

TOPOLOGICAL ENTROPY OF BUNIMOVICH STADIUM BILLIARDS

MICHAŁ MISIUREWICZ AND HONG-KUN ZHANG

ABSTRACT. We estimate from below the topological entropy of the Bunimovich stadium billiards. We do it for long billiard tables, and find the limit of estimates as the length goes to infinity.

1. INTRODUCTION

In this paper, we consider Bunimovich stadium billiards. This was the first type of billiards having convex (focusing) components of the boundary $\partial\Omega$, yet enjoying the hyperbolic behavior [6,7]. Such boundary consists of two semicircles at the ends, joined by segments of straight lines (see Figure 1). For those billiards, ergodicity, K-mixing and Bernoulli property were proved in [10] for the natural measure.



FIGURE 1. Bunimovich stadium.

We consider billiard maps (not the flow) for two-dimensional billiard tables. Thus, the phase space of a billiard is the product of the boundary of the billiard table and the interval $[-\pi/2, \pi/2]$ of angles of reflection. We will use the variables (r, φ) , where r parametrizes the table boundary by the arc length, and φ is the angle of reflection. We mentioned the natural measure: it is $c \cos \varphi dr d\varphi$, where c is the normalizing constant. This measure is invariant for the billiard map.

As we said, we want to study topological entropy of the billiard map. This means that we should look at the billiard as a topological dynamical system. However, existence of the natural measure resulted in most authors looking at the billiard as a measure preserving transformation. That is, all important properties of the billiard were proved only almost everywhere, not everywhere. Additionally, the billiard map is only piecewise continuous instead of continuous. Often it is even not defined everywhere. All this creates problems already at the level of definitions. We will discuss those problems in the next section.

2020 *Mathematics Subject Classification.* Primary 37C83, 37B40.

Key words and phrases. Bunimovich stadium billiard, topological entropy.

Research of Michał Misiurewicz was partially supported by grant number 426602 from the Simons Foundation.

In view of this complicated situation, we will not try to produce a comprehensive theory of the Bunimovich stadium billiards from the topological point of view, but present the results on their topological entropy that are independent of the approach. For this we will find a subspace of the phase space that is compact and invariant, and on which the billiard map is continuous. We will find the topological entropy restricted to this subspace. This entropy is a lower bound of the topological entropy of the full system, no matter how this entropy is defined. Finally, we will find the limit of our estimates as the length of the billiard table goes to infinity.

The reader who wants to learn more on other properties of the Bunimovich stadium billiards, can find it in many papers, in particular [2, 4, 6–9, 11]. While some of them contain results about topological entropy of those billiards, none of those results can be considered completely rigorous.

The paper is organized as follows. In Section 2 we discuss the problems connected with defining topological entropy for billiards. In Section 3 we produce symbolic systems connected with the Bunimovich billiards. In Section 4 we perform actual computations of the topological entropy.

2. TOPOLOGICAL ENTROPY OF BILLIARDS

Let $\mathcal{M} = \partial\Omega \times [-\pi/2, \pi/2]$ be the phase space of a billiard and let $\mathcal{F} : \mathcal{M} \rightarrow \mathcal{M}$ be the billiard map. We assume that the boundary of the billiard table is piecewise C^2 with finite number of pieces. In such a situation the map \mathcal{F} is piecewise continuous (in fact, piecewise smooth) with finitely many pieces. That is, \mathcal{M} is the union of finitely many open sets \mathcal{M}_i (of quite regular shape) and a singular set \mathcal{S} , which is the union of finitely many smooth curves, and on which the map is often even not defined. The map \mathcal{F} restricted to each \mathcal{M}_i is a diffeomorphism onto its image.

This situation is very similar as for piecewise continuous piecewise monotone interval maps. For those maps, the usual way of investigating them from the topological point of view is to use *coding*. We produce the *symbolic system* associated with our map by taking sequences of symbols (numbers enumerating pieces of continuity) according to the number of the piece to which the n -th image of our point belongs. On this symbolic space we have the shift to the left. In particular, the topological entropy of this symbolic system was shown to be equal to the usual Bowen's entropy of the underlying interval map (see [13]).

Thus, it is a natural idea to do the same for billiards. Thus, for a point $x \in \mathcal{M}$, whose trajectory is disjoint from \mathcal{S} , we take its *itinerary* (code) $\omega(x) = (\omega_n)$, where $\omega_n = i$ if and only if $\mathcal{F}(x) \in \mathcal{M}_i$. The problem is that the set of itineraries obtained in such a way is usually not closed (in the product topology). Therefore we have to take the closure of this set. Then the question one has to deal with is whether there is no essential dynamics (for example, invariant measures with positive entropy) on this extra set.

A rigorous approach to studying topological entropy for Sinai billiards can be found in the recent paper of Baladi and Demers [3]. The Sinai billiard maps are different from the Bunimovich stadium maps in the sense that the former have uniform hyperbolicity, and the transfer operators have spectra gaps. The stadium billiard is hyperbolic, but not uniformly. Moreover, here we have to deal with the trajectories that are bouncing between the straight line segments of the boundary.

To complete the list of problems, the coding with four pieces of the boundary seems to be not sufficient (as has been noticed in [4]).

The papers dealing with the topological entropy of Bunimovich stadium billiards use different definitions. In [4] and [11], topological entropy is explicitly defined as the exponential growth rate of the number of periodic orbits of a given period. In [8], first coding is performed in a different way, using rectangles defined by stable and unstable manifolds. This coding uses an infinite alphabet. Then various definitions of topological entropy for the obtained symbolic system are used. In [3], topological entropy is defined as the topological entropy of the corresponding symbolic system, that is, as the exponential growth rate of the number of nonempty cylinders of a given length in the symbolic system. As we mentioned, it is shown that the result is the same as when one is using the classical Bowen's definition for the original billiard map. In [2], topological entropy is not formally defined, but it seems that the authors think of the entropy of the symbolic system.

In this paper, we will be considering a subsystem of the full billiard map. This subsystem is a continuous map of a compact space to itself, and is conjugate to a subshift of finite type. Thus, whether we define the topological entropy of the full system as the entropy of the symbolic system or as the growth rate of the number of periodic orbit, our estimates will be always lower bounds for the topological entropy.

3. SUBSYSTEM AND CODING

We consider the Bunimovich stadium billiard table, with the radius of the semi-circles equal to 1, and the lengths of straight segments $\ell > 1$. The phase space of this billiard map will be denoted by \mathcal{M}_ℓ , and the map by \mathcal{F}_ℓ . The subspace of \mathcal{M}_ℓ consisting of points whose trajectories have no two consecutive collisions with the same semicircle will be denoted by \mathcal{K}_ℓ . The subspace of \mathcal{K}_ℓ consisting of points whose trajectories have no $N + 1$ consecutive collisions with the straight segments will be denoted by $\mathcal{K}_{\ell,N}$. We will show that if $\ell > 2N + 2$, then the map \mathcal{F}_ℓ restricted to $\mathcal{K}_{\ell,N}$ has very good properties.

In general, coding for \mathcal{F}_ℓ needs at least six symbols. They correspond to the four pieces of the boundary of the stadium, and additionally on the semicircles we have to specify the orientation of the trajectory (whether φ is positive or negative), see [4]. However, in \mathcal{K}_ℓ this additional requirement is unnecessary, because there are no multiple consecutive collisions with the same semicircle. This also implies that in \mathcal{K}_ℓ for a given ℓ the angle φ is uniformly bounded away from $\pm\pi/2$.

While in [2] the statements about generating partition are written in terms of measure preserving transformations, the sets of measure zero that have to be removed are specified. In \mathcal{K}_ℓ the only set that needs to be removed is the set of points whose trajectories are periodic of period 2, bouncing from the two straight line segments. However, this set carries no topological entropy, so we can ignore it. Thus, according to [2], the symbolic system corresponding to \mathcal{F}_ℓ on \mathcal{K}_ℓ is a closed subshift Σ_ℓ of a subshift of finite type with 4 symbols. We say that there is a *transition* from a state i to j if it is possible that $\omega_n = i$ and $\omega_{n+1} = j$. In our subshift here are some transitions that are forbidden: one cannot go from a symbol corresponding to a semicircle to the same symbol. There are of course also some transitions in many steps forbidden; they depend on ℓ .

For every element of Σ_ℓ there is a unique point of \mathcal{K}_ℓ with that itinerary. However, the same point of \mathcal{K}_ℓ may have more than one itinerary, because there are four points on the boundary of the stadium that belong to two pieces of the boundary each. Thus, the coding is not one-to-one, but this is unavoidable if we want to obtain a compact symbolic system. Another solution would be to remove codes of all trajectories that pass through any of four special points, and at the end take the closure of the symbolic space.

This problem disappears when we pass to $\mathcal{K}_{\ell,N}$ with $\ell > 2N + 2$. Namely, then the angle φ at any point of $\mathcal{K}_{\ell,N}$ whose first coordinate is on the straight line piece, is larger than $\pi/4$ in absolute value.

Let us look at the geometry of this situation. Let C be the right unit semicircle in \mathbb{R}^2 (without endpoints), $A \in C$, and let L_1, L_2 be half-lines emerging from A , reflecting from C (like a billiard flow trajectory) from inside at A (see Figure 2). Assume that for $i = 1, 2$ the angles between L_i and the horizontal lines are less than $\pi/4$, and that L_i intersects C only at A . Consider the argument $\arg(A)$ of A (as in polar coordinates on in the complex plane, where the origin is at the center of the circle).

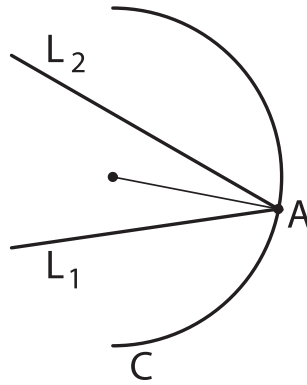


FIGURE 2. Situation from Lemma 3.1.

Lemma 3.1. *In the above situation, $|\arg(A)| < \pi/4$. Moreover, neither L_1 nor L_2 passes through an endpoint of C .*

Proof. If $|\arg(A)| \geq \pi/4$, then both lines L_1 and L_2 are on the same side of the origin, so the incidence and reflection angle cannot be the same. Therefore, $|\arg(A)| < \pi/4$.

Suppose that L_1 passes through the lower endpoint of C (the other cases are similar). Then $\arg(A) < 0$, so L_2 intersects the semicircle also at the point with argument

$$\arg(A) + (\arg(A) - (-\pi/2)) = 2\arg(A) + \pi/2,$$

which is a contradiction. \square

In view of the above lemma, the collision points on the semicircles cannot be too close to the endpoints of the semicircles (including endpoints themselves). Thus, the correspondence between $\mathcal{K}_{\ell,N}$ and its coding system $\Sigma_{\ell,N}$ is a bijection. Standard

considerations of topologies in both systems show that this bijection is a homeomorphism, say $\xi_{\ell,N} : \mathcal{K}_{\ell,N} \rightarrow \Sigma_{\ell,N}$. If σ is the left shift in the symbolic system, then by the construction we have $\xi_{\ell,N} \circ \mathcal{F}_\ell = \sigma \circ \xi_{\ell,N}$. In such a way we get the following lemma.

Lemma 3.2. *If $\ell > 2N + 2$, then the systems $(\mathcal{K}_{\ell,N}, \mathcal{F}_\ell)$ and $(\Sigma_{\ell,N}, \sigma)$ are topologically conjugate.*

We can modify our codings, in order to simplify further proofs. The first thing is to identify the symbols corresponding to two semicircles. This can be done due to the symmetry, and will result in producing symbolic systems Σ'_ℓ and $\Sigma'_{\ell,N}$, which are 2-to-1 factors of Σ_ℓ and $\Sigma_{\ell,N}$ respectively. Since the operation of taking a 2-to-1 factor preserves topological entropy, this will not affect our results.

With this simplification, Σ'_ℓ is a closed, shift-invariant subset of the phase space of a subshift of finite type $\tilde{\Sigma}$. Subshift of finite type $\tilde{\Sigma}$ looks as follows. There are three states, $0, A, B$ (where 0 corresponds to the semicircles), and the only forbidden transitions are from A to A and from B to B .

Then $\Sigma'_{\ell,N}$ is a closed, shift-invariant subset of Σ'_ℓ , where additionally n -step transitions involving only states A and B are forbidden if $n > N$. However, it pays to recode $\Sigma'_{\ell,N}$. Namely, we replace states A and B by $1, 2, \dots, N$ and $-1, -2, \dots, -N$. If $(\omega_n) \in \Sigma'_{\ell,N}$, and $\omega_k = \omega_{k+m+1} = 0$, while $\omega_n \in \{A, B\}$ for $n = k + 1, k + 2, \dots, k + m$, then for the recoded sequence (ρ_n) we have $\rho_k = \rho_{k+m+1} = 0$ and $(\rho_{k+1}, \rho_{k+2}, \dots, \rho_{k+m})$ is equal to $(1, 2, \dots, m)$ if $\omega_{k+1} = A$ and $(-1, -2, \dots, -m)$ if $\omega_{k+1} = B$.

Geometric meaning of the recoding is simple. We unfold the stadium by using reflections from the straight parts (see Figure 3). We will label the levels of the semicircles by integers. Our new coding translates to this picture as follows. We

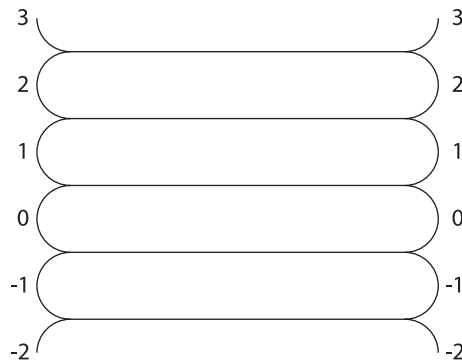


FIGURE 3. Unfolded stadium.

start at a semicircle, then go to a semicircle on the other side and m levels up or down, etc.

For symbolic systems, recoding in such a way amounts to the topological conjugacy of the original and recoded systems (see [12]). This means that the system $(\Sigma'_{\ell,N}, \sigma)$ is topologically conjugate to a subsystem of $\tilde{\Sigma}_N$, which is the subshift of

finite type defined as follows. The states are $-N, -N + 1, \dots, N - 1, N$. And the transitions are: from 0 to 0, 1 or -1 ; from i to $i + 1$ and 0 if $1 \leq i \leq N - 1$; from N to 0; from $-i$ to $-i - 1$ and 0 if $1 \leq i \leq N - 1$; and from $-N$ to 0.

Lemma 3.3. *If $\ell > 2N + 2$, then $(\Sigma'_{\ell, N}, \sigma)$ is topologically conjugate to $(\tilde{\Sigma}_N, \sigma)$.*

Proof. Both sets $\Sigma'_{\ell, N}$ and $\tilde{\Sigma}_N$ are closed and $\Sigma'_{\ell, N} \subset \tilde{\Sigma}_N$. Therefore, it is enough to prove that $\Sigma'_{\ell, N}$ is dense in $\tilde{\Sigma}_N$. For this we show that, for every sequence $(\rho_0, \rho_1, \dots, \rho_k)$ appearing as a block in an element of $\tilde{\Sigma}_N$, there is a point $(r_0, \varphi_0) \in \mathcal{K}_{\ell, N}$ for which after coding and recoding a piece of trajectory of length $k + 1$, we get $(\rho_0, \rho_1, \dots, \rho_k)$. By taking a longer sequence, we may assume that $\rho_0 = \rho_k = 0$.

Consider all candidates for such trajectories in the unfolded stadium, when we do not care whether the incidence and reflection angles are equal and we allow extra intersections with the semicircles. That is, we consider all curves that are unions of straight line segments from x_0 to x_1 to $x_2 \dots$ to x_m in the unfolded stadium, such that x_0 is in the left semicircle at level 0, x_1 is in the right semicircle at level n_1 , x_2 is in the left semicircle at level $n_1 + n_2$, etc. Here n_1, n_2, \dots are the numbers of non-zero elements of the sequence $(\rho_0, \rho_1, \dots, \rho_k)$ between a zero element and the next zero element, where we also take into account the signs of those non-zero elements. In other words, this curve is an approximate trajectory (of the flow) in the unfolded stadium that would have the recoded itinerary $(\rho_0, \rho_1, \dots, \rho_k)$. Additionally we require that x_0 and x_m are at the midpoints of their semicircles. The class of such curves is a compact space with the natural topology, so there is the longest curve in this class. We claim that this curve is a piece of the flow trajectory corresponding to the trajectory we are looking for.

If we look at the ellipse with foci at x_i and x_{i+2} to which x_{i+1} belongs, then x_{i+1} has to be a point of tangency of that ellipse and the semicircle. Since for the ellipse the angles of incidence and reflection are equal, the same is true for the semicircle.

Observe also that if x_{i+1} cannot be an endpoint of a semicircle. If it were, then an infinitesimally small movement of this point along the semicircle would result in both straight segments of the curve that end/begin at x_{i+1} to get longer.

It remains to prove that our curve intersects the semicircles only at the points x_i . This follows from the observation that if $\ell \geq 2N + 2$ then the angles between the segments of our curve and the straight parts of the billiard table boundary are smaller than $\pi/4$. Suppose that the segment from x_i to x_{i+1} intersects the semicircle C to which x_{i+1} belongs at some other point y (see Figure 4). Then x_{i+1} and y belong to the same half of C . Since at x_{i+1} the incidence and reflection angles of our curve are equal, the segment from x_{i+1} to x_{i+2} also intersects C at some other point, so x_{i+1} should belong to the other half of C , a contradiction. This completes the proof. \square

Remark 3.4. By Lemmas 3.2 and 3.3 (plus the way we obtained $\Sigma'_{\ell, N}$ from $\Sigma_{\ell, N}$), it follows that if $\ell > 2N + 2$ then the natural partition of $\mathcal{K}_{\ell, N}$ into four sets is a Markov partition.

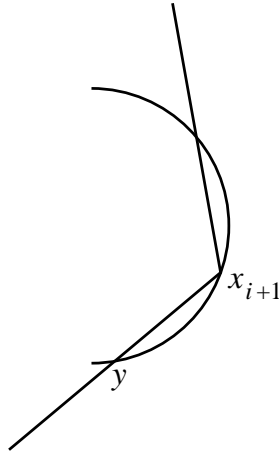


FIGURE 4. Two intersections.

4. COMPUTATION OF TOPOLOGICAL ENTROPY

In the preceding section we obtained some subshifts of finite type. Now we have to compute their topological entropies. If the alphabet of a subshift of finite type is $\{1, 2, \dots, k\}$, then we can write the *transition matrix* $M = (m_{ij})_{i,j=1}^n$, where $m_{ij} = 1$ if there is a transition from i to j and $m_{ij} = 0$ otherwise. Then the topological entropy of our subshift is the logarithm of the spectral radius of M (see [1,12]).

Lemma 4.1. *Topological entropy of the system (Σ'_ℓ, σ) is $\log(1 + \sqrt{2})$.*

Proof. The transition matrix of (Σ'_ℓ, σ) is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

The characteristic polynomial of this matrix is $-(1+x)(x^2 - 2x - 1)$, so the entropy is $\log(1 + \sqrt{2})$. □

In the case of larger, but not too complicated, matrices, in order to compute the spectral radius one can use the *rome method* (see [1,5]). For the transition matrices of $\tilde{\Sigma}_N$ this method is especially simple. Namely, if we look at the paths given by transitions, we see that 0 is a *rome*: all paths lead to it. Then we only have to identify the lengths of all paths from 0 to 0 that do not go through 0 except at the beginning and the end. The spectral radius of the transition matrix is then the largest zero of the function $\sum x^{-p_i} - 1$, where the sum is over all such paths and p_i is the length of the i -th path.

Lemma 4.2. *Topological entropy of the system $(\tilde{\Sigma}_N, \sigma)$ is the logarithm of the largest root of the equation*

$$(4.1) \quad (x^2 - 2x - 1) = -2x^{1-N}.$$

Proof. The paths that we mentioned before the lemma, are: one path of length 1 (from 0 directly to itself), and two paths of length $2, 3, \dots, N$ each. Therefore, our entropy is the logarithm of the largest zero of the function $2(x^{-N} + \dots + x^{-3} + x^{-2}) + x^{-1} - 1$. We have

$$x(1-x)(2(x^{-N} + \dots + x^{-3} + x^{-2}) + x^{-1} - 1) = (x^2 - 2x - 1) + 2x^{1-N},$$

so our entropy is the logarithm of the largest root of equation (4.1). \square

Now that we computed topological entropies of the subshifts of finite type involved, we have to go back to the definition of the topological entropy of billiards (and their subsystems). As we mentioned earlier, the most popular definitions either employ the symbolic systems or use the growth rate of the number of periodic orbits of the given period. For subshifts of finite type that does not make difference, because the exponential growth rate of the number of periodic orbits of a given period is the same as the topological entropy (if the systems are topologically mixing, which is the case here). As the first step, we get the following result, that follows immediately from Lemmas 3.2, 3.3 and 4.2.

Theorem 4.3. *If $\ell > 2N + 2$, then topological entropy of the system $(\mathcal{K}_{\ell, N}, \mathcal{F}_{\ell})$ is the logarithm of the largest root of the equation (4.1).*

Now, independently of which definition of the entropy $h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell})$ of $(\mathcal{K}_{\ell}, \mathcal{F}_{\ell})$ we choose, we get the next theorem.

Theorem 4.4. *We have*

$$(4.2) \quad \liminf_{\ell \rightarrow \infty} h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell}) \geq \log(1 + \sqrt{2}).$$

Proof. On one hand, $\mathcal{K}_{\ell, N}$ is a subset of \mathcal{K}_{ℓ} , so $h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell}) \geq h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell, N})$ for every N . Therefore, by Theorem 4.3,

$$\liminf_{\ell \rightarrow \infty} h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell}) \geq \lim_{N \rightarrow \infty} \log y_N,$$

where y_N is the largest root of the equation (4.1). The largest root of $x^2 - 2x - 1 = 0$ is $1 + \sqrt{2}$. In its neighborhood the right-hand side of (4.1) goes uniformly to 0 as $N \rightarrow \infty$. Thus, $\lim_{N \rightarrow \infty} y_N = 1 + \sqrt{2}$, so we get (4.2). \square

If we choose the definition of the entropy via the entropy of the corresponding symbolic system, then, taking into account Lemma 4.1, we get a stronger theorem.

Theorem 4.5. *We have*

$$(4.3) \quad \lim_{\ell \rightarrow \infty} h(\mathcal{F}_{\ell}|\mathcal{K}_{\ell}) = \log(1 + \sqrt{2}).$$

Of course, the same lower estimates hold for the whole billiard.

Remark 4.6. In this paper we studied the classical Bunimovich stadium billiards. However, our methods can be used for a larger class of billiards¹. In particular, the semicircles can be replaced by “caps” of similar shapes. The estimates that used the fact that the caps are semicircles, can be modified if the assumption that $\ell > 2N + 2$ is replaced by a stronger one. Another problem may appear if we

¹We thank the anonymous referee for pointing this out.

want to specify generating partitions. This may be important for the definition of topological entropy. However, in Theorem 4.4 we consider only a subsystem and provide the lower estimate for the entropy, so this theorem will hold for a large class of caps. Investigating how large such class may be and checking whether other modifications of the billiard table will give similar results, is a subject for a separate paper.

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Manuscript received January 29 2020

revised June 4 2020

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