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STOCHASTIC SEMILINEAR PARABOLIC EQUATIONS WITH MEASURES AS INITIAL DATA

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Abstract. It is studied the existence and uniqueness of a weak solution to stochastic differential equations with linear multiplicative Gaussian noise of the form

$$dX - \Delta X\,dt + \varphi(X)dt = X\,dW$$

with measure as initial data, where $\varphi : \mathbb{R} \to \mathbb{R}$ is a continuous and monotonically nondecreasing function with polynomial growth of order $p < \frac{d+1}{d}$.

1. Introduction

We consider here the stochastic differential equation

$$\begin{aligned} &dX - \Delta X\,dt + \varphi(X)dt = X\,dW && \text{in } (0,T)\times\mathcal{O} = \mathcal{Q},\\ &X(0) = x && \text{in } \mathcal{O},\\ &X = 0 && \text{on } \Sigma = (0,T)\times\partial\mathcal{O}, \end{aligned} \tag{1.1}$$

where $\mathcal{O}$ is a bounded and open domain of $\mathbb{R}^d$, $d \geq 1$, with smooth boundary $\partial\mathcal{O}$, W is a Wiener process in $L^2(\mathcal{O})$ and φ is a continuous, monotonically nonincreasing function. More precisely, the hypotheses below will be in effect everywhere in the following:

(i) W is a cylindrical Wiener process of the form

$$W(t,\xi) = \sum_{j=1}^{\infty} \mu_j e_j(\xi)\beta_j(t),\ t \geq 0,\ \xi \in \mathcal{O}, \tag{1.2}$$

where $\{\beta_j\}_{j=1}^{\infty}$ is an independent system of Brownian motions in a probability space $\{\Omega, \mathcal{F}, \mathbb{P}\}$, $\{e_j\}_{j=1}^{\infty} \subset C^2(\overline{\mathcal{O}})$ is an orthonormal basis in $L^2(\mathcal{O})$ and $\{\mu_j\}_{j=1}^{\infty} \subset \mathbb{R}$ are such that

$$\sum_{j=1}^{\infty} \mu_j^2 |e_j|_{\infty}^2 < \infty. \tag{1.3}$$

(ii) $\varphi : \mathbb{R} \to \mathbb{R}$ is continuous, monotonically nondecreasing, $\varphi(0) = 0$,

$$\varphi(r) \leq C|r|^p,\ \forall r \in \mathbb{R}, \tag{1.4}$$

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where $C>0$ and

$$0 \le p < \frac{d+2}{d}\,. \tag{1.5}$$

It is well known (see, e.g., [8], [11]) that, for each $x\in L^2(\mathcal{O})$, equation (1.1) has a unique strong solution $X\in L^2(\Omega;C([0,T];L^2(\mathcal{O}))\cap L^2((0,T)\times\Omega;H^1_0(\mathcal{O}))$. (Sharper results are given in [1].) Here we shall study equation (1.1) in the case where the initial data x is a bounded Radon measure on $\mathcal{O}$ (for instance, a Dirac measure δ or a linear combination of Dirac measures). This case is delicate because one cannot expect for (1.1) a strong solution in such a situation, but only a weak solution of distributional type. In the deterministic case, such a result was first obtained by H. Brezis and A. Friedman [4] and it is known that condition (1.5) is maximal.

As regards the stochastic parabolic equations with measure initial data by our knowledge only the case of linear heat equations with Lipschitzian multiplicative noise was studied so far. (See [6], [7].)

Notation. $L^q(\mathcal{O}),L^q(\mathcal{Q}))$, $1\le q\le\infty$, are standard spaces of q-integrable Lebesgue functions on $\mathcal{O}$ and, respectively, $\mathcal{Q}$ norms $|\cdot|_q$. Denote by $C(\overline{\mathcal{O}})$ the space of continuous functions on the closure $\overline{\mathcal{O}}$ of $\mathcal{O}$ and by $C_0(\overline{\mathcal{O}})=C_0$ the space $\{y\in C(\overline{\mathcal{O}});\ y=0 \text{ on } \partial\mathcal{O}\}$. By $C_0^{1,2}(\overline{\mathcal{Q}})$ denote the space $\{y\in C(\overline{\mathcal{Q}}); \frac{\partial y}{\partial t},\frac{\partial y}{\partial x_i},\frac{\partial^2 y}{\partial x_i\partial x_j}\in C(\overline{\mathcal{O}});\ i,j=1,...,d \text{ on } \Sigma\}$. By $\mathcal{D}'(\mathcal{Q})$ denote the space of distributions on $\mathcal{Q}$, that is, the dual of $C_0^\infty(\mathcal{Q})$. $W_0^{1,q}(\mathcal{O})=W_0^{1,q}$, $1\le q\le\infty$, is the Sobolev space $\{y\in L^q(\mathcal{O});\ \frac{\partial y}{\partial x_i}\in L^q(\mathcal{O}),\ i=1,...,d\}$ with the dual $W^{-1,q'}(\mathcal{O})$, $\frac{1}{q}+\frac{1}{q'}=1$. $\mathcal{M}_b(\mathcal{O})=\mathcal{M}_b$ is the space of bounded Radon measures on $\mathcal{O}$, that is, the dual $(C_0(\overline{\mathcal{O}}))^*$ of the space $C_0(\overline{\mathcal{O}})=C_0$. Given a Banach space Z, we denote by $C([0,T];Z)$ the space of all continuous functions $u:[0,T]\to Z$ and by $L^p(0,T;Z)$, $1\le p\le\infty$, the space of Bochner p-integrable Z-valued functions on $[0,T]$. The norm of the Banach space Z will be denoted by $\|\cdot\|_Z$ and the duality functional between Z and the dual space Z^* by ${}_{Z^*}\langle\cdot,\cdot\rangle_Z$. We set also $y_t=\frac{\partial y}{\partial t}$.

2. The main result

We begin with the definition of the weak solution for equation (1.1) with initial data $x\in\mathcal{M}_b(\mathcal{O})$. We denote by $(\mathcal{F}_t)_{t\ge0}$ the natural filtration corresponding to W.

Definition 2.1. The process $X:[0,T]\to L^1(\mathcal{O})$ is said to be a weak solution to equation (1.1) if X is $(\mathcal{F}_t)_{t\ge0}$-adapted and the following conditions hold.

$$X\in C([0,T];L^1(\mathcal{O})),\ \mathbb{P}\text{-a.s.}, \tag{2.1}$$

$$\lim_{t\to0}\int_{\mathcal{O}} X(t,\xi)\psi(\xi)d\xi = {}_{\mathcal{M}_b}\langle x,\psi\rangle_{C_0}\,,\ \forall\psi\in C_0, \tag{2.2}$$

$$\begin{aligned}\int_{\mathcal{Q}} X(t,\xi)(\mathcal{X}_t(t,\xi)+\Delta\mathcal{X}(t,\xi)dt\,d\xi &= \int_{\mathcal{Q}}\varphi(X(t,\xi)))\mathcal{X}(t,\xi))dt\,d\xi\\ &-\int_{\mathcal{Q}}\mathcal{X}(t,\xi)X(t,\xi)dW(t) - {}_{\mathcal{M}_b}\langle x,\mathcal{X}(0)\rangle_{C_0}\,,\\ &\forall\mathcal{X}\in C_0^{2,1}(\overline{\mathcal{Q}}),\mathcal{X}(T,0)=0,\end{aligned} \tag{2.3}$$

$$\varphi(X) \in L^1(\mathcal{Q}),\ \mathbb{P}\text{-a.s.,} \tag{2.4}$$

$$\mathbb{E}\int_0^T \int_{\mathcal{O}} |X(t,\xi)|^2 dt\, d\xi < \infty. \tag{2.5}$$

We note that under the above conditions on X the Itô integral arising in (2.3) is well defined.

We note that, if X is a strong solution to equation (1.1) (which is the case if $x \in L^2(\mathcal{O})$), then conditions (2.1)–(2.5) automatically hold.

Theorem 2.2 which follows is the main result.

Theorem 2.2. *Under hypotheses* (i), (ii)*, for each $x \in \mathcal{M}_b(\mathcal{O})$, there is a unique weak solution X to* (1.1)*, which satisfies*

$$\mathbb{E}\int_0^T \|X(t)\|^p_{W_0^{1,q}} dt \le C\|x\|^p_{\mathcal{M}_b}, \tag{2.6}$$

where $q \in \left[1, \frac{pd}{(p-1)d+p}\right)$. If $x \in L^1(\mathcal{O})$, then $X \in ([0,T]; L^1(\mathcal{O}))$.

It follows by (2.1)-(2.2) that the solution X is $\mathcal{M}_b$-valued continuous on $[0,T]$, where $\mathcal{M}_b = \mathcal{M}_b(\mathcal{O})$ is endowed with the standard weak-star topology. On the other hand, it follows that, on $(0,T]$, $t \to X(t)$ is $L^1(\mathcal{O})$-valued continuous $\mathbb{P}$-a.s. As mentioned earlier, the standard example is that where x is the Dirac mass.

3. Proof of Theorem 2.2

By the transformation

$$X = e^W y$$

one reduces equation (1.1) to the random parabolic equation (see, e.g., [1])

$$\begin{aligned} &\frac{\partial y}{\partial t} - e^{-W}\Delta(e^W y) + e^{-W}\eta(e^W y) + \mu y = 0 \text{ in } \mathcal{D}'(\mathcal{Q}),\\ &y(0,\cdot) = x \text{ in } \mathcal{O},\\ &y(t,\xi) = 0 \text{ on } \Sigma, \end{aligned} \tag{3.1}$$

where

$$\mu(\xi) = \frac{1}{2}\sum_{j=1}^{\infty} \mu_j^2 e_j^2(\xi),\ \xi \in \mathcal{O}.$$

Equivalently,

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + ay + b\cdot\nabla y + e^{-W}\varphi(e^W y) = 0 \text{ in } \mathcal{D}'(\mathcal{Q}),\\ &y(0,\cdot) = x \text{ in } \mathcal{O},\\ &y = 0 \text{ on } \Sigma, \end{aligned} \tag{3.1$'$}$$

where

$$\begin{aligned} a(t,\xi) &= \mu(\xi) - e^{-W}\Delta(e^W),\\ b(t,\xi) &= -2\nabla W(t,\xi). \end{aligned}$$

We fix $\omega \in \Omega$ and treat (3.1) (or, equivalently, (3.1)$'$) as a parabolic equation in $Q = (0,T) \times \mathcal{O}$. Our main purpose here is to prove the existence of a solution y to (3.1)$'$ for $x \in \mathcal{M}_b(\mathcal{O})$.

Taking into account hypotheses (i)–(ii) and the fact that the operator $z \to -\Delta z + \eta(z)$ is monotone, coercive and bounded from $V = H_0^1(\mathcal{O})$ to $V' = H^{-1}(\mathcal{O})$, we infer (see Theorem 1 in [1]) that (3.1) has, for each $x \in L^2(\mathcal{O})$, a unique $(\mathcal{F}_t)_{t\geq 0}$-adapted solution $y : [0,T] \times \mathcal{O} \times \Omega \to \mathbb{R}$ which satisfies

$$\begin{aligned} &y \in L^2((0,T)\times\Omega; H_0^1(\mathcal{O})), \frac{\partial y}{\partial t} \in L^2((0,T)\times\Omega; H^{-1}(\mathcal{O})) \\ &y \in C([0,T]; L^2(\mathcal{O})),\ \forall \omega \in \Omega. \end{aligned} \tag{3.2}$$

We also have

Lemma 3.1. *The following estimates hold $\mathbb{P}$-a.s., $\omega \in \Omega$,*

$$\|y\|_{L^\infty(0,T;L^1(\mathcal{Q}))} + \|e^{-W}\varphi(e^W y)\|_{L^1(\mathcal{Q})} + \|y\|^p_{L^p(\mathcal{Q})} \leq C_1(\omega)|x|_1, \tag{3.3}$$

$$\|y\|_{L^r(0,T;W^{1,q}(\mathcal{O}))} \leq C_2(\omega)|x|_1, \tag{3.4}$$

where $C_i \in L^\ell(\Omega),\ i=1,2,\ \forall \ell \geq 1,$ and

$$1 \leq r, q < \infty;\ \frac{2}{r} + \frac{d}{q} > d+1. \tag{3.5}$$

Proof. Estimate (3.3) follows in a standard way by multiplying equation (3.1)$'$ by sgn y or, more precisely, by $\zeta(y)$, where ζ is a smooth approximation of the signum function

$$\text{sgn } y = \frac{y}{|y|} \text{ for } y \neq 0, \quad \text{sgn } 0 = 0,$$

and integrating on $\mathcal{Q}$. Taking into account that $\frac{\partial}{\partial t}|y| = y_t \text{ sgn } y$ and

$$-\int_{\mathcal{O}} \Delta y \text{ sgn } y \, d\xi \geq 0, \quad \varphi(e^W y)\text{sgn } y = |\varphi(e^W y)|,$$

we get via Gronwall's lemma that (3.3) holds.

To get (3.4), we shall use a duality argument first used in [5] (see also [2]). Namely, we rewrite (3.1)$'$ as

$$\begin{aligned} &\frac{\partial y}{\partial t} - \Delta y + ay + b \cdot \nabla y = f \overset{\text{def}}{=\!=} e^{-W}\varphi(e^W y) &&\text{in } \mathcal{Q}, \\ &y(0) = x &&\text{in } \mathcal{O}, \\ &y = 0 &&\text{on } \Sigma, \end{aligned} \tag{3.6}$$

and consider the backward linear parabolic equation

$$\begin{aligned} &\frac{\partial z}{\partial t} + \Delta z - az + \text{div}(bz) = \text{div } h &&\text{in } \mathcal{Q}, \\ &z(T) = 0 &&\text{in } \mathcal{O}, \\ &z = 0 &&\text{on } \Sigma, \end{aligned} \tag{3.7}$$

where $h = \{h_i\}_{i=1}^d \in (C^\infty(\overline{\mathcal{O}}))^d$. Equation (3.7) has, for each such h, a unique solution $z \in L^2(0,T; H_0^1(\mathcal{O}) \cap H^2(\mathcal{O}))$ with $\frac{\partial z}{\partial t} \in L^2(\mathcal{Q})$. Moreover, we have (see [9], Theorem III 7.1, p. 181) that $z \in L^\infty(\mathcal{Q})$ and

$$\|z\|_{L^\infty(\mathcal{Q})} \le C_3(\omega)\|h\|_{(L^s(0,T;W_0^{1,\ell}(\mathcal{O})))^d},$$

where

$$\frac{2}{s} + \frac{d}{\ell} < 1, \quad 1 < s, \ell < \infty,$$

and

$$0 \le C_3(\omega) \le C(\sup\{|W(t,\xi)|;\ (t,\xi) \in \mathcal{Q}\} + 1). \tag{3.8}$$

If multiply (3.6) by z and take into account (3.3), (3.7), we get

$$\begin{aligned}\int_{\mathcal{Q}} y \operatorname{div} h \, d\xi \, dt = \int_{\mathcal{Q}} f\, z \, dt \, d\xi &\le C_3 \|f\|_{L^2(\mathcal{Q})} \|h\|_{(L^s(0,T;W^{1,\ell}))^d} \\ &\le C_3 C_1 |x|_1 \sum_{j=1}^d \|h_i\|_{L^s(0,T;W_0^{1,\ell})}.\end{aligned}$$

By duality, this yields (3.4), (3.5) as claimed. □

Proposition 3.2. *Let $x \in \mathcal{M}_b$. Then there is a unique $(\mathcal{F}_t)_{t\ge 0}$-adapted process $y : [0,T] \to L^1(\mathcal{O})$ which satisfies*

$$\begin{aligned}&\int_{\mathcal{Q}} (y(\mathcal{X}_t + e^W \Delta(e^{-W}\mathcal{X}) - \mu\mathcal{X}) - e^{-W}\varphi(e^W y)\mathcal{X}) dt \, d\xi \\ &\qquad = -{}_{\mathcal{M}_b}\langle x, \mathcal{X}(0)\rangle_{C_0}, \forall \mathcal{X} \in C_0^{2,1}(\overline{\mathcal{Q}}),\ \mathcal{X}(T) = 0,\end{aligned} \tag{3.9}$$

$$y \in L^\infty(0,T; L^1(\mathcal{O})) \cap C((0,T]; L^1(\mathcal{O})) \cap L^p(0,T; W_0^{1,q}), \tag{3.10}$$

$$\lim_{t\to 0} \int_{\mathcal{O}} y(t,\xi)\psi(\xi) d\xi = {}_{\mathcal{M}_b}\langle x, \psi\rangle_{C_0},\ \forall \psi \in C_0, \tag{3.11}$$

$$\mathbb{E}\int_{\mathcal{Q}} |\varphi(e^W y)| dt \, d\xi < \infty, \tag{3.12}$$

$$\mathbb{E}\int_{\mathcal{Q}} |y(t,\xi)|^2 dt \, d\xi < \infty, \tag{3.13}$$

where $q \in \left[1, \frac{pd}{(p-1)d+p}\right)$. Moreover, estimates (3.3), (3.4) *hold and, if $x \in L^1(\mathcal{O})$, then $y \in C([0,T]; L^1(\mathcal{O}))$. Such a function is called the weak solution to* (3.1).

Proof. We fix $x \in \mathcal{M}_b$ and consider a sequence $\{x_n\} \subset L^2(\mathcal{O})$ such that, for $n \to \infty$, $x_n \to x$ weak-star in $\mathcal{M}_b$, that is,

$$\lim_{n\to\infty} \int_{\mathcal{O}} x_n \mathcal{X} \, d\xi = {}_{\mathcal{M}_b}\langle x, \mathcal{X}\rangle_{C_0},\ \forall \mathcal{X} \in C_0, \tag{3.14}$$

$$\limsup_{n\to\infty} |x_n|_1 \le \|x\|_{\mathcal{M}_b}. \tag{3.15}$$

Let $y_n \in C([0,T];L^2(\mathcal{O})) \cap L^2(0,T;H_0^1(\mathcal{O})) \cap W^{1,2}([0,T];H^{-1}(\mathcal{O}))$ be the corresponding solution to equation $(3.1)'$. By Lemma 3.1, we have

$$\|y_n\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \|y_n\|_{L^p(0,T;W_0^{1,q}(\mathcal{O}))} + \|e^{-W}\varphi(e^W y_n)\|_{L^1(\mathcal{Q})} \le C(\omega)|x_n|_1,\ C \in \bigcap_{q>1} L^q(\Omega),$$

and, therefore,

$$\limsup_{n\to\infty} \left\{ \|y_n\|_{L^\infty(0,T;L^1(\mathcal{O}))} + \|y_n\|_{L^p(0,T;W_0^{1,q}(\mathcal{O}))} + \|e^{-W}\varphi(e^W y_n)\|_{L^1(\mathcal{Q})} \right\} \le C(\omega)\|x\|_{\mathcal{M}_b} \tag{3.16}$$

for

$$\frac{2}{p} + \frac{d}{q} > d+1. \tag{3.17}$$

Recalling that by the Rellich–Kondrachov theorem (see, e.g., [3], p. 285) $W_0^{1,q}(\mathcal{O})$ is compactly embedded in $L^{p^*}(\mathcal{O})$ for $1 < p^* < \frac{dq}{d-q}$, it follows by (3.17) that we have

$$W_0^{1,q}(\mathcal{O}) \subset L^p(\mathcal{O}),\ \text{for } \tfrac{1}{q} < \tfrac{1}{d} - \tfrac{1}{p} \tag{3.18}$$

with dense and compact embedding. In particular, it follows by (1.4), (3.16), (3.18) that there is $\alpha \in (1,\infty)$ such that

$$\limsup_{n\to\infty} \|e^{-W}\varphi(e^W y_n)\|_{L^\alpha(0,T;L^\alpha(\mathcal{O}))} < \infty \tag{3.19}$$

and

$$\limsup_{n\to\infty} \|e^{-W}\Delta(e^W y_n)\|_{L^p(0,T;W^{-1,q'}(\mathcal{O}))} < \infty, \tag{3.20}$$

where $\frac{1}{q'} = 1 - \frac{1}{q}$. By (1.1), we also have

$$\limsup_{n\to\infty} \left\| \frac{\partial y_n}{\partial t} \right\|_{L^\alpha(0,T;L^\alpha(\mathcal{O})+W^{-1,q'}(\mathcal{O}))} < \infty. \tag{3.21}$$

Taking into account that

$$W_0^{1,q}(\mathcal{O}) \subset L^p(\mathcal{O}) \subset L^\alpha(\mathcal{O}) + W^{-1,q'}(\mathcal{O})$$

with the compact embedding of $W_0^{1,q}(\mathcal{O})$ into $L^p(\mathcal{O})$, it follows by estimates (3.16), (3.21) that the sequence $\{y_n\}_{n=1}^\infty$ is compact in $L^p(0,T;L^p(\mathcal{O})) = L^p(\mathcal{Q})$ (see Theorem 5.1 in [10], Chapter 2).

Hence, on a subsequence, again denoted $\{n\}$, we have

$$y_n \longrightarrow y \text{ strongly in } L^p(\mathcal{Q}) \text{ weakly in } L^p(0,T;W_0^{1,q}(\mathcal{O})) \tag{3.22}$$

$$e^{-W}\varphi(e^W y_n) \longrightarrow e^{-W}\varphi(e^W y) \text{ strongly in } L^1(\mathcal{Q}) \tag{3.23}$$

$$\frac{dy_n}{dt} \longrightarrow \frac{dy}{dt} \text{ in } \mathcal{D}'(\mathcal{Q}) \tag{3.24}$$

$$e^{-W}\Delta(e^W y_n) \longrightarrow e^{-W}\Delta(e^W y) \text{ weakly in } L^p(0,T;W_0^{-1,q}(\mathcal{O})). \tag{3.25}$$

Moreover, multiplying the equation

$$
\begin{aligned}
&\frac{\partial y_n}{\partial t} - e^{-W}\Delta(e^W y_n) + e^{-W_n}\varphi(e^W y_n) + \mu, y_n(0) = x_n \text{ in } \mathcal{O},\\
&y_n = 0 \text{ on } \Sigma,
\end{aligned} \tag{3.26}
$$

by $\mathcal{X} \in C_0^{2,1}(\overline{\mathcal{Q}})$ with $\mathcal{X}(T) = 0$ and integrating on $\mathcal{Q}$, it follows by (3.23), (3.25) that y satisfies equation (3.9).

By estimates (3.16) and by (3.23), (3.24), it is also clear that y satisfies (3.10), (3.12), (3.13). (In fact, the later is implied by (3.16) if one takes into account that, under our conditions on q, it follows that $W_0^{1,q}(\mathcal{O}) \subset L^2(\mathcal{O})$.)

It remains to prove $\mathcal{M}_b(\mathcal{O})$-continuity condition (3.11). To this end, we note that by (3.26), written as (3.1)$'$ with $y = y_n$, we have

$$
y_n(t) = S(t)x_n - \int_0^t S(t-s)(ay_n(s) + b\cdot\nabla y_n(s) + e^{-W}\varphi(e^W y_n(s)))ds, \quad t \in [0,T], \tag{3.27}
$$

where $S(t)$ is the C_0-semigroup on $L^1(\mathcal{O})$ generated by the operator Δ with the domain $\{y \in L^1(\mathcal{O}); \Delta y \in L^1(\mathcal{O}) = 0 \text{ on } \partial\mathcal{O}\}$.

By (3.22), (3.23), it follows that

$$
\begin{aligned}
&\int_{\mathcal{O}} y(t,\xi)\psi(\xi)d\xi = \lim_{n\to\infty}\Big\{\int_{\mathcal{O}} S(t)x_n(\xi)\psi(\xi)d\xi\\
&-\int_0^t ds\int_{\mathcal{O}} S(t-s)(ay_n(s,\xi) + b\cdot\nabla y_n(s,\xi) + e^{-W}\varphi(e^W y_n(s,\xi)))\psi(\xi)d\xi\Big\}\\
&= {}_{\mathcal{M}_b}\langle S^*(t)x, \psi\rangle_{C_0}\\
&-\int_0^t ds\int_{\mathcal{O}} S(t-s)(ay(s,\xi) + b\cdot\nabla y(s,\xi) + e^{-W}\varphi(e^W y(s,\xi)))\psi(s)d\xi,\\
&\qquad\qquad \forall \psi \in C_0(\overline{\mathcal{O}}),
\end{aligned} \tag{3.28}
$$

where $S^*(t) : \mathcal{M}_b(\mathcal{O}) \to \mathcal{M}_b(\mathcal{O})$ is the dual of the semigroup $S(t)$. We note that S is not a continuous semigroup on $\mathcal{M}_b(\mathcal{O})$ but it is, however, continuous in t with respect to the weak-star topology of $\mathcal{M}_b(\mathcal{O})$. (See [12], p. 39.) Then, by (3.28), it follows that

$$
\lim_{t\to 0}\int_{\mathcal{O}} y(t,\xi)\psi(\xi)d\xi = {}_{\mathcal{M}_b}\langle x, \psi\rangle_{C_0}, \ \forall \psi \in C_0,
$$

as claimed.

Taking into account that, by (3.27), we have

$$
y(t) = S^*(t)x - \int_0^t S(t-s)(ay(s) + b\cdot\nabla y(s) + e^{-W}\varphi(e^W y(s)))ds, \ \forall t \in [0,T],
$$

and that $S^*(t) = S(t)$ on $(0,\infty)$, it follows also that, if $x \in L^1(\mathcal{O})$, then $y \in C([0,T]; L^1(\mathcal{O}))$. Since y_n are $(\mathcal{F}_t)_{t\ge 0}$-adapted and, as shown below, the limit y is itself an $(\mathcal{F}_t)_{t\ge 0}$-adapted. This completes the proof of the existence.

Uniqueness. Let y_1, y_2 be two weak solutions satisfying (3.10)–(3.12). We set $z = y_1 - y_2$ and $g = e^{-W}(\varphi(e^W y_1) - \varphi(e^W y_2))$. By (3.9),

$$\int_Q z(\mathcal{X}_t + \Delta\mathcal{X} - a\mathcal{X} + \operatorname{div}(b\mathcal{X}))dt\,d\xi = -\int_Q g\mathcal{X}\,dt\,d\xi, \quad \forall \mathcal{X} \in C_0^{2,1}(\overline{Q}),\ \mathcal{X}(T) = 0. \tag{3.29}$$

Formally, we see by (3.29) that, for $\mathcal{X} = \operatorname{sgn} z$, we get

$$|z(t)|_1 \le C\int_0^t |z(t)|_1 ds,$$

that is, $z \equiv 0$, because

$$\int_Q \Delta z \operatorname{sgn} z \le 0,\ \int_Q g \operatorname{sgn} z\,dt\,d\xi \ge 0,$$

by monotonicity of g, and a, b are smooth and bounded. This heuristic (formal) argument can be transformed in a rigorous one as in [3] if one take $\mathcal{X} = \zeta_\varepsilon(z_\varepsilon)$, where ζ_ε is a smooth approximation of the signum function and z_ε is a smooth approximation of z. The details are omitted. □

Proof of Theorem 2.2. Let y be the weak solution to equation (3.1) given by Proposition 3.2. By the uniqueness of the solution y, it follows that the sequence $\{y_n\}$ arising in (3.22)–(3.25) is independent on $\omega \in \Omega$ and, since y_n is $(\mathcal{F}_t)_{t\ge0}$-adapted, it follows that y is $(\mathcal{F}_t)_{t\ge0}$-adapted, too. Taking into account that by the Itô formula we have (see [1])

$$d(e^W y) = e^W dy + e^W y\,dW + \mu e^W,$$

we get

$$e^W y\,d\mathcal{X} = d(e^W y\mathcal{X}) - \mathcal{X}e^W dy - \mathcal{X}e^W y\,dW - \mu\mathcal{X}e^W y,\ \forall \mathcal{X} \in C_0^{2,1}(\overline{Q}),$$

and substituting in (3.9), it follows that X satisfies (2.3). As regards (2.1), (2.2), (2.4), (2.5), these follow by the corresponding properties of y given in Proposition 3.2. The uniqueness of the solution X follows as in the previous case by choosing an appropriate function $\mathcal{X}$ in (2.3). This completes the proof. □

Remark 3.3. Theorem 2.2 extends *mutatis mutandis* to equation (3.1) with additive Gaussian noise, that is,

$$\begin{aligned} &dX - \Delta X\,dt + \varphi(X)dt = dW \text{ in } (0,T)\times\mathcal{O},\\ &X(0) = x \text{ in } \mathcal{O}, \quad X = 0 \text{ on } \Sigma. \end{aligned} \tag{3.30}$$

Indeed, by the substitution $y = X - W$, one reduces equation (3.30) to the random parabolic equation

$$\begin{aligned} &y_t - \Delta y + \varphi(y + W) = \Delta W \text{ in } (0,T)\times\mathcal{O},\\ &y(0) = x \text{ in } \mathcal{O}, \quad y = 0 \text{ on } \Sigma. \end{aligned} \tag{3.31}$$

The treatment of equation (3.31) is similar to that of equation (3.1) if one takes into account that $\Delta W, \varphi(W)$ are regular and φ is monotonically nondecreasing. So, proceeding as above, one proves that (3.30) has, under assumptions (i), (ii), a unique weak solution X but we omit the details.

Remark 3.4. Inspecting the proof, we see that Theorem 2.2 remains true if φ is merely a maximal monotone (multivalued) graph in $\mathbb{R}$ satisfying the growth conditions (1.4)-(1.5). In particular, it is true for discontinuous monotonically nondecreasing functions φ of the form

$$\varphi(r) = \sum_{i=1}^{m} a_i H(r - r_i)\varphi_i(r),\ \forall r \in \mathbb{R},$$

where φ_i satisfy condition (1.4), $a_i \geq 0$, $\{r_i\}_{i=1}^m \subset \mathbb{R}$, and H is the Heviside function.

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