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CHAOTIC HYPOTHESIS AND THE SECOND LAW OF THERMODYNAMICS

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ABSTRACT. Based on the Gallavotti-Cohen chaotic hypothesis, we define the entropy for nonequilibrium chaotic systems of statistical mechanics to be the Kolmogorov-Sinai entropy of the Sinai-Ruelle-Bowen measure for a transitive uniformly hyperbolic system. For families of one-dimensional Markov transformations and expanding maps, we give a mathematically rigorous proof of the second law of thermodynamics: the strict increase of the entropy as a nonequilibrium system evolves to its unique equilibrium.

1. INTRODUCTION

In their 1995 seminal paper [10], Gallavotti and Cohen, extending Ruelle's idea of describing macroscopic behaviors of turbulent flows with the Sinai-Ruelle-Bowen (SRB) measure, proposed the following, now commonly referred as, the Gallavotti-Cohen Chaotic Hypothesis for general nonequilibrium statistical mechanics of many-particle systems: *A reversible many-particle system in a stationary state can be regarded as a transitive Anosov system for the purpose of computing the macroscopic properties of the system.* Gallavotti later further clarified the hypothesis in [9]: *Motions developing on the attracting set of a chaotic system can be regarded as a transitive hyperbolic system.*

The chaotic hypothesis has greatly motivated researchers in both areas of statistical mechanics and smooth dynamical systems to develop further the theory for nonequilibrium statistical mechanics. In statistical mechanics, the chaotic hypothesis enables the verification of Onsager's reciprocal relations and the fluctuation-dissipation theorem for models of nonequilibrium systems [9, 11, 22]. In smooth dynamical systems, Ruelle proved the smooth dependence of the SRB-measure for hyperbolic attractors and calculated its derivative formula, the linear response function [20] (see [1] for a review on this topic). Ruelle also formulated the entropy production rate and proved its nonnegativity for systems with hyperbolicity [21].

In this paper, motivated also by the chaotic hypothesis, we consider the following question: *since transitive Anosov systems are considered as mathematical models for thermodynamic systems of many particles*, *would it be possible to prove the second law of thermodynamics for such systems*? Let's use a simple example to illustrate this question. Assume that certain fluid or gas in an adiabatically insulated container is initially heated at one spot and then the source of the heat is removed.

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The heat distribution in the container starts to evolve to its natural equilibrium. Assume that the system throughout this process is modeled by transitive Anosov systems, can we prove that *the entropy of the system is strictly increasing as the system evolves to its equilibrium*?

To answer this question, we first need to clarify the meaning of several concepts, in particular, the entropy for a nonequilibrium system. We recall that in statistical mechanics, a thermodynamic system is called at its thermal equilibrium if its state is no longer dependent on either time or the initial state and it is entirely characterized by the macroscopic variables and external parameters such as the volume, the total energy, and the number of particles [2]. For simplicity, we restrict in our exposition to transitive Anosov systems. Other systems with various hyperbolicity will be addressed later in Section 2.

• *Family of transitive Anosov systems modelling thermodynamic systems evolving to equilibrium.* We need to specify families of Anosov systems that describe a thermodynamic system in all possible states: at equilibrium, near equilibrium, and far from equilibrium. Clearly, Anosov systems on different phase spaces should not be considered in the same family since we assume that macroscopic parameters of the container do not change in the process of the evolution to equilibrium and systems out of equilibrium will evolve gradually to equilibrium. A small perturbation of a system in the family should remain in the family and systems within the family should be homotopic: any two systems in the family can be connected via a smooth path. Thus, since Anosov systems are structurally stable, all maps in this family will have the same topological entropy. Assume that such a family of transitive Anosov systems is now defined and denoted by $\mathcal{A}(M)$, where M is the phase space, a compact Riemannian manifold.

• *Invariant measure for Anosov systems modelling stationary states of thermodynamic systems.* In order to define the entropy for each system in $\mathcal{A}(M)$, we need to specify the meaning of a stationary state for both equilibrium and nonequilibrium systems. Following Ruelle's idea, the stationary state for any $f \in \mathcal{A}(M)$ is defined to be the unique invariant probability measure, the **Sinai-Ruelle-Bowen (SRB) measure**, ρ_f satisfying the following two conditions [6, 24]:

(1) The time average of a macroscopic observable φ along an orbit starting from a typical initial point converges to the spatial average: for Lebesgue almost every $x \in M$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} \varphi(f^i(x)) = \int \varphi(x) d\rho_f. \tag{1.1}$$

(2) The conditional probability of ρ_f on unstable manifolds is absolutely continuous with respect to the Lebesgue measure on unstable manifolds.

For any transitive Anosov map, ρ_f exists and is unique [3].

• *Entropy of the SRB measure of an Anosov system defining the (global) entropy for thermodynamic systems both at equilibrium and out of equilibrium.* Even though there is still no universally accepted definition for the global entropy for a nonequilibrium system [9], the most reasonable choice for the entropy of a transitive Anosov system is the **Kolmogorov-Sinai entropy** of the probability measure

ρ_f. This choice of the entropy for a nonequilibrium system is different from Ruelle's [23]. In his consideration of the global entropy for nonequilibrium systems, Ruelle used the Gibbs entropy: $S(\mu) = -\int \ln \rho(x)\rho(x)dx$, where $\mu = \rho(x)dx$. The Kolmogorov-Sinai (KS) entropy of f, denoted by $H(f)$ reflects the complexity of the nonequilibrium system - still a chaotic system. It is the smooth dynamical system counterpart of the Boltzmann entropy S_B discussed in [16, 17] by Lebowitz.

For an Anosov system, $H(f)$ can be represented as the expected value of the expanding rate in unstable manifolds over the phase space with respect to the stationary state ρ_f:

$$H(f) = \int \log J^u f d\rho_f, \tag{1.2}$$

where $J^u f$ is the Jacobian of f along unstable manifolds [19]. The value of this entropy over the family $\mathcal{A}(M)$ is proven to be able to take any positive value not exceeding the topological entropy [12].

• *Mathematical formulation for the evolution of a nonequilibrium system to equilibrium.* The chaotic hypothesis is about mathematical modeling of a thermodynamic system at its stationary state. It does not give information on how a system evolves from its nonequilibrium state to the equilibrium state. To consider the second law of thermodynamics, we must make an assumption about the nature of this process. Since we have defined the entropy $H(f)$ and we know that $H(f)$ depends on f differentiably in an appropriate setting [20]. We propose the following *Entropy Maximization Hypothesis.* Let Φ_t denote the gradient flow of $H(f)$ in $\mathcal{A}(M)$. We assume that *the evolution of the system from a nonequilibrium state to its equilibrium state is governed by the gradient flow, i.e., a system $f_0 \in \mathcal{A}(M)$ not at equilibrium will evolve in the direction that maximizes its entropy.* This gradient flow will be denoted by $\Phi_t(\cdot)$. The orbit $\{f_t = \Phi_t(f_0), t \geq 0\}$ models the trajectory of the evolving system.

• *System at equilibrium.* We now define the concept of an **equilibrium system** for Anosov maps in $\mathcal{A}(M)$ corresponding to the concept of a thermal equilibrium in statistical mechanics. A system $f^* \in \mathcal{A}(M)$ is called an equilibrium system, or at equilibrium, if it is a critical value of the entropy functional $H(f)$, i.e., the derivative of the entropy $H(f)$ at f^* is zero in all permissible directions of perturbation. A system is said to be far from equilibrium if it is not an equilibrium and its entropy is much smaller than the maximum entropy of the system, the topological entropy.

We are now ready to formulate the mathematical statement equivalent to the second law of thermodynamics under both the chaotic hypothesis and the entropy maximization hypothesis in the context of transitive Anosov systems.

Given a nonequilibrium system f_0 in a family of transitive Anosov systems $\mathcal{A}(M)$, $H(f_t)$, the KS-entropy of the SRB measure of the system $f_t = \Phi_t(f_0)$, strictly increases in time $t \in [0, \infty)$.

Such a family $\mathcal{A}(M)$ will be called a **second-law** family or satisfying the second law.

In next section, we will give more detailed mathematical descriptions of aforementioned objects and examples of second-law families. We will show that the entropy functional $H(f)$ on these families has a unique critical value where it attains its maximal value.

2. Main theorems and conjectures

The Kolmogorov-Sinai (KS) entropy is defined for any measure preserving transformation on a probability space [19].

Definition 2.1. Assume T is a measure preserving transformation on a probability space $\{\Omega, \mathcal{F}, \mu\}$, $\mu(T^{-1}(A)) = \mu(A)$ for all $A \in \mathcal{F}$ and $\mathcal{P}$, a measurable partition of Ω. The KS entropy of T with respect to $\mathcal{P}$, denoted by $h_\mu(T, \mathcal{P})$ is given by the limit

$$h_\mu(T, \mathcal{P}) := -\lim_{n\to\infty} \sum_{p\in\bigvee_{j=0}^{n-1} T^{-j}\mathcal{P}} \frac{1}{n}\mu(p)\ln\mu(p), \tag{2.1}$$

where $\bigvee_{j=0}^{n-1} T^{-j}\mathcal{P}$ denotes the partition of Ω consisting of all subsets of the form $p_0 \cap T^{-1}p_1 \cap \cdots \cap T^{-(n-1)}p_{n-1}$, with $p_j \in \mathcal{P}, j = 0, 1, \ldots, n-1$.

The KS entropy of T with respect to μ is the supremum of $h_\mu(T, \mathcal{P})$ over all possible measurable partitions [19].

The Brin-Katok formula [4,6] gives us the meaning of the KS entropy when T is a continuous map on a compact metric space and the invariant measure μ is ergodic. The entropy measures the exponentially sensitive dependence of a typical orbit on its initial point. Let

$$B_n(x, \epsilon) = \{y : d(T^i(x), T^i(y)) \le \epsilon, i = 0, 1, \ldots n-1\} \tag{2.2}$$

be the Bowen ball. Then, for μ-almost x,

$$H_\mu(T) = \lim_{\epsilon\to 0}\lim_{n\to\infty} -\frac{1}{n}\ln\mu(B_n(x, \epsilon)). \tag{2.3}$$

Definition 2.2. A family of smooth dynamical systems on a phase space is called satisfying the second law of thermodynamics (a second-law family, for short) if the family is a path connected topological space with neighborhoods of each member identified with an open set of a given Banach space, and the following additional conditions are satisfied:

1. Every member of the family has a unique SRB measure.
2. The entropy functional defined by the Kolmogorov-Sinai entropy of each member with respect to its SRB measure is a positive and differentiable function in this family.
3. The entropy functional has a unique critical value in the family and its maximal value is attained at this critical value.

Remarks. 1. The third condition in this definition is slightly stronger than the one mentioned at the end of Section 1. It avoids using the entropy gradient flow and the entropy maximization hypothesis. If the third condition holds, then the entropy along any trajectory of the gradient flow of the entropy functional strictly increases, thus a second-law family.

2. Notice that we do not assume that the SRB measure itself is differentiable with respect to the system. Such a condition is not necessary and might be too restrictive for some families.

3. In statistical mechanics, the three basic laws (the zeroth, the first, and the second) are considered as axioms. In our mathematical models, however, the laws become mathematical statements that their validity can be checked. Under the entropy maximization hypothesis, the zeroth law says that given any initial state of the system f_0, the limit $\lim_{t\to\infty} \Phi_t(f_0)$ exists. The second law states that the entropy $H(\Phi_t(f_0))$ strictly increases in t. The first law about the conservation of total energy is not considered in this article since the concept of total energy is not defined for a hyperbolic dynamical system. One may propose that the total energy is represented by the system's topological entropy. While it is true that the topological entropy is the same for topologically conjugated systems, this representation seems to lack physical interpretation. Note that in the definition of the second-law family we do not assume that the zeroth law is satisfied. For mathematical models, these two laws can be studied separately and we focus on the second law.

We now formulate families of smooth dynamical system with hyperbolicity that satisfy the second law of thermodynamics. The proofs are given in the last section.

2.1. **Finite Markov Transformations of the unit interval.** Let $0 = a_0 < a_1 < \cdots < a_n = 1$ be a sequence of increasing points in $[0,1]$ and let $I_i = [a_{i-1}, a_i]$ be the ith subinterval of a partition of $[0,1]$. A *Markov transformation* f of $[0,1]$ is a piecewise continuous function monotonic in each subinterval I_i, $i = 1, 2, \ldots, n$. Let $f_i = f|_{I_i}$ and denote $f = \{f_i\}$. At two endpoints of I_i, f_i's values are chosen so that f_i is a continuous function on the closed interval I_i. A Markov transformation f is called $C^{1+\alpha}$ if for each i, the derivative of f_i is a Hölder continuous function on I_i and we assume $|f_i'(x)| > 1$ on I_i including the one-sided derivative at two endpoints. Any $C^{1+\alpha}$ Markov transformation f has a unique SRB measure which is absolutely continuous with respect to the Lebesgue measure and the density function is positive and Hölder continuous on $[0,1]$ [19]. Because of the uniqueness of the SRB measure for f, $\rho(x)dx$ is the only invariant probability measure under f that is absolutely continuous with respect to the Lebesgue measure. Conversely, given any positive Hölder continuous density function $\rho(x)$ on $[0,1]$, there exists a $C^{1+\alpha}$ Markov transformation f whose SRB measure has density function $\rho(x)$ (this can be done by using the Dacorogna-Moser theorem included in Section 3). Notice that $f = \{f_i\}_{i=1}^n$ preserving the the probability measure with a given density function $\rho(x)$ is equivalent to

$$\rho(y) = \sum_{i=1}^{n} \rho(f_i^{-1}(y))[f_i^{-1}]'(y), y \in [0,1] \tag{2.4}$$

Given any positive Hölder continuous density function $\rho(x)$ on $[0,1]$, the family of all $C^{1+\alpha}$ Markov transformations preserving the same probability measure on $[0,1]$ with a density function $\rho(x)$ is denoted by $\mathrm{MT}^{1+\alpha}_{n,\rho}([0,1])$, i.e.,

$$\begin{aligned}\mathrm{MT}^{1+\alpha}_{n,\rho}([0,1]) = \big\{ & f = \{f_i\}_{i=1}^n : 0 = a_0 < a_1 < \cdots < a_n = 1,\\ & I_i = [a_{i-1}, a_i], f_i(I_i) = [0,1], f_i \in C^{1+\alpha}(I_i), |f_i'(x)| > 1,\end{aligned}$$

$$\rho(y) = \sum_{i=1}^{n} \rho(f_i^{-1}(y))[f_i^{-1}]'(y), y \in [0,1].\}$$

The entropy of any map f in this family with respect to its SRB measure $\rho(x)dx$ is given by the formula [19]

$$H(f) = \sum_{i=1}^{n} \int_{I_i} \ln |f_i'(x)| \rho(x) dx. \tag{2.5}$$

It is differentiable in f. Thus, to prove that this family is a second-law family, we only need to show that $H(f)$ has no nontrivial critical values.

Theorem 2.3. *For any positive Hölder continuous density function $\rho(x)$ and $n \geq 2$, the family $\mathrm{MT}_{n,\rho}^{1+\alpha}([0,1])$ is a second-law family: For any given $f \in \mathrm{MT}_{n,\rho}^{1+\alpha}([0,1])$, there exists φ, a piecewise $C^{1+\alpha}$ function on $[0,1]$ such that for any ϵ sufficiently small, $f + \epsilon\varphi \in \mathrm{MT}_{n,\rho}^{1+\alpha}([0,1])$ and*

$$\frac{d}{d\epsilon} H(f + \epsilon\varphi)\big|_{\epsilon=0} \neq 0, \tag{2.6}$$

with the only exception when $\psi \circ f \circ \psi^{-1}$ is the piecewise linear Markov transformation L_n on $[0,1]$ with equal subintervals $|I_i| = \frac{1}{n}$, where $\psi(x) = \int_0^x \rho(x)dx$.

2.2. $C^{1+\alpha}$ **Expanding maps on the unit circle.** Just as Markov transformations, every $C^{1+\alpha}$ expanding map on the unit circle S^1 preserves a probability measure with a Hölder continuous density function $\rho(x) > 0$, i.e., its unique SRB measure is $\rho(x)dx$. The converse is still true. Given any Hölder continuous density function $\rho(x) > 0$ on S^1 and a positive integer $n \geq 2$, there exists a $C^{1+\alpha}$ expanding map of degree n whose SRB measure is $\rho(x)dx$ [19].

Given any Hölder continuous density function $\rho(x) > 0$ on S^1, define $\mathrm{EM}_{n,\rho}^{1+\alpha}(S^1)$ to be the family of all degree n $C^{1+\alpha}$ expanding maps on S^1 that preserve the same measure $\rho(x)dx$ and all maps are either orientation preserving or orientation reversing. The entropy of any map f in this family with respect to its SRB measure $\rho(x)dx$ is given by

$$H(f) = \int_{S^1} \ln |f'(x)| \rho(x) dx, \tag{2.7}$$

which is differentiable in f [19].

Theorem 2.4. *For any Hölder continuous density function $\rho(x) > 0$ on S^1 and any integer $n \geq 2$, the family $\mathrm{EM}_{n,\rho}^{1+\alpha}(S^1)$ is a second-law family: For any given $f \in \mathrm{EM}_{n,\rho}^{1+\alpha}(S^1)$, there exists φ, a $C^{1+\alpha}$ map on S^1 such that for any ϵ sufficient small, $f + \epsilon\varphi \in \mathrm{EM}_{n,\rho}^{1+\alpha}(S^1)$ and*

$$\frac{d}{d\epsilon} H(f + \epsilon\varphi)\big|_{\epsilon=0} \neq 0, \tag{2.8}$$

with the only exception when $\psi \circ f \circ \psi^{-1}$ is the degree n linear expanding map on S^1 where $\psi(x)$ is a $C^{1+\alpha}$ diffeomorphism of S^1 satisfying $\psi'(x) = \rho(x)$.

One may wonder how big these two families are: do they contain systems that are far from equilibrium? The answer is affirmative in terms of the values of the entropy. The infimum of the entropy over either family is proven to be zero [13].

Theorem 2.5. *The range of the entropy functional $H(f)$ on either $\mathrm{MT}^{1+\alpha}_{n,\rho}([0,1])$ or $\mathrm{EM}^{1+\alpha}_{n,\rho}(S^1)$ is $(0, \ln n]$.*

The proofs of Theorems 2.3 and 2.4 are surprisingly straightforward using only elementary techniques. Based on these two theorems, we remark that in our setting, the existence of a unique SRB measure of a given system does not tell us whether a system is at equilibrium or not. The existence of the SRB measure represents some kind of physical constraint of the system. In classical statistical mechanics of thermodynamics, the many-particle system's microscopic dynamics is described by a Hamiltonian system of many degrees. Thus, the Liouville measure (i.e., the Lebesgue measure for the symplectic manifold) is always preserved regardless whether the system is at its thermal equilibrium or not. Thus, the physical interpretation of an SRB measure with a nonconstant density function can be that the physical space is not homogeneous, for example, the heat conduction in a nonhomogeneous material with spatially dependent heat capacity, or a fluid flow in spatially nonhomogeneous porous medium.

Based on the understanding of these two second-law families, we observe that each second-law family needs to stay within the homotopy class of the equilibrium system. Since every expanding map on the unit circle is within the homotopy class of a linear expanding map of a given degree, a second-law family is much smaller than the entire homotopy class of a linear expanding map. Furthermore, since the entropy formula for smooth measure-preserving maps depends only on the unstable Jacobian, the uniqueness of the critical value seems unlikely to hold. Based on this understanding, we now further formulate second-law families of smooth hyperbolic dynamical systems on higher dimensional phase spaces in a weaker form. We omit the standard definition of the terms we use in the next families of smooth dynamical systems and refer readers to reference books such as [15] or [19].

2.3. $C^{1+\alpha}$ **Expanding maps on an n-dimensional torus.** We will restrict the phase space to an n-dimensional torus $\mathbb{T}^n = S^1 \otimes S^1 \otimes \cdots \otimes S^1$ since general Riemannian manifolds may not admit 'linear maps' on them. On the other hand, whether a family of $C^{1+\alpha}$ expanding maps on other more general compact Riemannian manifold satisfies the second law is still an interesting question.

Given any Hölder continuous density function $\rho(x) > 0$ on $\mathbb{T}^n$, let $\mathrm{EM}^{1+\alpha}_{k,\rho}(\mathbb{T}^n)$ denote the family of $C^{1+\alpha}$ expanding maps of degree k on $\mathbb{T}^n$ that preserve the volume form $\rho(x)dm$. Then, $\mathrm{EM}^{1+\alpha}_{k,\rho}(\mathbb{T}^n)$ is a second-law family in the following sense:

(1) Every member of the family has a unique SRB measure [19];

(2) The Kolmogorov-Sinai entropy of the SRB measure is a positive and differentiable function in this family [19].

(3) The range of the entropy functional is the interval $(0, \ln k]$ [13, 15].

(4) (Conjecture) Let $f_0 \in \mathrm{EM}^{1+\alpha}_{k,\rho}(\mathbb{T}^n)$ with the entropy $H(f_0) < \ln k$. Then, The entropy functional $H(\Phi_t(f_0))$ is a strictly increasing function in $t > 0$, where Φ_t is

the gradient flow of the entropy functional. Furthermore, $f_t = \Phi_t(f_0)$ converges to an equilibrium system $f_\infty \in \mathrm{EM}^{1+\alpha}_{k,\rho}(\mathbb{T}^n)$ with $H(f_\infty) = \ln k$.

Remark. Statements (1)-(2) are well-known facts. Statement (3) follows from the facts that all $C^{1+\alpha}$ expanding maps are topologically conjugate and homotopic to a linear expanding map of the same degree and that the infimum of the entropy is zero for this family [13] . By the Dacorogna-Moser theorem (see Section 3), we only need to prove Statement (4) for maps preserving the Lebesgue measure.

2.4. **Anosov maps on an n-dimensional torus.** Any Anosov map f on $\mathbb{T}^n$ is homotopic to a unique hyperbolic toral automorphism of $\mathbb{T}^n$ [7, 15] and we denote this hyperbolic toral automorphism by L. For a general transitive Anosov map f, its SRB measure ρ_f exists and is unique but usually singular, though its conditional measure on unstable manifolds are absolutely continuous with respect to the Lebesgue measure [19]. As f is perturbed, the underlying unstable manifolds are usually changed as well. It seems not helpful to make assumptions that the SRB measure is preserved when f is perturbed. The differentiability of the entropy functional

$$H(f) = \int_{\mathbb{T}^n} \ln J^u f d\rho_f \tag{2.9}$$

in f follows from Ruelle's pioneering work in [20] where the SRB measure ρ_f is shown to be differentiable in f. Let f_0 be a fixed Anosov map and f is $C^{1+\alpha}$-close to f_0 and $f \circ h_f = h_f \circ f_0$, where h_f is a homeomorphism close to the identity. We have that

$$H(f) = \int_{\mathbb{T}^n} \ln J^u f d\rho_f = \int_{\mathbb{T}^n} \ln J^u f(h_f) dh_f^*(\rho_f), \tag{2.10}$$

depends on f differentiably since both $\ln J^u f(h_f)$ and $h_f^*(\rho_f)$ are differentiable in f in a $C^r, r > 3$ neighborhood of f_0 .

We denote by $\mathrm{AM}^r_L(\mathbb{T}^n)$ the family of all $C^r, r > 3$ Anosov transitive maps on $\mathbb{T}^n$ that are homotopic to a hyperbolic toral automorphism L. We claim that $\mathrm{AM}^r_L(\mathbb{T}^n)$ is a second-law family in the following sense:

(1) Every member of the family has a unique SRB measure [19].

(2) The Kolmogorov-Sinai entropy of the SRB measure is a positive and differentiable function in this family [20].

(3) The range of the entropy functional is the interval $(0, H(L)]$ [12].

(4) (Conjecture) Let $f_0 \in \mathrm{AM}^r_L(\mathbb{T}^n)$ with the entropy $H(f_0) < H(L)$. Then, the entropy functional $H(\Phi_t(f_0))$ is a strictly increasing function in t, where Φ_t is the gradient flow of the entropy functional. Furthermore, $f_t = \Phi_t(f_0)$ converges to an equilibrium system $f_\infty \in \mathrm{AM}^r_L(\mathbb{T}^n)$, where $J^u f_\infty$, the Jacobian of f_∞ along the unstable manifold is a constant and $H(f_\infty) = H(L)$.

2.5. **Other uniformly hyperbolic systems.** The existence of the unique SRB measure extends to other uniformly hyperbolic systems. For hyperbolic attractors and Axiom A hyperbolic systems, the existence, uniqueness, and the smooth dependence of the SRB measure have all been proven [14, 20]. Thus, the gradient flow of the entropy is well defined for such families of diffeomorphisms. We believe that the two basic laws of thermodynamics should hold for these families: the zeroth law states that if the entropy is not at its maximum, then the dynamics defined by the gradient flow will take the system to its equilibrium and the second law says that the entropy is strictly increasing as the system reaches its equilibrium.

3. Proofs

For families of transformations preserving a smooth measure with a density function $\rho(x)$, we will show that we can simply assume that $\rho(x) = 1$ because of Dacorogna-Moser Theorem. Since the theorem is used multiple times, we include a simplified version below. For complete statements of the theorem see Theorem 1 of [5]. See also Levi's SIAM review article for a very short intuitive proof [18].

Dacorogna-Moser Theorem. *Let M be a compact Riemannian manifold and $\rho(x)$ a $C^{k+\alpha}$ positive probability density function on M, where $k \geq 0$ is an integer and $0 < \alpha < 1$. There exists a $C^{k+1,\alpha}$ diffeomorphism ψ of M such that the Jacobian of $\psi(x)$, $J(\psi)(x) = \rho(x)$.*

Lemma 3.1 ([19, Page 219])**.** *Assume that f preserves a volume form $\rho(x)dm$ on a compact manifold M. Let ψ be the diffeomorphism whose Jacobian is $\rho(x)$: $J(\psi)(x) = \rho(x)$. Then, $g = \psi \circ f \circ \psi^{-1}$ preserves the Lebesgue measure dm. Furthermore, the entropy of f with respect to $\rho(x)dm$ is the same as that of g with respect to dm.*

Based on these two results, we see that for any two probability densities $\rho_1(x), \rho_2(x)$, there is a smooth conjugacy, depending only on $\rho_1(x), \rho_2(x)$, between corresponding maps in two families $\mathrm{MT}^{1+\alpha}_{n,\rho_1}([0,1])$ and $\mathrm{MT}^{1+\alpha}_{n,\rho_2}([0,1])$ (or $\mathrm{EM}^{1+\alpha}_{n,\rho_1}(S^1)$ and $\mathrm{EM}^{1+\alpha}_{n,\rho_2}(S^1)$). Since the entropy with respect to the corresponding invariant measure does not change for smoothly conjugated maps, in the proof of Theorems 2.3 and 2.4, we may assume the density $\rho(x)$ is a constant.

3.1. Markov transformations on the unit interval.

We first prove the case when a Markov transformation has only one discontinuity:

Let $\mathrm{MT}^{1+\alpha}_{2,1}([0,1]), \alpha \in (0,1)$ be the family of Markov transformations satisfying the following conditions. (1) For each $f(x) \in \mathrm{MT}^{1+\alpha}_{2,1}([0,1])$, there is $b \in (0,1)$ such that

$$f(x) = \begin{cases} f_1(x), x \in [0,b] \\ f_2(x), x \in (b,1] \end{cases} , \tag{3.1}$$

where both $f_1(x)$ and $f_2(x)$ (extended to the domain $[b,1]$) are $C^{1+\alpha}$, onto $[0,1]$, and $|f_i'(x)| > 1, i = 1,2$. For $i = 1,2$, the sign of $f_i'(x)$ is the same for all transformations in the family.

(2) $f(x)$ preserves the Lebesgue measure: For any subinterval $I \subset [0,1]$, $\mu(f^{-1}(I)) = \mu(I)$.

Proposition 3.2. *For any $\alpha \in (0,1)$, the family of Markov transformations $\mathrm{MT}^{1+\alpha}_{2,1}([0,1])$ is a second-law family.*

Proof. Since the Lebesgue measure is preserved, it is the unique SRB measure for all maps in $\mathrm{MT}^{1+\alpha}_{2,1}([0,1])$. The entropy of $f(x)$ is thus given by [19] (Page 230)

$$H(f) = \int_0^b \ln|f_1'(x)|dx + \int_b^1 \ln|f_2'(x)|dx. \tag{3.2}$$

We denote two branches of the inverse function of $f(x)$ by $g_1(y)$ and $g_2(y)$.

For convenience, we will assume both branches are increasing functions for rest of the proof. The proof for other cases are the essentially same. Since f is measure preserving and differentiable, we have

$$1 = g_1'(y) + g_2'(y),\ y \in [0,1]. \tag{3.3}$$

Thus, using integration by substitution: $y = f_1(x), y = f_2(x)$ in two integrals of (3.2), respectively, in terms of $g_1(x)$, we have

$$H(g_1) := H(f) = -\int_0^1 g_1'(y)\ln g_1'(y) + (1-g_1'(y))\ln(1-g_1'(y))dy. \tag{3.4}$$

Notice that for each $\alpha \in (0,1)$, the family $\mathrm{MT}^{1+\alpha}_{2,1}([0,1])$ is a Banach manifold. For each map $f(x) \in \mathrm{MT}^{1+\alpha}_{2,1}([0,1])$, $f(x)$ is uniquely determined by $g_1(y) \in C^{1+\alpha}[0,1]$ as long as $g_1(0) = 0$, $0 < g'(y) < 1$. An open neighborhood of $f(x)$ is thus identified with an open neighborhood of the origin of the Banach space

$$\mathbf{B} := \{\varphi(y), \in C^{k+\alpha}[0,1], \varphi(0) = 0.\} \tag{3.5}$$

For any given $\varphi(y) \in \mathbf{B}$, there exists an $\epsilon_0 > 0$, such that for all $0 < \epsilon < \epsilon_0$, $g_\epsilon(y) = g_1(y) + \epsilon\varphi(y) \in C^{1+\alpha}[0,1]$ with $g_\epsilon(0) = 0$ and $0 < g_\epsilon'(y) < 1$.

Calculating the first order term in ϵ of $(g_1'(y)+\epsilon\varphi'(y))\ln(g_1'(y)+\epsilon\varphi'(y))$, we have

$$(g_1'(y) + \epsilon\varphi'(y))\ln(g_1'(y) + \epsilon\varphi'(y)) \tag{3.6}$$

$$= (g_1'(y) + \epsilon\varphi'(y))(\ln g_1'(y) + \ln(1 + \epsilon\varphi'(y)/g_1'(y)) \tag{3.7}$$

$$= g_1'(y)\ln g_1'(y) + (\ln g_1'(y) + 1)\varphi'(y)\epsilon + O(\epsilon^2). \tag{3.8}$$

The calculation of the first order term in ϵ of the other term is the same. Thus, we have the derivative of $H(g_1)$ in the direction of $\varphi(y)$

$$\frac{d}{d\epsilon}H(g + \epsilon\varphi)|_{\epsilon=0} = \int_0^1 \left[\ln(1 - g_1'(y)) - \ln g_1'(y)\right]\varphi'(y)dy. \tag{3.9}$$

By Lemma 3.3 proved next, this derivative is zero for all $\varphi(y) \in \mathbf{B}$ if and only if $g_1(x) = 1/2$, i.e, $g_1(y) = g_2(y)$. □

Lemma 3.3. (A) *Given any nonzero continuous function $f(x) \in C^0[0,1]$, there exists a C^∞ function $\varphi \in C^\infty[0,1]$ with $\varphi(0) = 0$ such that* $\int_0^1 f(x)\varphi'(x)dx \neq 0$.

(B) *Given any nonconstant function $f(x) \in C^1[0,1]$, there exists a C^∞ function $\varphi \in C^\infty[0,1]$ with $\varphi(0) = \varphi'(0) = \varphi(1) = \varphi'(1) = 0$ such that $\int_0^1 f(x)\varphi'(x)dx \neq 0$.*

Proof. (A) Since $f(x)$ is continuous and not identically zero, we may assume that there exist $0 < a < b < 1$ such that $f(x) > \epsilon_0 > 0, x \in [a,b]$ for some positive number ϵ_0. Let $g(x) = \epsilon_0$, when $x \in [a,b]$ and $g(x) = 0$ otherwise. We have $f(x) > g(x), x \in [a,b]$. Choose $\delta > 0$ small so that $a + 3\delta < b$. Let $0 \leq r(x) \leq$

$1 \in C^\infty[0,1]$ be a function satisfying conditions $r(x)=0$ when $x \in [0, a+\delta] \cup [b,1]$ and $r(x)=1$ when $x \in [a+2\delta, b-\delta]$. Let $\phi(x) = \int_0^x r(t)g(t)dt, x \in [0,1]$. We have $\phi(x) \in C^\infty[0,1], \varphi(0)=0$ and

$$\int_0^1 f(x)\varphi'(x)dx = \int_a^b f(x)r(x)g(x)dx \tag{3.10}$$

$$> \int_{a+2\delta}^{b-\delta} f(x)r(x)g(x)dx > (b-a-3\delta)\epsilon_0^2 > 0. \tag{3.11}$$

(B) Since $f(x)$ is not a constant, $f'(x)$ is not identically zero. Repeat the proof of Part (A) for $f'(x)$, but let $\varphi(x) = r(x)g(x)$, using integration by parts, we have

$$\int_0^1 f(x)\varphi'(x)dx = -\int_0^1 f'(x)\varphi(x)dx = -\int_a^b f'(x)r(x)g(x)dx \neq 0. \tag{3.12}$$

Since $\varphi(x) = 0$ for $x \in [0, a+\delta] \cup [b,1]$, we have that all kth other derivatives are zero at $x = 0,1$: $\varphi^{(k)}(0) = \varphi^{(k)}(1) = 0$, $k \geq 0$.

We now prove the case when a Markov transformation has multiple discontinuities.

Let $\mathrm{MT}_{n,1}^{1+\alpha}([0,1]), \alpha \in (0,1)$ be the family of Markov transformations with exactly n discontinuities and preserve the Lebesgue measure. □

Proposition 3.4. *For each $n \geq 2, \alpha \in (0,1)$, the family of Markov transformations $\mathrm{MT}_{n,1}^{1+\alpha}([0,1])$ is a second-law family.*

Proof. Let $f \in \mathrm{MT}_{n,1}^{1+\alpha}([0,1])$, $f_i = f|_{I_i}, I_i = [a_{i-1}, a_i]$. extended to $C^{1+\alpha}$ functions on I_i, and $g_i(y)$ is the inverse of f_i, $i = 1, 2, \ldots, n$.

The entropy formula is now given by

$$H(f) = \sum_{i=1}^n \int_{I_i} \ln |f_i'(x)| dx. \tag{3.13}$$

Make the same kind of change of variables: $x = g_i(y)$, or $y = f_i(x)$, $i = 1, 2, \ldots, n$. Each Markov transformation $f(x)$ is uniquely determined by the sequence of maps $g_i(y) : [0,1] \to [a_{i-1}, a_i], 1 \leq i \leq n$. Since f is measure preserving, we have

$$1 = \sum_{i=1}^n |g_i'(y)|. \tag{3.14}$$

In terms of $g_i(y)$, we have the entropy

$$H(f) = -\int_0^1 \sum_{i=1}^n |g_i'(y)| \ln |g_i'(y)| dy. \tag{3.15}$$

Notice that the open neighborhood of f is now identified with the direct product of $n-1$ copies of the Banach space $\mathbf{B} = \{\varphi \in C^{1+\alpha}[0,1], \varphi(0) = 0\}$. But we take the derivative in the direction of one nonzero component at a time. Pick any two adjacent branches $g_i(y), g_{i+1}(y), 1 \leq i \leq n-1$. All other branches remain unperturbed. Just as in the case of having exactly two branches in Proposition 3.2, given a perturbation of $g_i(y)$: $g_i(y) + \epsilon\varphi$, the perturbation for $g_{i+1}(y)$ is uniquely determined by $g_i(y) + \epsilon\varphi(y)$ since the rest branches are unperturbed. Calculating the first order term in ϵ as before, we have the derivative of $H(f)$ in the direction of $\varphi(x)$:

$$\frac{d}{d\epsilon}H(g_i+\epsilon\varphi)|_{\epsilon=0}=\int_0^1\left[\ln(1-\sum_{k\neq i+1,1\leq k\leq n-1}g_k'(y))-\ln g_i'(y)\right]\varphi'(y)dy.$$

This derivative is zero for all $\varphi\in\mathbf{B}$ if and only if

$$2g_i'(y)+\sum_{k\neq i,i+1,1\leq k\leq n-1}g_k'(y)=1, i=1,2,\dots,n-1. \tag{3.16}$$

We need to have another equation of the same nature from another pair: $g_1(y)$ and $g_n(y)$. However, since these two branches are not adjacent, when $g_1(y)$ is perturbed, $g_2(y)$ may have to change since $g_1(1)=g_2(0)$. But the entropy depends only on the derivatives of $g_i'(y)$. Thus, we can perturb $g_1(y)$ without changing the values of $g_k'(y)$ for $k=2,3,\dots,n-1$ and $g_n(y)$ is determined by $g_1(y)$. We thus obtain the last equation that we need

$$2g_1'(y)+g_2'(y)+\dots g_{n-1}'(y)=1. \tag{3.17}$$

Solving this system of n linear equations (3.16) and (3.17), we have $g_i'(y)=1/n$ for $i=1,2,\dots,n$. □

3.2. **Expanding maps on the unit circle.** The difference in the proof for an expanding map is that we need to impose more boundary conditions on the perturbation function φ. Again we first prove the theorem in the case when the degree of the map is two. Thus, each $C^{1+\alpha}$ expanding map on $S^1=[0,1]$ mod 1 can be identified with a Markov transformation with one discontinuity on the unit interval. We assume 0 is the fixed point of the expanding map. We consider only the orientation preserving expanding maps. For orientation reversing ones, the proof is the same.

A by-product of the proof is that we now know how to perturb an expanding map so that the entropy of its SRB measure strictly increases if it has not yet reached the maximum value of $\ln k$, where k is the map's degree. If we reverse the direction of the perturbation, we can reduce the entropy monotonically to any positive number as small as we wish.

Let $\mathrm{EM}_{2,1}^{1+\alpha}(S^1),\alpha\in(0,1)$ be the family of orientation preserving expanding maps satisfying the following conditions. We identify S^1 with the interval $[0,1]$, mod 1. For each $f(x)\in\mathrm{EM}_{2,1}^{1+\alpha}(S^1)$, there is $b\in(0,1)$ such that

$$f(x)=\begin{cases}f_1(x), x\in[0,b]\\ f_2(x), x\in[b,1]\end{cases}, \tag{3.18}$$

where both $f_1(x)$ and $f_2(x)$ are $C^{1+\alpha}$, onto $[0,1]$mod 1, $f_i'(x)>1, i=1,2$, the derivatives $f_1'(b^-)=f_2'(b^+)$ and $f_1'(0^+)=f_2'(1^-)$. $f(x)$ preserves the Lebesgue measure: $\mu(f^{-1}(I))=\mu(I)$ for any subinterval $I\subset[0,1]$.

Proposition 3.5. *The family of $C^{1+\alpha}$ measure-preserving degree 2 expanding maps on the unit circle, $\mathrm{EM}_{2,1}^{1+\alpha}(S^1)$ is a second-law family.*

Proof. The entropy formula for $f(x)$ is still the same given by

$$H(f) = \int_0^b \ln f_1'(x)dx + \int_b^1 \ln f_2'(x)dx. \tag{3.19}$$

We still denote two branches of the inverse function of $f(x)$ by $g_1(y)$ and $g_2(y)$ and we have $1 = g_1'(y) + g_2'(y), y \in [0,1]$ and

$$H(g_1) := H(f) = -\int_0^1 g_1'(x)\ln g_1'(x) + (1-g_1'(x))\ln(1-g_1'(x))dx. \tag{3.20}$$

Given any $C^{1+\alpha}$ expanding map f on S^1 preserving the Lebesgue measure, f is again uniquely determined by the first branch of its inverse map $g_1(y) : [0,1] \to [0,b]$: $g_2(y) = y - g_1(y) + b$. We see that $f \in \mathrm{EM}_{2,1}^{1+\alpha}(S^1)$ if and only if $g_1(y) \in C^{1+\alpha}[0,1]$, $g_1(0) = 0, 0 < g_1(1) < 1$, and $g_1'(0) = g_1'(1)$. Given a $C^{1+\alpha}$ function $\varphi(y) \in C^{1+\alpha}[0,1]$, $g_1(y) + \epsilon\varphi(y)$ determines a $f \in \mathrm{EM}_{2,1}^{1+\alpha}(S^1)$ for sufficiently small ϵ if and only if $\varphi(0) = 0, \varphi'(0) = \varphi'(1)$. So, $\mathrm{EM}_{2,1}^{1+\alpha}(S^1)$ is a Banach manifold modeled on the Banach space

$$\mathbf{B}' = \{\varphi \in C^{1+\alpha}[0,1], \varphi(0) = 0, \varphi'(0) = \varphi'(1)\}. \tag{3.21}$$

The derivative formula of $H(g_1)$ in the direction of $\varphi(x) \in \mathbf{B}'$ is also the same

$$\frac{d}{d\epsilon}H(g+\epsilon\varphi)|_{\epsilon=0} = \int_0^1 \left[\ln(1-g_1'(y)) - \ln g_1'(y)\right]\varphi'(y)dy. \tag{3.22}$$

We now apply Lemma 3.3 Part B. We conclude that $\ln(1-g_1'(y)) - \ln g_1'(y)$ must be a constant function of y: $\ln(1-g_1'(y)) - \ln g_1'(y) = C$. Thus $\frac{1-g_1'(y)}{g_1'(y)} = e^C$. So, $g_1'(y) = 1/(1+e^C)$ is a constant. This implies $g_1(y)$ must be linear so the corresponding expanding map must be a linear expanding map on S^1: i.e., $f(x) = 2x \bmod 1, x \in [0,1]$.

The proof for expanding maps of a general degree $m > 2$ is essentially identical to that for Markov transformations with multiple discontinuities. We leave the details to interested readers. ☐

Corollaries. Since the perturbation function $\varphi(y)$ can be chosen to be C^∞, smoother families of Markov transformations on the unit interval or expanding maps on the unit circle are also second-law families: for any Hölder continuous density function $\rho(x) > 0$, any integers $m \geq 2, k \geq 1$, and $\alpha \in (0,1)$, $\mathrm{MT}_{m,\rho}^{k+\alpha}([0,1])$ and $\mathrm{EM}_{m,\rho}^{k+\alpha}(S^1)$ are all second-law families.

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