Pure and Applied Functional Analysis
Volume 6, Number 2, 2021, 257–288

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON CIRCUMCENTER MAPPINGS INDUCED BY NONEXPANSIVE OPERATORS

HEINZ H. BAUSCHKE, HUI OUYANG, AND XIANFU WANG

Abstract. We introduce the circumcenter mapping induced by a set of (usually nonexpansive) operators. One prominent example of a circumcenter mapping is the celebrated Douglas–Rachford splitting operator. Our study is motivated by the Circumcentered–Douglas–Rachford method recently introduced by Behling, Bello Cruz, and Santos in order to accelerate the Douglas–Rachford method for solving certain classes of feasibility problems. We systematically explore the properness of the circumcenter mapping induced by reflectors or projectors. Numerous examples are presented. We also present a version of Browder's demiclosedness principle for circumcenter mappings.

1. Introduction

Throughout this paper, we assume that

$$\mathcal{H} \text{ is a real Hilbert space}$$

with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$. Let $m \in \mathbb{N} \smallsetminus \{0\}$, and let $T_1, \ldots, T_{m-1}, T_m$ be operators from $\mathcal{H}$ to $\mathcal{H}$. Set

$$\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\},$$

and denote the power set of $\mathcal{H}$ as $2^{\mathcal{H}}$. The associated set-valued operator $\mathcal{S} : \mathcal{H} \to 2^{\mathcal{H}}$ is defined by

$$(\forall x \in \mathcal{H}) \quad \mathcal{S}(x) = \{T_1x, \ldots, T_{m-1}x, T_mx\}.$$

Unless otherwise specified, we assume that

$$U_1, \ldots, U_m \text{ are closed affine subspaces of } \mathcal{H}, \text{ with } \bigcap_{i=1}^{m} U_i \neq \varnothing.$$

In this paper, we introduce the circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$ which maps every element $x \in \mathcal{H}$ to either empty set or the (unique if it exists) circumcenter of the finitely many elements in the nonempty set $\mathcal{S}(x)$. In fact, the circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$ is the composition $CC \circ \mathcal{S}$ where CC is the circumcenter operator defined in [4]. The domain of $CC_{\mathcal{S}}$ is defined to be $\operatorname{dom} CC_{\mathcal{S}} = \{x \in \mathcal{H} \mid CC_{\mathcal{S}}x \neq \varnothing\}$. We say the circumcenter mapping $CC_{\mathcal{S}}$ is *proper*, if $\operatorname{dom} CC_{\mathcal{S}} = \mathcal{H}$. Properness is an important property for algorithms where one wishes to consider sequences of the form $(CC_{\mathcal{S}}^k x)_{k\in\mathbb{N}}$.

2020 *Mathematics Subject Classification.* Primary 47H09,47H04; Secondary 41A50, 90C25 .
Key words and phrases. Browder's demiclosedness principle, circumcenter, circumcenter mapping, nonexpansive, projector, reflector.

The goal of this paper is to explore conditions sufficient for the circumcenter mapping to be proper. We also connect the circumcenter mapping to the celebrated demiclosedness principle by Felix Browder.

The CRM (Circumcentered–Reflection Method) operator C recently investigated by Behling, Bello Cruz, and Santos in [7, page 159] is a particular instance of a proper circumcenter mapping. The C–DRM (Circumcentered–Douglas–Rachford Method) operator C_T defined by Behling et al. in [6, Section 2] is CRM operator associated with only two linear subspaces. Hence, the C_T is a special case of our proper circumcenter mapping as well.

Behling et al. introduced in [6] the C–DRM which generates iterates by taking the intersection of bisectors of reflection steps to accelerate the Douglas–Rachford method to solve certain classes of feasibility problems. Our paper [4] and this paper are motivated by [6]. The proof of one of our main results, Theorem 4.3, is inspired by [6, Lemma 2]. We now discuss further results and the organization of this paper. In Section 2, we collect various results for subsequent use. In particular, facts on circumcenter operator defined in [4, Definition 3.4] are reviewed in Section 2.3. In Section 3, we introduce the circumcenter mapping $CC_{\mathcal{S}}$ induced by a set of operators $\mathcal{S}$. Based on some known results of circumcenter operator, we derive some sufficient conditions for the circumcenter mapping $CC_{\mathcal{S}}$ to be proper. When $\mathcal{S}$ consists of only three operators, we provide a sufficient and necessary condition for the $CC_{\mathcal{S}}$ to be proper. We also obtain conditions sufficient for continuity. Examples illustrating the tightness of our assumptions are provided as well. Section 3.4 contains the demiclosedness principle for certain circumcenter mappings. In Section 4, we consider the circumcenter of finite subsets drawn from the affine hull of compositions of reflectors. Inspired by [6, Lemma 2], we prove the properness of a certain class of circumcenter mappings induced by reflectors. We also provide improper examples. Two particular instances of $CC_{\mathcal{S}}$, one of which belongs to the class of C–DRM operators from [6] while the other is new, are considered. Comparing to the Douglas–Rachford Method (DRM) and the Method of Alternating projections (MAP), we find in preliminary numerical explorations that $(CC^k_{\mathcal{S}}x)_{k\in\mathbb{N}}$ can be used to solve best approximation problems. It is interesting that in general $CC_{\mathcal{S}}$ is neither continuous nor linear. In Section 5, the operators in $\mathcal{S}$ are chosen from the affine hull of the set of compositions of projectors. We provide both proper and improper examples of corresponding circumcenter mappings. The final Section 6 deals with reflectors and reflected resolvents.

Let us turn to notation. Let K and C be subsets of $\mathcal{H}$, $z \in \mathcal{H}$ and $\lambda \in \mathbb{R}$. Then $K + C = \{x + y \mid x \in K, y \in C\}$, $K + z = K + \{z\}$, and $\lambda K = \{\lambda x \mid x \in K\}$. The cardinality of the set K is denoted as $\operatorname{card}(K)$. The intersection of all the linear subspaces of $\mathcal{H}$ containing K is called the *span* of K, and is denoted by $\operatorname{span} K$. A nonempty subset K of $\mathcal{H}$ is an *affine subspace* of $\mathcal{H}$ if $(\forall \rho \in \mathbb{R})\ \rho K + (1-\rho)K = K$; moreover, the smallest affine subspace containing K is the *affine hull* of K, denoted $\operatorname{aff} K$. Assume that C is a nonempty closed, convex subset in $\mathcal{H}$. We denote by P_C the *projector* onto C. $\mathrm{R}_C := 2\,\mathrm{P}_C - \mathrm{Id}$ is the *reflector* associated with C. Let $T : \mathcal{H} \to \mathcal{H}$. The set of *fixed points* of T is $\operatorname{Fix} T = \{x \in \mathcal{H} \mid x = Tx\}$. Let $(x_k)_{k\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ and let $x \in \mathcal{H}$. We use $x_k \rightharpoonup x$ to indicate that $(x_k)_{k\in\mathbb{N}}$ converges weakly to x. The set $\mathbf{B}[x;r] := \{y \in \mathcal{H} \mid \|y - x\| \leq r\}$ is the closed ball centered at x of radius $r \geq 0$. For other notation not explicitly defined here, we refer the reader to [2].

2. Auxiliary results

In this section, we provide various results that will be useful in the sequel. We start with some facts about affine subspaces.

2.1. **Affine subspaces and related concepts.**

Definition 2.1 ([10, page 4])**.** An affine subspace C is said to be *parallel* to an affine subspace M if $C = M + a$ for some $a \in \mathcal{H}$.

Fact 2.2 ([10, Theorem 1.2])**.** *Every affine subspace C is parallel to a unique linear subspace L, which is given by*

$$(\forall y \in C) \quad L = C - y = C - C.$$

Definition 2.3 ([10, page 4])**.** The *dimension* of a nonempty affine subspace is defined to be the dimension of the linear subspace parallel to it.

Fact 2.4 ([10, page 7])**.** *Let $x_1, \ldots, x_m \in \mathcal{H}$. Then the affine hull is given by*

$$\operatorname{aff}\{x_1, \ldots, x_m\} = \Big\{\lambda_1 x_1 + \cdots + \lambda_m x_m \;\Big|\; \lambda_1, \ldots, \lambda_m \in \mathbb{R} \text{ and } \sum_{i=1}^{m} \lambda_i = 1\Big\}.$$

Fact 2.5 ([4, Lemma 2.6])**.** *Let $x_1, \ldots, x_m \in \mathcal{H}$, where $m \geq 2$. Then for every $i_0 \in \{2, \ldots, m\}$, we have*

$$\begin{aligned} &\operatorname{aff}\{x_1, \ldots, x_m\} \\ &= x_1 + \operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\} \\ &= x_{i_0} + \operatorname{span}\{x_1 - x_{i_0}, \ldots, x_{i_0-1} - x_{i_0}, x_{i_0+1} - x_{i_0}, \ldots, x_m - x_{i_0}\}. \end{aligned}$$

Definition 2.6 ([10, page 6])**.** Let $x_0, x_1, \ldots, x_m \in \mathcal{H}$. The $m+1$ vectors $x_0, x_1, \ldots, x_m$ are said to be affinely independent if $\operatorname{aff}\{x_0, x_1, \ldots, x_m\}$ is m-dimensional. We will also say $(x_0, x_1, \ldots, x_m) = (x_i)_{i \in \{0,1,\ldots,m\}}$ is affinely independent.

Fact 2.7 ([10, page 7])**.** *Let $x_1, x_2, \ldots, x_m \in \mathcal{H}$. Then $x_1, x_2, \ldots, x_m$ are affinely independent if and only if $x_2 - x_1, \ldots, x_m - x_1$ are linearly independent.*

2.2. **Projectors and reflectors.**

Our first result follows easily from the definitions.

Lemma 2.8. *Let C be a nonempty closed convex subset of $\mathcal{H}$. Then*

(i) $\mathrm{P}_C \mathrm{P}_C = \mathrm{P}_C$.
(ii) *If C is a closed affine subspace, then* $\mathrm{R}_C \mathrm{R}_C = \mathrm{Id}$.

Fact 2.9 ([9, Theorem 5.8])**.** *Let C be a closed linear subspace of $\mathcal{H}$. Then*

(i) $\mathrm{Id} = \mathrm{P}_C + \mathrm{P}_{C^\perp}$.
(ii) $C^\perp = \{x \in \mathcal{H} \mid \mathrm{P}_C(x) = 0\}$ *and* $C = \{x \in \mathcal{H} \mid \mathrm{P}_{C^\perp}(x) = 0\} = \{x \in \mathcal{H} \mid \mathrm{P}_C(x) = x\}$.

The following result is a mild extension [6, Proposition 1] and it is useful in the proof of Theorem 4.3.

Proposition 2.10. *Let C be a closed affine subspace of $\mathcal{H}$. Then the following hold:*

(i) *The projector P_C and the reflector R_C are affine operators.*

(ii) *Let x be in $\mathcal{H}$ and let p be in $\mathcal{H}$. Then*

$$p = \mathrm{P}_C x \iff p \in C \quad \text{and} \quad (\forall v \in C)\ (\forall w \in C) \quad \langle x - p, v - w \rangle = 0.$$

(iii) $(\forall x \in \mathcal{H})\ (\forall v \in C)\ \|x - \mathrm{P}_C x\|^2 + \|v - \mathrm{P}_C x\|^2 = \|x - v\|^2$.

(iv) $(\forall x \in \mathcal{H})\ (\forall y \in \mathcal{H})\ \|x - y\| = \|\mathrm{R}_C x - \mathrm{R}_C y\|$.

(v) $(\forall x \in \mathcal{H})\ (\forall v \in C)\ \|x - v\| = \|\mathrm{R}_C x - v\|$.

Proof. (i): P_C is affine by [2, Corollary 3.22(ii)]; this implies that $\mathrm{R}_C = 2\,\mathrm{P}_C - \mathrm{Id}$ is affine as well.

(ii): [2, Corollary 3.22(i)].

(iii): Indeed, for every $x \in \mathcal{H}$ and $v \in C$,

$$\begin{aligned} \|x - v\|^2 &= \|x - \mathrm{P}_C x - (v - \mathrm{P}_C x)\|^2 \\ &= \|x - \mathrm{P}_C x\|^2 - 2\langle x - \mathrm{P}_C x, v - \mathrm{P}_C x \rangle + \|v - \mathrm{P}_C x\|^2 \\ &= \|x - \mathrm{P}_C x\|^2 + \|v - \mathrm{P}_C x\|^2. \quad \text{(by (ii))} \end{aligned}$$

(iv): For every $x \in \mathcal{H}$, and for every $y \in \mathcal{H}$, by (ii),

$$\begin{aligned} &\langle \mathrm{P}_C x - \mathrm{P}_C y, \mathrm{P}_C x - x \rangle - \langle \mathrm{P}_C x - \mathrm{P}_C y, \mathrm{P}_C y - y \rangle = 0 \\ \iff &\langle \mathrm{P}_C x - \mathrm{P}_C y, \mathrm{P}_C x - \mathrm{P}_C y - (x - y) \rangle = 0 \\ \iff &\|x - y\|^2 = 4\|\mathrm{P}_C x - \mathrm{P}_C y\|^2 - 4\langle \mathrm{P}_C x - \mathrm{P}_C y, x - y \rangle + \|x - y\|^2 \\ \iff &\|x - y\|^2 = \|(2\,\mathrm{P}_C x - x) - (2\,\mathrm{P}_C y - y)\|^2 \\ \iff &\|x - y\| = \|\mathrm{R}_C x - \mathrm{R}_C y\|. \quad (\text{by } \mathrm{R}_C = 2\,\mathrm{P}_C - \mathrm{Id}) \end{aligned}$$

(v): Notice that $\operatorname{Fix} \mathrm{R}_C = C$ and then use (iv). ☐

2.3. **Circumcenters.** In this subsection, $\mathcal{P}(\mathcal{H})$ is the set of all nonempty subsets of $\mathcal{H}$ containing *finitely many* elements.

By [4, Proposition 3.3], we know that for every $K \in \mathcal{P}(\mathcal{H})$, there is at most one point $p \in \operatorname{aff}(K)$ such that $\{\|p - x\| \mid x \in K\}$ is a singleton. Hence, the following notion is well-defined.

Definition 2.11 (circumcenter operator ([4, Definition 3.4]))**.** The *circumcenter operator* is $CC\colon \mathcal{P}(\mathcal{H}) \to \mathcal{H} \cup \{\varnothing\}$ such that for every $K \in \mathcal{P}(\mathcal{H})$,

$$CC(K) = \begin{cases} p, & \text{if } p \in \operatorname{aff}(K) \text{ and } \{\|p - x\| \mid x \in K\} \text{ is a singleton;} \\ \varnothing, & \text{otherwise.} \end{cases}$$

In particular, when $CC(K) \in \mathcal{H}$, that is, $CC(K) \neq \varnothing$, we say that the circumcenter of K exists and we call $CC(K)$ the circumcenter of K.

Fact 2.12 ([4, Example 3.6])**.** *Let x_1, x_2 be in $\mathcal{H}$. Then*

$$CC\big(\{x_1, x_2\}\big) = \frac{x_1 + x_2}{2}.$$

Fact 2.13 ([4, Theorem 4.1]). *Set $K = \{x_1, \ldots, x_m\}$, where $x_1, \ldots, x_m$ are affinely independent. Then $CC(K) \in \mathcal{H}$, which means that $CC(K)$ is the unique point satisfying the following two conditions:*

(i) *$CC(K) \in \operatorname{aff}(K)$, and*
(ii) *$\{\|CC(K) - y\| \mid y \in K\}$ is a singleton.*

Moreover,

$$\begin{aligned} &CC(K) \\ &= x_1 + \frac{1}{2}(x_2 - x_1, \ldots, x_m - x_1) G(x_2 - x_1, \ldots, x_m - x_1)^{-1} \begin{pmatrix} \|x_2 - x_1\|^2 \\ \vdots \\ \|x_m - x_1\|^2 \end{pmatrix}, \end{aligned}$$

where $G(x_2 - x_1, \ldots, x_{m-1} - x_1, x_m - x_1)$ is the Gram matrix *of $x_2 - x_1, \ldots, x_{m-1} - x_1, x_m - x_1$, i.e.,*

$$\begin{aligned} &G(x_2 - x_1, \ldots, x_{m-1} - x_1, x_m - x_1) \\ &= \begin{pmatrix} \|x_2 - x_1\|^2 & \langle x_2 - x_1, x_3 - x_1\rangle & \cdots & \langle x_2 - x_1, x_m - x_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_{m-1} - x_1, x_2 - x_1\rangle & \langle x_{m-1} - x_1, x_3 - x_1\rangle & \cdots & \langle x_{m-1} - x_1, x_m - x_1\rangle \\ \langle x_m - x_1, x_2 - x_1\rangle & \langle x_m - x_1, x_3 - x_1\rangle & \cdots & \|x_m - x_1\|^2 \end{pmatrix}. \end{aligned}$$

Fact 2.14 ([4, Theorem 8.1]). *Suppose that $K = \{x, y, z\} \in \mathcal{P}(\mathcal{H})$ and that* $\operatorname{card}(K) = 3$. *Then x, y, z are affinely independent if and only if $CC(K) \in \mathcal{H}$.*

Combining Fact 2.12 and Fact 2.14, we obtain the following two results.

Corollary 2.15. *Let $K = \{x_1, x_2, x_3\} \in \mathcal{P}(\mathcal{H})$. Then $CC(K) \in \mathcal{H}$ if and only if exactly one of the following cases holds.*

(i) $\operatorname{card}\{x_1, x_2, x_3\} \leq 2$.
(ii) $\operatorname{card}\{x_1, x_2, x_3\} = 3$ *and if there is $\{\alpha, \beta\} \subseteq \mathbb{R}$ such that $\alpha(x_2 - x_1) + \beta(x_3 - x_1) = 0$, then $\alpha = 0$ and $\beta = 0$.*

Corollary 2.16. *Let a, b, c be in $\mathbb{R}$. Then there exists no $x \in \mathbb{R}$ such that $|x - a| = |x - b| = |x - c|$ if and only if* $\operatorname{card}\{a, b, c\} = 3$.

Fact 2.17 (scalar multiples ([4, Proposition 6.1])). *Let $K \in \mathcal{P}(\mathcal{H})$ and $\lambda \in \mathbb{R} \smallsetminus \{0\}$. Then $CC(\lambda K) = \lambda CC(K)$.*

Fact 2.18 (translations ([4, Proposition 6.3])). *Let $K \in \mathcal{P}(\mathcal{H})$ and $y \in \mathcal{H}$. Then $CC(K + y) = CC(K) + y$.*

Fact 2.19 ([4, Lemma 4.2]). *Let $K \in \mathcal{P}(\mathcal{H})$ and let $M \subseteq K$ be such that* $\operatorname{aff}(M) = \operatorname{aff}(K)$. *Suppose that $CC(K) \in \mathcal{H}$. Then $CC(K) = CC(M)$.*

Fact 2.20 ([4, Theorem 7.1]). *Let $K = \{x_1, \ldots, x_m\} \in \mathcal{P}(\mathcal{H})$. Suppose that $CC(K) \in \mathcal{H}$. Then the following hold.*

(i) *Let $t = \dim\big(\operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}\big)$, $\widetilde{K} = \{x_1, x_{i_1}, \ldots, x_{i_t}\} \subseteq K$ be such that $x_{i_1} - x_1, \ldots, x_{i_t} - x_1$ form a basis of* $\operatorname{span}\{x_2 - x_1, \ldots, x_m - x_1\}$.

Furthermore, let $\big((x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)})\big)_{k\geq 1} \subseteq \mathcal{H}^{t+1}$ *with*

$$\lim_{k\to\infty} (x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)}) = (x_1, x_{i_1}, \ldots, x_{i_t}),$$

and set $(\forall k \geq 1)$ $\widetilde{K}^{(k)} = \{x_1^{(k)}, x_{i_1}^{(k)}, \ldots, x_{i_t}^{(k)}\}$. *Then there exist* $N \in \mathbb{N}$ *such that for every* $k \geq N$, $CC(\widetilde{K}^{(k)}) \in \mathcal{H}$ *and*

$$\lim_{k\to\infty} CC(\widetilde{K}^{(k)}) = CC(\widetilde{K}) = CC(K).$$

(ii) *Suppose that* $x_1, \ldots, x_{m-1}, x_m$ *are affinely independent, and let*

$$\big((x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)})\big)_{k\geq 1} \subseteq \mathcal{H}^m$$

satisfy $\lim_{k\to\infty}(x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}) = (x_1, \ldots, x_{m-1}, x_m)$. *Assume that* $(\forall k \geq 1)$ $K^{(k)} = \{x_1^{(k)}, \ldots, x_{m-1}^{(k)}, x_m^{(k)}\}$. *Then*

$$\lim_{k\to\infty} CC(K^{(k)}) = CC(K).$$

Fact 2.21 ([4, Example 7.6])**.** *Suppose that* $\mathcal{H} = \mathbb{R}^2$. *Let* $x_1 = (-2, 0)$ *and* $x_2 = x_3 = (2, 0)$. *Let* $(\forall k \geq 1)$ $\big(x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\big) = \big((-2, 0), (2, 0), (2 - \frac{1}{k}, \frac{1}{4k})\big)$. *Then*

$$(\forall k \geq 1) \quad CC(\{x_1^{(k)}, x_2^{(k)}, x_3^{(k)}\}) = \big(0, -8 + \tfrac{2}{k} + \tfrac{1}{8k}\big).$$

3. Circumcenter mappings induced by operators

Suppose that $T_1, \ldots, T_{m-1}, T_m$ are operators from $\mathcal{H}$ to $\mathcal{H}$, with $m \in \mathbb{N} \smallsetminus \{0\}$ and that

$$\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\} \text{ and } (\forall x \in \mathcal{H})\ \mathcal{S}(x) = \{T_1 x, \ldots, T_{m-1}x, T_m x\}.$$

3.1. **Definition.**

Definition 3.1 (induced circumcenter mapping)**.** The *circumcenter mapping* $CC_{\mathcal{S}}$ *induced by* $\mathcal{S}$ is

$$CC_{\mathcal{S}} \colon \mathcal{H} \to \mathcal{H} \cup \{\varnothing\} \colon x \mapsto CC(\mathcal{S}(x)),$$

that is, $CC_{\mathcal{S}} = CC \circ \mathcal{S}$. The *domain* of $CC_{\mathcal{S}}$ is

$$\operatorname{dom} CC_{\mathcal{S}} = \{x \in \mathcal{H} \mid CC_{\mathcal{S}} x \neq \varnothing\}.$$

Recall that if $\operatorname{dom} CC_{\mathcal{S}} = \mathcal{H}$, then we say the circumcenter mapping $CC_{\mathcal{S}}$ induced by $\mathcal{S}$ is *proper*; otherwise, we will call $CC_{\mathcal{S}}$ *improper*.

Remark 3.2. By Definitions 2.11 and 3.1, for every $x \in \mathcal{H}$, if the circumcenter of the set $\mathcal{S}(x)$ defined in Definition 2.11 does not exist in $\mathcal{H}$, then $CC_{\mathcal{S}} x = \varnothing$. Otherwise, $CC_{\mathcal{S}} x$ is the unique point satisfying the two conditions below:

(i) $CC_{\mathcal{S}} x \in \operatorname{aff}(\mathcal{S}(x)) = \operatorname{aff}\{T_1 x, \ldots, T_{m-1}x, T_m x\}$, and
(ii) $\|CC_{\mathcal{S}} x - T_1 x\| = \cdots = \|CC_{\mathcal{S}} x - T_{m-1} x\| = \|CC_{\mathcal{S}} x - T_m x\|$.

3.2. **Basic properties.** We start with some examples.

Proposition 3.3. *Assume* $\mathcal{S} = \{T_1, T_2\}$. *Then* $CC_{\mathcal{S}}$ *is proper. Moreover,*

$$(\forall x \in \mathcal{H}) \quad CC_{\mathcal{S}}x = \frac{T_1x + T_2x}{2}.$$

Proof. Clear from Fact 2.12 and Definition 3.1. □

Corollary 3.4. *Let* $\mathcal{S} = \{T_1, T_2, T_3\}$ *and let* $x \in \mathcal{H}$. *Then* $x \notin \operatorname{dom} CC_{\mathcal{S}}$ *if and only if* $\operatorname{card}\{T_1x, T_2x, T_3x\} = 3$ *and there exists* $(\alpha, \beta) \in \mathbb{R}^2 \smallsetminus \{(0,0)\}$ *such that* $\alpha(T_2x - T_1x) + \beta(T_3x - T_1x) = 0$.

Proof. This follows from Corollary 2.15. □

Example 3.5. Assume that $\mathcal{H} = \mathbb{R}^2$. Set $U_1 = \mathbb{R} \cdot (1,0)$, $U_2 = \mathbb{R} \cdot (0,1)$, and let $\alpha \in \mathbb{R}$. Set $\mathcal{S} = \{\alpha \operatorname{Id}, R_{U_1}, R_{U_2}\}$. Then the following hold:

(i) If $\alpha = 0$, then $\operatorname{dom} CC_{\mathcal{S}} = \{(0,0)\}$.
(ii) If $\alpha = 1$ or $\alpha = -1$, then $\operatorname{dom} CC_{\mathcal{S}} = \mathbb{R}^2$, i.e., $CC_{\mathcal{S}}$ is proper.
(iii) If $\alpha \in \mathbb{R} \smallsetminus \{0, 1, -1\}$, then $\operatorname{dom} CC_{\mathcal{S}} = \big(\mathbb{R}^2 \smallsetminus (U_1 \cup U_2)\big) \cup \{(0,0)\}$.

Proposition 3.6. *Suppose that for every* $x \in \mathcal{H}$, *there exists a point* $p(x) \in \mathcal{H}$ *such that*

(i) $p(x) \in \operatorname{aff}\{T_1x, \ldots, T_{m-1}x, T_mx\}$, *and*
(ii) $\|p(x) - T_1x\| = \cdots = \|p(x) - T_{m-1}x\| = \|p(x) - T_mx\|$.

Then $CC_{\mathcal{S}}$ *is proper and*

$$(\forall x \in \mathcal{H}) \quad CC_{\mathcal{S}}x = p(x).$$

Proof. This follows from Remark 3.2. □

Proposition 3.7. *Suppose that for every* $x \in \mathcal{H}$, *there exists* $\mathrm{I}(x) \subseteq \mathrm{I} := \{1, \ldots, m\}$ *such that* $\operatorname{card}\big(\mathrm{I}(x)\big) = \operatorname{card}\big(\mathcal{S}(x)\big)$ *and* $(T_ix)_{i \in \mathrm{I}(x)}$ *is affinely independent. Then* $CC_{\mathcal{S}}$ *is proper.*

Proof. Let $x \in \mathcal{H}$. Since $\mathrm{I}(x) \subseteq \mathrm{I}$, we have $\{T_ix\}_{i \in \mathrm{I}(x)} \subseteq \mathcal{S}(x)$. The affine independence of $(T_ix)_{i \in \mathrm{I}(x)}$ yields $\operatorname{card}\big(\{T_ix\}_{i \in \mathrm{I}(x)}\big) = \operatorname{card}\big(\mathrm{I}(x)\big)$. Combining with $\operatorname{card}\big(\mathrm{I}(x)\big) = \operatorname{card}\big(\mathcal{S}(x)\big)$, we obtain that $\{T_ix\}_{i \in \mathrm{I}(x)} = \mathcal{S}(x)$, which implies that

$$CC_{\mathcal{S}}x = CC\big(\mathcal{S}(x)\big) = CC\big(\{T_ix\}_{i \in \mathrm{I}(x)}\big). \tag{3.1}$$

Using the assumption that $(T_ix)_{i \in \mathrm{I}(x)}$ is affinely independent, by Fact 2.13, we deduce that $CC\big(\{T_ix\}_{i \in \mathrm{I}(x)}\big) \in \mathcal{H}$. Combining with (3.1), we deduce that $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x \in \mathcal{H}$, i.e., $CC_{\mathcal{S}}$ is proper. □

The following example illustrates that the converse of Proposition 3.7 is not true in general.

Example 3.8. Let U be a closed linear subspace of $\mathcal{H}$ with $\{0\} \neq U \subsetneqq \mathcal{H}$. Denote by 0 also the zero operator: $(\forall x \in \mathcal{H})$ $0(x) = 0$. Set $\mathcal{S} = \{\operatorname{Id}, \mathrm{P}_U, \mathrm{P}_{U^\perp}, 0\}$. Then the following hold:

(i) $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}}x = \frac{x}{2}$; consequently, $CC_{\mathcal{S}}$ is proper.
(ii) $\big(\forall x \in \mathcal{H} \smallsetminus (U \cup U^\perp)\big)$ $x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x, 0(x)$ are pairwise distinct.

(iii) $(\forall x \in \mathcal{H})$ $\operatorname{Id} x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x, 0(x)$ are affinely dependent.

Proof. (i): Let $x \in \mathcal{H}$. By Proposition 2.10(iii) and by $0 \in U$ and $\mathrm{P}_U x \in U$, we deduce that $\|\frac{x}{2} - \mathrm{P}_U \frac{x}{2}\|^2 + \|\mathrm{P}_U \frac{x}{2}\|^2 = \|\frac{x}{2}\|^2$ and that $\|\frac{x}{2} - \mathrm{P}_U \frac{x}{2}\|^2 + \|\mathrm{P}_U x - \mathrm{P}_U \frac{x}{2}\|^2 = \|\frac{x}{2} - \mathrm{P}_U x\|^2$. Combining with the linearity of P_U, we obtain

$$\Big\|\frac{x}{2}\Big\| = \Big\|\frac{x}{2} - \mathrm{P}_U x\Big\|. \tag{3.2}$$

Similarly, by Proposition 2.10(iii) again, replace U in the above analysis by $U^\perp$ to yield that

$$\Big\|\frac{x}{2}\Big\| = \Big\|\frac{x}{2} - \mathrm{P}_{U^\perp} x\Big\|. \tag{3.3}$$

Combining (3.2) with (3.3), we obtain that

$$\Big\|\frac{x}{2}\Big\| = \Big\|\frac{x}{2} - 0(x)\Big\| = \Big\|\frac{x}{2} - x\Big\| = \Big\|\frac{x}{2} - \mathrm{P}_U x\Big\| = \Big\|\frac{x}{2} - \mathrm{P}_{U^\perp} x\Big\|. \tag{3.4}$$

Since $\frac{x}{2} = \frac{x}{2} + \frac{0}{2} \in \operatorname{aff}\{x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x, 0(x)\}$, (3.4) yields that $(\forall x \in \mathcal{H})$ $CC_{\mathcal{S}} x = \frac{x}{2}$.

(ii): In fact, by Fact 2.9(ii),

$$x = \mathrm{P}_U x \Longleftrightarrow x \in U; \tag{3.5a}$$

$$x = \mathrm{P}_{U^\perp} x \Longleftrightarrow x \in U^\perp; \tag{3.5b}$$

$$U \cap U^\perp = \{0\}. \tag{3.5c}$$

In addition, Combining (3.5) with Fact 2.9(i), we know that

$$\mathrm{P}_U x = \mathrm{P}_{U^\perp} x \Longrightarrow \mathrm{P}_U x = \mathrm{P}_{U^\perp} x = 0 \Longrightarrow x = \mathrm{P}_U x + \mathrm{P}_{U^\perp} x = 0 \in U \cup U^\perp.$$

Hence, for every $x \in \mathcal{H} \smallsetminus (U \cup U^\perp)$, $x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x, 0(x)$ are pairwise distinct.

(iii): Now for every $x \in \mathcal{H}$,

$$\begin{aligned}
& x = \mathrm{P}_U x + \mathrm{P}_{U^\perp} x \\
\Rightarrow\ & x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x \text{ are linear dependent} \\
\Leftrightarrow\ & x - 0, \mathrm{P}_U x - 0, \mathrm{P}_{U^\perp} x - 0 \text{ are linear dependent} \\
\Leftrightarrow\ & 0(x), \operatorname{Id} x, \mathrm{P}_U x, \mathrm{P}_{U^\perp} x \text{ are affinely dependent.} \quad \text{(by Fact 2.7)}
\end{aligned}$$

The proof is complete. ☐

The following theorem provides a way to verify the properness of $CC_{\mathcal{S}}$ where $\mathcal{S}$ contains three operators.

Theorem 3.9. *Suppose that $\mathcal{S} = \{T_1, T_2, T_3\}$. Then $CC_{\mathcal{S}}$ is proper if and only if for every $x \in \mathcal{H}$ with $\operatorname{card}\big(\mathcal{S}(x)\big) = 3$, the vectors $T_1 x, T_2 x, T_3 x$ are affinely independent.*

Proof. By Fact 2.14, for every $x \in \mathcal{H}$ with $\operatorname{card}\big(\mathcal{S}(x)\big) = 3$,

$$CC_{\mathcal{S}} x \in \mathcal{H} \Longleftrightarrow T_1 x, T_2 x, T_3 x \text{ are affinely independent.} \tag{3.6}$$

"$\Longrightarrow$": It follows directly from (3.6).

"$\Longleftarrow$": Assume that for every $x \in \mathcal{H}$ with $\operatorname{card}\big(\mathcal{S}(x)\big) = 3$, $T_1 x, T_2 x, T_3 x$ are affinely independent in $\mathcal{H}$. Let $x \in \mathcal{H}$. If $\operatorname{card}\big(\mathcal{S}(x)\big) = 3$, by (3.6) and the assumption, then $CC_{\mathcal{S}}x \in \mathcal{H}$. Assume card $\big(\mathcal{S}(x)\big) \leq 2$, by Proposition 3.3, $CC_{\mathcal{S}}x \in \mathcal{H}$. Altogether, $(\forall x \in \mathcal{H})$, $CC_{\mathcal{S}}x \in \mathcal{H}$, which means that $CC_{\mathcal{S}}$ is proper. □

Proposition 3.10. *Suppose that $\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\}$. Then the following hold:*

(i) $\cap_{i=1}^m \operatorname{Fix} T_i \subseteq \operatorname{Fix} CC_{\mathcal{S}}$.
(ii) *If* $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \cup_{i=1}^m \operatorname{Fix} T_i$, *then* $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} T_i$.
(iii) *If* $T_1 = \operatorname{Id}$, *then* $\cap_{i=1}^m \operatorname{Fix} T_i = \operatorname{Fix} CC_{\mathcal{S}}$.

Proof. (i): Let $x \in \cap_{i=1}^m \operatorname{Fix} T_i$. Then

$$(\forall i \in \{1, \ldots, m-1, m\}) \quad T_i x = x, \tag{3.7}$$

which yields that $\operatorname{aff}\{T_1x, \ldots, T_{m-1}x, T_mx\} = \operatorname{aff}\{x\} = \{x\}$. In addition, by (3.7),

$$\|x - T_1x\| = \cdots = \|x - T_{m-1}x\| = \|x - T_mx\| = 0.$$

Therefore, we obtain that $CC_{\mathcal{S}}x = x$, which means that $x \in \operatorname{Fix} CC_{\mathcal{S}}$. Hence, $\cap_{i=1}^m \operatorname{Fix} T_i \subseteq \operatorname{Fix} CC_{\mathcal{S}}$.

(ii): Let $x \in \operatorname{Fix} CC_{\mathcal{S}}$. By the assumption, there is $i_0 \in \{1, \ldots, m\}$ such that

$$x = T_{i_0}x. \tag{3.8}$$

Now $x \in \operatorname{Fix} CC_{\mathcal{S}}$, i.e., $x = CC_{\mathcal{S}}x$, implies that

$$\|x - T_1x\| = \cdots = \|x - T_{m-1}x\| = \|x - T_mx\|. \tag{3.9}$$

Combining (3.9) with (3.8), we obtain that

$$\|x - T_1x\| = \cdots = \|x - T_{m-1}x\| = \|x - T_mx\| = 0,$$

which means that $x \in \cap_{i=1}^m \operatorname{Fix} T_i$. Hence, $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \cap_{i=1}^m \operatorname{Fix} T_i$. Combining with (i), we deduce that $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} T_i$.

(iii): If $T_1 = \operatorname{Id}$, then $\operatorname{Fix} T_1 = \mathcal{H}$ and the result follows from (ii). □

Example 3.11. Assume that $\mathcal{H} = \mathbb{R}^2$. Set $T_1 = \mathrm{P}_{\mathbf{B}[(-2,0);1]}$, $T_2 = \mathrm{P}_{\mathbf{B}[(0,2);1]}$, $T_3 = \mathrm{P}_{\mathbf{B}[(2,0);1]}$, and $\mathcal{S} = \{T_1, T_2, T_3\}$. Then $CC_{\mathcal{S}}$ is proper. Moreover, $\varnothing = \cap_{i=1}^3 \operatorname{Fix} T_i \subsetneqq \operatorname{Fix} CC_{\mathcal{S}} = \{(0,0)\}$.

Proof. The properness of $CC_{\mathcal{S}}$ follows from Theorem 3.9 while the rest is a consequence of elementary manipulations. □

3.3. Continuity.

Proposition 3.12. *Assume that the elements of $\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\}$ are continuous operators and that $x \in \operatorname{dom} CC_{\mathcal{S}}$. Then the following hold:*

(i) *Let* $\widetilde{\mathcal{S}}_x = \{T_1, T_{i_1}, \ldots, T_{i_{d_x}}\} \subseteq \mathcal{S}$ *be such that*[1] $T_{i_1}x - T_1x, \ldots, T_{i_{d_x}}x - T_1x$ *is a basis of* $\operatorname{span}\{T_2x - T_1x, \ldots, T_mx - T_1x\}$. *Then for every* $(x^{(k)})_{k\in\mathbb{N}} \subseteq \mathcal{H}$ *satisfying* $\lim_{k\to\infty} x^{(k)} = x$, *there exists* $N \in \mathbb{N}$ *such that for every* $k \geq N$, $CC_{\widetilde{\mathcal{S}}_x}(x^{(k)}) \in \mathcal{H}$. *Moreover*

$$\lim_{k\to\infty} CC_{\widetilde{\mathcal{S}}_x}(x^{(k)}) = CC_{\widetilde{\mathcal{S}}_x}x = CC_{\mathcal{S}}x. \tag{3.10}$$

(ii) *If* $T_1x, \ldots, T_{m-1}x, T_mx$ *are affinely independent, then* $CC_{\mathcal{S}}$ *is continuous at* x.

Proof. (i): Let $(x^{(k)})_{k\in\mathbb{N}} \subseteq \mathcal{H}$ satisfying $\lim_{k\to\infty} x^{(k)} = x$. Now

$$\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\}, \quad \mathcal{S}(x) = \{T_1(x), \ldots, T_{m-1}x, T_m x\},$$
$$\widetilde{\mathcal{S}}_x = \{T_1, T_{i_1}, \ldots, T_{i_{d_x}}\}, \quad \widetilde{\mathcal{S}}_x(x) = \{T_1 x, T_{i_1} x, \ldots, T_{i_{d_x}} x\},$$
$$\widetilde{\mathcal{S}}_x(x^{(k)}) = \{T_1 x^{(k)}, T_{i_1} x^{(k)}, \ldots, T_{i_{d_x}} x^{(k)}\}.$$

By Definition 3.1, $CC_{\mathcal{S}} x \in \mathcal{H}$ means that $CC(\mathcal{S}(x)) \in \mathcal{H}$. By assumptions,

$$T_{i_1}x - T_1 x, \ldots, T_{i_{d_x}} x - T_1 x \text{ is a basis of } \operatorname{span}\{T_2 x - T_1 x, \ldots, T_m x - T_1 x\}.$$

Substituting the K, $\widetilde{K}$ and $\widetilde{K}^{(k)}$ in Fact 2.20(i) by the above $\mathcal{S}(x)$, $\widetilde{\mathcal{S}}_x(x)$ and $\widetilde{\mathcal{S}}_x(x^{(k)})$ respectively, we obtain the desired results.

(ii): This follows easily from (i). ☐

The next result summarizes conditions under which the proper circumcenter mapping $CC_{\mathcal{S}}$ is continuous at a point x.

Proposition 3.13. *Assume that the elements of $\mathcal{S} = \{T_1, \ldots, T_{m-1}, T_m\}$ are continuous operators and that $CC_{\mathcal{S}}$ is proper. Let $x \in \mathcal{H}$. The following assertions hold:*

(i) *If $T_1 x, \ldots, T_{m-1}x, T_m x$ are affinely independent, then $CC_{\mathcal{S}}$ is continuous at x.*
(ii) *If $T_1 x, \ldots, T_{m-1}x, T_m x$ are affinely dependent and $m \leq 2$, then $CC_{\mathcal{S}}$ is continuous at x.*

Proof. (i) follows from Proposition 3.12(ii) while (ii) is a consequence of Proposition 3.3. ☐

The following examples show that even when $T_1 x, \ldots, T_{m-1}x, T_m x$ are affinely dependent and $m \geq 3$, then $CC_{\mathcal{S}}$ may still be continuous at x.

Example 3.14. Suppose that U is a closed linear subspace of $\mathcal{H}$ such that $\{0\} \subsetneqq U \subsetneqq \mathcal{H}$. Set $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_U, \mathrm{R}_{U^\perp}\}$. Then the following hold:

(i) The vectors $x, \mathrm{R}_U x, \mathrm{R}_{U^\perp} x$ are affinely dependent for every $x \in U \cup U^\perp$.
(ii) $CC_{\mathcal{S}} \equiv 0$ which is thus proper and continuous on $\mathcal{H}$.

Proof. (i): For every $x \in U$ (respectively $x \in U^\perp$), $\mathrm{R}_U x = x$ (respectively $\mathrm{R}_{U^\perp} x = x$), which implies that $x, \mathrm{R}_U x, \mathrm{R}_{U^\perp} x$, which is $x, x, \mathrm{R}_{U^\perp} x$ (respectively $x, \mathrm{R}_U x, x$) are affinely dependent.

(ii): Since $\mathrm{Id} = \mathrm{P}_U + \mathrm{P}_{U^\perp}$ and $\mathrm{R}_U = 2\,\mathrm{P}_U - \mathrm{Id}$, we have

$$\begin{aligned}\frac{\mathrm{R}_U + \mathrm{R}_{U^\perp}}{2} &= \frac{(2\,\mathrm{P}_U - \mathrm{Id}) + (2\,\mathrm{P}_{U^\perp} - \mathrm{Id})}{2} \\ &= \frac{1}{2}\Big(2\,\mathrm{P}_U - \mathrm{Id} + 2(\mathrm{Id} - \mathrm{P}_U) - \mathrm{Id}\Big) = 0.\end{aligned}$$

Let $x \in \mathcal{H}$. Then $0 = \frac{\mathrm{R}_U x + \mathrm{R}_{U^\perp} x}{2} \in \operatorname{aff}\{x, \mathrm{R}_U x, \mathrm{R}_{U^\perp} x\}$. In addition, clearly $0 \in U \cap U^\perp$. In Proposition 2.10(v), substitute $C = U$, and let the point $v = 0$. We

[1]When $\mathcal{S}(x)$ is a singleton, then $\widetilde{\mathcal{S}}_x = \{T_1\}$ by the standard convention that $\varnothing$ is the basis of $\{0\}$.

get $\|x\| = \|\mathrm{R}_U x\|$. Similarly, in Proposition 2.10(v), substitute $C = U^\perp$ and let the point $v = 0$. We get $\|x\| = \|\mathrm{R}_{U^\perp} x\|$. Hence, we have

(a) $0 \in \operatorname{aff}\{x, \mathrm{R}_U x, \mathrm{R}_{U^\perp} x\}$ and
(b) $\|0 - x\| = \|0 - \mathrm{R}_U x\| = \|0 - \mathrm{R}_{U^\perp} x\|$,

which means that $(\forall x \in \mathcal{H})\ CC_S(x) = 0$. □

Example 3.15. Assume that $\mathcal{H} = \mathbb{R}^2$ and $\mathcal{S} = \{T_1, T_2, T_3\}$, where for every $(x, y) \in \mathbb{R}^2$,

$$T_1(x, y) = (x, y); \quad T_2(x, y) = (-x, y); \quad T_3(x, y) = \big(x, -\tfrac{1}{4}(x - 2)\big).$$

Then

(i) $T_1(x, y), T_2(x, y), T_3(x, y)$ are affinely independent $\Leftrightarrow 2x\big(-\frac{1}{4}(x-2)-y\big) \neq 0$;
(ii) $\big(\forall (x, y) \in \mathbb{R}^2\big) \quad CC_{\mathcal{S}}(x, y) = \big(0, \frac{1}{2}\big(y - \frac{1}{4}(x - 2)\big)\big)$.

Consequently, $CC_{\mathcal{S}}$ is proper and continuous.

The following example shows that even if the operators in $\mathcal{S}$ are continuous, we generally have

$$CC_{\mathcal{S}} \text{ is proper} \not\Rightarrow CC_{\mathcal{S}} \text{ is continuous.}$$

Example 3.16. Assume that $\mathcal{H} = \mathbb{R}^2$ and $\mathcal{S} = \{T_1, T_2, T_3\}$, where for every $(x, y) \in \mathbb{R}^2$,

$$T_1(x, y) = (2, 0); \quad T_2(x, y) = (-2, 0); \quad T_3(x, y) = \big(x, -\tfrac{1}{4}(x - 2)\big).$$

Then

(i) $CC_{\mathcal{S}}$ is proper;
(ii) Let $(\forall k \geq 1)\ (x^{(k)}, y^{(k)}) = (2 - \frac{1}{k}, 0)$. Then $CC_{\mathcal{S}}(x^{(k)}, y^{(k)}) \to (0, -8) \neq (0, 0) = CC_{\mathcal{S}}(2, 0)$. Consequently, $CC_{\mathcal{S}}$ is not continuous at the point $(2, 0)$.

Proof. (i): Let $(x, y) \in \mathbb{R}^2$. Now by Fact 2.7,

$$\begin{aligned}
&T_1(x, y), T_2(x, y), T_3(x, y) \text{ are affinely independent}\\
\Longleftrightarrow\ &T_2(x, y) - T_1(x, y), T_3(x, y) - T_1(x, y) \text{ are linearly independent}\\
\Longleftrightarrow\ &(-4, 0), \big(x - 2, -\tfrac{1}{4}(x - 2)\big) \text{ are linearly independent}\\
\Longleftrightarrow\ &\det(A) \neq 0, \text{ where } A = \begin{pmatrix} -4 & x - 2 \\ 0 & -\frac{1}{4}(x - 2) \end{pmatrix}\\
\Longleftrightarrow\ &x - 2 \neq 0.
\end{aligned}$$

Hence, by Corollary 2.15, when $x - 2 \neq 0$, we have $CC_{\mathcal{S}}(x, y) \in \mathcal{H}$. Actually, when $x - 2 = 0$, that is $x = 2$, then for every $y \in \mathbb{R}$,

$$T_1(2, y) = (2, 0), T_2(2, y) = (-2, 0), T_3(2, y) = \big(2, -\tfrac{1}{4}(2 - 2)\big) = (2, 0).$$

By Proposition 3.3, we know that $CC_{\mathcal{S}}(x, y) = (0, 0) \in \mathcal{H}$. Hence, $CC_{\mathcal{S}}$ is proper.

(ii): Let $(\overline{x}, \overline{y}) = (2, 0)$, and $(\forall k \geq 1)\ (x^{(k)}, y^{(k)}) = (2 - \frac{1}{k}, 0)$. By the analysis in (i) above, we know

$$CC_{\mathcal{S}}(\overline{x}, \overline{y}) = (0, 0). \tag{3.11}$$

On the other hand, since

$$\begin{aligned}\mathcal{S}(x^{(k)}, y^{(k)}) &= \Big\{T_1(x^{(k)}, y^{(k)}), T_2(x^{(k)}, y^{(k)}), T_3(x^{(k)}, y^{(k)})\Big\}\\ &= \Big\{(2,0), (-2,0), \big(2-\tfrac{1}{k}, \tfrac{1}{4k}\big)\Big\},\end{aligned}$$

and since, by Definition 3.1, $CC_{\mathcal{S}}(x^{(k)}, y^{(k)}) = CC(\mathcal{S}(x^{(k)}, y^{(k)}))$, we deduce that, by Fact 2.21,

$$CC_{\mathcal{S}}(x^{(k)}, y^{(k)}) = \big(0, -8 + \tfrac{2}{k} + \tfrac{1}{8k}\big). \tag{3.12}$$

Hence,

$$\lim_{k\to\infty} CC_{\mathcal{S}}(x^{(k)}, y^{(k)}) = (0,-8) \neq (0,0) \overset{(3.11)}{=} CC_{\mathcal{S}}(2,0)$$

and we are done. ☐

3.4. **The Demiclosedness Principle for circumcenter mappings.** Let $T\colon \mathcal{H} \to \mathcal{H}$ be nonexpansive. Then

$$\left.\begin{array}{c} x_k \rightharpoonup x \\ x_k - Tx_k \to 0 \end{array}\right\} \;\Rightarrow\; x \in \operatorname{Fix} T. \tag{3.13}$$

This well known implication (see [8, Theorem 3(a)]) is *Browder's Demiclosedness Principle*; it is a powerful tool in the study of nonexpansive mappings. (Technically speaking, (3.13) states that $\operatorname{Id} - T$ is demiclosed at 0, but because a shift of a nonexpansive mapping is still nonexpansive, it is demiclosed everywhere.) For the sake of brevity, we shall simply say that

"the demiclosedness principle holds for T" whenever (3.13) holds.

Clearly, the demiclosedness principle holds whenever T is weak-to-strong continuous, e.g., when T is continuous and $\mathcal{H}$ is finite-dimensional. The demiclosedness principle also holds for so-called subgradient projectors; see [5, Lemma 5.1] for details.

We now obtain a condition sufficient for the circumcenter mapping to satisfy the demiclosedness principle. Throughout, we assume $T_1, \ldots, T_m$ are mappings from $\mathcal{H}$ to $\mathcal{H}$.

Theorem 3.17. *Suppose that the demiclosedness principle holds for each element in $\mathcal{S} = \{T_1, T_2, \ldots, T_m\}$. In addition, assume that $CC_{\mathcal{S}}$ is proper and that the implication*

$$x_k - CC_{\mathcal{S}}x_k \to 0 \;\Rightarrow\; (\forall i \in \{1,\ldots,m\})\; CC_{\mathcal{S}}x_k - T_i x_k \to 0 \tag{3.14}$$

holds. Then the demiclosedness principle holds for $CC_{\mathcal{S}}$ and $\operatorname{Fix} CC_{\mathcal{S}} = \bigcap_{i=1}^m \operatorname{Fix} T_i$.

Proof. Let $x_k \rightharpoonup x$ and

$$x_k - CC_{\mathcal{S}}x_k \to 0. \tag{3.15}$$

By (3.15) and (3.14),

$$(\forall i \in \{1,\ldots,m\}) \quad CC_{\mathcal{S}}x_k - T_i x_k \to 0.$$

Hence,

$$(\forall i \in \{1,\ldots,m\}) \quad \|x_k - T_i x_k\| \leq \|x_k - CC_{\mathcal{S}} x_k\| + \|CC_{\mathcal{S}} x_k - T_i x_k\| \to 0.$$

Because the demiclosedness principle holds for each T_i, we deduce that $x \in \cap_{i=1}^m \operatorname{Fix} T_i \subseteq \operatorname{Fix} CC_{\mathcal{S}}$, where the last inclusion follows from Proposition 3.10(i). Therefore, $x - CC_{\mathcal{S}} x = 0$, which shows that the demiclosedness principle holds for $CC_{\mathcal{S}}$. To verify the remaining assertion, let $\bar{x} \in \operatorname{Fix} CC_{\mathcal{S}}$. For every $k \in \mathbb{N}$, substitute x_k by $\bar{x}$. Then using the assumption (3.14), we deduce that $(\forall i \in \{1,\ldots,m\})$ $CC_{\mathcal{S}}\bar{x} - T_i \bar{x} = 0$. Combining with $(\forall i \in \{1,\ldots,m\})$ $\|\bar{x} - T_i\bar{x}\| \leq \|\bar{x} - CC_{\mathcal{S}}\bar{x}\| + \|CC_{\mathcal{S}}\bar{x} - T_i\bar{x}\|$, we obtain that $\bar{x} \in \cap_{i=1}^m \operatorname{Fix} T_i$. Hence, $\operatorname{Fix} CC_{\mathcal{S}} \subseteq \cap_{i=1}^m \operatorname{Fix} T_i$. Therefore, the desired result follows from Proposition 3.10(i). ☐

Corollary 3.18. *Suppose that $T_1 = \operatorname{Id}$ and that $CC_{\mathcal{S}}$ is proper. Then the implication*

$$x_k - CC_{\mathcal{S}} x_k \to 0 \;\Rightarrow\; (\forall i \in \{1,\ldots,m\})\; CC_{\mathcal{S}} x_k - T_i x_k \to 0. \tag{3.16}$$

holds.

Proof. Since $CC_{\mathcal{S}}$ is proper, by Remark 3.2, $\|CC_{\mathcal{S}} x_k - x_k\| = \|CC_{\mathcal{S}} x_k - T_2 x_k\| = \cdots = \|CC_{\mathcal{S}} x_k - T_m x_k\|$, which implies that (3.16) is true. ☐

Proposition 3.19. *Suppose that $T_1 = \operatorname{Id}$, that for every $i \in \{2,\ldots,m\}$, the demiclosedness principle holds for T_i, that $\mathcal{S} = \{T_1, T_2, \ldots, T_m\}$, and that $CC_{\mathcal{S}}$ is proper. Then the demiclosedness principle holds for $CC_{\mathcal{S}}$ and $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} T_i$.*

Proof. Combine Theorem 3.17 with Corollary 3.18. ☐

We are now ready for the main result of this section.

Theorem 3.20 (a demiclosedness principle for circumcenter mappings)**.** *Suppose that $T_1 = \operatorname{Id}$, that each operator in $\mathcal{S} = \{T_1, T_2, \ldots, T_m\}$ is nonexpansive, and that $CC_{\mathcal{S}}$ is proper. Then the demiclosedness principle holds for $CC_{\mathcal{S}}$ and $\operatorname{Fix} CC_{\mathcal{S}} = \cap_{i=1}^m \operatorname{Fix} T_i$.*

Proof. Combine Browder's Demiclosedness Principle with Proposition 3.19. ☐

We now present (omitting its easy proof) another consequence of Proposition 3.19.

Corollary 3.21. *Suppose $\mathcal{H}$ is finite-dimensional, and $\mathcal{S} = \{T_1, \ldots, T_m\}$, where $T_1 = \operatorname{Id}$ and T_j is continuous for every $j \in \{2,\ldots,m\}$, and that $CC_{\mathcal{S}}$ is proper. Then the demiclosedness principle holds for $CC_{\mathcal{S}}$. In particular,*

$$\left.\begin{array}{c} x_k \to \overline{x} \\ x_k - CC_{\mathcal{S}} x_k \to 0 \end{array}\right\} \;\Rightarrow\; \overline{x} \in \cap_{j=1}^m \operatorname{Fix} T_j = \operatorname{Fix} CC_{\mathcal{S}}. \tag{3.17}$$

We now provide an example where the demiclosedness principle does not hold for $CC_{\mathcal{S}}$.

Example 3.22. Suppose that $\mathcal{H} = \mathbb{R}^2$. Set $L = \big\{(u,v) \in \mathcal{H} \;\big|\; v = -\frac{1}{4}u + \frac{1}{2}\big\}$. Assume that $\mathcal{S} = \{T_1, T_2, T_3\}$, where

$$\big(\forall (u,v) \in \mathcal{H}\big) \quad T_1(u,v) = (-2,0),\; T_2(u,v) = (2,0) \text{ and } T_3(u,v) = \mathrm{P}_L(u,v).$$

Set $\overline{x} = (0,-8)$ and $(\forall k \in \mathbb{N} \smallsetminus \{0\})$ $x_k = \big(\frac{1}{k}, -\frac{1}{4k} - 8\big)$. Then the following hold.

(i) $CC_{\mathcal{S}}$ is proper.
(ii) $\operatorname{Fix} CC_{\mathcal{S}} = \varnothing$.
(iii) $\lim_{k\to\infty} CC_{\mathcal{S}} x_k = \overline{x} = \lim_{k\to\infty} x_k$; consequently, $\lim_{k\to\infty} (x_k - CC_{\mathcal{S}} x_k) = 0$. (See also Figure 1.)
(iv) $\overline{x} \notin \operatorname{Fix} CC_{\mathcal{S}}$; consequently, the demiclosedness principle does not hold for $CC_{\mathcal{S}}$.

Proof. (i): Let $x \in \mathcal{H}$. If $T_3 x \in \mathbb{R} \cdot (1,0)$, then $T_3 x = (2,0)$ and so $CC_{\mathcal{S}} x = (0,0)$. Now assume that $T_3 x \notin \mathbb{R} \cdot (1,0)$. Then $T_1 x, T_2 x, T_3 x$ are affinely independent. Hence, by Theorem 3.9, $CC_{\mathcal{S}} x \in \mathcal{H}$. Altogether, $CC_{\mathcal{S}}$ is proper.

(ii): Since $T_1 x = (-2,0)$ and $T_2 x = (2,0)$, by definition of circumcenter mapping,

$$CC_{\mathcal{S}} x \in \mathbb{R} \cdot (0,1),$$

which implies if $x \in \operatorname{Fix} CC_{\mathcal{S}}$, then $x \in \mathbb{R} \cdot (0,1)$. Since $T_3(0,-8) = \mathrm{P}_L(0,-8) = (2,0)$, by Proposition 3.3, $CC_{\mathcal{S}}(0,-8) = (0,0) \neq (0,-8)$. Hence, $(0,-8) \notin \operatorname{Fix} CC_{\mathcal{S}}$. Let $x := (0,v) \in \big(\mathbb{R} \cdot (0,1)\big) \smallsetminus \{(0,-8)\}$. As seen in the proof of (i), the vectors $T_1 x, T_2 x, T_3 x$ are affinely independent. Hence, by definition of circumcenter mapping, in this case $CC_{\mathcal{S}} x$ is the intersection of $\mathbb{R} \cdot (0,1)$ and the perpendicular bisector of the two points $T_2 x, T_3 x$. Denote by $CC_{\mathcal{S}} x := (0,w)$. Some easy calculation yields that if $v > -8$, then $w > v$; if $v < -8$, then $w < v$, which means that $CC_{\mathcal{S}} x \neq x$. Altogether, $\operatorname{Fix} CC_{\mathcal{S}} = \varnothing$.

(iii): Let $k \in \mathbb{N} \smallsetminus \{0\}$. Since $x_k = \big(\frac{1}{k}, -\frac{1}{4k} - 8\big)$, by definition of T_3,

$$T_3 x_k = \big(2 - \tfrac{1}{k}, \tfrac{1}{4k}\big).$$

Hence

$$CC_{\mathcal{S}} x_k = CC\big\{(-2,0), (2,0), \big(2 - \tfrac{1}{k}, \tfrac{1}{4k}\big)\big\}.$$

By Example 3.16(ii), we obtain that

$$\lim_{k\to\infty} CC_{\mathcal{S}} x_k = (0,-8) = \overline{x} = \lim_{k\to\infty} x_k.$$

(iv): By (ii), $\overline{x} \notin \operatorname{Fix} CC_{\mathcal{S}}$. Therefore, the demiclosedness principle does not hold for $CC_{\mathcal{S}}$. ☐

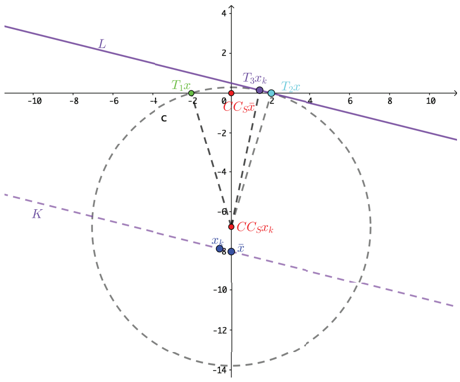

FIGURE 1. Example 3.22 illustrates that the demiclosedness principle may not hold for $CC_{\mathcal{S}}$.

Remark 3.23. Consider Example 3.22 where each T_i is a projector and thus firmly nonexpansive but $\operatorname{Fix} CC_{\mathcal{S}} = \varnothing$. Is it possible to obtain an example where the demiclosedness principle does not hold but yet $\operatorname{Fix} CC_{\mathcal{S}} \neq \varnothing$? We do not know the answer to this question.

4. Circumcenter mappings induced by reflectors

Recall that $m \in \mathbb{N} \smallsetminus \{0\}$ and that $U_1, \ldots, U_m$ are closed affine subspaces in the real Hilbert space $\mathcal{H}$ with $\cap_{i=1}^m U_i \neq \varnothing$. In the whole section, denote

$$\Omega = \Big\{ \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \ \Big| \ r \in \mathbb{N}, \text{ and } i_1, i_2, \ldots, i_r \in \{1, \ldots, m\} \Big\}.$$

By the empty product convention, $\prod_{j=1}^0 \mathrm{R}_{U_{i_j}} = \mathrm{Id}$. Hence, Ω is a set that contains the identity operator, Id, and all (finite) compositions of the operators in $\{\mathrm{R}_{U_1}, \ldots, R_{U_m}\}$.

4.1. **Proper circumcenter mappings induced by reflectors.** Let us start with a useful lemma.

Lemma 4.1. *Assume that* $\mathrm{Id} \in \mathcal{S} \subseteq \Omega$. *Let* $x \in \mathcal{H}$. *Then for every* $\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \in \mathcal{S}$,

$$(\forall u \in \cap_{i=1}^m U_i) \quad \|x - u\| = \|\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\|.$$

Proof. Let $u \in \cap_{i=1}^m U_i$. Because $U_1, \ldots, U_m$ are closed affine subspaces and $u \in \cap_{i=1}^m U_i \subseteq \cap_{j=1}^r U_{i_j}$, by Proposition 2.10(v), we have

$$\begin{aligned}
\|x - u\| &= \|\mathrm{R}_{U_{i_1}} x - u\| \\
\|\mathrm{R}_{U_{i_1}} x - u\| &= \|\mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\| \\
&\ \, \vdots \\
\|\mathrm{R}_{U_{i_{r-1}}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\| &= \|\mathrm{R}_{U_{i_r}} \mathrm{R}_{U_{i_{r-1}}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\|,
\end{aligned}$$

which yield

$$\|x - u\| = \|\mathrm{R}_{U_{i_r}} \mathrm{R}_{U_{i_{r-1}}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\|.$$

□

Proposition 4.2. *Assume that* $\mathrm{Id} \in \mathcal{S} \subseteq \Omega$. *Let* $x \in \mathcal{H}$. *Then for every* $u \in \cap_{i=1}^m U_i$,

(i) $\mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u) \in \mathrm{aff}(\mathcal{S}(x))$, *and*

(ii) *for every* $\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \in \mathcal{S}$,

$$\|\mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u) - x\| = \|\mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u) - \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x\|.$$

Proof. (i): Let $u \in \cap_{i=1}^m U_i$. Because $\mathrm{aff}(\mathcal{S}(x))$ is the translate of a finite-dimensional linear subspace, $\mathrm{aff}(\mathcal{S}(x))$ is a closed affine subspace. Hence, we know $\mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u)$ is well-defined. Clearly, $\mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u) \in \mathrm{aff}(\mathcal{S}(x))$, i.e., (i) is true.

(ii): Take $\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}}$ in $\mathcal{S}$. Since $\mathrm{Id}, \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \in \mathcal{S}$, we know $x, \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x \in \mathcal{S}(x) \subseteq \mathrm{aff}(\mathcal{S}(x))$. Denote $p = \mathrm{P}_{\mathrm{aff}(\mathcal{S}(x))}(u)$. Substituting $C = \mathrm{aff}(\mathcal{S}(x))$, $x = u$ and $v = x$ in Proposition 2.10(iii), we deduce

$$\|u - p\|^2 + \|x - p\|^2 = \|u - x\|^2. \tag{4.1}$$

Similarly, substitute $C = \operatorname{aff}(\mathcal{S}(x))$, $x = u$ and $v = \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x$ in Proposition 2.10(iii) to obtain

$$\|u-p\|^2 + \|\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - p\|^2 = \|u - \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x\|^2. \tag{4.2}$$

On the other hand, by Lemma 4.1, we know

$$\|x-u\| = \|\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x - u\|. \tag{4.3}$$

Combining (4.3) with (4.1) and (4.2), we yield

$$\|p-x\| = \|p - \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} x\|.$$

Since $\mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \in \mathcal{S}$ is arbitrary, thus (ii) holds. □

Combining Proposition 3.6 with Proposition 4.2, we deduce the theorem below which is one of the main results in this paper.

Theorem 4.3. *Assume that* $\mathrm{Id} \in \mathcal{S} \subseteq \Omega$. *Then the following hold:*

(i) *The circumcenter mapping* $CC_{\mathcal{S}} : \mathcal{H} \to \mathcal{H}$ *induced by* $\mathcal{S}$ *is* proper, *i.e., for every* $x \in \mathcal{H}$, $CC_{\mathcal{S}}x$ *is the unique point satisfying the two conditions below:*
  (a) $CC_{\mathcal{S}}x \in \operatorname{aff}(\mathcal{S}(x))$, *and*
  (b) $(\forall\, \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_1}} \in \mathcal{S})\ \|CC_{\mathcal{S}}x - x\| = \|CC_{\mathcal{S}}x - \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_1}} x\|$.

(ii) $(\forall x \in \mathcal{H})\ (\forall u \in \cap_{i=1}^m U_i)\ CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(u)$.

(iii) $(\forall x \in \mathcal{H})\ CC_{\mathcal{S}}x = \mathrm{P}_{\operatorname{aff}(\mathcal{S}(x))}(\mathrm{P}_{\cap_{i=1}^m U_i} x)$.

Proof. (i) and (ii): The required results follow from Propositions 3.6 and 4.2.
(iii): Since $\mathrm{P}_{\cap_{i=1}^m U_i} x \in \cap_{i=1}^m U_i$, the desired result comes from (ii). □

We now list several proper circumcenter mappings induced by reflectors; the properness of some of these mappings is derived from Theorem 4.3.

Example 4.4. Assume that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \ldots, \mathrm{R}_{U_m}\}$. By Theorem 4.3(i), $CC_{\mathcal{S}}$ is proper.

Example 4.5. Assume that

$$\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \mathrm{R}_{U_3}\mathrm{R}_{U_2}, \ldots, \mathrm{R}_{U_m}\mathrm{R}_{U_{m-1}}, \mathrm{R}_{U_1}\mathrm{R}_{U_m}\}.$$

By Theorem 4.3(i), $CC_{\mathcal{S}}$ is proper.

Example 4.6 (Behling et al. [6])**.** Assume that $m = 2$ and that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Then, by Theorem 4.3(i), $CC_{\mathcal{S}}$ is proper.

Example 4.7 (Behling et al. [7])**.** Assume that

$$\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \ldots, \mathrm{R}_{U_m} \cdots \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}.$$

Then, by Theorem 4.3(i), $CC_{\mathcal{S}}$ is proper.

Remark 4.8. In fact, the C–DRM operator C_T defined in [6, Section 2] is the $CC_{\mathcal{S}}$ operator of Example 4.6 while the CRM operator C defined in [7, page 159] is the operator $CC_{\mathcal{S}}$ from Example 4.7.

Example 4.9. Assume that $m = 2$ and that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. By Proposition 3.3,

$$CC_{\mathcal{S}} = \frac{\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1}}{2},$$

which is the well-known Douglas–Rachford splitting operator. Clearly, $CC_{\mathcal{S}}$ is proper.

Example 4.10. Assume that $m = 2$ and that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$. Then $CC_{\mathcal{S}}$ is proper. Moreover,

$$(\forall x \in U_1) \quad CC_{\mathcal{S}}x = \mathrm{P}_{U_2}x \quad \text{and} \quad (\forall x \in U_2) \quad CC_{\mathcal{S}}x = \mathrm{P}_{U_1}x.$$

Proof. The first assertion follows from Example 4.4. As for the remaining ones, note that

$$(\forall x \in U_1)\ \mathcal{S}(x) = \{x, \mathrm{R}_{U_2}x\} \text{ and } (\forall x \in U_2)\ \mathcal{S}(x) = \{x, \mathrm{R}_{U_1}x\}. \tag{4.4}$$

Combining (4.4) with Proposition 3.3, we obtain that

$$(\forall x \in U_1) \quad CC_{\mathcal{S}}x = \frac{x + \mathrm{R}_{U_2}x}{2} = \mathrm{P}_{U_2}x \quad \text{and}$$
$$(\forall x \in U_2) \quad CC_{\mathcal{S}}x = \frac{x + \mathrm{R}_{U_1}x}{2} = \mathrm{P}_{U_1}x.$$

The proof is complete. □

Example 4.11. Assume that $m = 2$ and that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Let $x \in \mathcal{H}$ and set $l = \mathrm{card}\{x, \mathrm{R}_{U_1}x, \mathrm{R}_{U_2}x, \mathrm{R}_{U_2}\mathrm{R}_{U_1}x\}$. Then exactly one of the following cases occurs.

(i) $l = 1$ and $CC_{\mathcal{S}}x = x$.

(ii) $l = 2$, say $S(x) = \{x_1, x_2\}$, where x_1 and x_2 are two distinct elements in $S(x)$, and $CC_{\mathcal{S}}x = \frac{x_1+x_2}{2}$.

(iii) $l = 3$, say $S(x) = \{x_1, x_2, x_3\}$, where x_1, x_2, x_3 are pairwise distinct elements in $S(x)$, and $CC_{\mathcal{S}}x = \frac{N}{D}$ where $N := \|x_2 - x_3\|^2\langle x_1 - x_3, x_1 - x_2\rangle x_1 + \|x_1 - x_3\|^2\langle x_2 - x_3, x_2 - x_1\rangle x_2 + \|x_1 - x_2\|^2\langle x_3 - x_1, x_3 - x_2\rangle x_3$ and $D := 2(\|x_2 - x_1\|^2\|x_3 - x_1\|^2 - \langle x_2 - x_1, x_3 - x_1\rangle^2)$.

(iv) $l = 4$ and

$$CC_{\mathcal{S}}x = x_1 + \frac{1}{2}\mathbf{u}\,G(x_2 - x_1, \ldots, x_{t_x-1} - x_1, x_{t_x} - x_1)^{-1}\mathbf{v}$$

where $\{x_1, \ldots, x_{t_x}\} \subseteq S(x)$ is such that $\dim(\mathrm{aff}\{x_1, x_2, \ldots, x_{t_x}\}) = \dim(\mathrm{aff}\, S(x))$ and $x_1, x_2, \ldots, x_{t_x}$ are affinely independent, and $\mathbf{u} := (x_2 - x_1, \ldots, x_{t_x} - x_1)$, $\mathbf{v} := (\|x_2 - x_1\|^2, \ldots, \|x_{t_x} - x_1\|^2)^T$ and

$$G(x_2 - x_1, \ldots, x_{t_x-1} - x_1, x_{t_x} - x_1)$$
$$= \begin{pmatrix} \|x_2 - x_1\|^2 & \langle x_2 - x_1, x_3 - x_1\rangle & \cdots & \langle x_2 - x_1, x_{t_x} - x_1\rangle \\ \vdots & \vdots & & \vdots \\ \langle x_{t_x-1} - x_1, x_2 - x_1\rangle & \langle x_{t_x-1} - x_1, x_3 - x_1\rangle & \cdots & \langle x_{t_x-1} - x_1, x_{t_x} - x_1\rangle \\ \langle x_{t_x} - x_1, x_2 - x_1\rangle & \langle x_{t_x} - x_1, x_3 - x_1\rangle & \cdots & \|x_{t_x} - x_1\|^2 \end{pmatrix}.$$

Proof. By Theorem 4.3(i), $CC_{\mathcal{S}}$ is proper. The rest follows from Facts 2.12 and 2.13. □

We now turn to the properness of $CC_{\widetilde{\mathcal{S}}}$ when $\mathrm{Id} \in \widetilde{\mathcal{S}} \subseteq \operatorname{aff}\Omega$.

Proposition 4.12. *Let $\alpha \in \mathbb{R}$. Assume that*

$$\widetilde{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha)\,\mathrm{Id} + \alpha\,\mathrm{R}_{U_1}, \ldots, (1-\alpha)\,\mathrm{Id} + \alpha\,\mathrm{R}_{U_m}\}, \tag{4.5}$$

and that

$$\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \ldots, \mathrm{R}_{U_m}\}. \tag{4.6}$$

Then $CC_{\widetilde{\mathcal{S}}}$ is proper. Moreover,

$$(\forall x \in \mathcal{H}) \quad CC_{\widetilde{\mathcal{S}}}x = \alpha CC_{\mathcal{S}}x + (1-\alpha)x \in \mathcal{H}. \tag{4.7}$$

Proof. If $\alpha = 0$, then $\widetilde{\mathcal{S}} = \{\mathrm{Id}\}$, by Definition 3.1,

$$(\forall x \in \mathcal{H}) \quad CC_{\widetilde{\mathcal{S}}}x = x = 0CC_{\mathcal{S}}x + (1-0)x \in \mathcal{H}.$$

Now assume $\alpha \neq 0$. Let $x \in \mathcal{H}$. For every $i \in \{1, \ldots, m\}$, thus

$$\begin{aligned}
&CC_{\widetilde{\mathcal{S}}}x \\
&= CC(\widetilde{\mathcal{S}}(x)) \quad \text{(by Definition 3.1)} \\
&= CC\Big(\big\{x, (1-\alpha)x + \alpha\,\mathrm{R}_{U_1} x, \ldots, (1-\alpha)x + \alpha\,\mathrm{R}_{U_m} x\big\}\Big) \quad \text{(by (4.5))} \\
&= CC\Big(\big\{0, \alpha(\mathrm{R}_{U_1} x - x), \ldots, \alpha(\mathrm{R}_{U_m} x - x)\big\} + x\Big) \\
&= CC\Big(\big\{0, \alpha(\mathrm{R}_{U_1} x - x), \ldots, \alpha(\mathrm{R}_{U_m} x - x)\big\}\Big) + x \quad \text{(by Fact 2.18)} \\
&= \alpha CC\Big(\big\{0, \mathrm{R}_{U_1} x - x, \ldots, \mathrm{R}_{U_m} x - x\big\}\Big) + x \quad \text{(by Fact 2.17 and } \alpha \neq 0) \\
&= \alpha CC\Big(\big\{x, \mathrm{R}_{U_1} x, \ldots, \mathrm{R}_{U_m} x\big\} - x\Big) + x \\
&= \alpha CC\Big(\big\{x, \mathrm{R}_{U_1} x, \ldots, \mathrm{R}_{U_m} x\big\}\Big) - \alpha x + x \quad \text{(by Fact 2.18)} \\
&= \alpha CC\big(\mathcal{S}(x)\big) + (1-\alpha)x \quad \text{(by (4.6))} \\
&= \alpha CC_{\mathcal{S}}x + (1-\alpha)x \in \mathcal{H}. \quad \text{(by Definition 3.1 and Theorem 4.3(i))}
\end{aligned}$$

The proof is complete. ☐

Proposition 4.13. *Assume that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$, set $T = \frac{\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1}}{2}$, which is the Douglas–Rachford splitting operator, and set $\widetilde{\mathcal{S}} = \{\mathrm{Id}, T, T^2\}$. Then the following hold:*

(i) $\operatorname{aff}\{\mathrm{Id}, T, T^2\} = \operatorname{aff}\mathcal{S}$.
(ii) *$CC_{\widetilde{\mathcal{S}}}$ is proper.*

Proof. (i): By Fact 2.5, $\operatorname{aff}\{\mathrm{Id}, T, T^2\} = \operatorname{aff}(\mathcal{S})$ if and only if $\mathrm{Id} + \operatorname{span}\{T - \mathrm{Id}, T^2 - \mathrm{Id}\} = \mathrm{Id} + \operatorname{span}\{\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}\}$. On the other hand,

$$T - \mathrm{Id} = \frac{\mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{Id}}{2} - \mathrm{Id} = \frac{\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}}{2}, \tag{4.8}$$

and

$$(4.9)\qquad \begin{aligned} T^2 - \mathrm{Id} &= T^2 - T + T - \mathrm{Id} = (T-\mathrm{Id})T + (T - \mathrm{Id}) \\ &= \frac{\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}}{2}\Big(\frac{\mathrm{R}_{U_2}\mathrm{R}_{U_1} + \mathrm{Id}}{2}\Big) + \frac{\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}}{2} \\ &= \frac{1}{4}(\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}) + \frac{1}{2}(\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}), \end{aligned}$$

which result in

$$(4.10)\qquad \Big(T-\mathrm{Id},\quad T^2-\mathrm{Id}\Big) = \Big(\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id},\quad \mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}\Big)\begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}.$$

Set $A = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{4} \end{pmatrix}$. Since $\det(A) = \frac{1}{8} \neq 0$, (4.10) yields

$$\mathrm{span}\{T-\mathrm{Id}, T^2-\mathrm{Id}\} = \mathrm{span}\{\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1} - \mathrm{Id}\}.$$

Altogether, the proof of (i) is complete.

(ii): If x, Tx, T^2x are affinely independent, by Fact 2.13, then $CC_{\widetilde{\mathcal{S}}}x \in \mathcal{H}$. Suppose x, Tx, T^2x are affinely dependent. By (4.10) above and $\det(A) \neq 0$, in this case, $x, \mathrm{R}_{U_2}\mathrm{R}_{U_1}x, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x$ are affinely dependent. Applying Theorem 4.3(i), we know $CC_{\mathcal{S}}x \in \mathcal{H}$. Hence, Fact 2.14 yields that

$$(4.11)\qquad \mathrm{card}\Big(\{x, \mathrm{R}_{U_2}\mathrm{R}_{U_1}, x\,\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x\}\Big) = \mathrm{card}\big(\mathcal{S}(x)\big) \leq 2.$$

If $Tx - x = 0$, by Proposition 3.3, $CC_{\widetilde{\mathcal{S}}}x = \frac{x+T^2x}{2}$. Now suppose $Tx - x \neq 0$. By (4.8), $\mathrm{R}_{U_2}\mathrm{R}_{U_1}x \neq x$. Therefore, by (4.11) and $\mathrm{R}_{U_2}\mathrm{R}_{U_1}x \neq x$, either $\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x = \mathrm{R}_{U_2}\mathrm{R}_{U_1}x$ or $\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x = x$. Suppose $\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x = \mathrm{R}_{U_2}\mathrm{R}_{U_1}x$. Multiply both sides by $\mathrm{R}_{U_1}\mathrm{R}_{U_2}$, by Lemma 2.8(ii), to deduce $\mathrm{R}_{U_2}\mathrm{R}_{U_1}x = x$, which contradicts with $\mathrm{R}_{U_2}\mathrm{R}_{U_1}x \neq x$. Suppose $\mathrm{R}_{U_2}\mathrm{R}_{U_1}\mathrm{R}_{U_2}\mathrm{R}_{U_1}x = x$, by (4.8) and (4.9), which implies, $Tx = T^2x$. Then by Proposition 3.3, we obtain $CC_{\widetilde{\mathcal{S}}}x = \frac{x+Tx}{2} \in \mathcal{H}$.

In conclusion, $(\forall x \in \mathcal{H})$ $CC_{\widetilde{\mathcal{S}}}x \in \mathcal{H}$, which means (ii) holds. ☐

4.2. **Improper circumcenter mappings induced by reflectors.**

Propositions 4.12 and 4.13 naturally prompt the following question: Is $CC_{\widetilde{\mathcal{S}}}$ proper for every $\widetilde{S}$ with $\mathrm{Id} \in \widetilde{S} \subseteq \mathrm{aff}\,\Omega$? The following examples provide negative answers.

Example 4.14. Assume that $m = 2$, that $U := U_1 = U_2 \subsetneqq \mathcal{H}$, and that $\{\alpha_1, \alpha_2\} \subseteq \mathbb{R}$. Assume further that $\widetilde{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1\mathrm{R}_U, (1-\alpha_2)\mathrm{Id} + \alpha_2\mathrm{R}_U\}$. Then $CC_{\widetilde{\mathcal{S}}}$ is improper if and only if $\alpha_1 \neq 0, \alpha_2 \neq 0$ and $\alpha_2 \neq \alpha_1$.

Proof. By Proposition 3.3, when $\alpha_1 = 0$ or $\alpha_2 = 0$, then $CC_{\widetilde{\mathcal{S}}}$ is proper.

For every $x \in \mathcal{H}$, if $\alpha_1 \neq 0$,

$$\begin{aligned} \big((1-\alpha_2)x + \alpha_2\mathrm{R}_U x\big) - x &= \alpha_2(\mathrm{R}_U x - x) \\ &= \frac{\alpha_2}{\alpha_1}\alpha_1(\mathrm{R}_U x - x) \\ &= \frac{\alpha_2}{\alpha_1}\Big(\big((1-\alpha_1)x + \alpha_1\mathrm{R}_U x\big) - x\Big), \end{aligned}$$

which implies that, by Fact 2.7,

$$x, (1-\alpha_1)x + \alpha_1 \mathrm{R}_U x, (1-\alpha_2)x + \alpha_2 \mathrm{R}_U x \quad \text{are affinely dependent.}$$

On the other hand, if $x \in \mathcal{H} \smallsetminus U$, then since $\alpha_1 \neq 0$, $\alpha_2 \neq 0$, $\alpha_1 \neq \alpha_2$ and $\operatorname{Fix} \mathrm{R}_U = U$, we obtain that

$$\begin{aligned} &\operatorname{card}\big\{x, (1-\alpha_1)x + \alpha_1 \mathrm{R}_U x, (1-\alpha_2)x + \alpha_2 \mathrm{R}_U x\big\} = 3 \\ \iff &\alpha_1 \neq 0, \alpha_2 \neq 0 \text{ and } \alpha_2 \neq \alpha_1. \end{aligned}$$

Therefore, we deduce the required result. ☐

Example 4.15. Assume that $m = 2$, that $U := U_1 = U_2 \subsetneqq \mathcal{H}$, and that $\{\alpha_1, \alpha_2\} \subseteq \mathbb{R}$. Assume further that $\widetilde{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U\big)\}$. Then $CC_{\widetilde{\mathcal{S}}}$ is improper if and only if $\alpha_1 \neq 0, \alpha_1 \neq \frac{1}{2}, \alpha_2 \neq 0$ and $\alpha_2 \neq \frac{\alpha_1}{2\alpha_1 - 1}$.

Proof. By Proposition 3.3, when $\alpha_1 = 0$ or $\alpha_2 = 0$, then $CC_{\widetilde{\mathcal{S}}}$ is proper.

Note that

$$\begin{aligned} &\big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U\big) \\ &= (1 - \alpha_1 - \alpha_2 + 2\alpha_2\alpha_1)\mathrm{Id} + (\alpha_1 + \alpha_2 - 2\alpha_2\alpha_1)\mathrm{R}_U \quad \text{(by Lemma 2.8(ii)).} \end{aligned}$$

Hence, for every $x \in \mathcal{H}$, if $\alpha_1 \neq 0$,

$$\begin{aligned} &\Big(\big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U\big)x\Big) - x \\ &= (\alpha_1 + \alpha_2 - 2\alpha_2\alpha_1)(\mathrm{R}_U x - x) \\ &= \frac{\alpha_1 + \alpha_2 - 2\alpha_2\alpha_1}{\alpha_1} \alpha_1 (\mathrm{R}_U x - x) \\ &= \frac{\alpha_1 + \alpha_2 - 2\alpha_2\alpha_1}{\alpha_1} \Big(\big((1-\alpha_1)x + \alpha_1 \mathrm{R}_U x\big) - x\Big), \end{aligned}$$

which implies, by Fact 2.7, that $x, (1-\alpha_1)x + \alpha_1 \mathrm{R}_U x, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U\big)x$ are affinely dependent. On the other hand, assume now $x \in \mathcal{H} \smallsetminus U$. Then $\operatorname{card}\Big\{x, (1-\alpha_1)x + \alpha_1 \mathrm{R}_U x, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_U\big)x\Big\} = 3$ if and only if $\alpha_1 \neq 0, \alpha_1 \neq \frac{1}{2}, \alpha_2 \neq 0$ and $\alpha_2 \neq \frac{\alpha_1}{2\alpha_1 - 1}$. Therefore, combining Corollary 3.4 with the results obtained above, we infer the desired result. ☐

The following example is a special case of Example 4.15.

Example 4.16. Assume that $m = 2$, that $U_1 \subsetneqq U_2 = \mathcal{H}$, and that $\{\alpha_1, \alpha_2\} \subseteq \mathbb{R}$. Assume further that $\widetilde{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_{U_2}\mathrm{R}_{U_1}, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_{U_2}\mathrm{R}_{U_1}\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_{U_2}\mathrm{R}_{U_1}\big)\}$. Then $CC_{\widetilde{\mathcal{S}}}$ is improper if and only if $\alpha_1 \neq 0, \alpha_1 \neq \frac{1}{2}, \alpha_2 \neq 0$ and $\alpha_2 \neq \frac{\alpha_1}{2\alpha_1 - 1}$.

Proof. Since $\mathrm{R}_{U_2} = \mathrm{R}_{\mathcal{H}} = \mathrm{Id}$, we deduce that

$$\widetilde{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_{U_1}, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{R}_{U_1}\big)\big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{R}_{U_1}\big)\}.$$

The desired result follows directly from Example 4.15. ☐

Notice that in Proposition 4.13 we showed that for $\widetilde{\mathcal{S}} = \{\mathrm{Id}, T, T^2\} = \{\mathrm{Id}, \frac{\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1}}{2}, \frac{\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1}}{2} \circ \frac{\mathrm{Id} + \mathrm{R}_{U_2}\mathrm{R}_{U_1}}{2}\}$, $CC_{\widetilde{\mathcal{S}}}$ is proper. The example above says that this result is not a conincidence.

4.3. **Particular circumcenter mappings in Euclidean spaces.**

4.3.1. *Application to best approximation.* Suppose that

$$\mathcal{S}_1 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\} \quad \text{and} \quad \mathcal{S}_2 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}.$$

By Example 4.4 and Example 4.6, we know $CC_{\mathcal{S}_1}$ and $CC_{\mathcal{S}_2}$ are proper. Hence, for every $x \in \mathcal{H}$, we are able to generate iterations $(CC^k_{\mathcal{S}_1} x)_{k \in \mathbb{N}}$ and $(CC^k_{\mathcal{S}_2} x)_{k \in \mathbb{N}}$.

In the following two examples, we choose two linear subspaces, U_1 and U_2, in $\mathbb{R}^3$ and one point $x_0 \in \mathbb{R}^3$. Then we count the iteration numbers needed for the four algorithms: the shadow sequence of the Douglas–Rachford method (DRM) (see, [1] for details), the sequence generated by the method of alternating projections (MAP), and the sequence generated by iterating $CC_{\mathcal{S}_1}$ and $CC_{\mathcal{S}_2}$ to find the best approximation point $\overline{x} = \mathrm{P}_{U_1 \cap U_2} x_0$.

Example 4.17. Assume that $\mathcal{H} = \mathbb{R}^3$, that U_1 is the line passing through the points $(0, 0, 0)$ and $(1, 0, 0)$, and that U_2 is the plane $\{(x, y, z) \mid x + y + z = 0\}$. Let $x_0 = (0.5, 0, 0)$. As Table 1 shows, both of the $CC_{\mathcal{S}_1}$ and $CC_{\mathcal{S}_2}$ are faster than DRM and MAP. (The results were obtained using `GeoGebra`.)

Algorithm	Iterations needed to find $\mathrm{P}_{U_1 \cap U_2} x_0$
Shadow DRM	12
MAP	12
Iterating $CC_{\mathcal{S}_1}$	1
Iterating $CC_{\mathcal{S}_2}$	1

TABLE 1. Iterations required for each algorithm. See Example 4.17 for details.

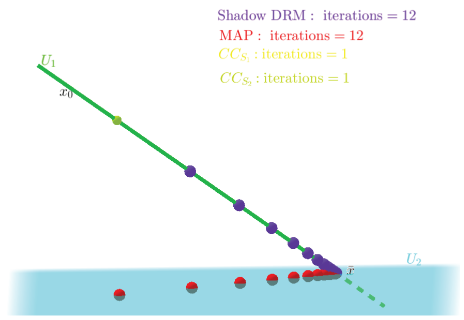

FIGURE 2. Example 4.17 compares iterations for a line and a plane.

Example 4.18. Assume that $\mathcal{H} = \mathbb{R}^3$, that $U_1 = \{(x, y, z) \mid x + y + z = 0\}$, and that $U_2 := \{(x, y, z) \mid -x + 2y + 2z = 0\}$. Set $x_0 = (-1, 0.5, 0.5)$. As Table 2 illustrates, $CC_{\mathcal{S}_2}$ is faster than the other methods, and $CC_{\mathcal{S}_1}$ performs no worse than DRM or MAP. (The results were obtained using `GeoGebra`.)

Algorithm	Iterations needed to find $\mathrm{P}_{U_1 \cap U_2} x_0$
Shadow DRM	5
MAP	6
Iterating $CC_{\mathcal{S}_1}$	5
Iterating $CC_{\mathcal{S}_2}$	2

TABLE 2. Iterations required for each algorithm. See Example 4.18 for details.

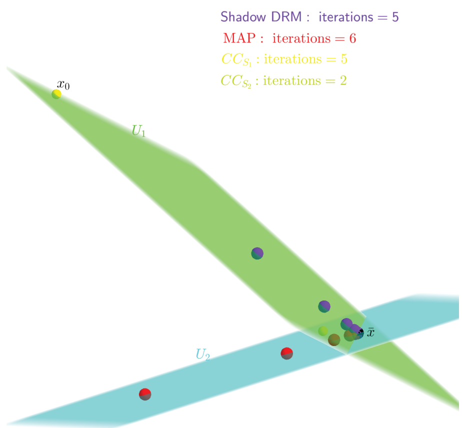

FIGURE 3. Example 4.18 compares iterations for two planes.

4.3.2. *Counterexamples.* The following two examples show that the circumcenter mapping induced by reflectors is in general neither linear nor continuous.

Example 4.19 (Discontinuity)**.** Suppose that $\mathcal{H} = \mathbb{R}^2$, set $U_1 = \mathbb{R} \cdot (1, 0)$, and set $U_2 := \mathbb{R} \cdot (1, 1)$. Suppose that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ or that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2} \mathrm{R}_{U_1}\}$. Let $\overline{x} = (1, 0)$ and let $(\forall k \in \mathbb{N})\ x_k = (1, \frac{1}{k+1})$. As Figure 4 illustrates, $CC_{\mathcal{S}} \overline{x} = \left(\frac{1}{2}, \frac{1}{2}\right)$ and $(\forall k \in \mathbb{N})\ CC_{\mathcal{S}} x_k = (0, 0)$. Hence,

$$\lim_{k \to \infty} CC_{\mathcal{S}} x_k = (0, 0) \neq \left(\tfrac{1}{2}, \tfrac{1}{2}\right) = CC_{\mathcal{S}} \overline{x},$$

which implies that $CC_{\mathcal{S}}$ is not continuous at $\overline{x}$. By Corollary 3.21, the demiclosedness principle holds for $CC_{\mathcal{S}}$.

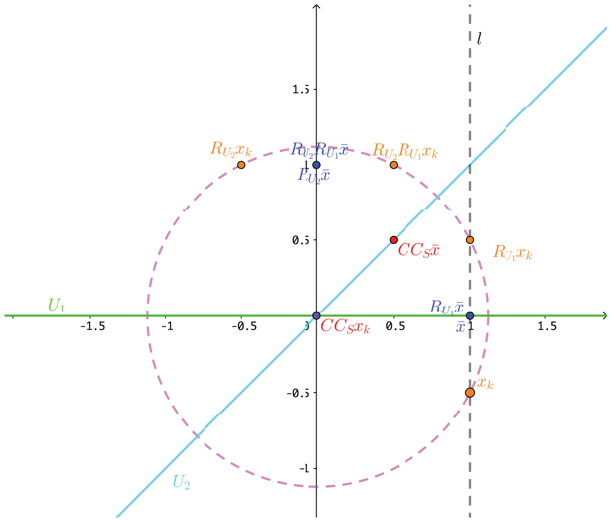

FIGURE 4. Example 4.19 provides a discontinuous $CC_{\mathcal{S}}$ in $\mathbb{R}^2$.

Example 4.20 (Nonlinearity)**.** Suppose that $\mathcal{H} = \mathbb{R}^2$, set $U_1 = \mathbb{R} \cdot (1,0)$ and set $U_2 = \mathbb{R} \cdot (1,1)$. Suppose that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ or that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Let $x = (1,0)$ and $y = (1,-1)$. As Figure 5 illustrates,

$$CC_{\mathcal{S}}x + CC_{\mathcal{S}}y = \left(\tfrac{1}{2}, \tfrac{1}{2}\right) + (0,0) \neq (0,0) = CC_{\mathcal{S}}(x+y),$$

which shows that $CC_{\mathcal{S}}$ is not linear. By Corollary 3.21, the demiclosedness principle holds for $CC_{\mathcal{S}}$.

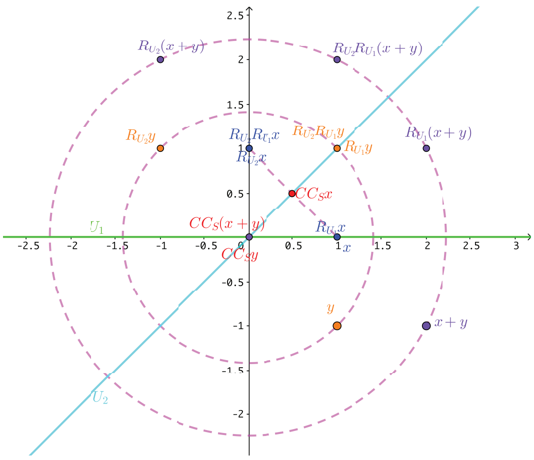

FIGURE 5. Example 4.20 presents a nonlinear $CC_{\mathcal{S}}$ in $\mathbb{R}^2$

5. CIRCUMCENTER MAPPINGS INDUCED BY PROJECTORS

In this section, we uphold the notations that

$$\Omega = \left\{ \mathrm{R}_{U_{i_r}} \cdots \mathrm{R}_{U_{i_2}} \mathrm{R}_{U_{i_1}} \;\middle|\; r \in \mathbb{N}, \text{ and } i_1, \ldots, i_r \in \{1, \ldots, m\} \right\}$$

and $\mathrm{Id} \in \mathcal{S} \subseteq \Omega$. In addition, set

$$\Theta = \Big\{ \mathrm{P}_{U_{i_r}} \cdots \mathrm{P}_{U_{i_2}} \mathrm{P}_{U_{i_1}} \ \Big|\ r \in \mathbb{N}, \text{ and } i_1, \ldots, i_r \in \{1, \ldots, m\} \Big\}.$$

By the empty product convention, $\prod_{j=1}^{0} \mathrm{P}_{U_{i_j}} = \mathrm{Id}$. Hence $\mathrm{Id} \in \Theta$. Specifically, we assume that

$$\mathrm{Id} \in \widehat{\mathcal{S}} \subseteq \operatorname{aff} \Theta.$$

5.1. **Proper circumcenter mappings induced by projectors.** First, we present some cases when $CC_{\widehat{\mathcal{S}}}$ is proper.

Proposition 5.1. *Let $\alpha \in \mathbb{R}$. Assume that*

$$\widehat{\mathcal{S}} = \{\mathrm{Id}, (1-\alpha)\mathrm{Id} + \alpha \mathrm{P}_{U_1}, \ldots, (1-\alpha)\mathrm{Id} + \alpha \mathrm{P}_{U_m}\},$$

and that

$$\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \ldots, \mathrm{R}_{U_m}\}.$$

Then $CC_{\widehat{\mathcal{S}}}$ is proper. Moreover,

$$(\forall x \in \mathcal{H}) \quad CC_{\widehat{\mathcal{S}}} x = \tfrac{\alpha}{2} CC_{\mathcal{S}} x + \big(1 - \tfrac{\alpha}{2}\big) x \in \mathcal{H}. \tag{5.1}$$

Proof. Apply Proposition 4.12 with α replaced by $\frac{\alpha}{2}$. □

Taking $\alpha = 1$ in Proposition 5.1, we deduce the next result.

Corollary 5.2. *Assume that $\widehat{\mathcal{S}} = \{\mathrm{Id}, \mathrm{P}_{U_1}, \ldots, \mathrm{P}_{U_{m-1}}, \mathrm{P}_{U_m}\}$. Then $CC_{\widehat{\mathcal{S}}}$ is proper, that is for every $x \in \mathcal{H}$, there exists unique $CC_{\widehat{\mathcal{S}}} x \in \mathcal{H}$ satisfying*

(i) $CC_{\widehat{\mathcal{S}}}(x) \in \operatorname{aff}\{x, \mathrm{P}_{U_1}(x), \ldots, \mathrm{P}_{U_{m-1}}(x), \mathrm{P}_{U_m}(x)\}$

(ii) $\|CC_{\widehat{\mathcal{S}}}(x) - x\| = \|CC_{\widehat{\mathcal{S}}}(x) - \mathrm{P}_{U_1}(x)\| = \cdots = \|CC_{\widehat{\mathcal{S}}}(x) - \mathrm{P}_{U_m}(x)\|$.

Proposition 5.3. *Assume that U_2 is linear and that $\widehat{\mathcal{S}} = \{\mathrm{Id}, \mathrm{P}_{U_1}, \mathrm{P}_{U_2}\mathrm{P}_{U_1}\}$. Then $CC_{\widehat{\mathcal{S}}}$ is proper.*

Proof. Let $x \in \mathcal{H}$. If $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) \leq 2$, by Proposition 3.3, $CC_{\widehat{\mathcal{S}}} x \in \mathcal{H}$. Now assume $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) = 3$. If $x, \mathrm{P}_{U_1} x, \mathrm{P}_{U_2}\mathrm{P}_{U_1} x$ are affinely independent, by Fact 2.14, $CC_{\widehat{\mathcal{S}}} x \in \mathcal{H}$.

Assume that

$$x, \mathrm{P}_{U_1} x, \mathrm{P}_{U_2}\mathrm{P}_{U_1} x \text{ are affinely dependent.} \tag{5.2}$$

Note that $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) = 3$ implies that $\mathrm{P}_{U_1} x - x \neq 0$; moreover, (5.2) yields that there exists $\alpha \neq 1$ such that

$$\mathrm{P}_{U_2}\mathrm{P}_{U_1} x - x = \alpha(\mathrm{P}_{U_1} x - x). \tag{5.3}$$

Because U_2 is linear subspace, P_{U_2} is linear. Applying to both sides of (5.3) the projector P_{U_2}, we obtain

$$\begin{aligned}
& \mathrm{P}_{U_2}\mathrm{P}_{U_2}\mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x = \alpha(\mathrm{P}_{U_2}\mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x) \\
\Longrightarrow\ & \mathrm{P}_{U_2}\mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x = \alpha(\mathrm{P}_{U_2}\mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x) \quad \text{(by Lemma 2.8(i))} \\
\Longrightarrow\ & (1-\alpha)\mathrm{P}_{U_2}\mathrm{P}_{U_1} x = (1-\alpha)\mathrm{P}_{U_2} x \\
\Longrightarrow\ & \mathrm{P}_{U_2}\mathrm{P}_{U_1} x = \mathrm{P}_{U_2} x. \quad (\alpha \neq 1)
\end{aligned}$$

Combining the implications above with $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) = 3$ and (5.2), we deduce that $x, \mathrm{P}_{U_1} x, \mathrm{P}_{U_2} x$ are pairwise distinct and affinely dependent. Applying Corollary 5.2 to $m = 2$, we obtain $CC(\{x, \mathrm{P}_{U_1} x, \mathrm{P}_{U_2} x\}) \in \mathcal{H}$. But this contradicts Fact 2.14. Therefore, $\operatorname{dom} CC_{\widehat{\mathcal{S}}} = \mathcal{H}$. ☐

Proposition 5.4. *Assume that U_2 is linear and that $\widehat{\mathcal{S}} = \{\mathrm{Id}, \mathrm{P}_{U_2}, \mathrm{P}_{U_2} \mathrm{P}_{U_1}\}$. Then $CC_{\widehat{\mathcal{S}}}$ is proper.*

Proof. Let $x \in \mathcal{H}$. Similarly to the proof in Proposition 5.3, we arrive at a contradiction for the case where $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) = 3$ and there exists $\alpha \neq 1$ such that

$$\mathrm{P}_{U_2} \mathrm{P}_{U_1} x - x = \alpha(\mathrm{P}_{U_2} x - x). \tag{5.4}$$

As in the proof of Proposition 5.3, we apply to both sides of (5.4) the projector P_{U_2}. Then

$$\begin{aligned}
&\mathrm{P}_{U_2} \mathrm{P}_{U_2} \mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x = \alpha(\mathrm{P}_{U_2} \mathrm{P}_{U_2} x - \mathrm{P}_{U_2} x) \\
\Longrightarrow\ & \mathrm{P}_{U_2} \mathrm{P}_{U_1} x - \mathrm{P}_{U_2} x = \alpha(\mathrm{P}_{U_2} x - \mathrm{P}_{U_2} x) = 0 \quad \text{(by Lemma 2.8(i))} \\
\Longrightarrow\ & \mathrm{P}_{U_2} \mathrm{P}_{U_1} x = \mathrm{P}_{U_2} x.
\end{aligned}$$

which contradicts $\operatorname{card}\big(\widehat{\mathcal{S}}(x)\big) = 3$. ☐

5.2. **Improper circumcenter mappings induced by projectors.** In view of Propositions 5.1, 5.3 and 5.4, we consider the following question:

Question 5.5. Suppose that $\{\alpha_1, \alpha_2\} \subseteq \mathbb{R} \smallsetminus \{0, 1\}$ and that at least one of α_1, α_2 is not 2.[2] Assume that

$$\widehat{\mathcal{S}} = \big\{ \mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_{U_1}, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{P}_{U_2}\big)\big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_{U_1}\big)\big\}$$

or that

$$\widehat{\mathcal{S}} = \big\{ \mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_{U_2}, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{P}_{U_2}\big)\big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_{U_1}\big)\big\}.$$

Is $CC_{\widehat{\mathcal{S}}}$ proper?

The following example demonstrates that the answer to Question 5.5 is negative.

Example 5.6. Assume that $m = 2$ and that $U := U_1 = U_2 \subsetneqq \mathcal{H}$ and $\{\alpha_1, \alpha_2\} \subseteq \mathbb{R}$. Assume further that $\widehat{\mathcal{S}} = \big\{ \mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_U, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{P}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_U\big)\big\}$. Then $CC_{\widehat{\mathcal{S}}}$ is improper if and only if $\alpha_1 \neq 0, \alpha_1 \neq 1, \alpha_2 \neq 0$ and $\alpha_2 \neq \frac{\alpha_1}{\alpha_1 - 1}$.

Proof. Since $\mathrm{R}_U = 2\mathrm{P}_U - \mathrm{Id}$, we deduce that

$$\begin{aligned}
\widehat{\mathcal{S}} &= \Big\{ \mathrm{Id}, (1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_U, \big((1-\alpha_2)\mathrm{Id} + \alpha_2 \mathrm{P}_U\big) \circ \big((1-\alpha_1)\mathrm{Id} + \alpha_1 \mathrm{P}_U\big) \Big\} \\
&= \Big\{ \mathrm{Id}, \mathrm{Id} + \alpha_1(\mathrm{P}_U - \mathrm{Id}), \big(\mathrm{Id} + \alpha_2(\mathrm{P}_U - \mathrm{Id})\big) \circ \big(\mathrm{Id} + \alpha_1(\mathrm{P}_U - \mathrm{Id})\big) \Big\} \\
&= \Big\{ \mathrm{Id}, \mathrm{Id} + \frac{\alpha_1}{2}(\mathrm{R}_U - \mathrm{Id}), \big(\mathrm{Id} + \frac{\alpha_2}{2}(\mathrm{R}_U - \mathrm{Id})\big) \circ \big(\mathrm{Id} + \frac{\alpha_1}{2}(\mathrm{R}_U - \mathrm{Id})\big) \Big\}.
\end{aligned}$$

The result now follows from the assumptions above and Example 4.15. ☐

[2]For $i \in \{1, 2\}$, when α_i is $0, 1$, or 2, then $(1-\alpha_i)\mathrm{Id} + \alpha_i \mathrm{P}_{U_1}$ is $\mathrm{Id}, \mathrm{P}_{U_1}$, or R_{U_1} respectively. In these special cases, the answer for Question 5.5 is positive (see Proposition 5.1 and Theorem 4.3(i)).

Next, we present further improper instances of $CC_{\widehat{\mathcal{S}}}$, where $\mathrm{Id} \in \widehat{\mathcal{S}} \subseteq \operatorname{aff}\Theta$.

Example 5.7. Assume that $\mathcal{H} = \mathbb{R}^2$, $m = 2$, $U_1 = \mathbb{R}\cdot(1,0)$, and that $U_2 = \mathbb{R}\cdot(1,2)$. Assume further that $\widehat{\mathcal{S}} = \{\mathrm{Id}, \mathrm{P}_{U_2}\mathrm{P}_{U_1}, \mathrm{P}_{U_2}\mathrm{P}_{U_1}\mathrm{P}_{U_2}\mathrm{P}_{U_1}\}$. Take $x = (2,4) \in U_2$. As Figure 6 illustrates, x, $\mathrm{P}_{U_2}\mathrm{P}_{U_1}x$, and $\mathrm{P}_{U_2}\mathrm{P}_{U_1}\mathrm{P}_{U_2}\mathrm{P}_{U_1}x$ are pairwise distinct and colinear. By Theorem 3.9, $CC_{\widehat{\mathcal{S}}}$ is improper.

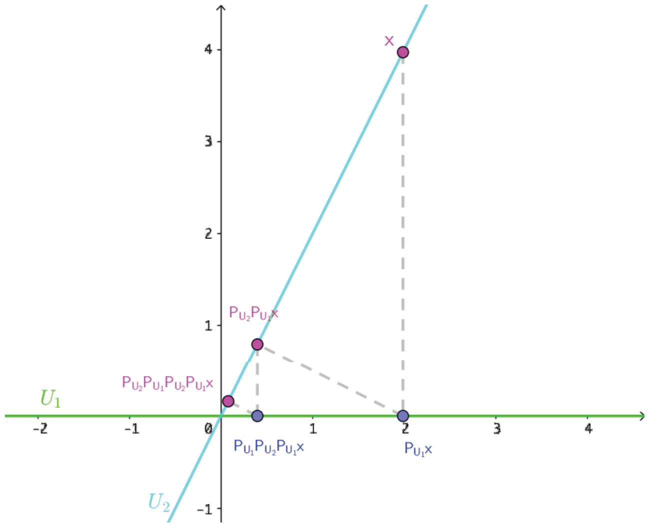

FIGURE 6. Example 5.7 illustrates $CC_{\widehat{\mathcal{S}}}x = \varnothing$ for the colinear case.

Example 5.8. Assume that $\mathcal{H} = \mathbb{R}^2$, that $m = 2$, that $U_1 = \mathbb{R}\cdot(1,0)$, and that $U_2 = \mathbb{R}\cdot(1,1)$. Assume further that $\widehat{\mathcal{S}} = \{\mathrm{Id}, \mathrm{P}_{U_1}, \mathrm{P}_{U_2}, \mathrm{P}_{U_2}\mathrm{P}_{U_1}\}$. Take $x = (4,2)$ and set $\mathcal{K} = \{\mathrm{Id}, \mathrm{P}_{U_1}, \mathrm{P}_{U_2}\}$. Clearly, $\mathrm{P}_{U_2}\mathrm{P}_{U_1}x - x \in \mathbb{R}^2 = \operatorname{span}\{\mathrm{P}_{U_1}x - x, \mathrm{P}_{U_2}x - x\}$, which implies that $\operatorname{aff}(\mathcal{K}(x)) = \operatorname{aff}(\widehat{\mathcal{S}}(x))$. By Fact 2.19, if $CC_{\widehat{\mathcal{S}}}x \in \mathcal{H}$, then $CC_{\widehat{\mathcal{S}}}x = CC_{\mathcal{K}}x$. As Figure 7 shows, $\|CC_{\mathcal{K}}x - x\| = \|CC_{\mathcal{K}}x - \mathrm{P}_{U_1}x\| = \|CC_{\mathcal{K}}x - \mathrm{P}_{U_2}x\| \neq \|CC_{\mathcal{K}}x - \mathrm{P}_{U_2}\mathrm{P}_{U_1}x\|$. Hence $CC_{\widehat{\mathcal{S}}}x = \varnothing$, which implies that $CC_{\widehat{\mathcal{S}}}$ is improper.

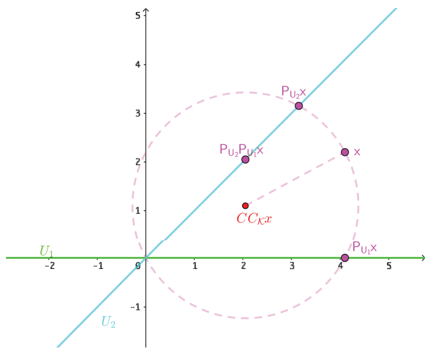

FIGURE 7. Example 5.8 illustrates $CC_{\widehat{\mathcal{S}}}x = \varnothing$ for the non-colinear case.

6. More improper circumcenter mappings induced by reflectors

In Theorem 4.3(i), to prove $CC_{\mathcal{S}}$ is proper, we required that

$$U_1, \ldots, U_m \text{ are closed affine subspaces in } \mathcal{H} \text{ with } \cap_{i=1}^m U_i \neq \varnothing, \tag{6.1}$$

and that

$$\mathrm{Id} \in \mathcal{S} \subseteq \Omega. \tag{6.2}$$

In Section 4.2, we have already seen that when the condition $\mathcal{S} \subseteq \Omega$ fails, the circumcenter mapping induced by reflectors $CC_{\mathcal{S}}$ may be improper. In the remaining part of this section, we consider two circumcenter mappings induced by reflectors, where $m = 2$ and $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ or $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. We construct additional improper circumcenter mappings with the conditions in (6.1) not being satisfied, which means that the conditions (6.1) and (6.2) are sharp.

6.1. **Inconsistent cases.** In this subsection, we focus on the case when $\cap_{i=1}^m U_i = \varnothing$. Let U and V be two nonempty, closed, convex (possibly nonintersecting) subsets of $\mathcal{H}$. A *best approximation pair* relative to (U, V) is

$$(a, b) \in U \times V \quad \text{such that} \quad \|a - b\| = \inf \|U - V\|.$$

In the reference [3], the authors used the Douglas–Rachford splitting operator $T = \frac{\mathrm{R}_V \mathrm{R}_U + \mathrm{Id}}{2}$ to find a best approximation pair relative to (U, V).

Fact 6.1 ([3, Theorem 3.13 and Remark 3.14(ii)]). *Let U be a closed affine subspace and let V be a nonempty, closed, convex set in $\mathcal{H}$ (U, V are possibly nonintersecting). Suppose that best approximation pairs relative to (U, V) exist. Set $T := \frac{\mathrm{R}_V \mathrm{R}_U + \mathrm{Id}}{2}$. Let $x_0 \in \mathcal{H}$ and set $x_n = T^n x_0$, for all $n \in \mathbb{N}$. Then*

$$\big((\mathrm{P}_V \mathrm{R}_U x_n, \mathrm{P}_U x_n)\big)_{n\in\mathbb{N}} \quad \textit{and} \quad \big((\mathrm{P}_V \mathrm{P}_U x_n, \mathrm{P}_U x_n)\big)_{n\in\mathbb{N}}$$

both converge weakly to best approximation pairs relative to (U, V).

The following examples show that even if both of U_1, U_2 are closed affine subspaces, when $U_1 \cap U_2 = \varnothing$, the operator $CC_{\mathcal{S}}$ may not be proper where $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ or $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. (Notice that in Example 6.2, U_1 is even a compact set.) Hence, we can not directly generalize Fact 6.1 by the circumcenter mapping induced by reflectors.

The results of the following examples in this section are easily from Corollary 3.4 and the proofs are omitted.

Example 6.2. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \{(2,0)\}$, and that $U_2 = \mathbb{R} \cdot (0,1)$. Set $\mathcal{S}_1 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ and $\mathcal{S}_2 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. Then

$$\operatorname{dom} CC_{\mathcal{S}_1} = \big(\mathbb{R}^2 \smallsetminus \mathbb{R} \cdot (1,0)\big) \cup \big\{(2,0), (0,0)\big\},$$
$$\operatorname{dom} CC_{\mathcal{S}_2} = \big(\mathbb{R}^2 \smallsetminus \mathbb{R} \cdot (1,0)\big) \cup \big\{(2,0), (4,0)\big\}.$$

6.2. **Non-affine cases.** One of the nice aspects of the Douglas–Rachford method is that it can be used for general convex sets. In this subsection, we assume that

$$\mathcal{S}_1 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\} \quad \text{or} \quad \mathcal{S}_2 = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}.$$

We shall present examples in which the operator $CC_{\mathcal{S}}$ is improper, with at least one of U_1 and U_2 not being an affine subspace while $U_1 \cap U_2 \neq \varnothing$.

Example 6.3. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \mathbb{R}^2_+$, and that $U_2 = (2,0)+\mathbb{R}\cdot(0,1)$. Then

$$\begin{aligned}\operatorname{dom} CC_{\mathcal{S}_1} &= \mathbb{R}^2 \smallsetminus \{(x,y) \mid x < 0 \text{ and } y \geq 0\},\\ \operatorname{dom} CC_{\mathcal{S}_2} &= \big(\mathbb{R}^2 \smallsetminus \{(x,y) \mid x < 0 \text{ and } y \geq 0\}\big) \cup \{(-2,y) \mid y \geq 0\}.\end{aligned}$$

In the remainder of this subsection, we revisit the examples used in [6] to show the potential of the Circumcentering Douglas–Rachford method, which are the iterations of the operator $CC_{\mathcal{S}_2}$.

Example 6.4. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \mathbf{B}[(0,0);1]$, and that $U_2 = (1,0) + \mathbb{R}\cdot(0,1)$. Then

$$\begin{aligned}\operatorname{dom} CC_{\mathcal{S}_1} &= \mathbb{R}^2 \smallsetminus \{(x,0) \mid x < -1\},\\ \operatorname{dom} CC_{\mathcal{S}_2} &= \mathbb{R}^2 \smallsetminus \{(x,0) \mid x < -3 \text{ or } -3 < x < -1\}.\end{aligned}$$

Example 6.5. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \mathbf{B}[(0,0);1]$, and that $U_2 = \mathbb{R}\cdot(0,1)$. Then

$$\begin{aligned}\operatorname{dom} CC_{\mathcal{S}_1} &= \mathbb{R}^2 \smallsetminus \{(x,0) \mid |x| > 1\},\\ \operatorname{dom} CC_{\mathcal{S}_2} &= \mathbb{R}^2 \smallsetminus \{(x,0) \mid |x| > 2 \text{ or } 1 < |x| < 2\}.\end{aligned}$$

Example 6.6. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \mathbf{B}[(-1,0);1]$, and that $U_2 = \mathbf{B}[(1,0);1]$. Then

$$\operatorname{dom} CC_{\mathcal{S}_1} = \mathbb{R}^2 \smallsetminus \{(x,0) \mid x < -2 \text{ or } x > 2\},$$

$$\{(x,0) \mid -6 \leq x \leq -4 \text{ or } x \geq -2\} \subseteq \operatorname{dom} CC_{\mathcal{S}_2},$$

$$\{(x,0) \mid x < -6 \text{ or } -4 < x < -2\} \subseteq \mathbb{R}^2 \smallsetminus \big(\operatorname{dom} CC_{\mathcal{S}_2}\big).$$

Example 6.7. Assume that $\mathcal{H} = \mathbb{R}^2$, that $U_1 = \mathbf{B}[(-1,0);2]$, and that $U_2 = \mathbf{B}[(1,0);2]$. Then

$$\operatorname{dom} CC_{\mathcal{S}_1} = \mathbb{R}^2 \smallsetminus \{(x,0) \mid x < -3 \text{ or } x > 3\},$$

$$\{(x,0) \mid -9 \leq x \leq -5 \text{ or } -3 \leq x \leq 3\} \subseteq \operatorname{dom} CC_{\mathcal{S}_2},$$

$$\{(x,0) \mid x < -9 \text{ or } -5 < x < -3 \text{ or } x > 3\} \subseteq \mathbb{R}^2 \smallsetminus \big(\operatorname{dom} CC_{\mathcal{S}_2}\big).$$

Finally, consider $U_1 = \{(x,y) \in \mathbb{R}^2 \mid (x+1)^2+y^2 = 4\}$ and $U_2 = \{(x,y) \in \mathbb{R}^2 \mid (x-1)^2 + y^2 = 4\}$. Note that neither U_1 nor U_2 is convex. For $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$ or $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$, one can show that $\operatorname{dom} CC_{\mathcal{S}} \subsetneqq \mathbb{R}^2$.

6.3. **Impossibility to extend to maximally monotone operators.** Assume that $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\}$ or $\mathcal{S} = \{\mathrm{Id}, \mathrm{R}_{U_1}, \mathrm{R}_{U_2}\mathrm{R}_{U_1}\}$. In order to show a counterexample where the definition of $CC_{\mathcal{S}}$ fails to be directly generalized to maximally monotone theory, we need the definition and facts below.

Definition 6.8 ([2, Definition 23.1])**.** Let $A : \mathcal{H} \to 2^{\mathcal{H}}$. The *resolvent* of A is

$$J_A = (\mathrm{Id} + A)^{-1}.$$

Fact 6.9 ([2, Corollary 23.11])**.** *Let $A : \mathcal{H} \to 2^{\mathcal{H}}$ be maximally monotone and let $\gamma \in \mathbb{R}_{++}$. Then the following hold.*

(i) *$J_{\gamma A} : \mathcal{H} \to \mathcal{H}$ and $\mathrm{Id} - J_{\gamma A} : \mathcal{H} \to \mathcal{H}$ are firmly nonexpansive and maximally monotone.*
(ii) *The* reflected resolvent

$$R_{\gamma A} : \mathcal{H} \to \mathcal{H} : x \mapsto 2J_{\gamma A}x - x.$$

is nonexpansive.

Fact 6.10 ([2, Corollary 20.28])**.** *Let $A : \mathcal{H} \to \mathcal{H}$ be monotone and continuous. Then A is maximally monotone.*

Fact 6.11 ([2, Corollary 20.26])**.** *Let C be a nonempty closed convex subset of $\mathcal{H}$. Then N_C is maximally monotone.*

Fact 6.12 ([2, Corollary 23.4])**.** *Let C be a nonempty closed convex subset of $\mathcal{H}$. Then*

$$J_{N_C} = (\mathrm{Id} + N_C)^{-1} = \mathrm{P}_C\,.$$

By Fact 6.12, $\mathrm{R}_{U_1} = 2\,\mathrm{P}_{U_1} - \mathrm{Id} = 2J_{N_{U_1}} - \mathrm{Id}$ and $\mathrm{R}_{U_2} = 2\,\mathrm{P}_{U_2} - \mathrm{Id} = 2J_{N_{U_2}} - \mathrm{Id}$. In these special cases, the reflectors are consistent with the corresponding reflected resolvent.

In the following examples, we replace the two maximally monotone operators N_{U_1}, N_{U_2} in the set $\mathcal{S} = \{\mathrm{Id}, 2J_{N_{U_1}} - \mathrm{Id}, 2J_{N_{U_2}} - \mathrm{Id}\}$ or $\mathcal{S} = \{\mathrm{Id}, 2J_{N_{U_1}} - \mathrm{Id}, (2J_{N_{U_2}} - \mathrm{Id}) \circ (2J_{N_{U_1}} - \mathrm{Id})\}$ by $\alpha\,\mathrm{Id}$ and $\beta\,\mathrm{Id}$ respectively, with $\alpha \geq 0$ and $\beta \geq 0$. By Fact 6.10, since $\alpha \geq 0$ and $\beta \geq 0$, we obtain that $\alpha\,\mathrm{Id}$ and $\beta\,\mathrm{Id}$ are maximally monotone operators. We shall characterize the improperness of the new operator $CC_{\mathcal{S}}$.

Example 6.13. Assume that $\{0\} \subsetneqq \mathcal{H}$. Set $A = \alpha\,\mathrm{Id}$ and $B = \beta\,\mathrm{Id}$, where $\alpha \geq 0$ and $\beta \geq 0$. Further set

$$\mathcal{S}_1 = \{\mathrm{Id}, R_A, R_B\} \quad \text{and} \quad \mathcal{S}_2 = \{\mathrm{Id}, R_A, R_B R_A\}.$$

Then $CC_{\mathcal{S}_1}$ is improper if and only if $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq \beta$. Moreover, $CC_{\mathcal{S}_2}$ is improper if and only if $\alpha \neq 0$, $\alpha \neq 1$, $\beta \neq 0$ and $\alpha \neq -\beta$.

Proof. The definitions yield

$$\begin{aligned}
J_A &= (A+\operatorname{Id})^{-1} = \big((\alpha+1)\operatorname{Id}\big)^{-1} = \frac{1}{\alpha+1}\operatorname{Id};\\
R_A &= 2J_A - \operatorname{Id} = \frac{2}{\alpha+1}\operatorname{Id} - \operatorname{Id} = \frac{1-\alpha}{\alpha+1}\operatorname{Id};\\
J_B &= (B+\operatorname{Id})^{-1} = (\big((\beta+1)\operatorname{Id}\big)^{-1} = \frac{1}{\beta+1}\operatorname{Id};\\
R_B &= 2J_B - \operatorname{Id} = \frac{2}{\beta+1}\operatorname{Id} - \operatorname{Id} = \frac{1-\beta}{\beta+1}\operatorname{Id}.
\end{aligned}$$

Let $x \in \mathcal{H} \smallsetminus 0$. Now

$$x = R_A x \iff x = \frac{1-\alpha}{\alpha+1}x \iff 1 = \frac{1-\alpha}{\alpha+1} \iff \alpha = 0; \tag{6.3a}$$

$$x = R_B x \iff x = \frac{1-\beta}{\beta+1}x \iff \beta = 0; \tag{6.3b}$$

$$R_A x = R_B x \iff \frac{1-\alpha}{\alpha+1}x = \frac{1-\beta}{\beta+1}x \iff \alpha = \beta; \tag{6.3c}$$

$$x = R_B R_A x \iff x = \frac{1-\alpha}{\alpha+1}\frac{1-\beta}{\beta+1}x \iff \alpha = -\beta; \tag{6.3d}$$

$$R_A x = R_B R_A x \iff \frac{1-\alpha}{\alpha+1}x = \frac{1-\alpha}{\alpha+1}\frac{1-\beta}{\beta+1}x \iff \alpha = 1 \text{ or } \beta = 0. \tag{6.3e}$$

"$\Longrightarrow$": According to the previous analysis, in both of the assertions, the contrapositive of the required results follow from Proposition 3.3.

"$\Longleftarrow$": Assume $\alpha \neq 0$, $\beta \neq 0$ and $\alpha \neq \beta$. Then

$$\begin{aligned}
\operatorname{aff}(\mathcal{S}_1(x)) &= \operatorname{aff}\{x, R_A x, R_B x\} = x + \operatorname{span}\{R_A x - x, R_B x - x\}\\
&= x + \operatorname{span}\Big\{\frac{-2\alpha}{\alpha+1}x, \frac{-2\beta}{\beta+1}x\Big\}\\
&= \mathbb{R}\cdot x.
\end{aligned}$$

Let $x \in \mathcal{H} \smallsetminus \{0\}$. We observe that

$$\begin{aligned}
&\Big(\exists y \in \operatorname{aff}(\mathcal{S}_1(x))\Big) \quad \|y - x\| = \|y - R_A x\| = \|y - R_B x\|\\
\iff &(\exists t \in \mathbb{R}) \quad \|tx - x\| = \|tx - \frac{1-\alpha}{\alpha+1}x\| = \|tx - \frac{1-\beta}{\beta+1}x\|\\
\iff &(\exists t \in \mathbb{R}) \quad |t-1| = \Big|t - \frac{1-\alpha}{\alpha+1}\Big| = \Big|t - \frac{1-\beta}{\beta+1}\Big|. \quad (\text{by } x \neq 0)
\end{aligned}$$

On the other hand, combining the assumptions with Corollary 2.16 and (6.3), we obtain that

$$(\nexists\, t \in \mathbb{R}) \quad |t-1| = \Big|t - \frac{1-\alpha}{\alpha+1}\Big| = \Big|t - \frac{1-\beta}{\beta+1}\Big|.$$

Hence,

$$(\forall x \in \mathcal{H} \smallsetminus \{0\}) \quad CC_{\mathcal{S}_1}x = \varnothing.$$

Assume $\alpha \neq 0$, $\alpha \neq 1$, $\beta \neq 0$ and $\alpha \neq -\beta$. A similar proof shows that for every $x \in \mathcal{H} \smallsetminus \{0\}$, there is no point $y \in \operatorname{aff}(\mathcal{S}_2(x))$, such that $\|y - x\| = \|y - R_A x\| = \|y - R_B R_A x\|$, which implies that $(\forall x \in \mathcal{H} \smallsetminus \{0\})\ CC_{\mathcal{S}_2} x = \varnothing$. ☐

Arguing similarly to the proof of the previous result, we also obtain the following result:

Example 6.14. Assume that $\{0\} \subsetneqq \mathcal{H}$. Let $\{a, b\} \subseteq \mathbb{R}$. Set $A \equiv a$, i.e., $(\forall x \in \mathcal{H})$ $Ax = a$, and $B \equiv b$. Furthermore, set

$$\mathcal{S}_1 = \{\mathrm{Id}, R_A, R_B\} \quad \text{and} \quad \mathcal{S}_2 = \{\mathrm{Id}, R_A, R_B R_A\}.$$

Then $CC_{\mathcal{S}_1}$ is improper if and only if $a \neq 0$, $b \neq 0$ and $a \neq b$. Moreover, $CC_{\mathcal{S}_2}$ is improper if and only if $a \neq 0$, $b \neq 0$ and $a \neq -b$.

The example above shows that there is no direct way to generalize the definition of $CC_{\mathcal{S}}$ to maximally monotone theory.

Acknowledgments

HHB and XW were partially supported by NSERC Discovery Grants. We thank a referee for her/his careful reading and constructive comments.

References

[1] H. H. Bauschke, J. Y. Bello Cruz, T. T. A. Nghia, H. M. Phan and X. Wang, *The rate of linear convergence of the Douglas-Rachford algorithm for subspaces is the cosine of the Friedrichs angle*, J. Approx. Theory **185** (2014), 63–79.

[2] H. H. Bauschke and P. L. Combettes, *Convex Analysis and Monotone Operator Theory in Hilbert Spaces*, CMS Books in Mathematics, Springer, second ed., 2017.

[3] H. H. Bauschke, P. L. Combettes and D. R. Luke, *Finding best approximation pairs relative to two closed convex sets in Hilbert spaces*, J. Approx. Theory **127** (2004), 178–192.

[4] H. H. Bauschke, H. Ouyang and X. Wang, *On circumcenters of finite sets in Hilbert spaces*, Linear Nonlinear Anal. **4** (2018), 271–295.

[5] H. H. Bauschke, C. Wang, X. Wang and J. Xu, *On subgradient projectors*, SIAM J. Optim. **25** (2015), 1064–1082.

[6] R. Behling, J. Y. Bello Cruz and L.-R. Santos, *Circumcentering the Douglas–Rachford method*, Numer. Algorithms **78** (2018), 759–776.

[7] R. Behling, J. Y. Bello Cruz and L.-R. Santos, *On the linear convergence of the circumcentered-reflection method*, Oper. Res. Lett. **46** (2018), 159–162.

[8] F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bulletin of the AMS. **74** (1968), 660–665.

[9] F. Deutsch, *Best Approximation in Inner Product Spaces*, CMS Books in Mathematics, Springer-Verlag, New York, 2012.

[10] R. T. Rockafellar, *Convex Analysis*, Princeton Landmarks in Mathematics, Princeton University Press, Princeton, NJ, 1997. Reprint of the 1970 original, Princeton Paperbacks.

Manuscript received November 28 2018
revised June 10 2019

Heinz H. Bauschke
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: heinz.bauschke@ubc.ca

Hui Ouyang
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: hui.ouyang@alumni.ubc.ca

Xianfu Wang
Mathematics, University of British Columbia, Kelowna, B.C. V1V 1V7, Canada
E-mail address: shawn.wang@ubc.ca