotes weak convergence). Take any $y_n \in F(x_n)$. Now since $F(D_0) \subseteq D_0$ we have $y_n \in D$. Also since $\overline{D_0}^w$ is weakly compact the Eberlein–Šmulian theorem guarantees that we may assume without loss of generality that $y_n \rightharpoonup y$ for some $y \in \overline{D_0}^w$. Note $y_n \in F(x_n)$, $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ implies $y \in F(x)$ since F has weakly sequentially closed graph. Thus $y \in F(x) \cap \overline{D_0}^w$ so $x \in F^{-1}(\overline{D_0}^w)$. As a result $\overline{D_0}^w \subseteq F^{-1}(\overline{D_0}^w)$ i.e. $F^\star(x) \neq \emptyset$ for each $x \in \overline{D_0}^w$.

Note $F^\star : \overline{D_0}^w \to K(\overline{D_0}^w)$ has weakly sequentially closed graph. Now Theorem 1.1 guarantees a $x \in \overline{D_0}^w$ with $x \in F^\star(x) \subseteq F(x)$. □

Theorem 2.9. *Let E be a Banach space, Q a nonempty closed convex subset of E, $x_0 \in Q$ and $F : Q \to K(Q)$ has weakly sequentially closed graph. Assume the following conditions hold:*

$$(2.13) \qquad \begin{cases} A \subseteq Q, \;\; A = co\,(\{x_0\} \cup F(A)), \;\; \text{for any} \\ \text{countable set} \;\; N \subseteq A \;\; \text{there exists a countable set} \\ P \subseteq A \;\; \text{with} \;\; \overline{co}\,(\{x_0\} \cup F(N)) \subseteq \overline{P^w} \end{cases}$$

and

$$(2.14) \qquad \begin{cases} A \subseteq Q, \;\; A = co\,(\{x_0\} \cup F(A)) \;\; \text{with} \;\; C \subseteq A \\ \text{countable and} \;\; \overline{C^w} = \overline{co}\,(\{x_0\} \cup F(C)), \\ \text{implies} \;\; \overline{C^w} \;\; \text{is weakly compact.} \end{cases}$$

Then F has a fixed point in Q.

Proof. Let $\mathbf{H}$ and D_0 be as in Theorem 2.7 and note $D_0 = co\,(\{x_0\} \cup F(D_0))$. A slight adjustment of the argument in Theorem 2.5 guarantees that $\overline{D_0}^w$ is weakly compact. Consider the map $F^\star : \overline{D_0}^w \to K(\overline{D_0}^w)$ given by $F^\star(x) = F(x) \cap \overline{D_0}^w$. Now apply Theorem 1.1. □

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