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A MULTIPLICITY THEOREM FOR NONCOERCIVE $(p,2)$ –EQUATIONS

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Abstract. We study a nonlinear Dirichlet problem driven by the sum of a $p-$Laplacian and of a Laplacian (a $(p,2)$ −equation). The reaction is $(p-1)$ −linear near $\pm\infty$ and linear near 0. First we obtain two nontrivial solutions of constant sign. Then by strengthening the regularity of the reaction and by using critical groups (Morse theory), we produce a third nontrivial smooth solution (three solution theorem).

1. Introduction

Let $\Omega \subseteq \mathbb{R}^N$ be a bounded domain with a C^2- boundary $\partial\Omega$. In this paper we study the following nonlinear, nonhomogeneous Dirichlet problem (a $(p,2)$-equation)

$$-\Delta_p u(z) - \Delta u(z) = f(z, u(z)) \text{ in } \Omega,\ u\mid_{\partial\Omega} = 0,\ 2 < p < \infty. \tag{1.1}$$

In this problem Δ_p denotes the $p-$Laplace differential operator defined by

$$\Delta_p u = div\left(|Du|^{p-2} Du\right), \text{ for all } u \in W_0^{1,p}(\Omega),$$

where $|\cdot|$ denotes the norm in $\mathbb{R}^N$. So, on the left hand side of problem (1.1) we have a combination of two differential operators of a different nature. One is the Laplacian (linear) and the other is the $p-$Laplacian (nonlinear). The resulting operator is nonhomogeneous and this is a source of difficulties in the analysis of problem (1.1).

Such operators arise in mathematical models of physical processes. In this direction we mention the works of Cherfils-Ilyasov [6] (reaction-diffusion systems) and Zhikov [28] (homogenization of composites consisting of two different materials with distinct hardening exponents, double phase problems).

In the reaction (right hand side of (1.1)), we have a Carathéodory function $f(z,x)$ (that is, for all $x \in \mathbb{R}$, $z \to f(z,x)$ is measurable and for a.a. $z \in \Omega$, $x \to f(z,x)$ is continuous).

We assume that $f(z,.)$ is $(p-1)$ −linear near $\pm\infty$, and stays above the principal eigenvalue $\widehat{\lambda}_1(p) > 0$ of $(-\Delta_p, W_0^{1,p}(\Omega))$. This makes the energy functional of the problem noncoercive. Near zero, $f(z,.)$ is linear and stays below the principal eigenvalue $\widehat{\lambda}_1(2) > 0$ of $\left(-\Delta, H_0^1(\Omega)\right)$. We mention that no global sign condition

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is imposed on $f(z,.)$. Under these general conditions, we produce two constant sign smooth solutions for problem (1.1). This way, we extend Theorem 1 of Zhang-Liang [27], where the authors produce a positive solution (without any regularity claim) under considerable more restrictive conditions on the reaction term $f(z,x)$. Then by strengthening the hypotheses on $f(z,.)$ (we assume that for a.a. $z \in \Omega$, $f(z,.) \in C^1(\mathbb{R})$) and by using tools from the theory of critical groups (Morse theory), we generate a third nontrivial smooth solution.

Three solutions theorems for Dirichlet $p-$Laplacian equations were proved by Gasinski-Papageorgiou [11], Guo-Liu [12], Liu [17], Liu-Liu [18], Papageor-giou-Papageorgiou [19], using a sign condition on the reaction. For $(p,2)$-equations there have been various recent multiplicity results, but for different settings. We mention the works of Aizicovici-Papageorgiou-Staicu [2], [3], [4], Liang-Han-Li [14], Papageorgiou-Rădulescu [20], Papageorgiou-Vetro-Vetro [22], Papageorgiou-Winkert [23], Sun [25], Sun-Zhang-Su [26].

2. Mathematical background

Let X be a Banach space and X^* be its topological dual. By $\langle .,. \rangle$ we denote the duality brackets for the pair (X^*, X). Also $\xrightarrow{w}$ will designate weak convergence in X.

The main spaces in the analysis of problem (1.1) are the Sobolev space $W_0^{1,p}(\Omega)$ and the Banach space $C_0^1(\overline{\Omega})$. By $\|.\|$ we will denote the norm of $W_0^{1,p}(\Omega)$. According to the Poincaré inequality (see, Gasinski-Papageorgiou [9], p.216), we can say that $\|u\| = \|Du\|_p$ for all $u \in W_0^{1,p}(\Omega)$.where $\|.\|_p$ stands for the L^p-norm.

The Banach space $C_0^1(\overline{\Omega}) = \left\{u \in C^1\left(\overline{\Omega}\right) : u(z) = 0 \text{ for all } z \in \partial\Omega\right\}$ is ordered with positive (order) cone

$$C_+ = \left\{u \in C_0^1\left(\overline{\Omega}\right) : u(z) \geq 0 \text{ for all } z \in \overline{\Omega}\right\}.$$

This cone has a nonempty interior given by

$$int\, C_+ = \left\{u \in C_+ : u(z) > 0 \text{ for all } z \in \Omega,\ \frac{\partial u}{\partial n} = (Du, n)_{\mathbb{R}^N} < 0 \text{ on } \partial\Omega\right\},$$

where $n(.)$ is the outward unit normal on $\partial\Omega$.

Recall that a function $\varphi \in C^1\left(W_0^{1,p}(\Omega)\right)$ is said to satisfy the *Cerami condition* (*C-condition*, for short) if the following property holds:

"*every sequence* $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ *such that* $\{\varphi(u_n)\}_{n\in\mathbb{N}} \subset \mathbb{R}$ *is bounded and*

$$(1 + \|u_n\|)\, \varphi'(u_n) \to 0 \text{ in } X^* \text{as } n \to \infty$$

admits a strongly convergent subsequence".

For $r \in (1,\infty)$, by $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega) := \left(W_0^{1,r}(\Omega\right)^*$ ($\frac{1}{r} + \frac{1}{r'} = 1$), we denote the nonlinear map defined by

$$\langle A_r(u), h\rangle = \int_\Omega |Du|^{r-2} (Du, Dh)_{\mathbb{R}^N}\, dz \text{ for all } u,\, h \in W_0^{1,r}(\Omega). \tag{2.1}$$

When $r = 2$, then we write $A_2 = A \in \mathcal{L}\left(H_0^1(\Omega), H_0^{-1}(\Omega)\right)$. The next lemma summarizes the main properties of A_r (see [9]).

Lemma 2.1. *The map* $A_r : W_0^{1,r}(\Omega) \to W^{-1,r'}(\Omega)$ *defined by* (2.1) *is bounded (that is, maps bounded sets to bounded sets), continuous, strictly monotone (hence maximal monotone, too), and of type* $(S)_+$, *that is, for every sequence* $\{u_n\}_{n\geq 1} \subseteq X$ *such that* $u_n \xrightarrow{w} u$ *and*

$$\limsup_{n\to\infty} \langle A_r(u_n), u_n - u\rangle \leq 0,$$

one has

$$u_n \to u \text{ in } W_0^{1,r}(\Omega) \text{ as } n \to \infty.$$

We will also use the spectra of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$ and of $\left(-\Delta, H_0^1(\Omega)\right)$. So, we consider the following nonlinear eigenvalue problem:

$$-\Delta_r u(z) = \widehat{\lambda}\,|u(z)|^{r-2}\,u(z) \text{ in } \Omega,\ u\,|_{\partial\Omega}= 0\ \ (1 < r < \infty). \tag{2.2}$$

We say that $\widehat{\lambda} \in \mathbb{R}$ is an eigenvalue for problem (2.2), if there exists a nontrivial solution $\widehat{u} \in W_0^{1,p}(\Omega)$, known as an *eigenfunction* corresponding to $\widehat{\lambda}$. The set of eigenvalues of (2.2) is denoted by $\widehat{\sigma}(r)$.

We know that problem (2.2) admits a smallest eigenvalue $\widehat{\lambda}_1(r) > 0$ which has the following properties:

- (a) $\widehat{\lambda}_1(r)$ is isolated (that is, there exists $\varepsilon > 0$ such that there are no eigenvalues in $\left(\widehat{\lambda}_1(r), \widehat{\lambda}_1(r) + \varepsilon\right)$);
- (b) $\lambda_1(r)$ is simple (that is, if $\widehat{u}$, $\widehat{v} \in W_0^{1,r}(\Omega)$ are eigenfunctions corresponding to $\widehat{\lambda}_1(r)$, then $\widehat{u} = \xi\widehat{v}$ with $\xi \in \mathbb{R}\backslash\{0\}$);
- (c) One has

$$\widehat{\lambda}_1(r) = \inf\left\{\frac{\|Du\|_r^r}{\|u\|_r^r} : u \in W_0^{1,r}(\Omega),\ u \neq 0\right\} > 0. \tag{2.3}$$

In (2.3) the infimum is achieved on the corresponding one-dimensional eigenspace (see (b)). The elements of this eigenspace do not change sign. By $\widehat{u}_1(r)$ we denote the positive L^r- normalized (that is, $\|\widehat{u}_1(r)\|_r = 1$) positive eigenfunction corresponding to $\widehat{\lambda}_1(r)$. The nonlinear regularity theory and the nonlinear maximum principle (see, for example, Gasinski-Papageorgiou [9], pp.737-738) imply that $\widehat{u}_1(r) \in int\ \ C_+$. These properties lead to the following straightforward lemma (see Gasinski-Papageorgiou ([10], Problem 5.67, p.857).

Lemma 2.2. *If* $\xi \in L^\infty(\Omega)$, $\xi(z) \leq \widehat{\lambda}_1(r)$ *for a.a.*$z \in \Omega$, $\xi \neq \widehat{\lambda}_1(r)$, *then*

$$\|Du\|_r^r - \int_\Omega \xi(z)\,|u(z)|^r\,dz \geq C_0\,\|u\|^r \text{ for some } C_0 > 0,\ \text{all } u \in W_0^{1,r}(\Omega).$$

The Lyusternik-Schnirelmann minimax scheme (see Gasinski-Papageorgiou [9], Section 5.5) generates a whole strictly increasing sequence $\left\{\widehat{\lambda}_k(r)\right\}_{k\geq 1}$ of eigenvalues of (2.2) such that $\widehat{\lambda}_k(r) \to +\infty$ as $k \to \infty$. These eigenvalues are known as *variational eigenvalues.* We do not known if this sequence exhausts the spectrum of (2.2).

Here we will use the variational eigenvalues obtained by using in the Lyusternik-Schnirelmann scheme, the Fadell-Rabinowitz cohomological index denoted by $ind\,(\cdot)$ (see [8]). We mention that if $\widehat{u}$ is an eigenfunction corresponding to any eigenvalue $\widehat{\lambda} \neq \widehat{\lambda}_1 (r)$, then $\widehat{u} \in C_0^1 \left(\overline{\Omega}\right)$ and $\widehat{u}$ is nodal (that is, sign changing).

If $r = 2$, then $\{\widehat{\lambda}_k (2)\}_{k\geq 1}$ exhausts the spectrum, all eigenvalues have finite dimensional eigenspaces $E(\widehat{\lambda}_k (2)) \subseteq C_0^1(\overline{\Omega})$ and

$$H_0^1 (\Omega) = \overline{\bigoplus_{k\geq 1} E\left(\widehat{\lambda}_k (2)\right)}$$

(orthogonal direct sum).

We will also use a weighted version of (2.2). So, let $\eta \in L^\infty (\Omega)$, $\eta (z) \geq 0$ for a.a.$z \in \Omega$, $\eta \neq 0$, and consider the following nonlinear eigenvalue problem

$$-\Delta_r u (z) = \widetilde{\lambda}\eta (z) |u (z)|^{r-2} u (z) \text{ in } \Omega,\ u \mid_{\partial\Omega}= 0. \tag{2.4}$$

The same results can be deduced for (2.4) and in this case the variational characterization of the principal eigenvalue $\widetilde{\lambda}_1 (r, m) > 0$ is

$$\widetilde{\lambda}_1 (r, \eta) = \inf \left\{ \frac{\|Du\|_r^r}{\int_\Omega \eta (z) |u (z)|^r dz} : u \in W_0^{1,r} (\Omega),\ u \neq 0 \right\}. \tag{2.5}$$

Again the infimum in (2.5) is realized on the corresponding one dimensional eigenspace. As before by $\widehat{u}_1 (r, \eta) \in int \ \ C_+$ we denote the positive, L^r- normalized eigenfunction corresponding to $\widetilde{\lambda}_1 (r, \eta) > 0$. The aforementioned properties lead easily to the following strict monotonicity property of the map $\eta \to \widetilde{\lambda}_1 (r, \eta)$:

Lemma 2.3. *If $\eta_1, \eta_2 \in L^\infty (\Omega)$, $0 \leq \eta_1 (z) \leq \eta_2 (z)$ for a.a.$z \in \Omega$, $\eta_1 \neq 0$, $\eta_2 \neq \eta_1$, then*

$$\widetilde{\lambda}_1 (r, \eta_2) < \widetilde{\lambda}_1 (r, \eta_1).$$

Now let us recall some basic facts about critical groups. So, let X be a Banach space, $\varphi \in C^1 (X, \mathbb{R})$ and $c \in \mathbb{R}$. We introduce the following sets:

$$\varphi^c = \{u \in X : \varphi (u) \leq c\},$$
$$K_\varphi = \left\{u \in X : \varphi' (u) = 0\right\} \text{ (the critical set of } \varphi\text{)},$$

and

$$K_\varphi^c = \{u \in K_\varphi : \varphi (x) = c\}.$$

Also, given a topological pair (Y_1, Y_2) with $Y_2 \subset Y_1 \subset X$ and $k \in \mathbb{N}_0$, by $H_k (Y_1, Y_2)$ we denote the k^{th}- relative singular homology group for the pair (Y_1, Y_2) with integer coefficients.

Now, if $u \in K_\varphi^c$ is isolated, then the *critical groups of* φ at u are defined by

$$C_k (\varphi, u) = H_k (\varphi^c \cap U, (\varphi^c \cap U) \setminus \{u\}) \text{ for all } k \in \mathbb{N}_0.$$

Here U is an isolating neighborhood of u, that is, $K_\varphi \cap \varphi^c \cap U = \{u\}$. The excision property of singular homology implies that this definition of critical groups is independent of the choice of the isolating neighborhood U.

If $\varphi \in C^1(X,\mathbb{R})$ satisfies the $C-$condition and $\inf \varphi(K_\varphi) > -\infty$, then we can define the *critical groups of* φ *at infinity* by

$$C_k(\varphi,\infty) = H_k(X,\varphi^c) \text{ for all } k \in \mathbb{N}_0 \text{ and with } c < \inf \varphi(K_\varphi).$$

This definition is independent of the choice of the level $c < \inf \varphi(K_\varphi)$ (see Papageorgiou-Rădulescu-Repovs [21], p.402). If $K_\varphi = \{u_0\}$, then $C_k(\varphi,\infty) = C_k(\varphi,u_0)$ for all $k \in \mathbb{N}_0$. Also, if $C_{k_0}(\varphi,\infty) \neq 0$, then we can find $u_0 \in K_\varphi$ such that $C_k(\varphi,u_0) \neq 0$.

Finally, let us outline some basic notation. If $x \in \mathbb{R}$, then we set $x^\pm = \max\{\pm x, 0\}$ and then for $u \in W_0^{1,p}(\Omega)$, we define $u^\pm(.) = u(.)^\pm$. We know that

$$u^\pm \in W_0^{1,p}(\Omega), \; u = u^+ - u^- \text{ and } |u| = u^+ + u^-.$$

Given a measurable function $g : \Omega\times\mathbb{R} \to\mathbb{R}$ (for example a Carathéodory function), by $N_g(\cdot)$ we denote the Nemytski (superposition) map corresponding to g, that is,

$$N_g(u)(\cdot) = g(\cdot, u(\cdot)) \text{ for all } u \in W_0^{1,p}(\Omega).$$

By $|\cdot|_N$ we denote the Lebesgue measure on $\mathbb{R}^N$, by $\sigma(p) \subseteq (0,\infty)$ the spectrum of $\left(-\Delta_p, W_0^{1,p}(\Omega)\right)$, and by $\delta_{k,m}$ the Kronecker symbol, defined by

$$\delta_{k,m} = \begin{cases} 1 & \text{if } k = m, \\ 0 & \text{if } k \neq m, \end{cases}$$

where k, $m \in \mathbb{N}_0$.

3. Solutions of constant sign

In this section we produce two constant sign smooth solutions for problem (1.1) (a positive solution and a negative solution). The hypotheses on the reaction $f(z,x)$ are the following:

$(\mathbf{H}_f)_1$: $f : \Omega \times \mathbb{R} \to\mathbb{R}$ is a Carathéodory function such that $f(z,0) = 0$ for a.a. $z \in \Omega$ and

(*i*) there exists a function $a \in L^\infty(\Omega)$ such that

$$|f(z,x)| \leq a(z)\left(1 + |x|^{p-1}\right) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R};$$

(*ii*) there exists $\eta \in L^\infty(\Omega)$ such that

$$\eta(z) \geq \widehat{\lambda}_1(p) \text{ for a.a. } z \in \Omega, \; \eta \neq \widehat{\lambda}_1(p),$$

$$\liminf_{x\to\pm\infty} \frac{f(z,x)}{|x|^{p-2}x} \geq \eta(z) \text{ uniformly for a.a. } z \in \Omega;$$

(*iii*) there exists a function $\theta \in L^\infty(\Omega)$ such that

$$\theta(z) \leq \widehat{\lambda}_1(2) \text{ for a.a. } z \in \Omega, \; \theta \neq \widehat{\lambda}_1(2),$$

$$\limsup_{x\to 0} \frac{f(z,x)}{x} \leq \theta(z) \text{ uniformly for a.a. } z \in \Omega.$$

Remark. We stress that no global sign condition is imposed on $f(z,\cdot)$, in contrast to most works in the literature producing solutions with fixed signs (see the references mentioned in the Introduction).

Proposition 3.1. *If hypotheses* $(\mathbf{H}_f)_1$ *hold, then problem* (1.1) *has at least two nontrivial constant sign solutions* $u_0 \in int \quad C_+$ *and* $v_0 \in -int \quad C_+$.

Proof. Let $F(z,x) = \int_0^x f(z,s)\,ds$ and consider the the C^1-functional $\varphi_+ : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_+(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F\left(z,u^+\right)dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

Hypotheses $(\mathbf{H}_f)_1(i), (iii)$ imply that given $\varepsilon > 0$, we can find $C_\varepsilon > 0$ such that

$$F(z,x) \le \frac{1}{2}[\theta(z)+\varepsilon]x^2 + C_\varepsilon |x|^p \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}. \tag{3.1}$$

Then, for $u \in W_0^{1,p}(\Omega)$, we have

$$\begin{aligned}\varphi_+(u) &\ge \frac{1}{2}\left[\|Du\|_2^2 - \int_\Omega \theta(z)u^2dz - \varepsilon\|u\|^2\right] - C_1\|u\|^p \\ &\qquad \text{for some } C_1 = C_1(\varepsilon) \ \text{(see (3.1))} \\ &\ge \frac{1}{2}[C_2-\varepsilon]\|u\|^2 - C_1\|u\|^p \\ &\qquad \text{for some } C_2 > 0 \text{ (see Lemma 2.2).}\end{aligned}$$

Choosing $\varepsilon \in (0, C_2)$, we obtain

$$\varphi_+(u) \ge C_3\|u\|^2 - C_1\|u\|^p \text{ for some } C_3 > 0, \text{ all } u \in W_0^{1,p}(\Omega). \tag{3.2}$$

Since $p > 2$, from (3.2) it follows that there exists $\rho \in (0,1)$ small such that

$$\varphi_+(u) > 0 = \varphi_+(0) \text{ for all } u \text{ with } 0 < \|u\| \le \rho. \tag{3.3}$$

Hypotheses $(\mathbf{H}_f)_1(i), (ii)$ imply that given $\varepsilon > 0$, we can find $\widehat{C}_\varepsilon > 0$ such that

$$F(z,x) \ge \frac{1}{p}[\eta(z)-\varepsilon]|x|^p - \widehat{C}_\varepsilon \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R}.$$

Then for $t > 0$, we have

$$\begin{aligned}&\varphi_+(t\widehat{u}_1(p)) \\ &= \frac{t^p}{p}\|D\widehat{u}_1(p)\|_p^p + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 - \int_\Omega F(z,t\widehat{u}_1(p))\,dz \\ &\le \frac{t^p}{p}\widehat{\lambda}_1(p) + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 - \frac{t^p}{p}\int_\Omega \eta(z)\,\widehat{u}_1(p)^p\,dz + \frac{\varepsilon t^p}{p} + C_4 \\ &\text{for some } C_4 = C_4(\varepsilon) > 0 \text{ (recall that } \|\widehat{u}_1(p)\|_p = 1) \\ &= \frac{t^p}{p}\left[\int_\Omega \left(\widehat{\lambda}_1(p) - \eta(z)\right)\widehat{u}_1(p)^p\,dz + \varepsilon\right] + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 + C_4.\end{aligned}$$

Since $\widehat{u}_1(p) \in int \quad C_+$, by the hypothesis on $\eta(\cdot)$ (see $(\mathbf{H}_f)_1(ii)$), we have

$$\beta_0 = \int_\Omega \left(\widehat{\lambda}_1(p) - \eta(z)\right)\widehat{u}_1(p)^p\,dz > 0.$$

Therefore

$$\varphi_+(t\widehat{u}_1(p)) \le \frac{t^p}{p}[-\beta_0+\varepsilon] + \frac{t^2}{2}\|D\widehat{u}_1(p)\|_2^2 + C_4 \text{ for all } t > 0.$$

Choosing $\varepsilon \in (0, \beta_0)$ and recalling that $2 < p$, we see that

$$\varphi_+ (t\widehat{u}_1 (p)) \to -\infty \text{ as } t \to +\infty. \tag{3.4}$$

Claim. φ_+ satisfies the $C-$condition.

Consider a sequence $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p} (\Omega)$ such that $\{\varphi_+ (u_n)\}_{n\geq 1} \subseteq \mathbb{R}$ is bounded and

$$(1 + \|u_n\|) (\varphi_+)' (u_n) \to 0 \text{ in } W^{-1,p'} (\Omega) = W_0^{1,p} (\Omega)^* \text{ as } n \to \infty. \tag{3.5}$$

From (3.5) we have

$$\begin{aligned} &\left|\langle A_p (u_n), h\rangle + \langle A (u_n), h\rangle - \textstyle\int_\Omega f (z, u_n^+) h dz\right| \leq \tfrac{\varepsilon_n \|h\|}{1+\|u_n\|} \\ &\qquad \text{for all } h \in W_0^{1,p} (\Omega), \text{ with } \varepsilon_n \to 0^+. \end{aligned} \tag{3.6}$$

In (3.6) we choose $h = -u_n^- \in W_0^{1,p} (\Omega)$. Then

$$\left\|Du_n^-\right\|_p^p + \left\|Du_n^-\right\|_2^2 \leq \varepsilon_n \text{ for all } n \in \mathbb{N},$$

hence

$$u_n^- \to 0 \text{ in } W_0^{1,p} (\Omega). \tag{3.7}$$

We show that $\{u_n^+\}_{n\geq 1} \subseteq W_0^{1,p} (\Omega)$ is bounded. Arguing by contradiction, assume that at least for a subsequence we have

$$\left\|u_n^+\right\| \to \infty, \text{ as } n \to \infty. \tag{3.8}$$

We set $y_n = \frac{u_n^+}{\|u_n^+\|}$, $n \in \mathbb{N}$. Then

$$\|y_n\| = 1, \; y_n \geq 0, \text{ for all } n \in \mathbb{N}.$$

So, we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p} (\Omega) \text{ as n} \to \infty. \tag{3.9}$$

Using (3.7) in (3.6), we obtain

$$\begin{aligned} &\left|\langle A_p (u_n^+), h\rangle + \langle A (u_n^+), h\rangle - \textstyle\int_\Omega f (z, u_n^+) h dz\right| \\ &\leq \varepsilon_n' \|h\| \text{ for all } h \in W_0^{1,p} (\Omega), \text{with } \varepsilon_n' \to 0^+. \end{aligned}$$

hence

$$\begin{aligned} &\left|\langle A_p (y_n), h\rangle + \tfrac{1}{\|u_n^+\|^{p-2}} \langle A (y_n), h\rangle - \textstyle\int_\Omega \tfrac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} h dz\right| \leq \tfrac{\varepsilon_n' \|h\|}{\|u_n^+\|^{p-1}} \\ &\qquad \text{for all } h \in W_0^{1,p} (\Omega), \text{all } n \in \mathbb{N}. \end{aligned} \tag{3.10}$$

From hypothesis $(\mathbf{H}_f)_1 (i)$ and (3.8) it is clear that

$$\left\{\frac{N_f (u_n^+)}{\|u_n^+\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'} (\Omega) \text{ is bounded.}$$

So, if in (3.10) we choose $h = y_n - y \in W_0^{1,p} (\Omega)$, pass to the limit as $n \to \infty$ and use (3.8), then

$$\lim_{n\to\infty} \langle A_p (y_n), y_n - y\rangle = 0,$$

hence

$$y_n \to y \text{ in } W_0^{1,p}(\Omega) \text{ and } \|y\| = 1,\ y \geq 0 \tag{3.11}$$

(see Lemma 2.1). The boundedness of $\left\{ \frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \right\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ and hypothesis $(\mathbf{H}_f)_1\,(iii)$ imply that by passing to a subsequence if necessary, we have

$$\frac{N_f(u_n^+)}{\|u_n^+\|^{p-1}} \xrightarrow{w} \widehat{\eta}(z)\, y^{p-1} \text{ in } L^{p'}(\Omega) \tag{3.12}$$

with

$$\eta(z) \leq \widehat{\eta}(z) \leq C_5 \text{ for a.a. } z \in \Omega, \text{ and some } C_5 > 0$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 16).

So, if in (3.10) we pass to the limit as $n \to \infty$ and use (3.11), (3.8) and (3.12) we obtain

$$\langle A_p(y), h\rangle = \int_\Omega \widehat{\eta}(z)\, y^{p-1} h dz \text{ for all } h \in W_0^{1,p}(\Omega),$$

therefore

$$-\Delta_p y(z) = \widehat{\eta}(z)\, y(z)^{p-1} \text{ for } a.a.z \in \Omega,\ y\,|_{\partial\Omega} = 0. \tag{3.13}$$

Using Lemma 2.3, we have

$$\widetilde{\lambda}_1(p, \widehat{\eta}) \leq \widetilde{\lambda}_1(p, \eta) < \widetilde{\lambda}_1\left(p, \widehat{\lambda}_1(p)\right) = 1,$$

hence y must be nodal (see (3.11)). This contradicts (3.11). Therefore

$$\left\{u_n^+\right\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded,}$$

and consequently

$$\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded (see (3.7)).}$$

So, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W_0^{1,p}(\Omega) \text{ as n} \to \infty. \tag{3.14}$$

Since $\{N_f(u_n^+)\}_{n\geq 1} \subseteq L^{p'}(\Omega)$ is bounded (see hypothesis $(\mathbf{H}_f)_1\,(i)$ and (3.14)), if in (3.6) we choose $h = u_n - u \in W_0^{1,p}(\Omega)$ and pass to the limit as $n \to \infty$, we obtain

$$\lim_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u_n), u_n - u\rangle\right] = 0,$$

hence

$$\limsup_{n\to\infty} \left[\langle A_p(u_n), u_n - u\rangle + \langle A(u), u_n - u\rangle\right] \leq 0,$$

(since $A(\cdot)$ is monotone), therefore

$$\limsup_{n\to\infty} \langle A_p(u_n), u_n - u\rangle \leq 0 \text{ (cf. (3.14))},$$

and consequently

$$u_n \to u \text{ in } W_0^{1,p}(\Omega),$$

(see Lemma 2.1). We conclude that φ_+ satisfies the $C-$condition. This proves the Claim.

Then (3.3), (3.4) and the Claim permit the use of mountain pass theorem (see for example, Gasinski-Papageorgiou [9], p.648). So, we can find $u_0 \neq 0$ such that

$$\langle A_p(u_0), h\rangle + \langle A(u_0), h\rangle = \int_\Omega f\left(z, u_0^+\right) h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{3.15}$$

In (3.15) we choose $h = -u_0^- \in W_0^{1,p}(\Omega)$ and obtain

$$u_0 \geq 0, \ u_0 \neq 0.$$

From (3.15) we have

$$-\Delta_p u_0(z) - \Delta u_0(z) = f(z, u_0(z)) \text{ for } a.a.z \in \Omega, \ u_0 \mid_{\partial\Omega} = 0. \tag{3.16}$$

From (3.16) and Theorem 7.1, p.286 of Ladyzhenskaia-Uraltseva [13], we have $u_0 \in L^\infty(\Omega)$. Applying Theorem 1 of Lieberman [16], we infer that

$$u_0 \in C_+ \setminus \{0\}.$$

Hypotheses $(\mathbf{H}_f)_1$ imply that if $\rho = \|u_0\|_\infty$, then we can find $\widehat{\xi}_\rho > 0$ such that

$$f(z,x) + \widehat{\xi}_\rho x^{p-1} \geq 0 \text{ for } a.a.z \in \Omega, \text{ all } 0 \leq x \leq \rho.$$

From (3.16), we have

$$\Delta_p u_0(z) + \Delta u_0(z) \leq \widehat{\xi}_\rho u_0(z)^{p-1} \text{ for } a.a.z \in \Omega,$$

hence

$$u_0 \in int\, C_+$$

(see Pucci-Serrin [24], pp.111, 120).

Similarly, using the C^1-functional $\varphi_- : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\varphi_-(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F_-\left(z, -u^-\right) dz \text{ for all } u \in W_0^{1,p}(\Omega)$$

and reasoning as above, we produce a negative smooth solution $v_0 \in -int \quad C_+$. ☐

4. Three solution theorem

In this section, using critical groups, we produce a third nontrivial smooth solution for problem (1.1). As we already mentioned in the Introduction, this requires more regularity on the reaction $f(z,\cdot)$. The new hypotheses on the function $f(z,x)$ are the following:

$(\mathbf{H}_f)_2$: $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a measurable function such that for a.a. $z \in \Omega$, $f(z,0) = 0$, $f(z,\cdot) \in C^1(\mathbb{R})$ and

(i) there exists $a \in L^\infty(\Omega)$ such that

$$\left|f_x'(z,x)\right| \leq a(z)\left(1 + |x|^{r-1}\right) \text{ for a.a. } z \in \Omega, \text{ all } x \in \mathbb{R},$$

with $p \leq r < p^*$, where p^* is the critical Sobolev exponent corresponding to p, i.e.,

$$p^* = \begin{cases} \frac{Np}{N-p} & \text{if } p < N, \\ +\infty & \text{if } p \geq N. \end{cases}$$

(ii) there exists an integer $m \geq 2$ such that

$$\lim_{x \to \pm\infty} \frac{f(z,x)}{|x|^{p-2}x} = \widehat{\lambda}_m(p), \quad \lim_{x \to \pm\infty} [f(z,x)x - pF(z,x)] = +\infty$$

uniformly for a.a. $z \in \Omega$;

(iii)

$$f'_x(z,0) = \lim_{x \to 0} \frac{f(z,x)}{x} \text{ uniformly for a.a. } z \in \Omega$$

and

$$f'_x(z,0) \leq \widehat{\lambda}_1(2) \text{ for a.a. } z \in \Omega, \; f'_x(\cdot,0) \neq \widehat{\lambda}_1(2).$$

We introduce the energy (Euler) functional $\varphi : W_0^{1,p}(\Omega) \to \mathbb{R}$ for problem (1.1) defined by

$$\varphi(u) = \frac{1}{p}\|Du\|_p^p + \frac{1}{2}\|Du\|_2^2 - \int_\Omega F(z,u)\,dz \text{ for all } u \in W_0^{1,p}(\Omega).$$

We have $\varphi \in C^2(W_0^{1,p}(\Omega))$. Also, u_0, $v_0 \in K_\varphi$ and of course we assume that K_φ is finite (otherwise we already have an infinity of nontrivial smooth solutions and so we are done).

Proposition 4.1. *If hypotheses* $(\mathbf{H}_f)_2$ *hold, then*

$$C_k(\varphi,u_0) = C_k(\varphi,v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0.$$

Proof. We consider the homotopy $h_+ : [0,1] \times W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$h_+(t,u) = (1-t)\varphi(u) + t\varphi_+(u) \text{ for all } t \in [0,1], \; u \in W_0^{1,p}(\Omega).$$

Suppose we could find $\{t_n\}_{n\geq1} \subseteq [0,1]$ and $\{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t, \; u_n \to u_0 \text{ in } W_0^{1,p}(\Omega), \; \left(h'_+\right)_u(t_n,u_n) = 0 \text{ for all } n \in \mathbb{N}. \tag{4.1}$$

We have

$$\langle A_p(u_n),h\rangle + \langle A(u_n),h\rangle = \int_\Omega \left[(1-t_n)f(z,u_n) + t_n f\left(u_n^+\right)\right] h dz \text{ for all } h \in W_0^{1,p}(\Omega). \tag{4.2}$$

In (4.2) we choose $h = u_n \in W_0^{1,p}(\Omega)$ and use (4.1) to infer that

$$\{u_n\}_{n\geq1} \subseteq W_0^{1,p}(\Omega) \text{ is bounded.}$$

Then from Ladyzhenskaia-Uraltseva ([13], p.286) we know that there exists $C_6 > 0$ such that

$$\|u_n\|_\infty \leq C_6 \text{ for all } n \in \mathbb{N}.$$

So, invoking Theorem 1 of Lieberman [16], we can find $\alpha \in (0,1)$ and $C_7 > 0$ such that

$$u_n \in C_0^{1,\alpha}\left(\overline{\Omega}\right) \text{ and } \|u_n\|_{C_0^{1,\alpha}(\overline{\Omega})} \leq C_7 \text{ for all } n \in \mathbb{N}.$$

Exploiting the compact embedding of $C_0^{1,\alpha}\left(\overline{\Omega}\right)$ into $C_0^1\left(\overline{\Omega}\right)$, we infer that

$$u_n \to u_0 \text{ in } C_0^1\left(\overline{\Omega}\right) \text{ (see (4.1))}.$$

Recall that $u_0 \in int \quad C_+$ (see Proposition 3.1). It follows that $u_n \in int \quad C$ for all $n \geq n_0$, hence

$$\{u_n\}_{n \geq n_0} \subseteq K_\varphi,$$

a contradiction (recall that K_φ is finite). Therefore (4.1) cannot occur, and then invoking the homotopy invariance property of critical groups (see Gasinski-Papageorgiou [10], Theorem 5.125, p.836), we have

$$C_k(\varphi, u_0) = C_k(\varphi_+, u_0) \text{ for all } k \in \mathbb{N}_0. \tag{4.3}$$

From the proof of Proposition 3.1 we know that $u_0 \in int\, C_+$ is a critical point of φ_+ of mountain pass type. Therefore Theorem 6.5.8, p.432 of Papageorgiou-Rădulescu-Repovš [21], implies that

$$C_1(\varphi_+, u_0) \neq 0$$

hence

$$C_1(\varphi, u_0) \neq 0 \text{ (see (4.3)).} \tag{4.4}$$

Recall that $\varphi \in C^2\left(W_0^{1,p}(\Omega)\right)$. So, according to Aizicovici-Papageorgiou-Staicu [2] (see the proof of Theorem 3) and on account of (4.4), we have

$$C_k(\varphi, u_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0.$$

Similarly using the functional φ_- and the homotopy

$$h_-(t,u) = (1-t)\varphi(u) + t\varphi_-(u) \text{ for all } t \in [0,1],\ u \in W_0^{1,p}(\Omega),$$

we show that

$$C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0.$$

□

Proposition 4.2. *If hypotheses $(\mathbf{H}_f)_2$ hold, then $C_m(\varphi,\infty) \neq 0$.*

Proof. Let $\lambda \in (\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)) \backslash \widehat{\sigma}(p)$ and consider the C^1−functional $\psi : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by

$$\psi(u) = \frac{1}{p}\|Du\|_p^p - \frac{\lambda}{p}\|u\|_p^p \text{ for all } u \in W_0^{1,p}(\Omega).$$

We consider the homotopy $h(t,u)$ defined by

$$h(t,u) = (1-t)\varphi(u) + t\psi(u) \text{ for all } t \in [0,1],\ u \in W_0^{1,p}(\Omega).$$

Claim. We can find $\eta \in \mathbb{R}$ and $\widehat{\delta} > 0$ such that

$$h_t(u) := h(t,u) \leq \eta \Longrightarrow (1+\|u\|)\left\|(h_t)'(u)\right\|_* \geq \widehat{\delta} \text{ for all } t \in [0,1].$$

We argue indirectly. So, suppose the Claim is not true. Since $h(\cdot,\cdot)$ maps bounded sets to bounded sets (see hypothesis $(\mathbf{H}_f)_2(i)$), we can find $\{t_n\}_{n\geq 1} \subseteq [0,1]$ and $\{u_n\}_{n\geq 1} \subseteq W_0^{1,p}(\Omega)$ such that

$$t_n \to t,\ \|u_n\| \to \infty,\ h_{t_n}(u_n) \to -\infty \text{ and } (1+\|u_n\|)(h_{t_n})'(u_n) \to 0. \tag{4.5}$$

Then we have

$$\left|\langle A_p(u_n), h\rangle + (1-t_n)\langle A(u_n), h\rangle - \int_\Omega \left[(1-t_n) f(z,u_n) + t_n\lambda |u_n|^{p-2} u_n\right] hdz\right| \leq \frac{\varepsilon_n \|h\|}{1+\|u_n\|} \text{ for all } h\in W_0^{1,p}(\Omega), \text{ with } \varepsilon_n \to 0. \tag{4.6}$$

Let $y_n = \frac{u_n}{\|u_n\|}$, $n\in\mathbb{N}$. Then $\|y_n\| = 1$ for all $n\in\mathbb{N}$ and so we may assume that

$$y_n \xrightarrow{w} y \text{ in } W_0^{1,p}(\Omega). \tag{4.7}$$

From (4.6) it follows that

$$\begin{aligned}&\left|\langle A_p(y_n), h\rangle + \frac{(1-t_n)}{\|u_n\|^{p-2}}\langle A(y_n), h\rangle\right.\\ &\left.-\int_\Omega \left[(1-t_n)\frac{N_f(u_n)}{\|u_n\|^{p-1}} + t_n\lambda |y_n|^{p-2} y_n\right] hdz\right|\\ &\leq \frac{\varepsilon_n\|h\|}{(1+\|u_n\|)\|u_n\|^{p-1}} \text{ for all } h\in W_0^{1,p}(\Omega), \text{ all } n\in\mathbb{N}.\end{aligned} \tag{4.8}$$

Hypotheses $(\mathbf{H}_f)_2(i), (ii)$ imply that

$$\left\{\frac{N_f(u_n)}{\|u_n\|^{p-1}}\right\}_{n\geq 1} \subseteq L^{p'}(\Omega) \text{ is bounded.}$$

So, passing to a subsequence if necessary and using hypothesis $(\mathbf{H}_f)_2(ii)$, we obtain

$$\frac{N_f(u_n)}{\|u_n\|^{p-1}} \xrightarrow{w} \widehat{\lambda}_m(p)|y|^{p-2} y \text{ in } L^{p'}(\Omega) \tag{4.9}$$

(see Aizicovici-Papageorgiou-Staicu [1], proof of Proposition 30).

Choosing $h = y_n - y \in W_0^{1,p}(\Omega)$ in (4.6), passing to the limit as $n\to\infty$ and using (4.7) and (4.9) as before (see the proof of Proposition 3.1), we obtain

$$\limsup_{n\to\infty}\langle A_p(y_n), y_n - y\rangle \leq 0,$$

which implies (see Lemma 2.1)

$$y_n \to y \text{ in } W_0^{1,p}(\Omega), \text{ hence } \|y\| = 1. \tag{4.10}$$

So, if in (4.8) we pass to the limit as $n\to\infty$ and use $(4.10), (4.9), (4.5)$ (recall also that $p>2$), we have

$$\left|\langle A_p(y), h\rangle = \int_\Omega \lambda_t |y|^{p-2} yhdz\right| \text{ for all } h\in W_0^{1,p}(\Omega),$$

with

$$\lambda_t = (1-t)\widehat{\lambda}_m(p) + t\lambda,$$

hence

$$-\Delta_p y(z) = \lambda_t |y(z)|^{p-2} y(z) \text{ for } a.a. z\in\Omega,\ y|_{\partial\Omega} = 0. \tag{4.11}$$

If $\lambda_t \notin \widehat{\sigma}(p)$, then from (4.11) we infer that $y=0$, a contradiction (see (4.10)). If $\lambda_t\in\widehat{\sigma}(p)$, then from (4.10) we see that if

$$E_0 = \{z\in\Omega : y(z)\neq 0\}$$

then $|E_0|_N > 0$. We have

$$|u_n(z)| \to +\infty \text{ for all } z\in E_0,$$

hence

$$f(z,u_n(z))\,u_n(z)-pF(z,u_n(z))\to+\infty \text{ for a.a. } z\in E_0$$

(see hypothesis $(\mathbf{H}_f)_2\,(ii)$), therefore

$$\int_{E_0}[f(z,u_n(z))\,u_n(z)-pF(z,u_n(z))]\,dz\to+\infty \tag{4.12}$$

(by Fatou's lemma). On account of hypotheses $(\mathbf{H}_f)_2\,(i)\,,(ii)$, we see that

$$-C_8\le f(z,x)\,x-pF(z,x) \text{ for a.a. } z\in\Omega, \text{ all } x\in\mathbb{R}, \text{ some } C_8>0. \tag{4.13}$$

Hence we have

$$\begin{aligned}&\int_\Omega[f(z,u_n)\,u_n-pF(z,u_n)]\,dz\\&=\int_{E_0}[f(z,u_n)\,u_n-pF(z,u_n)]\,dz+\int_{\Omega\setminus E_0}[f(z,u_n)\,u_n-pF(z,u_n)]\,dz\\&\ge\int_{E_0}[f(z,u_n)\,u_n-pF(z,u_n)]\,dz-C_8\,|\Omega|_N \text{ (see (4.13))}\end{aligned}$$

therefore

$$\int_\Omega[f(z,u_n)\,u_n-pF(z,u_n)]\,dz\to+\infty \text{ (see (4.12)).} \tag{4.14}$$

From (4.5) we have

$$\begin{aligned}\|Du_n\|_p^p+\tfrac{p}{2}\|Du_n\|_2^2-\textstyle\int_\Omega[(1-t_n)\,pF(z,u_n)+\lambda t_n\,|u_n|^p]\,dz\\\le-1 \text{ for all } n\ge n_0.\end{aligned} \tag{4.15}$$

Also, if in (4.6) we choose $h=u_n\in W_0^{1,p}(\Omega)$, then

$$\begin{aligned}-\|Du_n\|_p^p-\|Du_n\|_2^2+\textstyle\int_\Omega[(1-t_n)\,f(z,u_n)\,u_n+\lambda t_n\,|u_n|^p]\,dz\\\le\varepsilon_n \text{ for all } n\in\mathbb{N}.\end{aligned} \tag{4.16}$$

Adding (4.15) and (4.16) and recalling that $p>2$, we obtain

$$(1-t_n)\int_\Omega[f(z,u_n)\,u_n-pF(z,u_n)]\,dz\le0 \text{ for all } n\ge n_1\ge n_0. \tag{4.17}$$

If $t=1$, then $\lambda_t=\lambda\notin\widehat{\sigma}(p)$. So, from (4.11) we infer that $y=0$, contradicting (4.6) . Hence $t\ne1$ and so we may assume that $t_n<1$ for all $n\ge n_1$. From (4.17) we obtain

$$\int_\Omega[f(z,u_n)\,u_n-pF(z,u_n)]\,dz\le0 \text{ for all } n\ge n_1. \tag{4.18}$$

Comparing (4.18) and (4.14) , we have a contradiction. This proves the Claim.

On account of the Claim, and using Theorem 5.1.21, p.334 of Chang [5] (see also Proposition 3.2 of Liang-Su [15]), we have

$$C_k(h_0,\infty)=C_k(h_1,\infty) \text{ for all } k\in\mathbb{N}_0,$$

therefore

$$C_k(\varphi,\infty)=C_k(\psi,\infty) \text{ for all } k\in\mathbb{N}_0. \tag{4.19}$$

Consider the following sets

$$D_q = \left\{ u \in W_0^{1,p}(\Omega) : \|Du\|_p^p < \lambda \|u\|_p^p, \ \|u\| = q \right\}, \ (q > 0),$$
$$C = \left\{ u \in W_0^{1,p}(\Omega) : \|Du\|_p^p \geq \lambda \|u\|_p^p \right\}.$$

These are symmetric sets and $D_q \cap C = \varnothing$. Let

$$\partial B_q = \left\{ u \in W_0^{1,p}(\Omega) : \|u\| = q \right\}.$$

This is a C^1- Banach manifold and so it is locally contractible. Note that $D_q \subseteq \partial B_q$ is relatively open. Hence D_q is locally contractible. Similarly the set $W_0^{1,p}(\Omega) \backslash C$ is open and so locally contractible. Recall that $ind(\cdot)$ denotes the Fadell-Rabinowitz cohomological index (see [8]). Since $\lambda \in \left(\widehat{\lambda}_m(p), \widehat{\lambda}_{m+1}(p)\right) \backslash \widehat{\sigma}(p)$ we have

$$ind(D_q) = ind\left(W_0^{1,p}(\Omega) \backslash C\right) = m.$$

According to Theorem 3.6 of Cingolani-Degiovanni [7], the sets D_q and C link in dimension m. So, invoking Theorem 3.2 of [7], we have

$$C_m(\psi, 0) \neq 0. \tag{4.20}$$

But note that $K_\psi = \{0\}$ (recall that $\lambda \notin \widehat{\sigma}(p)$). Therefore

$$C_k(\psi, 0) = C_k(\psi, \infty) \text{ for all } k \in \mathbb{N}_0,$$

hence

$$C_m(\psi, \infty) \neq 0 \text{ (see (4.20))},$$

and we conclude that

$$C_m(\varphi, \infty) \neq 0 \text{ (see (4.19))}.$$

□

Now we are ready to produce a third nontrivial smooth solution distinct from u_0 and v_0.

Proposition 4.3. *If hypotheses* $(\mathbf{H}_f)_2$ *hold, then problem* (1.1) *has a third nontrivial solution* $y_0 \in C_0^1\left(\overline{\Omega}\right)$ *with* $y_0 \notin \{u_0, v_0\}$.

Proof. We already have two constant sign solutions $u_0 \in int\, C_+$, $v_0 \in -int\, C_+$ (see Proposition 3.1). From Proposition 4.1, we have

$$C_k(\varphi, u_0) = C_k(\varphi, v_0) = \delta_{k,1}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.21}$$

As in the proof of Proposition 3.1, we can show that $u = 0$ is a local minimizer of φ. Therefore

$$C_k(\varphi, 0) = \delta_{k,0}\mathbb{Z} \text{ for all } k \in \mathbb{N}_0. \tag{4.22}$$

According to Proposition 4.3, we have $C_m(\varphi, \infty) \neq 0$. So, we can find $y_0 \in K_\varphi$ such that

$$C_m(\varphi, y_0) \neq 0. \tag{4.23}$$

By (4.21), (4.22) and (4.23) and since $m \geq 2$, we infer that

$$y_0 \notin \{0, u_0, v_0\}.$$

Moreover, by the nonlinear regularity theory, we have

$$y_0 \in C_0^1\left(\overline{\Omega}\right).$$

So, $y_0 \in C_0^1\left(\overline{\Omega}\right)$ is the third nontrivial smooth solution of (1.1), distinct from u_0, v_0. □

We can state the following multiplicity (three solutions) theorem for problem (1.1).

Theorem 4.4. *If hypotheses* $(\mathbf{H}_f)_2$ *hold, then problem* (1.1) *has at least three nontrivial solutions*

$$u_0 \in int\, C_+,\ v_0 \in -int\, C_+,\ y_0 \in C_0^1\left(\overline{\Omega}\right).$$

Remark. It is an interesting open problem whether under the hypotheses of the above theorem, one can produce a nodal solution.

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