



EXISTENCE AND NON-EXISTENCE RESULTS FOR A CLASS OF SEMILINEAR NONLOCAL OPERATORS WITH EXTERIOR CONDITION

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ABSTRACT. We consider a class of semilinear nonlocal problems with vanishing exterior condition and establish an Ambrosetti-Prodi type phenomenon when the nonlinear term satisfies certain conditions. Our technique makes use of the probabilistic tools and heat kernel estimates.

1. INTRODUCTION

In a seminal work [2] Ambrosetti and Prodi consider the problem

$$(AP) \quad -\Delta u + f(u) = h(x) \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{on } \partial D,$$

for a bounded $C^{2,\alpha}$ domain D and study existence of solutions for the above problem. The authors have shown that provided f is strictly convex with $f(0) = 0$ and

$$0 < \lim_{s \rightarrow -\infty} f'(s) < \lambda_1 < \lim_{s \rightarrow \infty} f'(s) < \lambda_2,$$

where λ_1, λ_2 are the first two eigenvalues of $-\Delta$, there exists a C^1 manifold \mathcal{M}_1 in $C^\alpha(\bar{D})$ which splits the space $C^\alpha(\bar{D})$ into two open sets \mathcal{M}_0 and \mathcal{M}_2 with the following property: (AP) has no solution for $h \in \mathcal{M}_0$, exactly one solution for $h \in \mathcal{M}_1$ and exactly two solutions for $h \in \mathcal{M}_2$. Following this fundamental observation, much work has been done in the direction of relaxing the conditions or generalizing it to non-linear partial equations or systems. In [3] Berger and Podolak propose a useful reformulation of the above problem as follows.

$$(BP) \quad -\Delta u = f(u) + \rho \Phi_1 + h(x) \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{on } \partial D,$$

where Φ_1 is the principal eigenfunction of the Laplace operator in D . Under suitable conditions, it is shown in [3] that for a real $\rho^* = \rho^*(h)$, (BP) has no solution for $\rho > \rho^*$, it has exactly one solution for $\rho = \rho^*$ and exactly two solutions for $\rho < \rho^*$. For further developments on Ambrosetti-Prodi type problems we refer to

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[1, 10, 14, 15, 18, 21] and references therein. There are also some recent works on Ambrosetti-Prodi problems involving fractional Laplacian operators, see [6, 19].

The goal of this article is to generalize the above results to a wider class of operators such as $\Psi(-\Delta)$. By $-\Psi(-\Delta)$ we denote the generator of a subordinate Brownian motion where the subordinator having Laplace exponent given by Ψ . See Example 2.2 below for some interesting examples of $\Psi(-\Delta)$. More precisely, given a bounded $C^{1,1}$ domain D we consider the problem

$$\begin{cases} \Psi(-\Delta) u = f(x, u) + \rho \Phi_1 + h(x) & \text{in } D, \\ u = 0 & \text{in } D^c, \end{cases}$$

where f, h are given continuous functions and f satisfies Ambrosetti-Prodi type conditions (see Assumption [AP] below). One of our main results (Theorem 2.5) can be informally stated as follows.

There exists a real ρ^ such that the above problem does not have any solution for $\rho > \rho^*$, at least one solution for $\rho = \rho^*$ and at least two solutions for $\rho < \rho^*$.*

The central idea of the proof remains the same as in [6, 13, 21] where one has to construct a minimal solution for certain values of ρ , and find bounds $\|u\|_{L^\infty(D)}$ and then use a degree theory argument to get to the conclusion. Some key tools required for this methodology to work are (1) *refined maximum principle* (see Theorem 2.4 below), (2) boundary behaviour of the solutions and (3) Hopf’s lemma. Recently, a version of refined maximum principle is obtained in [5] whereas the boundary behaviour of the solution has been obtained by [17]. As a substitute to the Hopf’s lemma we use the heat kernel estimates from [7, 9]. With these tools in hand we employ more technical arguments, compare to the existing literature, to obtain our results.

2. SETTING AND STATEMENT OF MAIN RESULT

2.1. Subordinate Brownian motion. A Bernstein function is a non-negative completely monotone function, i.e., an element of the set

$$\mathcal{B} = \left\{ f \in C^\infty((0, \infty)) : f \geq 0 \text{ and } (-1)^n \frac{d^n f}{dx^n} \leq 0, \text{ for all } n \in \mathbb{N} \right\}.$$

In particular, Bernstein functions are increasing and concave. We will make use below of the subset

$$\mathcal{B}_0 = \left\{ f \in \mathcal{B} : \lim_{u \downarrow 0} f(u) = 0 \right\}.$$

Let \mathcal{M} be the set of Borel measures μ on $\mathbb{R} \setminus \{0\}$ with the property that

$$\mu((-\infty, 0)) = 0 \quad \text{and} \quad \int_{\mathbb{R} \setminus \{0\}} (y \wedge 1) \mu(dy) < \infty.$$

Notice that, in particular, $\int_{\mathbb{R} \setminus \{0\}} (y^2 \wedge 1) \mu(dy) < \infty$ holds, thus μ is a Lévy measure supported on the positive semi-axis. It is well-known then that every Bernstein function $\Psi \in \mathcal{B}_0$ can be represented in the form

$$(2.1) \quad \Psi(u) = bu + \int_{(0, \infty)} (1 - e^{-yu}) \mu(dy)$$

with $b \geq 0$, moreover, the map $[0, \infty) \times \mathcal{M} \ni (b, \mu) \mapsto \Psi \in \mathcal{B}_0$ is bijective. Ψ is said to be a complete Bernstein function (see [20, Chapter 6]) if there exists a Bernstein function $\tilde{\Psi}$ such that

$$\Psi(u) = u^2 \mathcal{L}(\tilde{\Psi})(u), \quad u > 0,$$

where \mathcal{L} stands for the Laplace transformation. It is known that every complete Bernstein function is also a Bernstein function. Also, for a complete Bernstein function the Lévy measure $\mu(dy)$ has a completely monotone density with respect to the Lebesgue measure. The class of complete Bernstein functions is large, including important cases such as (i) $u^{\alpha/2}$, $\alpha \in (0, 2]$; (ii) $(u + m^{2/\alpha})^{\alpha/2} - m$, $m \geq 0, \alpha \in (0, 2)$; (iii) $u^{\alpha/2} + u^{\beta/2}$, $0 < \beta < \alpha \in (0, 2]$; (iv) $\log(1 + u^{\alpha/2})$, $\alpha \in (0, 2]$, (v) $u^{\alpha/2}(\log(1 + u))^{\beta/2}$, $\alpha \in (0, 2), \beta \in (0, 2 - \alpha)$, (vi) $u^{\alpha/2}(\log(1 + u))^{-\beta/2}$, $\alpha \in (0, 2], \beta \in [0, \alpha)$. On the other hand, the Bernstein function $1 - e^{-u}$ is not a complete Bernstein function. For a detailed discussion of Bernstein functions we refer to the monograph [20].

Bernstein functions are closely related to subordinators, and we will use this relationship below. Recall that a one-dimensional Lévy process $(S_t)_{t \geq 0}$ on a probability space $(\Omega_S, \mathcal{F}_S, \mathbb{P}_S)$ is called a subordinator whenever it satisfies $S_s \leq S_t$ for $s \leq t$, \mathbb{P}_S -almost surely. A basic fact is that the Laplace transform of a subordinator is given by a Bernstein function, i.e.,

$$(2.2) \quad \mathbb{E}_{\mathbb{P}_S}[e^{-uS_t}] = e^{-t\Psi(u)}, \quad t \geq 0,$$

holds, where $\Psi \in \mathcal{B}_0$. In particular, there is a bijection between the set of subordinators on a given probability space and Bernstein functions with vanishing right limits at zero; to emphasize this, we will occasionally write $(S_t^\Psi)_{t \geq 0}$ for the unique subordinator associated with Bernstein function Ψ . Corresponding to the examples of Bernstein functions above, the related processes are (i) $\alpha/2$ -stable subordinator, (ii) relativistic $\alpha/2$ -stable subordinator, (iii) sums of independent subordinators of different indices, (iv) geometric $\alpha/2$ -stable subordinators (specifically, the Gamma-subordinator for $\alpha = 2$), etc. The non-complete Bernstein function mentioned above describes the Poisson subordinator.

Let $(B_t)_{t \geq 0}$ be \mathbb{R}^d -valued a Brownian motion on Wiener space $(\Omega_W, \mathcal{F}_W, \mathbb{P}_W)$, running twice as fast as standard d -dimensional Brownian motion, and let $(S_t^\Psi)_{t \geq 0}$ be an independent subordinator. The random process

$$\Omega_W \times \Omega_S \ni (\omega_1, \omega_2) \mapsto B_{S_t(\omega_2)}(\omega_1) \in \mathbb{R}^d$$

is called subordinate Brownian motion under $(S_t^\Psi)_{t \geq 0}$. For simplicity, we will denote a subordinate Brownian motion by $(X_t)_{t \geq 0}$, its probability measure for the process starting at $x \in \mathbb{R}^d$ by \mathbb{P}^x , and expectation with respect to this measure by \mathbb{E}^x . Note that the characteristic exponent of $(X_t)_{t \geq 0}$ is given by $\Psi(|x|^2)$. It is also known that the Lévy measure of X has a density $y \mapsto j(|y|)$ where $j : (0, \infty) \rightarrow (0, \infty)$ is given by

$$(2.3) \quad j(r) = \int_0^\infty (4\pi t)^{-d/2} e^{-\frac{r^2}{4t}} \mu(dt),$$

and

$$(2.4) \quad \Psi(|z|^2) = \int_{\mathbb{R}^d \setminus \{0\}} (1 - \cos(y \cdot z)) j(|y|) dy.$$

We would be interested in the following class of Bernstein functions.

Definition 2.1. The function Ψ is said to satisfy a

- (i) weak lower scaling (WLSC) property with parameters $\underline{\mu} > 0$, $\underline{c} \in (0, 1]$ and $\underline{\theta} \geq 0$, if

$$\Psi(\gamma u) \geq \underline{c} \gamma^{\underline{\mu}} \Psi(u), \quad u > \underline{\theta}, \gamma \geq 1.$$

- (ii) weak upper scaling (WUSC) property with parameters $\bar{\mu} > 0$, $\bar{c} \in [1, \infty)$ and $\bar{\theta} \geq 0$, if

$$\Psi(\gamma u) \leq \bar{c} \gamma^{\bar{\mu}} \Psi(u), \quad u > \bar{\theta}, \gamma \geq 1.$$

Example 2.2. Some important examples of Ψ satisfying WLSC and WUSC include the following cases with the given parameters, respectively:

- (i) $\Psi(u) = u^{\alpha/2}$, $\alpha \in (0, 2]$, with $\underline{\mu} = \frac{\alpha}{2}$, $\underline{\theta} = 0$, and $\bar{\mu} = \frac{\alpha}{2}$, $\bar{\theta} = 0$.
- (ii) $\Psi(u) = (u + m^{2/\alpha})^{\alpha/2} - m$, $m > 0$, $\alpha \in (0, 2)$, with $\underline{\mu} = \frac{\alpha}{2}$, $\underline{\theta} = 0$ and $\bar{\mu} = 1$, $\bar{\theta} = 0$.
- (iii) $\Psi(u) = u^{\alpha/2} + u^{\beta/2}$, $\alpha, \beta \in (0, 2]$, with $\underline{\mu} = \frac{\alpha}{2} \wedge \frac{\beta}{2}$, $\underline{\theta} = 0$ and $\bar{\mu} = \frac{\alpha}{2} \vee \frac{\beta}{2}$, $\bar{\theta} = 0$.
- (iv) $\Psi(u) = u^{\alpha/2} (\log(1 + u))^{-\beta/2}$, $\alpha \in (0, 2]$, $\beta \in [0, \alpha)$ with $\underline{\mu} = \frac{\alpha - \beta}{2}$, $\underline{\theta} = 0$ and $\bar{\mu} = \frac{\alpha}{2}$, $\bar{\theta} = 0$.
- (v) $\Psi(u) = u^{\alpha/2} (\log(1 + u))^{\beta/2}$, $\alpha \in (0, 2)$, $\beta \in (0, 2 - \alpha)$, with $\underline{\mu} = \frac{\alpha}{2}$, $\underline{\theta} = 0$ and $\bar{\mu} = \frac{\alpha + \beta}{2}$, $\bar{\theta} = 0$.

The following condition will be imposed on $(X_t)_{t \geq 0}$.

Assumption 1. Ψ satisfies both WLSC and WUSC properties with respect to some parameters $(\underline{\mu}, \underline{c}, \underline{\theta})$ and $(\bar{\mu}, \bar{c}, \bar{\theta})$, respectively, with $\bar{\mu} < 1$. Moreover, for some positive constant ϱ we have

$$(2.5) \quad j(r + 1) \geq \varrho j(r), \quad \text{for all } r \geq 1,$$

where j is given by (2.3).

It is obvious that $\bar{\mu} \geq \underline{\mu}$. If Ψ is complete Bernstein and satisfies for some $\alpha \in (0, 1)$ that $\Psi(r) \asymp r^\alpha \ell(r)$, as $r \rightarrow \infty$, for some locally bounded and slowly varying function ℓ , then (2.5) holds [16, Theorem 13.3.5]. Many results of this article would be valid without Assumption 1. However, to establish compactness of certain operators (see Theorem 2.3 or Lemma 3.8 below) we use some estimates from [17] which uses Assumption 1.

For our analysis we also require the renewal function V of the properly normalized ascending ladder-height process of $X_t^{(1)}$, where $X_t^{(1)}$ denotes the first coordinate of X_t . The ladder-height process is a subordinator with Laplace exponent

$$\tilde{\Psi}(\xi) = \exp \left\{ \frac{1}{\pi} \int_0^\infty \frac{\log \Psi(\xi^2 \zeta^2)}{1 + \zeta^2} d\zeta \right\}, \quad \xi \geq 0,$$

and $V(x)$ is its potential measure of the half-line $(-\infty, x)$. The Laplace transform of V is given by

$$\int_0^\infty V(x)e^{-\xi x} dx = \frac{1}{\xi \tilde{\Psi}(\xi)}, \quad \xi > 0.$$

It is also known that $V = 0$ for $x \leq 0$, V is continuous and strictly increasing in $(0, \infty)$ and $V(\infty) = \infty$ (see [12] for more details). From [7, Lemma 1.2] it is known that for some universal constant C , dependent only on the dimension d , we have

$$(2.6) \quad C^{-1} \Psi(r^{-2}) \leq \frac{1}{V^2(r)} \leq C \Psi(r^{-2}) \quad r > 0.$$

2.2. Main results. Let D be a $\mathcal{C}^{1,1}$ open bounded set. By τ we denote the exit time of $(X_t)_{t \geq 0}$ from D . Given a function $U \in \mathcal{C}(\bar{D})$ called potential, the corresponding Feynman-Kac semigroup is given by

$$(2.7) \quad T_t^{D,U} f(x) = \mathbb{E}^x \left[e^{-\int_0^t U(X_s) ds} f(X_t) \mathbf{1}_{\{t < \tau\}} \right], \quad t > 0, x \in D, f \in L^2(D).$$

It is shown in [4, Lem 3.1] that $T_t^{D,U}$, $t > 0$, is a Hilbert-Schmidt operator on $L^2(D)$ with continuous integral kernel in $(0, \infty) \times D \times D$. Moreover, every operator T_t has the same purely discrete spectrum, independent of t , whose lowest eigenvalue is the principal eigenvalue λ^* having multiplicity one, and the corresponding principal eigenfunction $\Phi \in L^2(D)$ is strictly positive in D . Since the boundary of D is regular by [8, proof of Lemma 2.9] we also have from [4, Lem. 3.1] that $\Phi \in \mathcal{C}_0(D)$, where $\mathcal{C}_0(D)$ denotes the class of continuous functions on \mathbb{R}^d vanishing in D^c . Since Φ is an eigenfunction in semigroup sense, we have for all $t > 0$ that

$$(2.8) \quad e^{-\lambda^* t} \Phi(x) = T_t^{D,U} \Phi(x) = \mathbb{E}^x \left[e^{-\int_0^t U(X_s) ds} \Phi(X_t) \mathbf{1}_{\{t < \tau\}} \right], \quad x \in D.$$

Moreover, λ in (2.8) is an eigenvalue of the operator $\Psi(-\Delta) + U$ with Dirichlet exterior condition. By λ_U^* we denote the principal eigenvalue corresponding to the potential U and $\lambda^* = \lambda_0^*$. Let $\Phi_1 \in \mathcal{C}_0(D)$ be the positive eigenfunction corresponding to the eigenvalue λ^* . We normalize Φ_1 to satisfy $\|\Phi_1\|_\infty = 1$. In this paper we are interested in the existence and multiplicity of solutions of

$$(P_\rho) \quad \begin{cases} \Psi(-\Delta) u = f(x, u) + \rho \Phi_1 + h(x) & \text{in } D, \\ u = 0 & \text{in } D^c, \end{cases}$$

where $h \in \mathcal{C}(\bar{D})$ and f is continuous function satisfying some appropriate condition. In what follows by a solution of

$$(2.9) \quad \begin{cases} \Psi(-\Delta) u = g & \text{in } D, \\ u = 0 & \text{in } D^c, \end{cases}$$

for $g \in \mathcal{C}(\bar{D})$ we mean semigroup or potential theoretic solution. More precisely, the solution of (2.9) is given by

$$u(x) = \int_D g(y) G^D(x, y) dy = \mathbb{E}^x \left[\int_0^\tau g(X_s) ds \right],$$

where G^D denotes the Green function of $(X_t^D)_{t \geq 0}$, the killed process of X upon D . From the strong Markov property it is easily seen that

$$(2.10) \quad u(x) = \mathbb{E}^x \left[\int_0^{t \wedge \tau} g(X_s) ds \right] + \mathbb{E}^x [u(X_{t \wedge \tau})] \quad t \geq 0.$$

It can also be shown that the solution of (2.9) is also a viscosity solution of (2.9) (see [17]).

Our first result concerns with the existence of solution.

Theorem 2.3. *Suppose that Assumption 1 holds. Let $U, g \in \mathcal{C}(\bar{D})$ and $\lambda_U^* > 0$. Then there exists a unique $u \in \mathcal{C}_0(D)$ satisfying*

$$(2.11) \quad \Psi(-\Delta) u + Uu = g \text{ in } D, \quad u = 0 \text{ in } D^c.$$

We also need the following refined maximum principle.

Theorem 2.4. *Let Assumption 1 hold. Suppose that $U \in \mathcal{C}(\bar{D})$ and $\lambda_U^* > 0$. Let $u \in \mathcal{C}_b(\mathbb{R}^d)$ be a viscosity solution of $\Psi(-\Delta) u + Uu = g_1$ and $v \in \mathcal{C}_b(\mathbb{R}^d)$ be a viscosity solution of $\Psi(-\Delta) v + Uv = g_2$ in D for some $g_1, g_2 \in \mathcal{C}(\bar{D})$ with $g_1 \leq g_2$. Furthermore, assume that $u = v = 0$ in D^c . Then we have either $u < v$ in D or $u = v$ in \mathbb{R}^d .*

We impose the following Ambrosetti-Prodi type condition on f .

Assumption [AP]. Let $f : \bar{D} \times \mathbb{R} \rightarrow \mathbb{R}$ be such that

(1) both $f(x, u)$ and $D_u f(x, u)$ are continuous in $(x, u) \in \bar{D} \times \mathbb{R}$;

(2) there exist $U_1, U_2 \in \mathcal{C}(\bar{D})$ with $U_1 \geq U_2$ such that

$$(2.12) \quad \lambda_{U_1}^* > 0 \quad \text{and} \quad \lambda_{U_2}^* < 0,$$

$$(2.13) \quad f(x, q) \geq -U_1(x)q - C \quad \text{for all } q \leq 0, \quad x \in \bar{D},$$

$$(2.14) \quad f(x, q) \geq -U_2(x)q - C \quad \text{for all } q \geq 0, \quad x \in \bar{D},$$

(3) f has at most linear growth, i.e., there exists a constant $C > 0$ such that

$$|f(x, q)| \leq C(1 + |q|),$$

for all $(x, q) \in \bar{D} \times \mathbb{R}$.

In what follows, we assume with no loss of generality that $f(x, 0) = 0$, otherwise h can be replaced by $h - f(\cdot, 0)$. The condition $U_1 \geq U_2$ is imposed for some technical reason. As well known this condition is not required when $\Psi(r) = r^s$ for $s \in (0, 1]$ (see [6] and references therein). It should be observed that due to our Assumption [AP](2) we have $f(x, q) \geq -U_1(x)q - C$ for $q \in \mathbb{R}$.

Now we are ready to state our main result on the nonlocal Ambrosetti-Prodi problem.

Theorem 2.5. *Let Assumption 1 and [AP] hold. Then there exists $\rho^* = \rho^*(h) \in \mathbb{R}$ such that for $\rho < \rho^*$ the Dirichlet problem (P_ρ) has at least two solutions, at least one solution for $\rho = \rho^*$, and no solution for $\rho > \rho^*$.*

3. PROOFS

We prove Theorem 2.3-2.5 in this section. The following result would play a key role in our proofs.

Lemma 3.1. *Let $u \in C_0(D)$ be a solution of*

$$\Psi(-\Delta) u = g \quad \text{in } D,$$

for some $g \in \mathcal{C}(\bar{D})$. Consider $U \in \mathcal{C}(\bar{D})$. Then for any $t \geq 0$ we have

$$(3.1) \quad \mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} u(X_t) \mathbb{1}_{\{t < \tau\}} \right] - u(x) \\ = \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{\int_0^s U(X_p) dp} (U(X_s) u(X_s) - g(X_s)) ds \right], \quad x \in D.$$

Proof. Define

$$\psi(t) = \mathbb{E}^x \left[e^{\int_0^{t \wedge \tau} U(X_s) ds} u(X_{t \wedge \tau}) \right] = \mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} u(X_t) \mathbb{1}_{\{t < \tau\}} \right].$$

From [4, Lemma 3.1] it follows that ψ is continuous in $[0, \infty)$. We fix $t \geq 0$ and consider $h > 0$. Then

$$(3.2) \quad \psi(t+h) - \psi(t) = \mathbb{E}^x \left[e^{\int_0^{(t+h) \wedge \tau} U(X_s) ds} u(X_{(t+h) \wedge \tau}) \right] - \mathbb{E}^x \left[e^{\int_0^{t \wedge \tau} U(X_s) ds} u(X_{t \wedge \tau}) \right] \\ = \mathbb{E}^x \left[\left(e^{\int_0^{(t+h) \wedge \tau} U(X_s) ds} - e^{\int_0^{t \wedge \tau} U(X_s) ds} \right) u(X_{(t+h) \wedge \tau}) \right] \\ + \mathbb{E}^x \left[e^{\int_0^{t \wedge \tau} U(X_s) ds} (u(X_{(t+h) \wedge \tau}) - u(X_{t \wedge \tau})) \right] \\ = \mathbb{E}^x \left[\underbrace{e^{\int_0^{t \wedge \tau} U(X_s) ds} \left(e^{\int_{t \wedge \tau}^{(t+h) \wedge \tau} U(X_s) ds} - 1 \right)}_{A_1(h)} u(X_{(t+h) \wedge \tau}) \right] \\ + \mathbb{E}^x \left[\underbrace{e^{\int_0^{t \wedge \tau} U(X_s) ds} (\mathbb{E}^{X_{t \wedge \tau}}[u(X_{h \wedge \tau})] - u(X_{t \wedge \tau}))}_{A_2(h)} \right],$$

where in the last line we used strong Markov property. Since $u(X_{(t+h) \wedge \tau}) = 0$ on $\{t \geq \tau\}$, it follows that

$$A_1(h) = \mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} \left(e^{\int_{t \wedge \tau}^{(t+h) \wedge \tau} U(X_s) ds} - 1 \right) u(X_{(t+h) \wedge \tau}) \mathbb{1}_{\{t < \tau\}} \right],$$

and therefore, applying dominated convergence theorem we obtain

$$(3.3) \quad \lim_{h \rightarrow 0} \frac{A_1(h)}{h} = \mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} U(X_t) u(X_t) \mathbb{1}_{\{t < \tau\}} \right].$$

From (2.10) we get that

$$\mathbb{E}^{X_{t \wedge \tau}}[u(X_{h \wedge \tau})] - u(X_{t \wedge \tau}) = -\mathbb{E}^{X_{t \wedge \tau}} \left[\int_0^{h \wedge \tau} g(X_s) ds \right],$$

since both the sides vanishes on the set $\{t \geq \tau\}$. Thus again applying dominated convergence theorem we find

$$(3.4) \quad \lim_{h \rightarrow 0} \frac{A_2(h)}{h} = -\mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} g(X_t) \mathbb{1}_{\{t < \tau\}} \right].$$

Hence using (3.2), (3.3) and (3.4) we obtain

$$\psi'_+(t) = \mathbb{E}^x \left[e^{\int_0^t U(X_s) ds} (U(X_t)u(X_t) - g(X_t)) \mathbb{1}_{\{t < \tau\}} \right].$$

It also follows from [4, Lemma 3.1] that $t \mapsto \psi'_+(t)$ is continuous. Hence ψ is in $\mathcal{C}^1(0, \infty)$ and by fundamental theorem of calculus we have

$$\begin{aligned} \psi(t) - u(x) &= \psi(t) - \psi(0) = \int_0^t \mathbb{E}^x \left[e^{\int_0^s U(X_p) dp} (U(X_s)u(X_s) - g(X_s)) \mathbb{1}_{\{s < \tau\}} \right] ds \\ &= \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{\int_0^s U(X_p) dp} (U(X_s)u(X_s) - g(X_s)) ds \right]. \end{aligned}$$

This proves (3.1). □

Let us now prove Theorem 2.3.

Proof of Theorem 2.3. The main idea in proving (2.11) is to use Schauder's fixed point theorem. Consider a map $\mathcal{T} : \mathcal{C}_0(D) \rightarrow \mathcal{C}_0(D)$ defined such that for every $\psi \in \mathcal{C}_0(D)$, $\mathcal{T}\psi = \varphi$ is the unique solution of

$$(3.5) \quad \Psi(-\Delta) \varphi = g - U\psi \quad \text{in } D, \quad \text{and} \quad \varphi = 0 \quad \text{in } D^c.$$

Denoting $\bar{\phi} = [\Psi(r^{-2})]^{-\frac{1}{2}}$ and using [17, Theorem 1.1] we obtain that

$$(3.6) \quad \|\mathcal{T}\psi\|_{\mathcal{C}^{\bar{\phi}}(D)} \leq c_1(\|g\|_\infty + \|U\psi\|_\infty),$$

for a constant $c_1 = c_1(D, d, s)$ where

$$(3.7) \quad \|h\|_{\mathcal{C}^{\bar{\phi}}(D)} = \|h\|_{L^\infty(D)} + \sup_{x \neq y, x, y \in D} \frac{|h(x) - h(y)|}{\bar{\phi}(x - y)}.$$

Thus using (2.6) and (3.6) we have

$$|\mathcal{T}\psi(x) - \mathcal{T}\psi(y)| \leq c_2(\|g\|_\infty + \|U\psi\|_\infty)V(|x - y|).$$

This implies that \mathcal{T} is a compact linear operator. It is also easy to see that \mathcal{T} is continuous.

In a next step we show that the set

$$\mathcal{B} = \{\varphi \in \mathcal{C}_0(D) : \varphi = \mu\mathcal{T}\varphi \text{ for some } \mu \in [0, 1]\}$$

is bounded in $\mathcal{C}_0(D)$. For every $\varphi \in \mathcal{B}$ we have

$$(3.8) \quad \Psi(-\Delta) \varphi = \mu g - \mu U\varphi \quad \text{in } D, \quad \text{and} \quad \varphi = 0 \quad \text{in } D^c,$$

for some $\mu \in [0, 1]$. From (3.8) and Lemma 3.1 we see that

$$(3.9) \quad \varphi(x) = \mathbb{E}^x \left[e^{-\int_0^t \mu U(X_s) ds} \varphi(X_t) \mathbb{1}_{\{t < \tau\}} \right] + \mu \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s \mu U(X_p) dp} g(X_s) ds \right], \quad t \geq 0.$$

To show boundedness of \mathcal{B} it suffices to show that for a constant c_2 , independent of μ , we have

$$(3.10) \quad \sup_{x \in D} |\varphi(x)| \leq c_2 \sup_{x \in D} |g(x)|.$$

Once (3.10) is established, the existence of a fixed point of \mathcal{T} follows by Schauder's fixed point theorem. Since every solution of (3.5) is a semigroup solution and $\lambda^* > 0$,

the uniqueness of the solution follows from [5, Th. 4.2] and Lemma 3.1. To obtain (3.10) recall from [5, Cor. 4.1] that

$$(3.11) \quad \lambda_{\mu V}^* = - \lim_{t \rightarrow \infty} \frac{1}{t} \log \mathbb{E}^x \left[e^{-\int_0^t \mu U(X_s) ds} \mathbf{1}_{\{\tau > t\}} \right], \quad x \in D.$$

Recall that $\lambda^* > 0$ is the principal eigenvalue corresponding to the potential $U = 0$. Then from the concavity of the map $\mu \mapsto \lambda_{\mu U}^*$ (see [5, Lem. 4.3]) it follows that

$$\lambda_{\mu U}^* \geq \lambda_U^* \wedge \lambda_0^* = 2\delta > 0.$$

Hence by using (3.11) and the continuity of $\mu \mapsto \lambda_{\mu U}^*$, we find constants $c_3 > 0, \mu_0 > 1$, such that for every $\mu \in [0, \mu_0]$ we have

$$(3.12) \quad \mathbb{E}^x \left[e^{-\int_0^t \mu U(X_s) ds} \mathbf{1}_{\{\tau > t\}} \right] \leq c_3 e^{-\delta t}, \quad t \geq 0, \quad x \in D.$$

We rewrite (3.9) as

$$\varphi(x) = \mathbb{E}^x \left[e^{-\int_0^t \mu U(X_s) ds} \varphi(X_t) \mathbf{1}_{\{\tau > t\}} \right] + \int_0^t T_s^{\mathbb{D}, \mu U} g(x) ds,$$

where $T^{\mathbb{D}, \mu U}$ is given by (2.7). Letting $t \rightarrow \infty$, using (3.12) and Hölder inequality, it is easily seen that the first term at the right hand side of the above vanishes. Again by (3.12), we have for $x \in D$

$$\left| T_s^{\mathbb{D}, \mu V} g(x) \right| \leq c_3 \sup_{x \in \mathbb{D}} |g| e^{-\delta s}, \quad s \geq 0.$$

Thus finally we obtain

$$\sup_{x \in \mathbb{D}} |\varphi(x)| \leq \frac{c_3}{\delta} \sup_{x \in \mathbb{D}} |g(x)|,$$

yielding (3.10). □

Next we prove the comparison result Theorem 2.4.

Proof of Theorem 2.4. Using Lemma 3.1 we see that

$$u(x) = \mathbb{E}^x \left[e^{-\int_0^t U(X_s) ds} u(X_t) \mathbf{1}_{\{t < \tau\}} \right] + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U(X_p) dp} g_1(X_s) ds \right], \quad t \geq 0,$$

and,

$$v(x) = \mathbb{E}^x \left[e^{-\int_0^t U(X_s) ds} v(X_t) \mathbf{1}_{\{t < \tau\}} \right] + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U(X_p) dp} g_2(X_s) ds \right], \quad t \geq 0.$$

Denoting $w = v - u$ and using the above expressions we obtain

$$w(x) \geq \mathbb{E}^x \left[e^{-\int_0^t U(X_s) ds} w(X_t) \mathbf{1}_{\{t < \tau\}} \right], \quad t \geq 0.$$

From [5, Theorem 4.2] it then follows that either $w > 0$ in D or $w = 0$ in \mathbb{R}^d . Hence the proof. □

Remark 3.2. The condition $u = v = 0$ in D^c in Theorem 2.4 is not necessary. In fact, the same argument as above can be used to establish comparison principle provided $u \leq v$ in D^c .

The rest of the article is devoted to the proof of Theorem 2.5. The central strategy of the proof can be grouped in following three steps.

- (1) We find a ρ_1 such that for every $\rho \leq \rho_1$ there exists a (minimal) solution of (P_ρ) . We do this in Lemma 3.3 and 3.4.
- (2) Next we find $\rho_2 > \rho_1$ such that (P_ρ) does not have any solution for $\rho \geq \rho_2$. This is the content of Lemma 3.5, 3.6 and 3.7
- (3) Finally, we proceed along the lines of [11] with suitable modifications to find the bifurcation point ρ^* .

Let us begin by establishing existence of sub/super-solutions, which will be used for constructing a minimal solution.

Lemma 3.3. *Let Assumptions 1 and [AP] hold. Then we have the following.*

- (1) For every $\rho \in \mathbb{R}$ there exists $\underline{u} \in \mathcal{C}_0(D)$ satisfying $\underline{u} \leq 0$ in D and

$$\Psi(-\Delta) \underline{u} = f(x, \underline{u}) + \rho\Phi_1 + h(x) + g(x) \quad \text{in } D,$$

for some nonpositive $g \in \mathcal{C}(\bar{D})$.

- (2) There exists $\rho_1 < 0$ such that for every $\rho \leq \rho_1$ there exists $\bar{u} \in \mathcal{C}_0(D)$ satisfying $\bar{u} \geq 0$ in D and

$$\Psi(-\Delta) \bar{u} = f(x, \bar{u}) + \rho\Phi_1 + h(x) + g(x) \quad \text{in } D,$$

for some nonnegative $g \in \mathcal{C}(\bar{D})$.

- (3) We can construct \underline{u} to satisfy $\underline{u} \leq \hat{u}$, for every solution $\hat{u} \in \mathcal{C}_0(D)$ of

$$\Psi(-\Delta) \hat{u} = f(x, \hat{u}) + \rho\Phi_1 + h(x) + g(x) \quad \text{in } D,$$

with $g \geq 0$.

Proof. Consider $\rho \in \mathbb{R}$. Let $C_1 = 2 \sup_{\bar{D}} |h| + 2|\rho| + C$, where C is the same constant as in (2.13)-(2.14). Since $\lambda_{U_1}^* > 0$ by (2.12), it follows from Theorem 2.3 that there exists a unique $\underline{u} \in \mathcal{C}_0(D)$ satisfying

$$(3.13) \quad \Psi(-\Delta) \underline{u} = -U_1 \underline{u} - C_1 + h(x) + \rho\Phi_1 \quad \text{in } D.$$

By our choice of C_1 we see that

$$\Psi(-\Delta) \underline{u} + U_1 \underline{u} = -C_1 + h(x) + \rho\Phi_1 \leq 0,$$

and hence, by Theorem 2.4 we have $\underline{u} \leq 0$ in \mathbb{R}^d . Therefore, by making use of (2.13) and choosing $g(x) = -f(x, \underline{u}) - U_1 \underline{u} - C_1$ we get that

$$\Psi(-\Delta) \underline{u} = f(x, \underline{u}) + h(x) + \rho\Phi_1 + g(x) \quad \text{in } D, \quad \text{and} \quad \underline{u} = 0 \quad \text{in } D^c.$$

This proves part (1).

Now we proceed to establish (2). Due to Assumption [AP] there exists a constant C_1 satisfying $f(x, q) \leq C_1(1+q)$, for all $(x, q) \in \bar{D} \times [0, \infty)$. We consider the unique function $\bar{u} \in \mathcal{C}_0(D)$ satisfying

$$(3.14) \quad \Psi(-\Delta) \bar{u} = h^+ + C_1 \quad \text{in } D.$$

Therefore

$$|u(x)| = \left| \mathbb{E}^x \left[\int_0^\tau h^+(X_s) + C_1 ds \right] \right| \leq (\|h\|_\infty + C_1) \mathbb{E}^x[\tau].$$

Thus by Assumption 1 and [8, Theorem 4.6 and Lemma 7.5] we obtain

$$(3.15) \quad |u(x)| \leq c_1 V(\delta_D(x)), \quad x \in \bar{D},$$

for some constant c_1 , dependent on D , where $\delta_D(x) = \text{dist}(x, D^c)$. Again

$$\bar{u}(x) \geq C_1 \mathbb{E}^x[\tau] > 0 \quad \text{for } x \in D.$$

Let $p^D(t, x, y)$ be the transition density of the killed process X^D in D . In fact, one can write

$$p^D(t, x, y) = p(t, |x - y|) - \mathbb{E}^x[p(t - \tau, |X_\tau - y|)\mathbb{1}_{\{\tau < t\}}].$$

Using [7, Theorem 4.5] (see also [9]) we know that for some positive constants κ_1, r we have for $x, y \in D$

$$(3.16) \quad p^D(t, x, y) \geq \kappa_1 \mathbb{P}^x(\tau > t/2) \mathbb{P}^y(\tau > t/2)p(t \wedge V^2(r), |x - y|), \quad t \geq 0,$$

$$(3.17) \quad \mathbb{P}^x(\tau > t) \geq \kappa_1 \left(\frac{V(\delta_D(x))}{\sqrt{t \wedge V(r)}} \wedge 1 \right).$$

Now recall that $\Psi(-\Delta) \Phi_1 = \lambda^* \Phi_1$ in D , and $\Phi_1 > 0$ in D . Let $D_1 \Subset D$. Fixing $t = 2$ and using (2.8) we get that

$$\begin{aligned} \Phi_1(x) &= e^{2\lambda^*} \mathbb{E}^x [\Phi_1(X_t)\mathbb{1}_{\{2 < \tau\}}] \\ &= e^{2\lambda^*} \int_D \Phi_1(y)p^D(2, x, y) \, dy \\ &\geq e^{2\lambda^*} \int_{D_1} \Phi_1(y)p^D(2, x, y) \, dy \\ &\geq \kappa_1 e^{2\lambda^*} \min_{D_1} \Phi_1 \mathbb{P}^x(\tau > 1) \int_{D_1} \mathbb{P}^y(\tau > 1)p(1 \wedge V^2(r), |x - y|) \, dy \\ &\geq \kappa_2 p(1 \wedge V^2(r), \text{diam } D) \mathbb{P}^x(\tau > 1) \int_{D_1} \mathbb{P}^y(\tau > 1) \, dy, \end{aligned}$$

for some constant κ_2 , where in the fourth inequality we use (3.16). Now using (3.17) we can find a constant $\kappa_3 > 0$ satisfying

$$\Phi_1(x) \geq \kappa_3 V(\delta_D(x)), \quad x \in D.$$

Combining the above with (3.15) and choosing $-\rho_1 > 0$ large, we find for every $\rho \leq \rho_1$ that

$$-\rho\Phi_1(x) \geq C_1 c_1 V(\delta_D(x)) \geq C_1 \bar{u}(x), \quad \text{for } x \in D.$$

Hence using (3.14) and choosing $g(x) = -f(x, \bar{u}) - \rho\Phi_1 + C_1 + h^- \geq 0$ for $\rho \leq \bar{\rho}_1$ we have

$$\Psi(-\Delta) \bar{u} = f(x, \bar{u}) + \rho\Phi + h + g \quad \text{in } D.$$

This proves (2).

Now we come to (3). Since $f(x, q) \geq -U_1q - C$, by Assumption [AP], applying Lemma 3.1 we obtain that

$$\begin{aligned}
 \hat{u}(x) &= \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} \hat{u}(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\
 &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (f(x, \hat{u}) + \rho\Phi + h + g + U_1\hat{u})(X_s) ds \right] \\
 (3.18) \quad &\geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} \hat{u}(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\
 &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (\rho\Phi + h - C)(X_s) ds \right].
 \end{aligned}$$

Also using (3.13) and Lemma 3.1 we have

$$\begin{aligned}
 \underline{u}(x) &= \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} \underline{u}(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\
 (3.19) \quad &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (\rho\Phi + h - C_1)(X_s) ds \right].
 \end{aligned}$$

By our choice of C_1 , we obtain from (3.18) and (3.19) that

$$w(x) \geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} w(X_t) \mathbb{1}_{\{t < \tau\}} \right], \quad t \geq 0,$$

for $w = \hat{u} - \underline{u}$. Since $\lambda_{U_1}^* > 0$, we obtain from [5, Theorem 4.2] that $w \geq 0$ in \mathbb{R}^d . Hence the result. \square

Using Lemma 3.3 we can now prove the existence of a minimal solution applying monotone iteration scheme.

Lemma 3.4. *Suppose that the conditions of Lemma 3.3 hold. Then for $\rho \leq \rho_1$, where ρ_1 is same value as in Lemma 3.3, there exists $u \in \mathcal{C}_0(D)$ satisfying*

$$(3.20) \quad \Psi(-\Delta) u = f(x, u) + \rho\Phi_1 + h(x) \quad \text{in } D.$$

Moreover, the above u can be chosen to be minimal in the sense that if $\tilde{u} \in \mathcal{C}_0(D)$ is another solution of (3.20), then $\tilde{u} \geq u$ in \mathbb{R}^d .

Proof. The proof is based on the standard monotone iteration method. Denote by $m = \min_{\bar{D}} \underline{u}$ and $M = \max_{\bar{D}} \bar{u}$. Let $\theta > 0$ be a Lipschitz constant for $f(x, \cdot)$ on the interval $[m, M]$, i.e.,

$$|f(x, q_1) - f(x, q_2)| \leq \theta |q_1 - q_2| \quad \text{for } q_1, q_2 \in [m, M], \quad x \in \bar{D}.$$

Denote $F(x, u) = f(x, u) + \rho\Phi(x) + h(x)$. Consider the solutions of the following family of problems:

$$\begin{aligned}
 (3.21) \quad \Psi(-\Delta) u^{(n+1)} + \theta u^{(n+1)} &= F(x, u^{(n)}) + \theta u^{(n)} \quad \text{in } D, \\
 u^{(n+1)} &= 0 \quad \text{in } D^c.
 \end{aligned}$$

By Theorem 2.3, (3.21) has a unique solution. We claim that

$$(3.22) \quad \underline{u} = u^{(0)} \leq u^{(n)} \leq u^{(n+1)} \leq \bar{u} \quad \text{for all } n \geq 1.$$

Denote $w^{(n)} = u^{(n)} - u^{(n-1)}$. Then using Lemma 3.1 it is easily seen that

$$w^{(n+1)}(x) = \mathbb{E}^x \left[e^{-\theta t} w^{(n+1)}(X_t) \mathbb{1}_{\{t < \tau\}} \right]$$

$$(3.23) \quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\theta s} \left(F(X_s, u^{(n)}) - F(X_s, u^{(n-1)}) + \theta(u^{(n)} - u^{(n-1)}) \right) ds \right]$$

We note that for $n = 0$ the right most term in (3.23) vanishes. Therefore,

$$w^{(1)}(x) \geq \mathbb{E}^x \left[e^{-\theta t} w^{(1)}(X_t) \mathbf{1}_{\{t < \tau\}} \right] \quad t \geq 0.$$

From [5, Theorem 4.2] we find $w^{(1)} \geq 0$. Note that if $u^{(n)} - u^{(n-1)} \geq 0$ we have

$$w^{(n+1)}(x) \geq \mathbb{E}^x \left[e^{-\theta t} w^{(n+1)}(X_t) \mathbf{1}_{\{t < \tau\}} \right] \quad t \geq 0,$$

and therefore, we can apply induction to obtain $\underline{u} = u^{(0)} \leq u^{(n)} \leq u^{(n+1)}$. Denoting $v^n = \bar{u} - u^{(n)}$ we again write

$$\begin{aligned} v^{(n+1)}(x) &\geq \mathbb{E}^x \left[e^{-\theta t} v^{(n+1)}(X_t) \mathbf{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\theta s} \left(F(X_s, \bar{u}) - F(X_s, u^{(n)}) + \theta(\bar{u} - u^{(n)}) \right) ds \right]. \end{aligned}$$

Again employing an induction argument we have $u^{(n)} \leq \bar{u}$. This proves our claim (3.22). Therefore, the right hand side of (3.21) is bounded uniformly in n . Hence by [17, Theorem 1.1] we obtain

$$|u^{(n)}(x) - u^{(n)}(y)| \leq \kappa V(|x - y|) \quad x, y \in D, \quad n \geq 1.$$

This gives equicontinuity to the family $\{u^{(n)}\}_{n \geq 1}$. Hence by Arzelà-Ascoli theorem we get that $u^{(n)} \uparrow u$ uniformly in \mathbb{R}^d . Thus we obtain a solution u by passing to the limit in (3.21).

To establish minimality we consider a solution \tilde{u} of (3.20) in $C_0(D)$. From Lemma 3.3(3) we see that $\underline{u} \leq \tilde{u}$ in \mathbb{R}^d . Thus \bar{u} can be replaced by \tilde{u} , and the above argument shows that $u \leq \tilde{u}$. \square

Now we derive a priori bounds on the solutions of (P_ρ) . Our first result bounds the negative part of solutions u of (P_ρ) .

Lemma 3.5. *Suppose that Assumption 1 and [AP](2) hold. There exists a constant $\kappa = \kappa(d, \Psi, D, U_1)$, such that for any solution u of (P_ρ) with $\rho \geq -\hat{\rho}$, $\hat{\rho} > 0$, we have*

$$\sup_D |u^-| \leq \kappa(C + \hat{\rho} + \|h\|_\infty),$$

where C is same constant as in (2.13).

Proof. Let u be a solution to (P_ρ) for some $\rho \geq -\hat{\rho}$. Denote by $w = u \wedge 0$. Then by Lemma 3.1 we get

$$\begin{aligned} u(x) &= \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} u(X_t) \mathbf{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (f(x, u) + \rho\Phi + h + U_1 u)(X_s) ds \right] \\ &\geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} u(X_t) \mathbf{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (\rho\Phi + h - C)(X_s) ds \right] \end{aligned}$$

$$\begin{aligned} &\geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} w(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (-\hat{\rho}\Phi_1 - \|h\|_\infty - C) ds \right] \end{aligned}$$

since the right hand side of the above display is non-positive we have

$$(3.24) \quad \begin{aligned} w(x) &\geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} w(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (-\hat{\rho}\Phi_1 - \|h\|_\infty - C) ds \right], \end{aligned}$$

for $t \geq 0$ and $x \in D$. Let $v \in \mathcal{C}_0(D)$ be the unique solution of

$$(3.25) \quad \Psi(-\Delta)v + U_1v = -\hat{\rho}\Phi_1 - \|h\| - C \quad \text{in } D.$$

This is assured by Theorem 2.3. Using Lemma 3.1 we see that

$$\begin{aligned} v(x) &= \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} v(X_t) \mathbb{1}_{\{t < \tau\}} \right] \\ &\quad + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_1(X_p) dp} (-\hat{\rho}\Phi_1 - \|h\|_\infty - C) ds \right]. \end{aligned}$$

Combining with (3.24) we find

$$(3.26) \quad (w - v)(x) \geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} (w - v)(X_t) \mathbb{1}_{\{t < \tau\}} \right] \quad t \geq 0, x \in D.$$

Since $\lambda_{U_1}^* > 0$, using (3.26) and [5, Theorem 4.2] we obtain that $w \geq v$ in \mathbb{R}^d . From (3.25) and [5, Th. 4.7] we obtain a constant $\kappa = \kappa(d, \Psi, D, U_1)$ satisfying

$$\sup_{x \in \bar{D}} |v| \leq \kappa(C + \hat{\rho} + \|h\|_\infty)$$

holds. Thus $u^- = -w \leq \kappa(C + \hat{\rho} + \|h\|_\infty)$, for $x \in D$, and the result follows. \square

Our next result provides a lower bound on the growth of the solution for large ρ .

Lemma 3.6. *Let Assumption 1 and [AP](1)-(2) hold. For every $\hat{\rho} > 0$ there exists $C_3 > 0$ such that for every solution u of (P_ρ) with $\rho \geq -\hat{\rho}$ we have*

$$\rho^+ \leq C_3(1 + \|u^+\|_\infty) \leq C_3(1 + \|u\|_\infty).$$

Proof. Let $\varphi = u - \frac{\rho}{\lambda^*} \Phi_1$. Then we have $\varphi \in \mathcal{C}_0(D)$. Also,

$$\Psi(-\Delta)\varphi(x) = f(x, u) + \rho\Phi_1 + h - \rho\Phi_1 = f(x, u) - h$$

In particular,

$$\varphi(x) = \mathbb{E}^x \left[\int_0^\tau (f(X_s, u(X_s)) - h(X_s)) ds \right], \quad x \in D.$$

By our assumption on f and Lemma 3.5 we can find a constant $C_4 = C_4(\|h\|_\infty, \|U_1\|_\infty, C, \hat{\rho})$ satisfying

$$\begin{aligned} f(x, u) - h &\geq -U_1(x)u - C - \|h\|_\infty \\ &\geq -U_1(x)u^+ - \|U_1u^-\|_\infty - C - \|h\|_\infty \\ &\geq -C_4(u^+(x) + 1). \end{aligned}$$

It then follows that with a constant C_5 , dependent on $\text{diam } D$,

$$\sup_D (-\varphi)^+ \leq C_5 C_4 (1 + \|u^+\|_\infty)$$

holds. Pick $x \in D$ such that $\Phi_1(x) = 1$; this is possible since $\|\Phi_1\|_\infty = 1$ by assumption. It gives

$$\frac{\rho}{\lambda^*} - u(x) \leq (-\varphi(x))^+ \leq C_5 C_4 (1 + \|u^+\|_\infty),$$

which, in turn, implies

$$\rho \leq \lambda^* (C_4 C_5 + (1 + C_4 C_5) \|u^+\|_\infty),$$

proving the claim. □

One may notice that we have not used the second condition in (2.12) so far. The next result makes use of this condition to establish an upper bound on the growth of u .

Lemma 3.7. *Suppose that Assumption 1 and [AP] hold. For each $\hat{\rho} > 0$ there exists C_0 such that for every solution u of (P_ρ) , for $\rho \geq -\hat{\rho}$, we have*

$$(3.27) \quad \|u\|_\infty \leq C_0.$$

In particular, there exists $\rho_2 > 0$ such that (P_ρ) does not have any solution for $\rho \geq \rho_2$.

Proof. Suppose, to the contrary, that there exists a sequence $(\rho_n, u_n)_{n \in \mathbb{N}}$ satisfying (P_ρ) with $\rho_n \geq -\hat{\rho}$ and $\|u_n\|_\infty \rightarrow \infty$. From Lemma 3.5 it follows that $\|u_n^+\|_\infty = \|u_n\|_\infty$. Define $v_n = \frac{u_n}{\|u_n\|_\infty}$. Then

$$(3.28) \quad \Psi(-\Delta) v_n = H_n(x) = \frac{1}{\|u_n\|_\infty} (f(x, u_n) + \rho_n \Phi_1 + h) \quad \text{in } D.$$

Since $\|H_n\|_\infty$ is uniformly bounded by Lemma 3.6, it follows by [17, Theorem 1.1] that

$$\sup_n \|v_n\|_{\mathcal{C}^{\bar{\phi}}(D)} \leq \kappa_1,$$

for some constant κ_1 and $\|\cdot\|_{\mathcal{C}^{\bar{\phi}}(D)}$ is given by (3.7). Hence we can extract a subsequence of $(v_n)_{n \in \mathbb{N}}$, denoted by the original sequence, such that it converges to a continuous function $v \in \mathcal{C}_0(D)$ in $\mathcal{C}(\mathbb{R}^d)$. Denote

$$G_n(x) = \frac{1}{\|u_n\|_\infty} (f(x, u_n(x)) + h(x) + U_2(x)u_n(x) + \rho_n \Phi_1(x)),$$

$$I_n(x) = \frac{1}{\|u_n\|_\infty} (f(x, -u_n^-(x)) + h(x) - U_2(x)u_n^-(x) - C + (\rho_n \wedge 0)\Phi_1(x)).$$

It then follows from (2.14) that $G_n \geq I_n$ and $I_n \rightarrow 0$ uniformly by Lemma 3.6. Using (3.28) and Lemma 3.1, we get

$$(3.29) \quad v_n(x) = \mathbb{E}^x \left[e^{-\int_0^t U_2(X_s) ds} v_n(X_t) \mathbf{1}_{\{t < \tau\}} \right] + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_2(X_p) dp} G_n(X_s) ds \right]$$

$$\geq \mathbb{E}^x \left[e^{-\int_0^t U_2(X_s) ds} v_n(X_t) \mathbf{1}_{\{t < \tau\}} \right] + \mathbb{E}^x \left[\int_0^{t \wedge \tau} e^{-\int_0^s U_2(X_p) dp} I_n(X_s) ds \right].$$

Letting $n \rightarrow \infty$ in (3.29) and using the uniform convergence of I_n and v_n , we obtain

$$(3.30) \quad v(x) \geq \mathbb{E}^x \left[e^{-\int_0^t U_2(X_s) ds} v(X_t) \mathbf{1}_{\{t < \tau_D\}} \right] \quad \text{for all } x \in D, t \geq 0.$$

Since $\|v\|_\infty = 1$ and $v \geq 0$ in \mathbb{R}^d , it is easily seen from (3.30) that $v > 0$ in D . Hence by [5, Prop. 4.1] it follows that $\lambda_{U_2}^* \geq 0$, contradicting (2.12). This proves the first part of the result. The second part follows by Lemma 3.6 and (3.27). \square

With the above results in hand, we can now proceed to prove Theorem 2.5. Define

$$\mathcal{A} = \{ \rho \in \mathbb{R} : (P_\rho) \text{ has a solution} \}.$$

By Lemma 3.4 we have that $\mathcal{A} \neq \emptyset$, and Lemma 3.7 imply that \mathcal{A} is bounded from above. Define $\rho^* = \sup \mathcal{A}$. Note that if $\rho' < \rho^*$, then $\rho' \in \mathcal{A}$. Indeed, there is $\tilde{\rho} \in (\rho', \rho^*) \cap \mathcal{A}$ and the corresponding solution $u^{(\tilde{\rho})}$ of $(P_{\tilde{\rho}})$ with $\rho = \tilde{\rho}$ is a super-solution at level ρ' , i.e.,

$$\Psi(-\Delta) u^{(\tilde{\rho})} = f(x, u^{(\tilde{\rho})}) + \rho' \Phi_1 + h(x) + g(x) \quad \text{in } D, \quad \text{and} \quad u^{(\tilde{\rho})} = 0 \quad \text{in } D^c,$$

where $g(x) = (\tilde{\rho} - \rho') \Phi_1 \geq 0$. Using Lemma 3.3(3) and from the proof of Lemma 3.4 we have a minimal solution of $(P_{\rho'})$ with $\rho = \rho'$. Next we show that there are at least two solutions for $\rho < \rho^*$.

Recall that $\delta_D : \bar{D} \rightarrow [0, \infty)$ is the distance function from the set D^c . We can assume that δ_D is a positive \mathcal{C}^1 -function in D . For a sufficiently small $\varepsilon > 0$, to be chosen later, consider the Banach space

$$\mathfrak{X} = \left\{ \psi \in \mathcal{C}_0(D) : \left\| \frac{\psi}{V(\delta_D)} \right\|_{\mathcal{C}^\varepsilon(D)} < \infty \right\}.$$

In fact, it is sufficient to consider any ε strictly smaller than the parameter α in [17, Th. 1.2]. It should be observed that for every $\psi \in \mathfrak{X}$ we can extend $\psi \cdot [V(\delta_D)]^{-1}$ up to the boundary ∂D continuously.

For $\rho \in \mathbb{R}$ and $m \geq 0$ we define a map $K_\rho : \mathfrak{X} \rightarrow \mathfrak{X}$ as follows. For $v \in \mathfrak{X}$, $K_\rho v = u$ is the unique solution (see Theorem 2.3) to the Dirichlet problem

$$\Psi(-\Delta) u + mu = f(x, v) + \rho \Phi_1 + h(x) + mv \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{in } D^c.$$

It follows from [17, Th. 1.2]

$$\left\| \frac{\psi}{V(\delta_D)} \right\|_{\mathcal{C}^\alpha(D)} < \infty,$$

for $\alpha > \varepsilon$, and thus $u \in \mathfrak{X}$. In fact, using the above estimate it can be easily shown that K_ρ is continuous and compact.

Lemma 3.8. *Let $\rho < \rho^*$. Then there exists $m \geq 0$ and an open $\mathcal{O} \subset \mathfrak{X}$, containing the minimal solution, satisfying $\deg(I - K_\rho, \mathcal{O}, 0) = 1$.*

Proof. We borrow some of the arguments of [11](see also [6]) with a suitable modification. Pick $\bar{\rho} \in (\rho, \rho^*)$ and let \bar{u} be a solution of $(P_{\bar{\rho}})$ with $\rho = \bar{\rho}$. It then follows that

$$\Psi(-\Delta) \bar{u} = f(x, \bar{u}) + \rho \Phi_1 + h(x) + g(x) \quad \text{in } D \quad \text{and} \quad \bar{u} = 0 \quad \text{in } D^c,$$

for $\bar{g}(x) = (\bar{\rho} - \rho)\Phi_1$ and by Lemma 3.4(1) we have a classical subsolution

$$\Psi(-\Delta) \underline{u} = f(x, \underline{u}) + \rho\Phi_1 + h(x) + \underline{g}(x) \quad \text{in } D \quad \text{and} \quad u = 0 \quad \text{in } D^c,$$

with $\underline{g} \leq 0$. Then Lemma 3.3(3) supplies $\underline{u} \leq \bar{u}$ in \mathbb{R}^d , hence the minimal solution u of (P_ρ) satisfies $\underline{u} \leq u \leq \bar{u}$ in \mathbb{R}^d . Note that for every $\psi \in \mathfrak{X}$, the ratio $\frac{\psi}{V(\delta_D)}$ is continuous up to the boundary. Define

$$\mathcal{O} = \left\{ \psi \in \mathfrak{X} : \underline{u} < \psi < \bar{u} \text{ in } D, \frac{\underline{u}}{V(\delta_D)} < \frac{\psi}{V(\delta_D)} < \frac{\bar{u}}{V(\delta_D)} \text{ on } \partial D, \|\psi\|_{\mathfrak{X}} < r \right\},$$

where the value of r will be chosen later. It is clear that \mathcal{O} is bounded, open and convex. Also, if we choose r large enough, then the minimal solution u belongs to \mathcal{O} . Indeed, note that for $w = u - \underline{u}$

$$(3.31) \quad \Psi(-\Delta) w = f(x, u) - f(x, \underline{u}) - \underline{g} \quad \text{in } D.$$

Define

$$U(x) = \begin{cases} \left(\frac{f(x, \underline{u}(x)) - f(x, u(x))}{\underline{u}(x) - u(x)} \right)^- & \text{if } u(x) \neq \underline{u}(x), \\ (D_u f(u(x), x))^- & \text{if } u(x) = \underline{u}(x). \end{cases}$$

By Assumption [AP](1) we have $U \in \mathcal{C}(\bar{D})$. Also note that

$$f(x, u) - f(x, \underline{u}) + U(x)w = \left(\frac{f(x, \underline{u}(x)) - f(x, u(x))}{\underline{u}(x) - u(x)} \right)^+ w \geq 0,$$

since $w \geq 0$. Now applying Lemma 3.1 to (3.31) we obtain that

$$(3.32) \quad w(x) \geq \mathbb{E}^x \left[e^{-\int_0^t U_1(X_s) ds} w(X_t) \mathbf{1}_{\{t < \tau\}} \right], \quad t \geq 0.$$

Using estimate (3.16) it is obvious that $w > 0$ in D . Choose $t = 2$, $D_1 \Subset D$ and use (3.32) to obtain

$$\begin{aligned} w(x) &= e^{-2\|U\|_\infty} \mathbb{E}^x [w(X_t) \mathbf{1}_{\{2 < \tau\}}] \\ &= e^{-2\|U\|_\infty} \int_D w(y) p^D(2, x, y) dy \\ &\geq e^{-2\|U\|_\infty} \int_{D_1} w(y) p^D(2, x, y) dy \\ &\geq \kappa_1 e^{-2\|U\|_\infty} \min_{D_1} w \mathbb{P}^x(\tau > 1) \int_{D_1} \mathbb{P}^y(\tau > 1) p(1 \wedge V^2(r), |x - y|) dy \\ &\geq \kappa_2 p(1 \wedge V^2(r), \text{diam } D) \mathbb{P}^x(\tau > 1) \int_{D_1} \mathbb{P}^y(\tau > 1) dy, \end{aligned}$$

for some constant κ_2 , where in the fourth inequality we use (3.16). Now using (3.17) we can find a constant $\kappa_3 > 0$ satisfying

$$w(x) \geq \kappa_3 V(\delta_D(x)), \quad x \in D.$$

This of course, implies

$$\min_{\partial D} \left(\frac{u}{V(\delta_D)} - \frac{\underline{u}}{V(\delta_D)} \right) > 0.$$

Similarly, we can compare also u and \bar{u} .

We define m to be a Lipschitz constant of $f(x, \cdot)$ in the interval $[\min \underline{u}, \max \bar{u}]$. Also, define

$$\tilde{f}(x, q) = f(x, (\underline{u}(x) \vee q) \wedge \bar{u}(x)) + m(\underline{u}(x) \vee q) \wedge \bar{u}(x).$$

Note that f is bounded and Lipschitz continuous in q , and also non-decreasing in q . We define another map $\tilde{K}_\rho : \mathfrak{X} \rightarrow \mathfrak{X}$ as follows: for $v \in \mathfrak{X}$, $\tilde{K}_\rho v = \tilde{u}$ is the unique solution of

$$(3.33) \quad \Psi(-\Delta) \tilde{u} + m\tilde{u} = \tilde{f}(x, v) + \rho\Phi + h \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{in } D^c.$$

It is easy to check that \tilde{K}_ρ is a compact mapping. Since the right hand side of (3.33) is bounded, using again [17, Th. 1.2], we find r satisfying

$$\sup \left\{ \|\tilde{K}_\rho v\|_{\mathfrak{X}} : v \in \mathfrak{X} \right\} < r.$$

We fix this choice of r . We now show that $\tilde{K}_\rho v \in \mathcal{O}$ for all $v \in \mathfrak{X}$. Let $\tilde{u} = \tilde{K}_\rho v$. Then

$$\Psi(-\Delta)(u - \underline{u}) = -m(u - \underline{u}) + \tilde{f}(x, v) - m\underline{u} - f(x, \underline{u}) - \underline{g}$$

Since

$$\tilde{f}(x, v) - m\underline{u} - f(x, \underline{u}) - \underline{g} \geq \tilde{f}(x, \underline{u}) - m\underline{u} - f(x, \underline{u}) = 0,$$

from Lemma 3.1 we note that for $w = \tilde{u} - \underline{u}$

$$(3.34) \quad w(x) \geq \mathbb{E}^x \left[e^{-mt} w(X_t) \mathbf{1}_{\{t < \tau\}} \right].$$

Thus letting $t \rightarrow \infty$ in (3.34) we have obtain $w \geq 0$. Since $\underline{g} \leq 0$, it follows from (3.33) that w can not be identically 0. Hence again applying (3.34) we obtain $w > 0$ in D . Repeating the arguments as above (see below (3.32)) we also have

$$\min_{\partial D} \left(\frac{\tilde{u}}{V(\delta_D)} - \frac{\underline{u}}{V(\delta_D)} \right) > 0.$$

The other estimates with respect to \bar{u} can be obtained similarly. Finally, this implies that $\tilde{K}_\rho v \in \mathcal{O}$, for all $v \in \mathfrak{X}$. Moreover, $0 \notin (I - \tilde{K}_\rho)(\partial D)$. Then by the homotopy invariance property of degree we find that $\deg(I - \tilde{K}_\rho, \mathcal{O}, 0) = 1$ (see for instance, [11]). Since \tilde{K}_ρ coincides with K_ρ in \mathcal{O} , we obtain $\deg(I - K_\rho, \mathcal{O}, 0) = 1$. \square

Similarly as before, define $\mathcal{S}_\rho : \mathfrak{X} \rightarrow \mathfrak{X}$ such that for $v \in \mathfrak{X}$, $u = \mathcal{S}_\rho v$ is given by the unique solution of

$$\Psi(-\Delta) u = f(x, v) + \rho\Phi_1 + h(x) \quad \text{in } D, \quad \text{and} \quad u = 0 \quad \text{in } D^c.$$

Then the standard homotopy invariance of degree (w.r.t. m) gives that $\deg(I - \mathcal{S}_t, \mathcal{O}, 0) = 1$. This observation will be helpful in concluding the proof below.

Proof of Theorem 2.5. Using Lemma 3.8 we can now complete the proof by using [11, 13]. Recall the map \mathcal{S}_ρ defined above, and fix $\rho < \rho^*$. Denote by \mathcal{O}_R a ball of radius R in \mathfrak{X} . From Lemma 3.7 and [17, Theorem 1.2] we find that

$$\deg(I - \mathcal{S}_{\tilde{\rho}}, \mathcal{O}_R, 0) = 0 \quad \text{for all } R > 0, \tilde{\rho} \geq \rho_2.$$

Using again Lemmas 3.7 and [17, Th. 1.2], we obtain that for every $\hat{\rho}$ there exists a constant R such that

$$\|u\|_{\mathfrak{X}} < R$$

for each solution u of (P_ρ) with $\tilde{\rho} \geq -\hat{\rho}$. Fixing $\hat{\rho} > |\rho|$ and the corresponding choice of R , it then follows from homotopy invariance that $\deg(I - \mathcal{S}_\rho, \mathcal{O}_R, 0) = 0$. We can choose R large enough so that $\mathcal{O} \subset \mathcal{O}_R$ where \mathcal{O} is from Lemma 3.8. Since $\deg(I - \mathcal{S}_\rho, \mathcal{O}, 0) = 1$, as seen above, using the excision property of degree we conclude that there exists a solution of (P_ρ) in $\mathcal{O}_R \setminus \mathcal{O}$. Hence for every $\rho < \rho^*$ there exist at least two solutions of (P_ρ) . The existence of a solution at $\rho = \rho^*$ follows from the a priori estimates in Lemma 3.7, the estimate in [17, Theorem. 1.1], and the stability property of the semigroup solutions. This completes the proof of Theorem 2.5. \square

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