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EQUILIBRIUM AND GIBBS MEASURES FOR FLOWS

LUIS BARREIRA AND CARLLOS HOLANDA

Abstract. We construct equilibrium and Gibbs measures in the context of the nonadditive thermodynamic formalism for flows. More precisely, we consider the class of almost additive families of potentials and after establishing an appropriate version of the classical variational principle for the topological pressure, we obtain the existence and uniqueness of equilibrium and Gibbs measures for hyperbolic flows and families with bounded variation.

1. Introduction

We first recall some of the main components of the classical thermodynamic formalism. The notion of the topological pressure $P(\phi)$ of a continuous function ϕ with respect to a map $f\colon X \to X$ was introduced by Ruelle in [12] for expansive maps and by Walters in [14] in the general case. They also established a variational principle for the topological pressure:

$$P(\phi) = \sup_{\mu} \left(h_\mu(f) + \int_X \phi \, d\mu \right),$$

with the supremum taken over all f-invariant probability measures μ on X, denoting by $h_\mu(f)$ the Kolmogorov–Sinai entropy of f with respect to μ. An f-invariant probability measure μ on X is called an *equilibrium measure* for ϕ if

$$P(\phi) = h_\mu(f) + \int_X \phi \, d\mu.$$

These measures and particularly their Gibbs property play an important role in the dimension theory and multifractal analysis of dynamical systems. We refer the reader to the books [3, 6, 9, 13] for details and further references.

The nonadditive thermodynamic formalism was introduced in [1] as a generalization of the classical thermodynamic formalism, essentially replacing the topological pressure $P(\phi)$ by the topological pressure $P(\Phi)$ of a sequence of continuous functions $\Phi = (\phi_n)_{n\in\mathbb{N}}$. This formalism contains as a particular case a new formulation of the subadditive thermodynamic formalism introduced by Falconer in [7]. Moreover, for additive sequences it recovers the notion of topological pressure introduced by Pesin and Pitskel' in [10] as well as the notions of lower and upper capacity topological pressures introduced by Pesin in [8] for an arbitrary set. The nonadditive thermodynamic formalism also plays a corresponding role in the dimension theory of dynamical systems. In particular, [1] includes a version of the variational

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principle for the topological pressure (for discrete time), although with restrictive assumptions on the sequence Φ. This justifies the interest in looking for more general classes of sequences of functions for which it is still possible to establish a variational principle, including in the case of flows.

Our main objective is precisely to consider a new class of families for which it is still possible not only to establish a variational principle for the topological pressure, but also to discuss the existence and uniqueness of equilibrium and Gibbs measures. This is the class of almost additive families: a family of functions $(a_t)_{t\geq 0}$ is said to be *almost additive* with respect to a flow $(\phi_t)_{t\in\mathbb{R}}$ if there exists a constant $C>0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t,s \geq 0$. In particular, we establish the following variational principle for the topological pressure. We denote by $\mathcal{M}$ the set of all Φ-invariant probability measures on X and we refer to Section 2 for the notion of tempered variation.

Theorem 1.1. *Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation such that $\sup_{t\in[0,s]}\|a_t\|_\infty < \infty$ for some $s>0$. Then*

$$P(a) = \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right). \tag{1.1}$$

To the possible extent we follow the proof of Theorem 4.3.1 in [4]. In order to obtain the lower bound for the topological pressure, we first show that

$$P(a) \geq h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu$$

for each ergodic measure $\mu \in \mathcal{M}$. Since we need to use Birkhoff's ergodic theorem with respect to the time-1 map, but μ need not be ergodic with respect this map, we consider an ergodic decomposition of μ with respect to the time-1 map.

We also consider the particular case of hyperbolic flows and we establish the existence and uniqueness of the equilibrium measure of an almost additive family of continuous functions with bounded variation as well as its Gibbs property. We say that a Φ-invariant measure μ on X is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (1.1) is attained at μ, that is, if

$$P(a) = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.$$

The notion of a Gibbs measure requires introducing the somewhat technical notion of a Markov system (see Section 3.1). Our main result is the following theorem.

Theorem 1.2. *Let Λ be a hyperbolic set for a topologically mixing C^1 flow Φ and let a be an almost additive family of continuous functions on Λ with bounded variation such that $P(a)=0$ and $\sup_{t\in[0,s]}\|a_t\|_\infty < \infty$ for some $s>0$. Then:*

1. *there exists a unique equilibrium measure for a;*
2. *there exists a unique invariant Gibbs measure for a;*
3. *the two measures are equal and are ergodic.*

Note that there is no loss of generality in assuming that $P(a) = 0$. Indeed, let $b = (b_t)_{t\ge 0}$ be an almost additive family of continuous functions on Λ with bounded variation such that $\sup_{t\in[0,s]} \|b_t\|_\infty < \infty$ for some $s > 0$. Then let $a = (a_t)_{t\ge 0}$ be the family of continuous functions on Λ defined by

$$a_t = b_t - P(b)t$$

for each $t \ge 0$. Clearly, a is almost additive, has bounded variation and satisfies $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ and $P(a) = 0$. For each Φ-invariant probability measure μ on Λ we have

$$\frac{1}{t}\int_X a_t\, d\mu = \frac{1}{t}\int_X b_t\, d\mu - P(b).$$

This readily implies that a and b have the same equilibrium measures. The idea of the proof of Theorem 1.2 is to consider an almost additive sequence

$$c_n(x) = a_{\tau_n(x)}(x)$$

on the base $Z \subset \Lambda$ of a Markov system, where $\tau_n(x)$ is the nth-return time to Z (see Section 3.1 for details). The desired result can then be obtained from a corresponding result on the base.

To the possible extent, and up to the need of some nontrivial modifications in the case of flows, our arguments are inspired by work in [2] for discrete time.

2. Variational principle

In this section we consider the nonadditive topological pressure for a flow and we establish a version of the variational principle for an almost additive family of continuous functions.

We first recall the notion of nonadditive topological pressure for a flow. Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a continuous flow on a compact metric space (X, d). Moreover, let $a = (a_t)_{t\ge 0}$ be a family of continuous functions $a_t \colon X \to \mathbb{R}$ with tempered variation. This means that

$$\lim_{\varepsilon\to 0} \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a,\varepsilon)}{t} = 0,$$

where

$$\gamma_t(a,\varepsilon) = \sup\bigl\{|a_t(y) - a_t(x)| : y \in B_t(x,\varepsilon) \text{ for some } x \in X\bigr\}$$

taking

$$B_t(x,\varepsilon) = \bigl\{y \in X : d(\phi_s(y), \phi_s(x)) < \varepsilon \text{ for } s \in [0,t]\bigr\}. \tag{2.1}$$

Given $\varepsilon > 0$, we say that a set $\Gamma \subset X \times \mathbb{R}_0^+$ *covers* $Z \subset X$ if

$$\bigcup_{(x,t)\in\Gamma} B_t(x,\varepsilon) \supset Z$$

and we write

$$a(x,t,\varepsilon) = \sup\bigl\{a_t(y) : y \in B_t(x,\varepsilon)\bigr\} \quad \text{for } (x,t) \in \Gamma.$$

For each $Z \subset X$ and $\alpha \in \mathbb{R}$, let

$$M(Z,a,\alpha,\varepsilon) = \lim_{T\to+\infty} \inf_\Gamma \sum_{(x,t)\in\Gamma} \exp(a(x,t,\varepsilon) - \alpha t), \tag{2.2}$$

with the infimum taken over all countable sets $\Gamma \subset X \times [T, +\infty)$ covering Z. When α goes from $-\infty$ to $+\infty$, the quantity in (2.2) jumps from $+\infty$ to 0 at a unique value and so one can define

$$P(a|_Z, \varepsilon) = \inf\bigl\{\alpha \in \mathbb{R} : M(Z, a, \alpha, \varepsilon) = 0\bigr\}.$$

Moreover, the limit

$$P(a|_Z) = \lim_{\varepsilon \to 0} P(a|_Z, \varepsilon)$$

exists and is called the *nonadditive topological pressure* of the family a on the set Z. For simplicity of the notation, we shall also write $P(a|_X) = P(a)$.

Now we establish a version of the variational principle for the topological pressure of an almost additive family of continuous functions. We recall that a family $a = (a_t)_{t \geq 0}$ of functions $a_t \colon X \to \mathbb{R}$ is said to be *almost additive* (with respect to a flow Φ) if there exists a constant $C > 0$ such that

$$-C + a_t + a_s \circ \phi_t \leq a_{t+s} \leq a_t + a_s \circ \phi_t + C$$

for every $t, s \geq 0$. We denote by $\mathcal{M}$ the set of all Φ-invariant probability measures on X, that is, the probability measures μ on X such that

$$\mu(\phi_t(A)) = \mu(A)$$

for any Borel set $A \subset X$ and any $t \in \mathbb{R}$. Moreover, for each $\mu \in \mathcal{M}$, let $h_\mu(\Phi)$ be the Kolmogorov–Sinai entropy of Φ with respect to μ.

Theorem 2.1. *Let Φ be a continuous flow on a compact metric space X and let a be an almost additive family of continuous functions with tempered variation such that* $\sup_{t \in [0,s]} \|a_t\|_\infty < \infty$ *for some $s > 0$. Then*

$$\begin{aligned} P(a) &= \sup_{\mu \in \mathcal{M}} \left(h_\mu(\Phi) + \int_X \lim_{t \to \infty} \frac{a_t(x)}{t} \, d\mu(x) \right) \\ &= \sup_{\mu \in \mathcal{M}} \left(h_\mu(\Phi) + \lim_{t \to \infty} \frac{1}{t} \int_X a_t \, d\mu \right). \end{aligned} \tag{2.3}$$

Proof. To the possible extent we follow the proof of Theorem 4.3.1 in [4] while we also highlight the differences. Since a is almost additive, we have

$$a_{t+s} + C \leq (a_t + C) + a_s \circ \phi_t + C$$

for $s, t \geq 0$. Thus, $(a_n + C)_{n \in \mathbb{N}}$ is subadditive and it follows from Kingman's subadditive ergodic theorem that for each measure $\mu \in \mathcal{M}$ the limit

$$\tilde{a}(x) = \lim_{n \to \infty} (a_n(x)/n)$$

exists for μ-almost every $x \in X$. Now let $[x]$ be the integer part of the real number x. Again since a is almost additive, we have

$$-C + a_{[t]} + a_{t-[t]} \circ \phi_{[t]} \leq a_t \leq a_{[t]} + a_{t-[t]} \circ \phi_{[t]} + C \tag{2.4}$$

for $t > 0$. Taking $N \in \mathbb{N}$ such that $1/N < s$ (with s as in the statement of the theorem), we obtain

$$\begin{aligned}\left|\frac{a_t(x)}{t} - \frac{a_{[t]}(x)}{t}\right| &\le \left|\frac{(a_{t-[t]} \circ \phi_{[t]})(x)}{t}\right| + \frac{C}{t} \\ &\le \frac{\sup_{t\in[0,1]} \|a_t\|_\infty}{t} + \frac{C}{t} \\ &\le \frac{N \sup_{t\in[0,1/N]} \|a_t\|_\infty}{t} + \frac{NC}{t} \\ &\le \frac{N \sup_{t\in[0,s]} \|a_t\|_\infty}{t} + \frac{NC}{t}.\end{aligned}$$

Taking the limit when $t \to \infty$ gives

$$\lim_{t\to\infty} \left|\frac{a_t(x)}{t} - \frac{a_{[t]}(x)}{t}\right| = 0. \tag{2.5}$$

Since

$$\lim_{t\to\infty} \frac{a_{[t]}(x)}{t} = \lim_{t\to\infty} \frac{[t]}{t}\frac{a_{[t]}(x)}{[t]} = \lim_{t\to\infty} \frac{a_{[t]}(x)}{[t]} = \tilde{a}(x),$$

it follows from (2.5) that

$$\lim_{t\to\infty} \frac{a_t(x)}{t} = \tilde{a}(x)$$

for μ-almost every $x \in X$. Moreover,

$$-C[t] + \sum_{k=0}^{[t]-1} a_1 \circ \phi_k \le a_{[t]} \le \sum_{k=0}^{[t]-1} a_1 \circ \phi_k + C[t]$$

and so $|a_{[t]}/[t]| \le \|a_1\|_\infty + C$. Hence, it follows from Lebesgue's dominated convergence theorem that $a_{[t]}/[t] \to \tilde{a}$ in $L^1(X, \mu)$ when $t \to \infty$ and

$$\lim_{t\to\infty} \frac{1}{[t]} \int_X a_{[t]}\, d\mu = \int_X \tilde{a}\, d\mu = \int_X \lim_{t\to\infty} \frac{a_{[t]}}{[t]}\, d\mu. \tag{2.6}$$

Finally, by (2.4) we have

$$\begin{aligned}\left|\frac{1}{t}\int_X a_t\, d\mu - \frac{[t]}{t}\frac{1}{[t]}\int_X a_{[t]}\, d\mu\right| &\le \left|\frac{1}{t}\int_X a_{t-[t]} \circ \phi_{[t]}\, d\mu\right| + \frac{C}{t}\mu(X) \\ &\le \frac{1}{t}\mu(X) \sup_{s\in[0,1]} \|a_s\|_\infty + \frac{C}{t}\mu(X)\end{aligned}$$

and so, using (2.6), we obtain

$$\lim_{t\to\infty} \frac{1}{t}\int_X a_t\, d\mu = \lim_{t\to\infty} \frac{[t]}{t}\frac{1}{[t]}\int_X a_{[t]}\, d\mu = \int_X \lim_{t\to\infty} \frac{a_t(x)}{t}\, d\mu(x).$$

This shows that the two limits in (2.3) exist and are equal.

Now we establish the inequality

$$P(a) \le \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t}\int_X a_t\, d\mu\right). \tag{2.7}$$

Given $x \in X$, we define a probability measure on X by

$$\mu_{x,t} = \frac{1}{t}\int_0^t \delta_{\phi_s(x)}\,ds,$$

where δ_y is the probability measure concentrate on y. Let also $V(x)$ be the set of all sublimits of the family $(\mu_{x,t})_{t>0}$. The following result can be obtained as in the proof of Theorem 10.1.5 in [4].

Lemma 2.2. *Given $x \in X$ and $\mu \in V(x)$, there exists an increasing sequence $(t_n)_{n\in\mathbb{N}}$ such that*

$$\lim_{n\to\infty}\frac{a_{t_n}(x)}{t_n} = \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu.$$

We also need the following technical property (see [6] for a corresponding result in the additive case).

Lemma 2.3. *Let $\Gamma \subset X \times \{1\}$ be a finite cover of X. For the open cover $\mathcal{V} = \{V_1, \dots, V_r\}$ of X, where $V_j = B_1(x_j, \varepsilon/2)$ with $(x_j, 1) \in \Gamma$, there exist $m, T \in \mathbb{N}$ with T arbitrary large and a sequence $U = V_{i_1}\cdots V_{i_T}$ such that:*

(1) $x \in \bigcap_{r=1}^T \phi_{-r+1}V_{i_r}$ *and*

$$a_T(x) \le T\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + \delta\right);$$

(2) *there exists a subset $V \in (\mathcal{V}^m)^k$ of U of length $km \ge T - m$ such that $H(V) \le m(h_\mu(\Phi) + \delta)$.*

Proof of the lemma. By Lemma 2.2, given $\delta > 0$, we have

$$\left|\frac{a_{t_n}(x)}{t_n} - \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right| < \delta$$

for any sufficiently large n. So one can take T arbitrarily large such that

$$a_T(x) \le T\left(\lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu + \delta\right)$$

and the first property follows. The second property can be obtained as in the proof of Lemma 4.3.2 in [4]. $\square$

Given $\delta > 0$, $m \in \mathbb{N}$ and $u \in \mathbb{R}$, let $X_{m,u}$ be the set of points $x \in X$ satisfying the two properties in Lemma 2.3 for some measure $\mu \in V(x)$ with

$$u - \delta \le \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu \le u + \delta.$$

Moreover, let n_T be the number of all sequences $U \in \mathcal{V}^T$ with these two properties for some point $x \in X_{m,u}$. Taking

$$c = \sup_{\mu\in\mathcal{M}}\left(h_\mu(\Phi) + \lim_{t\to\infty}\frac{1}{t}\int_X a_t\,d\mu\right)$$

and

$$\alpha > c + 3\delta + \overline{\lim_{t\to+\infty}}\,\frac{\gamma_t(a,\varepsilon)}{t},$$

one can proceed as in the proof Theorem 4.3.1 in [4] to obtain

$$n_T \leq \exp[T\,(h_\mu(\Phi) + 2\delta)]$$

for any sufficiently large T and so also

$$M(X_{m,u}, a, \alpha, \varepsilon) = 0 \quad \text{and} \quad \alpha > P(a|_{X_{m,u}}, \varepsilon).$$

Finally, taking points $u_1, \ldots, u_r$ such that for each $u \in [\inf \tilde{a}, \sup \tilde{a}]$ there exists $j \in \{1, \ldots, r\}$ with $|u - u_j| < \delta$, we have

$$X = \bigcup_{m\in\mathbb{N}} \bigcup_{i=1}^{r} X_{m,u_i}$$

and so

$$\begin{aligned} P(a) = \overline{\lim_{\varepsilon\to 0}}\, P(a, \varepsilon) &= \overline{\lim_{\varepsilon\to 0}} \sup_{m,i} P(a|_{X_{m,u_i}}, \varepsilon) \\ &\leq c + \overline{\lim_{\varepsilon\to 0}}\, \overline{\lim_{t\to\infty}} \frac{\gamma_t(a, \varepsilon)}{t} + 3\delta = c + 3\delta. \end{aligned}$$

Since δ is arbitrary, we conclude that $P(a) \leq c$ and so inequality (2.7) holds.

To obtain the reverse inequality $P(a) \geq c$, to the possible extent we adapt corresponding arguments in the proof of Lemma 4.3.5 in [4], although this step requires an additional ingredient. The reason is that we are considering an ergodic measure μ with respect to the flow, but we need to use the ergodic theorem with respect to the time-1 map. Since μ need not be ergodic with respect this map, we consider an ergodic decomposition of μ with respect to the time-1 map.

Lemma 2.4. *For each ergodic measure $\mu \in \mathcal{M}$, we have*

$$P(a) \geq h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.$$

Proof of the lemma. Given $\varepsilon > 0$, there exist $\delta \in (0, \varepsilon)$, a measurable partition $\xi = \{C_1, \ldots, C_m\}$ of X and an open cover $\mathcal{V} = \{V_1, \ldots, V_k\}$ of X for some $k \geq m$ such that:

(1) $\operatorname{diam} C_j \leq \varepsilon$, $\overline{V_i} \subset C_i$ and $\mu(C_i \setminus V_i) < \delta^2$ for $i = 1, \ldots, m$;
(2) the set $E = \bigcup_{i=m+1}^{k} V_i$ has measure $\mu(E) < \delta^2$.

We consider a measure ν in the ergodic decomposition of μ with respect to the time-1 map ϕ_1. The ergodic decomposition is described by a measure τ in the space $\mathcal{M}'$ of ϕ_1-invariant probability measures that is concentrated on the ergodic measures (with respect to ϕ_1). Note that $\nu(E) < \delta$ for ν in a set $\mathcal{M}_\delta \subset \mathcal{M}'$ of positive τ-measure such that $\tau(\mathcal{M}_\delta) \to 1$ when $\delta \to 0$.

For each $x \in X$ and $n \in \mathbb{N}$, let $s_n(x)$ be the number of integers $l \in [0, n)$ such that $\phi_1^l(x) \in E$. By Birkhoff's ergodic theorem, since ν is ergodic for ϕ_1 we have

$$\lim_{n\to+\infty} \frac{s_n(x)}{n} = \lim_{n\to+\infty} \frac{1}{n} \sum_{j=0}^{n-1} \chi_E(\phi_1^j(x)) = \int_X \chi_E \, d\nu = \nu(E) \tag{2.8}$$

for ν-almost every $x \in X$. On the other hand, by Lemma 2.2, there exists an increasing sequence of integers $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{a_{t_n}(x)}{t_n} = \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \tag{2.9}$$

for μ-almost every $x \in X$. By (2.8) and (2.9) together with Egorov's theorem, there exist $\nu \in \mathcal{M}_\delta$, $N_1 \in \mathbb{N}$ and a measurable set $A_1 \subset X$ with $\nu(A_1) \geq 1 - \delta$ such that

$$\frac{s_n(x)}{n} < 2\delta \quad \text{and} \quad \left| \frac{a_n(x)}{n} - \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right| < \delta \tag{2.10}$$

for every $x \in A_1$ and $n > N_1$. For the partition

$$\xi_n := \bigvee_{j=0}^{n} \phi_1^{-j}(\xi),$$

one can use the Shannon–McMillan–Breiman theorem and again Egorov's theorem to conclude that there exist $N_2 \in \mathbb{N}$ and a measurable set $A_2 \subset X$ with $\nu(A_2) \geq 1-\delta$ such that

$$\nu(\xi_n(x)) \leq \exp\left[(-h_\nu(\phi_1, \xi) + \delta)n\right] \tag{2.11}$$

for every $x \in A_2$ and $n > N_2$, where $\xi_n(x)$ is the element of ξ_n containing x. We take $N = \max\{N_1, N_2\}$ and $A = A_1 \cap A_2$. Then $\nu(A) \geq 1 - 2\delta$ and by construction, (2.10) and (2.11) hold for every $x \in A$ and $n > N$.

Let Δ be a Lebesgue number of the cover $\mathcal{V}$ and take $\overline{\varepsilon} > 0$ with $2\overline{\varepsilon} < \Delta$. Given $\alpha \in \mathbb{R}$, take $\overline{N} \geq N$ such that for each $n \geq \overline{N}$ there exists a set $\Gamma \subset X \times [n, +\infty)$ covering X with

$$\left| \sum_{(x,t)\in\Gamma} \exp(a(x,t,\overline{\varepsilon}) - \alpha t) - M(X, a, \alpha, \overline{\varepsilon}) \right| < \delta.$$

Without loss of generality, we also assume that $\overline{N}$ is so large such that

$$\frac{\gamma_l(a, \overline{\varepsilon})}{l} \leq \varlimsup_{t\to+\infty} \frac{\gamma_t(a, \overline{\varepsilon})}{t} + \delta$$

for all $l \geq \overline{N}$. Moreover, given $l \in \mathbb{N}$, let

$$\Gamma_l = \left\{(x,t) \in \Gamma : B_l(x, \overline{\varepsilon}) \cap A \neq \emptyset\right\}$$

and let $B_l = \bigcup_{(x,t)\in\Gamma} B_t(x, \overline{\varepsilon})$. Following arguments in [4], it follows from the first inequality in (2.10) and (2.11) that

$$\operatorname{card} \Gamma_l \geq \nu(B_l \cap A) \exp\left[h_\nu(\phi_1, \xi)l - (1 + 2\log \operatorname{card} \xi)l\delta\right]$$

for $l \in \mathbb{N}$. This implies that (see [4])

$$P(a, \overline{\varepsilon}) \geq h_\nu(\phi_1, \xi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu - \varlimsup_{t\to+\infty} \frac{\gamma_t(a, \overline{\varepsilon})}{t}.$$

Now we consider measurable partitions ξ_l and open covers $\mathcal{V}_l$ as before with $\varepsilon = 1/l$. For each l, take $\overline{\varepsilon}_l > 0$ such that $2\overline{\varepsilon}_l < 1/l$ is a Lebesgue number of the cover $\mathcal{V}_l$. Since $\operatorname{diam} \xi_l \to 0$ when $l \to +\infty$, we have

$$\lim_{l\to+\infty} h_\nu(\phi_1, \xi_l) = h_\nu(\phi_1).$$

Since the family a has tempered variation, we obtain

$$\begin{aligned}
P(a) &= \lim_{l\to+\infty} P(a, \overline{\varepsilon}_l) \\
&\geq \lim_{l\to+\infty} h_\nu(\phi_1, \xi_l) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu - \lim_{l\to+\infty} \overline{\lim_{t\to+\infty}} \frac{\gamma_t(a, \overline{\varepsilon}_l)}{t} \\
&= h_\nu(\phi_1) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.
\end{aligned}$$

Integrating with respect to ν gives

$$P(a) \geq \int_{\mathcal{M}_\delta} h_\nu(\phi_1) \, d\tau(\nu) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu$$

and letting $\delta \to 0$ yields the inequality

$$\begin{aligned}
P(a) &\geq \int_{\mathcal{M}'} h_\nu(\phi_1) \, d\tau(\nu) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \\
&= h_\mu(\phi_1) + \int_Z b \, d\mu = h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.
\end{aligned}$$

This completes the proof of the lemma. $\square$

Now we consider the set

$$X_\mu = \big\{x \in X : V(x) = \{\mu\}\big\}.$$

When $\mu \in \mathcal{M}$ is ergodic, X_μ is a nonempty Φ-invariant set and $\mu(X_\mu) = 1$. Hence, it follows from Lemma 2.4 that

$$\begin{aligned}
P(a) &\geq P(a|_{X_\mu}) \\
&\geq h_\mu(\Phi|_{X_\mu}) + \lim_{t\to\infty} \frac{1}{t} \int_{X_\mu} a_t \, d\mu \\
&= h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu.
\end{aligned}$$

When $\mu \in \mathcal{M}$ is arbitrary, we can decompose X into ergodic components and use the previous argument to show that

$$P(a) \geq \sup_{\mu\in\mathcal{M}} \left(h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu \right).$$

This completes the proof of the theorem. $\square$

We say that a Φ-invariant measure μ_a is an *equilibrium measure* for the almost additive family a (with respect to the flow Φ) if the supremum in (2.3) is attained at μ_a, that is, if

$$P(a) = h_{\mu_a}(\Phi) + \lim_{t\to\infty} \frac{1}{t} \int_X a_t \, d\mu_a. \tag{2.12}$$

The following result gives a criterion for the existence of equilibrium measures in this context.

Theorem 2.5. *Let Φ be a continuous flow on a compact metric space X such that the map $\mu \mapsto h_\mu(\Phi)$ is upper semicontinuous. Then each almost additive family a of continuous functions with tempered variation such that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$ has at least one equilibrium measure.*

Proof. Since $a_n + C$ is a subadditive sequence , the real sequence $\int_X (a_n + C)\,d\mu$ is also subadditive. Then

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu &= \lim_{n\to\infty} \frac{1}{n}\int_X (a_n + C)\,d\mu \\ &\le \frac{1}{n}\int_X (a_n + C)\,d\mu \\ &= \frac{1}{n}\int_X a_n\,d\mu + \frac{C}{n}.\end{aligned} \tag{2.13}$$

Similarly, the sequence $\int_X (a_n - C)\,d\mu$ is supadditive and so

$$\lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu \ge \frac{1}{n}\int_X a_n\,d\mu - \frac{C}{n}. \tag{2.14}$$

It follows from (2.13) and (2.14) that

$$\left|\lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu - \frac{1}{n}\int_X a_n\,d\mu\right| \le \frac{C}{n}.$$

Now let μ_m be a sequence of measures converging to μ. Then

$$\left|\lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu_m - \frac{1}{n}\int_X a_n\,d\mu_m\right| \le \frac{C}{n}$$

for every $m, n \in \mathbb{N}$. Letting $m \to \infty$ and then $n \to \infty$, we obtain

$$\lim_{m\to\infty}\lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu_m = \lim_{n\to\infty} \frac{1}{n}\int_X a_n\,d\mu.$$

This shows that the map

$$\mu \mapsto \lim_{t\to\infty} \frac{1}{t}\int_X a_t\,d\mu$$

is continuous for each almost additive family a. Together with the upper semicontinuity of the map $\mu \mapsto h_\mu(\Phi)$, this implies that the map

$$\mu \mapsto h_\mu(\Phi) + \lim_{t\to\infty} \frac{1}{t}\int_X a_t\,d\mu$$

is upper semicontinuous. Hence, in view of the compactness of $\mathcal{M}$ there exists a measure $\mu_a \in \mathcal{M}$ satisfying (2.12). □

3. Hyperbolic flows

In this section we consider the particular case of hyperbolic flows and we describe a general condition for the uniqueness of the equilibrium measure of an almost additive family of continuous functions with tempered variation as well as for its Gibbs property.

3.1. **Basic notions.** Let $\Phi = (\phi_t)_{t\in\mathbb{R}}$ be a C^1 flow on a smooth manifold M. A compact Φ-invariant set $\Lambda \subset M$ is called a *hyperbolic set for* Φ if there exists a splitting

$$T_\Lambda M = E^s \oplus E^u \oplus E^0$$

and constants $c > 0$ and $\lambda \in (0,1)$ such that for each $x \in \Lambda$:

(1) the vector $(d/dt)\phi_t(x)|_{t=0}$ generates $E^0(x)$;
(2) for each $t \in \mathbb{R}$ we have

$$d_x\phi_t E^s(x) = E^s(\phi_t(x)) \quad \text{and} \quad d_x\phi_t E^u(x) = E^u(\phi_t(x));$$

(3) $\|d_x\phi_t v\| \le c\lambda^t\|v\|$ for $v \in E^s(x)$ and $t > 0$;
(4) $\|d_x\phi_{-t} v\| \le c\lambda^t\|v\|$ for $v \in E^u(x)$ and $t > 0$.

Given a hyperbolic set Λ for a flow Φ, for each $x \in \Lambda$ and any sufficiently small $\varepsilon > 0$ we define

$$A^s(x) = \big\{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to +\infty\big\}$$

and

$$A^u(x) = \big\{y \in B(x,\varepsilon) : d(\phi_t(y), \phi_t(x)) \searrow 0 \text{ when } t \to -\infty\big\}.$$

Moreover, let

$$V^s(x) \subset A^s(x) \quad \text{and} \quad V^u(x) \subset A^u(x)$$

be the largest connected components containing x. These are smooth manifolds, called respectively *(local) stable and unstable manifolds* of size ε at the point x, satisfying:

(1) $T_xV^s(x) = E^s(x)$ and $T_xV^u(x) = E^u(x)$;
(2) for each $t > 0$ we have

$$\phi_t(V^s(x)) \subset V^s(\phi_t(x)) \quad \text{and} \quad \phi_{-t}(V^u(x)) \subset V^u(\phi_{-t}(x));$$

(3) there exist $\kappa > 0$ and $\mu \in (0,1)$ such that for each $t > 0$ we have

$$d(\phi_t(y), \phi_t(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^s(x)$$

and

$$d(\phi_{-t}(y), \phi_{-t}(x)) \le \kappa\mu^t d(y,x) \quad \text{for } y \in V^u(x).$$

We recall that a set Λ is said to be *locally maximal* (with respect to a flow Φ) if there exists an open neighborhood U of Λ such that

$$\Lambda = \bigcap_{t\in\mathbb{R}} \phi_t(U).$$

Given a locally maximal hyperbolic set Λ and a sufficiently small $\varepsilon > 0$, there exists $\delta > 0$ such that if $x, y \in \Lambda$ satisfy $d(x,y) \le \delta$, then there exists a unique $t = t(x,y) \in [-\varepsilon, \varepsilon]$ such that

$$[x,y] := V^s(\phi_t(x)) \cap V^u(x)$$

is a single point in Λ.

Now we make some preparations to introduce the notion of a Markov system. Consider an open smooth disk $D \subset M$ of dimension $\dim M - 1$ that is transverse to Φ and take $x \in D$. Let $U(x)$ be an open neighborhood of x diffeomorphic to $D \times (-\varepsilon, \varepsilon)$. Then the projection $\pi_D \colon U(x) \to D$ defined by $\pi_D(\phi_t(y)) = y$ is differentiable. We say that a closed set $R \subset \Lambda \cap D$ is a *rectangle* if $R = \overline{\operatorname{int} R}$ and $\pi_D([x,y]) \in R$ for $x, y \in R$.

Consider rectangles $R_1, \ldots, R_k \subset \Lambda$ (each contained in some open smooth disk transverse to the flow) such that

$$R_i \cap R_j = \partial R_i \cap \partial R_j \quad \text{for } i \neq j.$$

Let $Z = \bigcup_{i=1}^k R_i$. We assume that there exists $\varepsilon > 0$ such that:

(1) $\Lambda = \bigcup_{t \in [0,\varepsilon]} \phi_t(Z)$;

(2) whenever $i \neq j$, either

$$\phi_t(R_i) \cap R_j = \emptyset \quad \text{for all } t \in [0,\varepsilon]$$

or

$$\phi_t(R_j) \cap R_i = \emptyset \quad \text{for all } t \in [0,\varepsilon].$$

We define the function $\tau \colon \Lambda \to \mathbb{R}_0^+$ by

$$\tau(x) = \min\bigl\{t > 0 : \phi_t(x) \in Z\bigr\},$$

and the *transfer map* $T \colon \Lambda \to Z$ by

$$T(x) = \phi_{\tau(x)}(x). \tag{3.1}$$

The restriction T_Z of T to Z is invertible and we have $T^n(x) = \phi_{\tau_n(x)}(x)$, where

$$\tau_n(x) = \sum_{i=0}^{n-1} \tau(T^i(x)).$$

The collection $R_1, \ldots, R_k$ is said to be a *Markov system* for Φ on Λ if

$$T(\operatorname{int}(V^s(x) \cap R_i)) \subset \operatorname{int}(V^s(T(x)) \cap R_j)$$

and

$$T^{-1}(\operatorname{int}(V^u(T(x)) \cap R_j)) \subset \operatorname{int}(V^u(x) \cap R_i)$$

for every $x \in \operatorname{int} T(R_i) \cap \operatorname{int} R_j$ and $i, j = 1, \ldots, k$. By work of Bowen [5] and Ratner [11], any locally maximal hyperbolic set Λ has Markov systems of arbitrarily small diameter and the function τ is Hölder continuous on each domain of continuity.

Given a Markov system $R_1, \ldots, R_k$ for a flow Φ on a locally maximal hyperbolic set Λ, we consider the $k \times k$ matrix A with entries

$$a_{ij} = \begin{cases} 1 & \text{if } \operatorname{int} T(R_i) \cap R_j \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

where T is the map in (3.1). We also consider the set

$$\Sigma_A = \bigl\{(\cdots i_{-1} i_0 i_1 \cdots) : a_{i_n i_{n+1}} = 1 \text{ for } n \in \mathbb{Z}\bigr\} \subset \{1, \ldots, k\}^{\mathbb{Z}}$$

and the shift map $\sigma \colon \Sigma_A \to \Sigma_A$ defined by

$$\sigma(\cdots i_0 \cdots) = (\cdots j_0 \cdots),$$

where $j_n = i_{n+1}$ for each $n \in \mathbb{Z}$. We denote by Σ_n the set of Σ_A-admissible sequences of length n, that is, the finite sequences $(i_1 \cdots i_n)$ for which there exists

$(\cdots j_0 j_1 j_2 \cdots) \in \Sigma_A$ such that $(i_1 \dots i_n) = (j_1 \cdots j_n)$. Finally, we define a *coding map* $\pi\colon \Sigma_A \to Z$ by

$$\pi(\cdots i_0 \cdots) = \bigcap_{n\in\mathbb{Z}} R_{i_{-n}\cdots i_n},$$

where

$$R_{i_{-n}\cdots i_n} = \bigcap_{l=-n}^{n} \overline{T_Z^{-l} \operatorname{int} R_{i_l}}.$$

The following properties hold:

(1) $\pi \circ \sigma = T \circ \pi$;
(2) π is Hölder continuous and onto;
(3) π is one-to-one on a full measure set with respect to any ergodic measure of full support and on a residual set.

Given $\beta > 1$, we equip Σ_A with the distance d_β defined by

$$d_\beta(\omega, \omega') = \begin{cases} \beta^{-n} & \text{if } \omega \neq \omega', \\ 0 & \text{if } \omega = \omega', \end{cases}$$

where $n = n(\omega, \omega') \in \mathbb{N} \cup \{0\}$ is the smallest integer such that $i_n(\omega) \neq i_n(\omega')$ or $i_{-n}(\omega) \neq i_{-n}(\omega')$. One can always choose β so that $\tau \circ \pi$ is Lipschitz.

Now let ν be a T_Z-invariant probability measure on Z. One can show that ν induces a Φ-invariant probability measure μ on Λ such that

$$\int_\Lambda g\, d\mu = \frac{\int_Z \int_0^{\tau(x)} (g \circ \phi_s)(x)\, ds\, d\nu}{\int_Z \tau\, d\nu} \tag{3.2}$$

for any continuous function $g\colon \Lambda \to \mathbb{R}$. In fact, any Φ-invariant probability measure μ on Λ is of this form for some T_Z-invariant probability measure ν on Z. Abramov's entropy formula says that

$$h_\mu(\Phi) = \frac{h_\nu(T_Z)}{\int_Z \tau\, d\nu}. \tag{3.3}$$

By (3.2) and (3.3) we obtain

$$h_\mu(\Phi) + \int_\Lambda g\, d\mu = \frac{h_\nu(T_Z) + \int_Z I_g\, d\nu}{\int_Z \tau\, d\nu},$$

where

$$I_g(x) = \int_0^{\tau(x)} (g \circ \phi_s) ds.$$

3.2. **Technical preparations.** In this section we make a few technical preparations. We start by considering the sequence of functions $c_n\colon Z \to \mathbb{R}$ defined by

$$c_n(x) = a_{\tau_n(x)}(x). \tag{3.4}$$

Lemma 3.1. *The sequence $c = (c_n)_{n\in\mathbb{N}}$ is almost additive with respect to the map T_Z.*

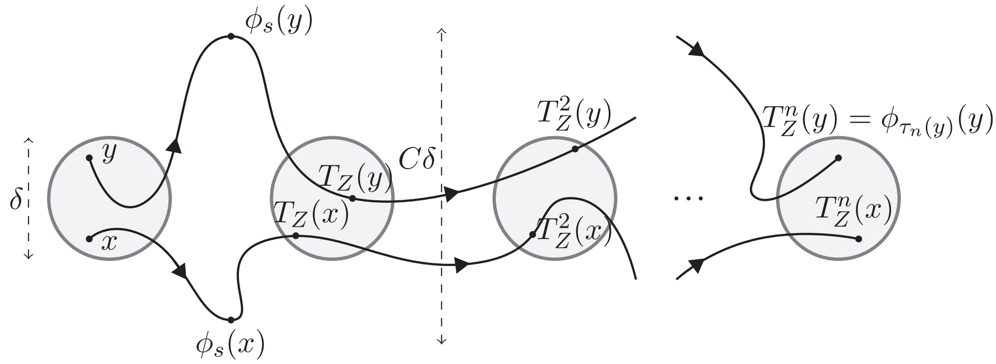

FIGURE 1. $d(\phi_s(x), \phi_s(y)) < C\delta$ for $s \in [0, \tau_n(y)]$.

Proof. Notice that

$$c_{n+m}(x) = a_{\tau_{n+m}(x)}(x) = a_{\tau_n(x)+\tau_m(T^n(x))}(x) \tag{3.5}$$

for $n, m \in \mathbb{N}$. Since a is almost additive with respect to Φ, by (3.5) we have

$$\begin{aligned} c_{n+m}(x) &\le a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(\phi_{\tau_n(x)}(x)) + C \\ &= a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(T^n(x)) + C \\ &= c_n(x) + c_m(T^n(x)) + C. \end{aligned}$$

Similarly, we have also

$$\begin{aligned} c_{n+m}(x) &\ge a_{\tau_n(x)}(x) + a_{\tau_m(T^n(x))}(T^n(x)) - C \\ &= c_n(x) + c_m(T^n(x)) - C. \end{aligned}$$

This shows that c is an almost additive sequence with respect to T_Z. $\square$

Now we consider the sets $B_t(x, \varepsilon)$ in (2.1) with $X = \Lambda$.

Lemma 3.2. *Given $\delta > 0$, there exists a Markov system $R_1, \ldots, R_k$ and a constant $C > 0$ such that*

$$R_{i_{-n} \cdots i_n} \subset B_{\tau_n(y)}(y, C\delta)$$

for every $n \in \mathbb{N}$ and $y \in R_{i_{-n} \cdots i_n}$.

Proof. Since the rectangles of a Markov system can have arbitrarily small diameters, for each $\delta > 0$ there exist $R_1, \ldots, R_k$ such that

$$R_j \subset B(z, \delta) \quad \text{for every } z \in R_j. \tag{3.6}$$

Given $x, y \in R_{i_{-n} \cdots i_n}$, we have $T^k(x), T^k(y) \in R_{i_k}$ for $k \in \{0, \ldots, n\}$. On the other hand, by (3.6),

$$R_{i_k} \subset B(T^k(y), \delta)$$

and so $d(T^k(x), T^k(y)) < \delta$ for $k \in \{0, 1, \ldots, n\}$. Finally, by the uniform continuity of $(t, x) \mapsto \phi_t(x)$ on compact sets, there exists $C = C(\delta) > 0$ (independent of n) such that

$$d(\phi_s(x), \phi_s(y)) < C\delta \quad \text{for } s \in [0, \tau_n(y)]$$

(see Figure 1). This yields the desired statement. $\square$

Given $\delta > 0$ and a Markov system as in Lemma 3.2, we consider the numbers

$$V_n(c) = \sup\bigl\{|c_n(x) - c_n(y)| : x, y \in R_{i_1\cdots i_n}\bigr\}$$

for $n \in \mathbb{N}$. We recall that a family of functions a is said to have *bounded variation* if for every $\kappa > 0$ there exists $\varepsilon > 0$ such that

$$|a_t(x) - a_t(y)| < \kappa \quad \text{whenever } y \in B_t(x, \varepsilon).$$

We shall always assume that $C\delta < \varepsilon$.

Lemma 3.3. *If a has bounded variation and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, then $\sup_{n\in\mathbb{N}} V_n(c) < \infty$ (in particular, c has tempered variation).*

Proof. Take $x, y \in R_{i_{-n}\cdots i_n}$ and $\omega, \omega' \in \Sigma_A$ such that $T(x) = \pi(\sigma(\omega))$ and $T(y) = \pi(\sigma(\omega'))$. Choosing $\beta > 1$ so that $\tau \circ \pi$ is Lipschitz, say with Lipschitz constant $L > 0$, one can write

$$\begin{aligned}|\tau_n(x) - \tau_n(y)| &= \left|\sum_{l=0}^{n-1} \tau(T^l(x)) - \sum_{l=0}^{n-1} \tau(T^l(y))\right| \\ &\le \sum_{l=0}^{n-1} |(\tau \circ \pi)(\sigma^l(\omega)) - (\tau \circ \pi)(\sigma^l(\omega'))| \\ &\le \sum_{l=0}^{n-1} L d_\beta(\sigma^l(\omega), \sigma^l(\omega')).\end{aligned}$$

This implies that there exists $D > 0$ (independent of x, y and n) such that

$$|\tau_n(x) - \tau_n(y)| \le D. \tag{3.7}$$

Assuming without loss of generality that $\tau_n(x) > \tau_n(y)$, since the family a is almost additive, we have

$$a_{\tau_n(x)}(x) \le a_{\tau_n(x)-\tau_n(y)}(x) + a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) + C.$$

Together with (3.7), this implies that

$$\begin{aligned}c_n(x) - c_n(y) &= a_{\tau_n(x)}(x) - a_{\tau_n(y)}(y) \\ &\le |a_{\tau_n(x)-\tau_n(y)}(x)| + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(y)| + C \\ &\le \sup_{l\in[0,D]} \|a_l\|_\infty + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(y)| + C \\ &\le \sup_{l\in[0,D]} \|a_l\|_\infty + |a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - a_{\tau_n(y)}(x)| \\ &\quad + |a_{\tau_n(y)}(x) - a_{\tau_n(y)}(y)| + C.\end{aligned} \tag{3.8}$$

Since a is almost additive and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$, we have

$$\sup_{l\in[0,D]} \|a_l\|_\infty \le M \tag{3.9}$$

for some constant $M > 0$. Moreover, by the definition of bounded variation,

$$|a_{\tau_n(y)}(x) - a_{\tau_n(y)}(y)| \le \kappa. \tag{3.10}$$

Now note that

$$y = \phi_{-\tau_n(y)}(\phi_{\tau_n(y)}(y)) = \phi_{-\tau_n(y)}(T^n(y))$$

and define

$$z := \phi_{-\tau_n(y)}(\phi_{\tau_n(x)}(x)) = \phi_{-\tau_n(y)}(T^n(x)).$$

Since $T^n(x), T^n(y) \in R_{i_n}$ and the map $p \mapsto \phi_{-\tau_n(y)}(p)$ is uniformly continuous on compact sets, there exists $\delta' > 0$ (depending only on δ) such that $d(y,z) < \delta'$. Thus,

$$d(x,z) \le d(x,y) + d(y,z) < \delta + \delta'.$$

By the uniform continuity of the map $(t,x) \mapsto \phi_t(x)$ on the set $[0, \tau_n(y)] \times \Lambda$, there exists $\varepsilon > 0$ such that

$$d(\phi_s(x), \phi_s(z)) < \varepsilon \quad \text{for } s \in [0, \tau_n(y)].$$

Again by the bounded variation property, we have

$$|a_{\tau_n(y)}(z) - a_{\tau_n(y)}(x)| \le \kappa. \tag{3.11}$$

By (3.8) together with (3.9), (3.10) and (3.11), we obtain

$$c_n(x) - c_n(y) \le M + 2\kappa + C.$$

Similarly, using the inequality

$$a_{\tau_n(x)}(x) \ge a_{\tau_n(x)-\tau_n(y)}(x) + a_{\tau_n(y)}(\phi_{\tau_n(x)-\tau_n(y)}(x)) - C,$$

one can show that

$$c_n(x) - c_n(y) \ge -(M + 2\kappa + C).$$

This yields the desired statement. ☐

We continue with an auxiliary result. Assume that $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$.

Lemma 3.4. *For every Φ-invariant measure μ on Λ induced by an ergodic T_Z-invariant measure ν on Z, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu = \lim_{n\to+\infty} \frac{1}{n} \int_Z c_n \, d\nu \Big/ \int_Z \tau \, d\nu. \tag{3.12}$$

Proof. Since ν is ergodic, the measure μ is also ergodic and so by Kingman's sub-additive ergodic theorem, we have

$$\lim_{n\to\infty} \frac{a_{\tau_n(x)}(x)}{\tau_n(x)} = \lim_{t\to\infty} \frac{a_t(x)}{t} = \lim_{t\to\infty} \frac{1}{t} \int_\Lambda a_t \, d\mu =: \eta$$

for ν-almost every $x \in Z$. It follows from Egorov's theorem that given $\varepsilon > 0$, there exists a measurable set $Z_\varepsilon \subset Z$ with $\nu(Z_\varepsilon) > 1 - \varepsilon$ on which c_n/τ_n converges uniformly to η. We assume in addition that

$$\left| \int_{Z_\varepsilon} \frac{\tau_n}{n} \, d\nu - \int_Z \tau \, d\nu \right| < \varepsilon \quad \text{and} \quad \left| \int_{Z_\varepsilon} \frac{c_n}{n} \, d\nu - \int_Z \frac{c_n}{n} \, d\nu \right| < \varepsilon \tag{3.13}$$

for all $n \in \mathbb{N}$, which is possible since $\int_Z \tau_n \, d\nu = n \int_Z \tau \, d\nu$ and since

$$\frac{c_n(x)}{n} = \frac{a_{\tau_n(x)}(x)}{\tau_n(x)} \cdot \frac{\tau_n(x)}{n}$$

is a product of uniformly bounded sequences (because $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$). In particular, there exists $N \in \mathbb{N}$ such that

$$(\eta - \varepsilon)\tau_n(x) < c_n(x) < (\eta + \varepsilon)\tau_n(x)$$

for all $x \in Z_\varepsilon$ and $n > N$. Together with (3.13) this gives

$$(\eta-\varepsilon)\left(\int_Z \tau\,d\nu-\varepsilon\right)-\varepsilon<\int_Z \frac{c_n}{n}\,d\nu<(\eta+\varepsilon)\left(\int_Z \tau\,d\nu+\varepsilon\right)+\varepsilon$$

for all $n > N$. Taking $n \to \infty$ and then $\varepsilon \to 0$, we obtain property (3.12). □

It follows from Lemma 3.4 that

$$h_\mu(\Phi)+\lim_{t\to\infty}\frac{1}{t}\int_\Lambda a_t\,d\mu=\left(h_\nu(T_Z)+\lim_{n\to\infty}\frac{1}{n}\int_Z c_n\,d\nu\right)\Big/\int_Z \tau\,d\nu.$$

Note that in Theorem 2.1 one can replace $\mathcal{M}$ by the set of ergodic measures (see Lemma 2.4). Hence, $P_\Phi(a) = 0$ if and only if $P_{T_Z}(c) = 0$. Moreover, if $P_\Phi(a) = 0$, then μ is an equilibrium measure for a if and only if ν is an equilibrium measure for c.

3.3. **Existence of Gibbs measures.** Now we introduce the notion of a Gibbs measure for a flow in the present context. A measure μ on a hyperbolic set Λ for a flow Φ is called a *Gibbs measure* for a if it is induced by a measure ν on $Z = \bigcup_{i=1}^k R_i$ satisfying

$$K^{-1} \le \frac{\nu(R_{i_{-n}\cdots i_n})}{\exp\left[-2nP_{T_Z}(c)+c_{2n}(x)\right]} \le K$$

for some constant $K \ge 1$, for every $n \in \mathbb{N}$ and $x \in R_{i_{-n}\cdots i_n}$ (recall that $c_n(x) = a_{\tau_n(x)}(x)$ for $n \in \mathbb{N}$ and $x \in Z$). Considering the sets

$$\tilde{R}_{i_1\cdots i_n}=\bigcup_{i_{-n}\cdots i_0} R_{i_{-n}\cdots i_n},$$

one can verify that this is equivalent to require that

$$\tilde{K}^{-1} \le \frac{\nu(\tilde{R}_{i_1\cdots i_n})}{\exp\left[-nP_{T_Z}(c)+c_n(x)\right]} \le \tilde{K} \tag{3.14}$$

for some constant $\tilde{K} \ge 1$, for every $n \in \mathbb{N}$ and $x \in \tilde{R}_{i_1\cdots i_n}$. If the measure ν is also invariant, then

$$P_{T_Z}(c)-\frac{c_n(x)}{n}-\frac{\log\tilde{K}}{n}\le-\frac{1}{n}\log\nu(\tilde{R}_{i_1\cdots i_n})\le P_{T_Z}(c)-\frac{c_n(x)}{n}+\frac{\log\tilde{K}}{n},$$

which implies that

$$h_\nu(T_Z,x):=\lim_{n\to\infty}-\frac{1}{n}\log\nu(\tilde{R}_{i_1\cdots i_n})=P_{T_Z}(c)-\lim_{n\to\infty}\frac{c_n(x)}{n}.$$

By the Shannon–McMillan–Breiman theorem, we have

$$\begin{aligned}h_\nu(T_Z)&=\int_Z h_\nu(T_Z,x)\,d\nu(x)\\&=P_{T_Z}(c)-\int_Z \lim_{n\to\infty}\frac{c_n(x)}{n}\,d\nu(x)\\&=P_{T_Z}(c)-\lim_{n\to\infty}\frac{1}{n}\int_Z c_n\,d\nu\end{aligned}$$

and so

$$P_{T_Z}(c) = h_\nu(T_Z) + \lim_{n\to\infty} \frac{1}{n} \int_Z c_n \, d\nu.$$

This shows that any invariant Gibbs measure satisfying (3.14) is an equilibrium measure for c with respect to the map T_Z.

Finally, we state our result on the existence of Gibbs measures.

Theorem 3.5. *Let Λ be a hyperbolic set for a topologically mixing C^1 flow Φ and let a be an almost additive family of continuous functions on Λ with bounded variation such that $P_\Phi(a) = 0$ and $\sup_{t\in[0,s]} \|a_t\|_\infty < \infty$ for some $s > 0$. Then:*

(1) *there exists a unique equilibrium measure for a;*
(2) *there exists a unique invariant Gibbs measure for a;*
(3) *the two measures are equal and are ergodic.*

Proof. We always consider a Markov system with sufficiently small diameter as in Lemmas 3.2 and 3.3. Let c be the sequence in (3.4).

Lemma 3.6 (see [4, Theorem 10.1.4])**.** *We have*

$$P_{T_Z}(c) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{i_1\cdots i_n} \exp c_n(x_{i_1\cdots i_n})$$

for any $x_{i_1\cdots i_n} \in R_{i_1\cdots i_n}$, for each $(i_1 \cdots i_n) \in \Sigma_n$ and $n \in \mathbb{N}$.

Now let

$$h_{i_1\cdots i_n} = \max\bigl\{\exp c_n(y) : y \in R_{i_1\cdots i_n}\bigr\} \quad \text{and} \quad H_n = \sum_{i_1\cdots i_n} h_{i_1\cdots i_n}.$$

Moreover, we define a probability measure ν_n in the algebra generated by the sets $R_{i_1\cdots i_n}$ by

$$\nu_n(R_{i_1\cdots i_n}) = h_{i_1\cdots i_n}/H_n$$

for each $(i_1 \cdots i_n) \in \Sigma_n$ and we extend it to the Borel σ-algebra $\mathcal{B}$ of Z. Let $\mathcal{M}_Z(c)$ be the set of all sublimits of the sequence $(\nu_n)_{n\in\mathbb{N}}$. Since Z is compact, $\mathcal{M}_Z(c)$ is sequentially compact and so it is nonempty.

Lemma 3.7 (see [4, Lemmas 10.1.10 and 10.1.11])**.** *The following two properties hold:*

(1) *each $\nu \in \mathcal{M}_Z(c)$ is a Gibbs measure for c with respect to T_Z;*
(2) *any Gibbs measure for c with respect to T_Z is ergodic.*

Using the former properties, one can now proceed as in the proof of Theorem 10.1.9 in [4] to show that:

(1) there exists a unique equilibrium measure ν_c for T_Z;
(2) there exists a unique invariant Gibbs measure for T_Z;
(3) the two measures coincide and are ergodic.

The measure ν_c induces a Φ-invariant probability measure μ_a on Λ by (3.2). Moreover, any Φ-invariant probability measure μ on Λ is of this form for some T_Z-invariant probability measure ν on Z.

Since $P_\Phi(a) = 0$, the measure μ is an equilibrium measure for a with respect to Φ if and only if ν is an equilibrium measure for c with respect to T_Z. Furthermore, one can verify that μ is ergodic with respect to Φ if and only if ν is ergodic with respect to T_Z. This readily implies the three properties in the theorem (note that μ_a is the unique equilibrium measure for a). $\square$

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L. Barreira
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: barreira@math.tecnico.ulisboa.pt

C. Holanda
Departamento de Matemática, Instituto Superior Técnico, Universidade de Lisboa, 1049-001 Lisboa, Portugal
E-mail address: carllosholanda@tecnico.ulisboa.pt