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A THOULESS-LIKE EFFECT IN THE DYSON HIERARCHICAL MODEL WITH CONTINUOUS SYMMETRY

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Abstract. In this paper we study Dyson's classical r-component hierarchical model with a Hamiltonian function which has a continuous $O(r)$-symmetry, $r \geq 2$. This is a one-dimensional ferromagnetic model with a long range interaction potential $U(i,j) = -l(d(i,j))d^{-2}(i,j)$, where $d(i,j)$ denotes the hierarchical distance. We are interested in the case when $l_n = l(2^n)$, $n = 1, 2, \ldots$, is an increasing sequence, with a sub-exponential growth as $n \to \infty$. For a class of free measures, we prove a conjecture of Dyson. This conjecture states that the convergence of the series $l_1^{-1} + l_2^{-1} + \cdots$ is a necessary and sufficient condition of the existence of phase transition in the model under consideration, and the spontaneous magnetization vanishes at the critical point, i.e., there is no Thouless' effect. We find, however, that the distribution of the normalized mean spin at the critical temperature T_c tends to the uniform distribution on the unit sphere in $\mathbb{R}^r$ as the volume tends to infinity, a phenomenon which resembles the Thouless effect. We prove that the limit distribution of the normalized mean spin is Gaussian for $T > T_c$, and it is non-Gaussian for $T \leq T_c$. We also show that the density of the limit distribution of the normalized mean spin for $T \leq T_c$ is a nice analytic function which can be found from the unique solution of a nonlinear fixed point integral equation. Finally, we determine some critical asymptotics and show that the divergence of the correlation length and magnetic susceptibility is super-polynomial as $T \to T_c$.

1. Introduction. Formulation of the main results

In this paper we investigate Dyson's hierarchical vector-valued model with *continuous symmetry*. The model consists of spin variables $\sigma(j) \in \mathbb{R}^r$, $j \in \mathbb{N} = \{1, 2, \dots\}$, where $r \geq 2$. We define the *hierarchical distance* $d(\cdot,\cdot)$ on $\mathbb{N}$ as

$$d(j,k) = 2^{n(j,k)-1} \quad \text{for } j \neq k$$

with

$$n(j,k) \;=\; \{\min n\colon \text{ there is an integer } l \text{ such that } (l-1)2^n < j,k \leq l2^n\} \quad \text{if} \quad j \neq k,$$

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and $d(j,j) = 0$. The Hamiltonian of the ferromagnetic Dyson's hierarchical r-component model in the volume $V_n = \{1, 2, \dots, 2^n\}$ is

$$\mathcal{H}_n(\sigma) = -\sum_{1\le j<k\le 2^n} \frac{l(d(j,k))}{d^2(j,k)}\,\sigma(j)\sigma(k), \tag{1.1}$$

where $\sigma(j)\sigma(k)$ denotes a scalar product in $\mathbb{R}^r$, and $l(t)$ is a positive function. In this paper we will be interested in the case when $l(t)$ is a positive increasing function such that

$$\lim_{t\to\infty} l(t) = \infty; \qquad \lim_{t\to\infty} \frac{l(t)}{t^\varepsilon} = 0, \quad \text{for all } \varepsilon > 0.$$

Since the hierarchical distance $d(j,k)$ for $j \ne k$ takes the values 2^n, $n = 0, 1, 2, \dots$, only, we consider the function $l(t)$ for $t = 2^n$ only and define

$$l_n = l(2^n).$$

Let $\nu(d\mathbf{x})$ be a probability measure on $\mathbb{R}^r$. Then the Gibbs measure in V_n at a temperature $T > 0$ with free boundary conditions and the free measure $\nu(d\mathbf{x})$ is defined as

$$\mu_n(d\mathbf{x}; T) = Z_n^{-1}(T) \exp\{-\beta\mathcal{H}_n(\mathbf{x})\} \prod_{j=1}^{2^n} \nu(d\mathbf{x}_j), \quad \beta = T^{-1}.$$

We will assume that the free measure $\nu(d\mathbf{x})$ is invariant with respect to the group $O(r)$ of orthogonal transformations, i.e., $\nu(UA) = \nu(A)$ for all $U \in O(r)$ and all Borel sets $A \in B(\mathbb{R}^r)$. Then the Gibbs measure $\mu_n(d\mathbf{x}; T)$ is $O(r)$-invariant as well,

$$\begin{aligned}\mu_n(UA_1, \dots, UA_{2^n}; T) \;&=\; \mu_n(A_1, \dots, A_{2^n}; T), \quad \text{for all } U \in O(r),\\ &\qquad A_j \in B(\mathbb{R}^r),\ j = 1, \dots, 2^n.\end{aligned}$$

In [12], Dyson proved the following theorem (see also [13]). Assume that $r = 3$, and $\nu(d\mathbf{x})$ is a uniform measure on the unit sphere in $\mathbb{R}^3$. This is the classical Heisenberg hierarchical model.

Theorem 1.1. (see [12]). *The classical Heisenberg hierarchical model has a phase transition if*

$$B = \sum_{n=1}^{\infty} l_n^{-1} < \infty. \tag{1.2}$$

It has a long-range order so long as $\beta > B$.

Dyson also formulated the following conjecture (see [12]): "It also seems likely that for sequences l_n which are positive and increasing with n the condition (1.2) is necessary for a phase transition in Heisenberg hierarchical models." The goal of this paper is to *prove* Dyson's conjecture for a class of hierarchical models and to study the *limit distribution of the normalized mean spin* both below and above the critical temperature if condition (1.2) holds. Dyson's proof is a clever application of *correlation inequalities*. Our approach is based on an analytical study of the *renormalization group transformation* for the hierarchical models.

The renormalization group (RG) approach to the Dyson hierarchical models was initiated in the works of Bleher and Sinai [8]– [10] (see also the monograph [18] and the review [2], and references therein). The Dyson hierarchical models are of a great interest because for this model the RG transformation reduces to a nonlinear integral equation, and this allows a study of critical phenomena unavailable in other models. The works of Bleher and Sinai were concerned with the critical phenomena and phase transitions in the scalar Dyson hierarchical models. They were extended to the study of critical phenomena and phase transitions in the vector Dyson hierarchical models with continuous symmetry in the works of Bleher and Major [3]– [7]. The present paper is a continuation of the works [3]– [7].

We apply a perturbation technique which works if the free measure $\nu(d\mathbf{x})$ is a *small perturbation* of the Gaussian measure. Hence, we cannot treat the case when $\nu(d\mathbf{x})$ is a uniform measure on the unit sphere. On the other hand, we will consider *arbitrary* spin dimension $r \geq 2$. We will focus on free measures $\nu(d\mathbf{x})$, which have a density function $p(\mathbf{x})$ on $\mathbb{R}^r$ such that $p(\mathbf{x})$ is close, in an appropriate sense, to the density function

$$p_0(\mathbf{x}) = C(\kappa) \exp\left\{-\frac{|\mathbf{x}|^2}{2} - \kappa \frac{|\mathbf{x}|^4}{4}\right\} \tag{1.3}$$

with a sufficiently small parameter $\kappa > 0$. Precise conditions on $p(\mathbf{x})$ are given below. We also will assume some regularity conditions about the sequence $l_n = l(2^n)$ (see below).

We are investigating the following question. Let $p_n(\mathbf{x}, T)$ denote the density function of the mean spin $2^{-n} \sum_{j=1}^{2^n} \sigma(j)$, where $(\sigma(1), \ldots, \sigma(2^n))$ is a $\mu_n(T)$-distributed random vector. Because of the rotational invariance of the model, the function $p_n(\mathbf{x}, T)$ is a function of $|\mathbf{x}|$. We are interested in the limit behaviour of the function $p_n(\mathbf{x}, T)$ as $n \to \infty$, with an appropriate normalization. In our papers [3]– [7] this problem was considered for the Hamiltonian

$$\mathcal{H}_n(\sigma) = -\sum_{1 \leq j < k \leq 2^n} \frac{1}{d^\alpha(j,k)} \sigma(j)\sigma(k),$$

where $1 < \alpha < 2$. Observe that if $\alpha \leq 1$ then the thermodynamic limit of the model does not exist, and if $\alpha \geq 2$ then there is no phase transition, hence the range $1 < \alpha < 2$ is natural. We distinguished in [3]– [7] the three cases for α:

(i) $1 < \alpha < 3/2$, (ii) $\alpha = 3/2$, and (iii) $3/2 < \alpha < 2$.

The difference between these cases appears in the asymptotic behavior of $p_n(\mathbf{x}, T)$ at small T. When T is small the spontaneous magnetization $M(T)$ is positive, and the function $p_n(\mathbf{x}, T)$ is concentrated in a narrow spherical shell near the sphere $|\mathbf{x}| = M(T)$. The question is what the width of this shell is and what the limiting shape of $p_n(\mathbf{x}, T)$ is like along the radius after an appropriate rescaling. In case (i), the width is of the order of $2^{-n/2}$ and the limit shape of $p_n(\mathbf{x}, T)$ is Gaussian (see [3]). In case (ii), there is a logarithmic correction in the asymptotics of the width, but the limit shape is still Gaussian (see [6]). In case (iii), the width of the shell has a nonstandard asymptotics of the order of $2^{-n(2-\alpha)}$, and the limit shape of $p_n(\mathbf{x}, T)$ along the radius (after a rescaling) is a non-Gaussian function which is a solution of a nonlinear integral equation (see [5] and the review [4]). In the present paper we are interested in the marginal potential $l(d(j,k))/d^2(j,k)$, with an extra factor $l(t)$ of a sub-polynomial growth.

Before formulating the main results we would like to discuss the importance of Dyson's condition (1.2). In the case of the Ising hierarchical model ($r = 1$), Dyson proved in [12] that there exists a "weakest" interaction function $l(t)$ for which the hierarchical model (1.1) has a phase transition. This function is $l(t) = \log\log t$, which corresponds to $l_n = \log n$. Dyson has proved that if

$$\lim_{n\to\infty} \frac{l_n}{\log n} = 0,$$

then the spontaneous magnetization is equal to zero for all temperatures $T > 0$. On the other hand, if

$$\frac{l_n}{\log n} > \varepsilon \quad \text{for all } n > 0 \text{ with some } \varepsilon > 0,$$

then the spontaneous magnetization is positive at sufficiently low temperatures $T > 0$. In the borderline model, when

$$l_n = J\log n, \qquad J > 0,$$

Dyson proved that the spontaneous magnetization $M(T)$ has a jump at the critical temperature T_c. The existence of the jump for the 1D Ising model with long-range interaction was first predicted by Thouless (see [19], and also the works [21] of Anderson, Yuval and [16] of Hamann and references therein) for the translationally invariant Ising model with the interaction

$$H(\sigma) = -\sum_{j,k} \frac{\sigma(j)\sigma(k)}{(j-k)^2}. \tag{1.4}$$

This phenomenon (the jump of $M(T)$ at $T = T_c$) is called the *Thouless effect*. The existence of a phase transition in the ferromagnetic one-dimensional Ising model with $1/(j-k)^2$ interaction energy was proved by Fröhlich and Spencer in [15]. A rigorous proof of the existence of the Thouless effect in the Ising model with the inverse square interaction (1.4) was given by Aizenman, J. Chayes, L. Chayes, and Newman [1]. Simon proved in [17] the absence of continuous symmetry breaking in the one-dimensional r-component Heisenberg model with the interaction (1.4), in the case when $r \geq 2$.

Dyson formulated a general heuristic principle in [12] which tells us when one should expect the Thouless effect in a 1D long-range ferromagnetic model: It should occur for the "weakest" interaction (if it exists) for which a phase transition appears. Dyson wrote that in the hierarchical model "in the Ising case, there exists a borderline model $l_n = \log n$ which is the 'weakest' ferromagnet for which a transition occurs, and this borderline model shows a Thouless effect. In the Heisenberg case there exists no borderline model, since there is no 'most slowly converging' series (1.2). Thus we do not expect to find a Thouless effect in any one-dimensional Heisenberg hierarchical ferromagnet." This conjecture of Dyson, about the absence of a Thouless effect in the Heisenberg case, plays a very essential role in our investigation. We show that in the class of the r-component hierarchical models under consideration, the spontaneous magnetization $M(T)$ approaches zero as T

approaches the critical temperature, i.e., there is no Thouless effect. On the other hand, we observe a phenomenon which resembles the Thouless effect: at $T = T_c$ the rescaled distribution

$$\bar{M}_n^r(T_c)p_n(M_n(T_c)\mathbf{x}, T_c)\, d\mathbf{x}, \qquad \bar{M}_n(T) = \left(\int_{\mathbb{R}^r} |\mathbf{x}|^2 p_n(\mathbf{x}, T)\, d\mathbf{x}\right)^{1/2},$$

approaches, as $n \to \infty$, a uniform measure on the unit sphere in $\mathbb{R}^r$, $r \geq 2$. Thus, although the spontaneous magnetization $M(T_c) = \lim_{n\to\infty} \bar{M}_n(T_c)$ is equal to zero at the critical point, the distribution of the normalized mean spin converges to a uniform measure on the unit sphere. This is a "remnant" of the spontaneous magnetization at the critical temperature T_c.

To formulate our results we will need some conditions on the sequence $l_n = l(2^n)$. We need different conditions on l_n in different theorems. We formulate the conditions we shall later apply.

Conditions on the sequence l_n, $n = 0, 1, 2, \ldots$. Let us introduce the notation

$$c_n = \frac{l_n}{l_{n-1}}, \quad n = 0, 1, \ldots, \text{ with } \quad l_{-1} = 1.$$

Condition 1.

$$l_0 = 1; \quad 1 \leq c_n \leq 1.01 \quad \text{for all } n; \qquad \lim_{n\to\infty} c_n = 1. \tag{1.5}$$

Remark. The requirement $l_0 = 1$ is not a real condition, it can be reached by a rescaling of the temperature. We use it just for a normalization.

Condition 2.

$$\lim_{n\to\infty} l_n \sum_{j=n}^{\infty} l_j^{-1} = \infty.$$

Moreover, the above condition is uniform in the following sense: For all $\varepsilon > 0$ there are some numbers $K(\varepsilon) > 0$ and $L(\varepsilon) > 0$ such that

$$l_n \sum_{j=n}^{n+K(\varepsilon)} l_j^{-1} \geq \varepsilon^{-1}$$

for all $n > L(\varepsilon)$.

Condition 3.

$$\sup_{1<n<\infty} \sum_{k=1}^{n} \left(l_k \sum_{j=k}^{n} l_j^{-1}\right)^{-2} < \infty.$$

Condition 4.

$$\sum_{n=1}^{\infty} l_j^{-1} > 400\, \kappa^{-1}.$$

Condition 5.

$$\frac{l_n}{l_{n+k}} > \bar{\eta} \quad \text{for all } \; n = 0, 1, 2, \ldots, \quad \text{and all } k = 1, \ldots, L.$$

The numbers $\kappa, \bar\eta > 0$, and $L \in \mathbb{N}$ in these conditions will be chosen later. An example of sequences l_n satisfying Conditions 1–5 is given in the following proposition.

Proposition 1.2. *The sequence*

$$l_n = (1+an)^\lambda, \qquad a > 0,\ \lambda > 1, \tag{1.6}$$

satisfies Conditions 2 and 3 for all $a > 0$ and $\lambda > 1$. There exists a number $a_0 = a_0(\lambda) > 0$ such that this sequence satisfies Condition 1 for all $0 < a < a_0$, a number $a_1 = a_1(\kappa, \lambda) > 0$ such that this sequence satisfies Condition 4 for all $0 < a < a_1$, and finally there exists a number $a_2 = a_2(\bar\eta, L) > 0$ such that this sequence satisfies Condition 5 for all $0 < a < a_2$.

Thus, for all $\lambda > 1$ there exists a number

$$a_3 = a_3(\lambda, \kappa, \bar\eta, L) = \min\{a_0(\lambda), a_1(\kappa,\lambda), a_2(\bar\eta, L)\} > 0$$

such that for all $0 < a < a_3$, the sequence (1.6) satisfy Conditions 1—5. We prove Proposition 1.2 in Appendix B below. Now we describe the class of initial densities we shall consider.

Class of initial densities. We say that a probability density $p(\mathbf{x})$ on $\mathbb{R}^r$ belongs to the class $\mathcal{P}_\kappa$ if

$$p(\mathbf{x}) = C(1+\varepsilon(|\mathbf{x}|^2)) \exp\left(-\frac{|\mathbf{x}|^2}{2} - \kappa\frac{|\mathbf{x}|^4}{4}\right), \tag{1.7}$$

where $C > 0$ is a norming factor, and

$$\|\varepsilon(t)\|_{C^4(\mathbb{R}^1)} < 0.01. \tag{1.8}$$

Now we formulate our main results. We denote by $p_n(\mathbf{x}, T)$ the distribution of the mean spin $2^{-n}[\sigma(1) + \cdots + \sigma(2^n)]$ with respect to the Gibbs measure $\mu_n(d\mathbf{x}; T)$ and put

$$\bar M_n(T) = \left(\int_{\mathbb{R}^r} |\mathbf{x}|^2 p_n(\mathbf{x}, T)\, d\mathbf{x}\right)^{1/2}. \tag{1.9}$$

By $\tilde p_n(\mathbf{x}, T)$ we denote the rescaled density function

$$\tilde p_n(\mathbf{x}, T) = \bar M_n^r(T) p_n(\bar M_n(T)\mathbf{x}, T) \tag{1.10}$$

and by $\tilde\nu_{n,T}(d\mathbf{x})$ the corresponding probability distribution

$$\tilde\nu_{n,T}(d\mathbf{x}) = \tilde p_n(\mathbf{x}, T)\, d\mathbf{x}. \tag{1.11}$$

Formulation of the main results. We fix a sufficiently small positive number η which will be the same through the whole paper. For instance, $\eta = 10^{-100}$ is a good choice. Define the following number $N = N(\eta)$:

$$N = \min\{n \colon l_n > \eta^{-1}\}. \tag{1.12}$$

Assume that an arbitrary number $\bar\eta$ in the interval $0 < \bar\eta \le \eta$ is fixed. (The number $\bar\eta$ appears in Condition 5).

Theorem 1.3. (Necessity of Dyson's condition). *Let us consider the case when*

$$\sum_{n=1}^{\infty} l_n^{-1} = \infty.$$

Then there exists a number $\kappa_0 = \kappa_0(N)$ such that for all $0 < \kappa < \kappa_0$ the following statements hold.

Assume that the density $p(\mathbf{x}) = \frac{\nu(d\mathbf{x})}{d\mathbf{x}}$ belongs to the class $\mathcal{P}_\kappa$ and the sequence $\{l_n,\ n \geq 0\}$ satisfies Conditions 1–3. Then there exists a constant $L = L(\bar{\eta}, \kappa)$ such that if the sequence $\{l_n,\ n \geq 0\}$ satisfies Condition 5, then for all $T > 0$, there exists the limit

$$\lim_{n\to\infty} 2^n \bar{M}_n^2(T) = \chi(T) > 0. \tag{1.13}$$

In particular, the spontaneous magnetization satisfies the relation

$$M(T) = \lim_{n\to\infty} \bar{M}_n(T) = 0.$$

In addition, the distribution $\tilde{\nu}_{n,T}(d\mathbf{x})$ tends weakly to the r-dimensional standard normal distribution as $n \to \infty$.

To formulate our results for the case when the Dyson condition (1.2) holds, we define a function $\bar{p}_n(t, T)$ by the formula

$$p_n(\mathbf{x}, T) = C_n(T)^{-1} \bar{p}_n(|\mathbf{x}|, T), \tag{1.14}$$

for $t = |\mathbf{x}| > 0$ and $\bar{p}_n(t, T) = 0$ for $t < 0$. The norming constant $C_n(T)$ is chosen in such a way that $\bar{p}_n(t, T)$ is a probability density function, i.e.

$$\int_0^\infty \bar{p}_n(t, T)\, dt = 1.$$

We will call $\bar{p}_n(t, T)$ the *probability density of the mean spin distribution along the radius.*

In Parts 2 and 3 we will describe the limit behaviour of an appropriate rescaling of the probability density $\bar{p}_n(t, T)$ for $T = T_c$ and $T < T_c$. Then we will formulate a Corollary which gives a good asymptotics for the norming constants $C_n(T)$ in (1.14). In such a way we get a good asymptotics for the probability density functions $p_n(\mathbf{x}, T)$ for $T \leq T_c$. To do this we introduce the notations

$$\begin{aligned} \hat{M}_n(T) &= \int_{-\infty}^{\infty} t \bar{p}_n(t, T)\, dt, \\ V_n(T) &= \left(\int_{-\infty}^{\infty} (t - \hat{M}_n(T))^2 \bar{p}_n(t, T)\, dt \right)^{1/2}, \end{aligned} \tag{1.15}$$

and the *rescaled* probability density

$$\pi_n(t, T) = V_n(T) \bar{p}_n \left(\hat{M}_n(T) + V_n(T)\, t, T \right) \tag{1.16}$$

which can be rewritte in an equivalent form as

$$\bar{p}_n(t, T) = \frac{1}{V_n(T)} \pi_n \left(\frac{t - \hat{M}_n(T)}{V_n(T)}, T \right). \tag{1.17}$$

Observe that, in general, $\hat{M}_n(T)$ and $\bar{M}_n(T)$, which is defined in (1.9), are different, but as we will see later,

$$\lim_{n\to\infty}[\hat{M}_n(T) - \bar{M}_n(T)] = 0.$$

Our aim is to prove that in the case when the Dyson condition (1.2) holds, there exists a critical temperature T_c such that the spontaneous magnetization $M(T) = \lim_{n\to\infty} \hat{M}_n(T)$ is positive for $T < T_c$ and it is zero for $T \geq T_c$. For $T < T_c$ the density function $\bar{p}_n(t,T)$ is concentrated near the point $t = \hat{M}_n(T)$, and the function $\pi_n(t,T)$ represents a rescaled distribution of $\bar{p}_n(t,T)$ near this point. We want to prove that $\pi_n(t,T)$ tends to a limit $\pi(t)$ as $n \to \infty$. It turns out that this limit does exist, and the limit function $\pi(t)$ is a nice analytic function, although it is non-Gaussian. The function $\pi(t)$ is expressed in terms of a solution of a nonlinear fixed point equation, and the next proposition concerns the existence of such a solution. Introduce the space of probability densities $p(t)$ on the line

$$\mathcal{A} = \left\{p(t)\colon\ \int_{-\infty}^{\infty} e^{\varepsilon|t|}p(t)\,dt < \infty \text{ for some } \varepsilon = \varepsilon(p(t)) > 0\right\}.$$

Consider also the subspace $\mathcal{A}_0 \subset \mathcal{A}$,

$$\mathcal{A}_0 = \left\{p(t)\colon\ p(t) \in \mathcal{A},\ \int_{-\infty}^{\infty} tp(t)\,dt = 0\right\}.$$

Proposition 1.4. *There exists a unique probability density function $g \in \mathcal{A}_0$ which satisfies the following fixed point equation:*

$$\begin{aligned} g(t) = \frac{2}{\pi^{\frac{r-1}{2}}} \int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} & e^{-|\mathbf{v}|^2} g\left(t - \frac{r-1}{4} - u + \frac{|\mathbf{v}|^2}{2}\right) \\ & \times g\left(t - \frac{r-1}{4} + u + \frac{|\mathbf{v}|^2}{2}\right) du\,d\mathbf{v}. \end{aligned} \tag{1.18}$$

The density $g(t)$ can be extended to an entire function on the complex plane, and for real t it satisfies the estimate

$$0 < g(t) < C_\varepsilon \exp\{-(2-\varepsilon)|t|\}, \quad \textit{for all } \varepsilon > 0. \tag{1.19}$$

For a proof of Proposition 1.4 see the proof of Lemmas 12 and 13 in [5]. It is worth noticing that the Fourier transform of g,

$$\tilde{g}(\xi) = \int_{-\infty}^{\infty} e^{i\xi t} g(t)\,dt,$$

solves the equation

$$\tilde{g}(\xi) = \frac{e^{\frac{i\xi(r-1)}{4}}\tilde{g}^2(\frac{\xi}{2})}{\left(1 + \frac{i\xi}{2}\right)^{\frac{r-1}{2}}}. \tag{1.20}$$

Using the probability density $g(t)$ of Proposition 1.4, we introduce a probability density $\pi(t)$ on the line of the form

$$\pi(t) = ce^{-2bt/3} g(bt - a), \tag{1.21}$$

where the numbers $b > 0$, $c > 0$, and a are chosen in such a way that

$$\int_{-\infty}^{\infty} \pi(t)\, dt = 1, \qquad \int_{-\infty}^{\infty} t\,\pi(t)\, dt = 0, \qquad \int_{-\infty}^{\infty} t^2 \pi(t)\, dt = 1. \tag{1.22}$$

Observe that such a, b, c exist and are unique. Indeed, after the change of variable $u = bt - a$, the second equation in (1.22) gives a as

$$a = -\frac{\int_{-\infty}^{\infty} ue^{-2u/3} g(u)\, du}{\int_{-\infty}^{\infty} e^{-2u/3} g(u)\, du}. \tag{1.23}$$

Then the first and third equations determine b and c uniquely from the system,

$$\begin{cases} \dfrac{c}{b} \displaystyle\int_{-\infty}^{\infty} e^{-2(u+a)/3} g(u)\, du = 1, \\ \dfrac{c}{b^3} \displaystyle\int_{-\infty}^{\infty} (u+a)^2 e^{-2(u+a)/3} g(u)\, du = 1. \end{cases} \tag{1.24}$$

Estimate (1.19) secures the convergence of the integrals in (1.23) and (1.24).

Now we formulate

Theorem 1.5. *Assume that*

$$\sum_{n=1}^{\infty} l_n^{-1} < \infty. \tag{1.25}$$

Then there exists a number $\kappa_0 = \kappa_0(N)$, where N is defined in (1.12), such that for all $0 < \kappa < \kappa_0$ the following statements hold.

Assume that the density $p(\mathbf{x}) = \frac{\nu(d\mathbf{x})}{d\mathbf{x}}$ belongs to the class $\mathcal{P}_\kappa$, and the sequence $\{l_n,\ n \geq 0\}$ satisfies Conditions 1–4. Then there exists a constant $L = L(\bar{\eta}, \kappa)$ such that if the sequence $\{l_n,\ n \geq 0\}$ satisfies Condition 5, then there exists a critical temperature $T_c > 0$ with the following properties:

(1) *If $T > T_c$ then*

$$\lim_{n\to\infty} 2^n \bar{M}_n^2(T) = \chi(T) > 0, \tag{1.26}$$

and the distribution $\tilde{\nu}_{n,T}(d\mathbf{x})$ tends weakly, as $n \to \infty$, to the r-dimensional standard normal distribution. The function $\chi(T)$ in (1.26) satisfies the following estimates near the critical point: There exists a temperature $T_0 > T_c$ and numbers $C_2 > C_1 > 0$ such that for all $T_0 > T > T_c$ there exists a number $\bar{n}(T)$ such that

$$\begin{aligned} &C_1 \sum_{k=\bar{n}(T)}^{\infty} l_k^{-1} < T - T_c \leq C_2 \sum_{k=\bar{n}(T)}^{\infty} l_k^{-1}, \\ &C_1 \frac{2^{\bar{n}(T)}}{l_{\bar{n}(T)}} < \chi(T) < C_2 \frac{2^{\bar{n}(T)}}{l_{\bar{n}(T)}}. \end{aligned} \tag{1.27}$$

(The number $\xi(T) = 2^{\bar{n}(T)}$ is the correlation length.)

(2) *At $T = T_c$, $\lim_{n\to\infty} M_n(T_c) = 0$ (there is no Thouless' effect), and moreover*

$$\lim_{n\to\infty} L_n^{-1} M_n(T_c) = 1, \tag{1.28}$$

where $M_n(T)$ is defined in (2.12), and

$$L_n = \left(\frac{r-1}{6} \sum_{j=n}^{\infty} l_j^{-1} \right)^{1/2}. \tag{1.29}$$

(Condition (1.25) implies that $\lim_{n\to\infty} L_n = 0$.)

Let us define the rescaled version $\rho_n(t)$ of the probability density function $\bar{p}_n(t, T_c)$ as

$$\rho_n(t) = \frac{\hat{M}_n(T_c)}{d_n} \bar{p}_n \left(\hat{M}_n(T_c) \left(1 + \frac{t}{d_n} \right), T_c \right), \tag{1.30}$$

where $\hat{M}_n(T)$ is defined in (1.15), and

$$d_n = \frac{(r-1) l_n}{2b} \sum_{k=n}^{\infty} l_k^{-1}. \tag{1.31}$$

(Observe that $\lim_{n\to\infty} d_n = \infty$ by Condition 2 on $\{l_n\}$.) The function $\rho_n(t)$ is defined on the half-line $[-d_n, \infty)$. Then

$$\lim_{n\to\infty} \|\rho_n(t) - \pi(t)\| = 0, \tag{1.32}$$

where the probability density $\pi(t)$ is defined in equations (1.21), (1.22) and

$$\|f(t)\| = \sum_{j=0}^{2} \sup_{t \ge -d_n} \left\{ e^{|t|/3} \left| \frac{d^j f(t)}{dt^j} \right| \right\}. \tag{1.33}$$

(3) *If $T < T_c$, then the numbers $\hat{M}_n(T)$ and $V_n(T)$ defined in formula (1.15) satisfy the following relations: The limit*

$$\lim_{n\to\infty} \hat{M}_n(T) = M(T) > 0 \tag{1.34}$$

exists, and

$$C_1 |T - T_c|^{1/2} < M(T) < C_2 |T - T_c|^{1/2}. \tag{1.35}$$

In addition,

$$\lim_{n\to\infty} l_n V_n(T) = \gamma(T) = \frac{bT}{3M(T)} > 0 \tag{1.36}$$

with the number b appeared in formula (1.21), and

$$\lim_{n\to\infty} \|\pi_n(t, T) - \pi(t)\| = 0, \tag{1.37}$$

where the probability densities $\pi_n(t, T)$ and $\pi(t)$ are defined in equations (1.16) and (1.21), (1.22), respectively, and $\|f(t)\|$ is defined in (1.33), with $d_n = \frac{\hat{M}_n(T)}{V_n(T)}$.

Theorems 1.3 and 1.5 are the central results of the present paper. Let us make some remarks about Theorem 1.5. Relations (1.26) and (1.28) imply that

$$M(T) = \lim_{n\to\infty} \hat{M}_n(T) = 0, \quad \text{for all } T \geq T_c,$$

i.e. the spontaneous magnetization $M(T)$ vanishes at $T \geq T_c$. Relation (1.35) implies that

$$\lim_{T\to T_c^-} M(T) = 0,$$

with the classical critical exponent $1/2$ for the magnetization.

The number $\bar{n}(T)$ in (1.27) is very important for our investigation in the subsequent sections. It shows how many iterations of the recursive equation (renormalization group transformation) is needed to reach the "high temperature region" (see Section 3 below for precise definitions). The quantity $\xi(T) = 2^{\bar{n}(T)}$ is the *correlation length.* Usually the correlation length has a power-like asymptotics $\xi(T) \asymp |T-T_c|^{-\nu}$ as $T \to T_c$ where ν is the critical exponent of the correlation length (see, e.g., [14] or [20]). It follows from (1.27) that in the case under consideration, $\xi(T)$ grows super-polynomially as $T \to T_c^+$. For instance, if l_n is a sequence determined by equation (1.6) then $\xi(T)$ grows like $\exp\left[C_0(T-T_c)^{1/(\lambda-1)}\right]$. Similarly, (1.27) implies that the magnetic susceptibility $\chi(T)$ diverges super-polynomially as $T \to T_c^+$.

Relation (1.36) shows that the mean square deviation of the mean spin along the radius behaves, when $n \to \infty$, as

$$V_n(T) \sim \frac{bT}{3M(T)l_n}, \qquad T < T_c,$$

so that it goes to zero very slowly as $n \to \infty$ (comparing with the standard behavior of $C2^{-n/2}$). In fact, it goes to zero sub-polynomially with respect to the number of spins 2^n. And according to (1.31), at $T = T_c$ the scaled mean square deviation of the mean spin along the radius, d_n^{-1}, goes to zero even slower, than at $T < T_c$, namely,

$$d_n^{-1} \sim \frac{2b}{r-1}\left(l_n \sum_{k=n}^{\infty} l_k^{-1}\right)^{-1}, \quad T = T_c.$$

On the other hand, observe that by (1.32) and (1.37) the limit distribution density $\pi(t)$ of the normalized mean spin along the radius is the same for all $T < T_c$ and for $T = T_c$ as well.

Let us say some words about our methods. The questions we investigate in this paper lead to a problem of the following type: We have a starting probability density function $p_0(\mathbf{x}, T)$ which depends on a parameter T, the temperature, and we apply the powers of an appropriately defined nonlinear operator $\mathbf{Q}$ to it. This operator $\mathbf{Q}$ is the renormalization group operator. We want to describe the behavior of the sequence of functions $p_n(\mathbf{x}, T) = \mathbf{Q}^n p_0(\mathbf{x}, T)$, $n = 1, 2, \ldots$. In particular, we want to understand how the behavior of this sequence of functions $p_n(\mathbf{x}, T)$, $n = 1, 2, \ldots$, depends on the parameter T. Our investigation shows that if the function $p_n(\mathbf{x}, T)$ is essentially concentrated around the origin, then a negligible error is committed when $p_{n+1}(\mathbf{x}, T) = \mathbf{Q}p_n(\mathbf{x}, T)$ is replaced by the convolution of the function $p_n(\mathbf{x}, T)$

with itself, and this is the case for all n if the parameter T is large. The replacement of the operator $\mathbf{Q}$ by the convolution is called the *high temperature approximation.*

On the other hand, if the function $p_n(\mathbf{x},T)$ is essentially concentrated in a narrow shell far from the origin, and this is the case for all n if the parameter T is small, then another good approximation of the function $p_{n+1}(\mathbf{x},T)=\mathbf{Q}_n p_n(\mathbf{x},T)$ is possible. This is called the *low temperature approximation.* The high temperature approximation actually means the application of the standard methods of classical probability theory. The low temperature approximation applied in this paper is a natural modification of the methods in our paper [5] where a similar problem was investigated. But in the present paper we have to make a more careful and detailed analysis. The reason for it is that while in [5] it was enough to investigate only very low temperatures T, now we have to follow carefully when the high and when the low temperature approximation is applicable. Moreover, — and this is a most important part of this paper, — to describe the behavior of the functions $p_n(\cdot,T)$ for all temperatures T we have to follow the behavior of these functions also in the case when neither the high nor the low temperature approximation is applicable. This is the so called *intermediate region.* (See Section 3 for precise definitions.)

We study the intermediate region in Section 5. There we show that if the function $p_n(\mathbf{x},T)$ "is not very far from the origin", namely, the low temperature approximation is not applicable for it, then the functions $p_{n+k}(\mathbf{x},T)$ are getting closer and closer to the origin as the index $n+k$ is increasing. Moreover, after finitely many steps k the high temperature approximation is already applicable, and the number of steps k we need to get into this situation can be bounded by a constant independent of the parameter T. The proof given in Section 5 contains arguments essentially different from the rest of the paper. Here we heavily exploit that the numbers $c_n=\frac{l_n}{l_{n-1}}$ are very close to one. Informally speaking, the sequence of numbers c_n-1 behaves like a small parameter, and this "small parameter" enables us to handle our model near the critical temperature.

The setup of the rest of the paper is the following. In Section 2 we give an analytic reformulation of the problem and connect Dyson's condition (1.2) with an approximate recursive formula for some quantities $M_n(T)$ related to the spontaneous magnetization (see (2.20) below). In Section 3 we introduce a notion of low and high temperature regions together with an intermediate region. Then we formulate the basic auxiliary theorems about the characterization of these regions. In Sections 4, 5, and 6 we prove the main estimates concerning the low temperature region, the intermediate region, and the high temperature region, respectively. In Section 7 we prove the convergence of the recursive iterations to the fixed point for all $T<T_c$. Finally, in Section 8 we prove Theorem 3.4 concerning some asymptotics near the critical point T_c and derive Theorems 1.3 and 1.5 from the auxiliary theorems.

2. Analytic reformulation of the problem. Strategy of the proof

The hierarchical structure of the Hamiltonian (1.1) leads to the following *recursive equation* for the density functions $p_n(\mathbf{x},T)$ (see, e.g., Appendix A to the paper [5]):

$$p_{n+1}(\mathbf{x},T)=C_n(T)\int_{\mathbb{R}^r}\exp\left(\frac{l_n}{T}(\mathbf{x}^2-\mathbf{u}^2)\right)p_n(\mathbf{x}-\mathbf{u},T)p_n(\mathbf{x}+\mathbf{u},T)\,d\mathbf{u} \tag{2.1}$$

for $n \geq 0$, where $p_0(\mathbf{x}, T) = p_0(\mathbf{x})$ is defined in (1.7),

$$l_n = l(2^n),$$

and $C_n(T)$ is an appropriate norming constant which turns $p_{n+1}(\mathbf{x}, T)$ into a density function. We are interested in the asymptotic behaviour of the functions $p_n(\mathbf{x}, T)$ as $n \to \infty$. For the sake of simplicity we will assume that $\varepsilon(t) = 0$ in (1.7), so that $p_0(\mathbf{x})$ coincides with (1.3). All the proofs below are easily extended to the case of nonzero $\varepsilon(t)$ satisfying estimate (1.8).

Define

$$c_n = \frac{l_n}{l_{n-1}}, \quad n = 0, 1, \dots \text{ with } \quad l_{-1} = 1, \tag{2.2}$$

$$A_n = 1 + \sum_{j=1}^{\infty} \frac{c_{n+1}}{2} \cdots \frac{c_{n+j}}{2} = 1 + l_n^{-1} \sum_{j=1}^{\infty} 2^{-j} l_{n+j}, \qquad n = 0, 1, \dots. \tag{2.3}$$

Then

$$l_n = \prod_{j=0}^{n} c_j, \quad n \geq 0, \tag{2.4}$$

and

$$l_n A_n = l_n + \frac{l_{n+1} A_{n+1}}{2}. \tag{2.5}$$

Indeed, by (2.3),

$$l_n A_n = l_n + \sum_{j=1}^{\infty} 2^{-j} l_{n+j} = \sum_{j=0}^{\infty} 2^{-j} l_{n+j},$$

hence

$$l_n A_n - l_n = \sum_{j=1}^{\infty} 2^{-j} l_{n+j} = \frac{1}{2} \sum_{j=0}^{\infty} 2^{-j} l_{n+1+j} = \frac{l_{n+1} A_{n+1}}{2}, \tag{2.6}$$

and (2.5) follows.

Define

$$q_n(\mathbf{x}, T) = \Lambda_n(T)^{-1} \exp\left(\frac{A_n l_n \mathbf{x}^2}{2}\right) p_n(\sqrt{T}\,\mathbf{x}, T), \tag{2.7}$$

where $\Lambda_n(T) > 0$ is a norming constant such that

$$\int_{\mathbb{R}^r} q_n(\mathbf{x}, T)\, d\mathbf{x} = 1.$$

Let

$$c^{(n)} = (1 + A_n)\, l_n, \qquad n = 0, 1, 2, \dots \tag{2.8}$$

Then it follows from equations (2.1) and (2.5) that

$$q_{n+1}(\mathbf{x}, T) = \frac{1}{Z_n(T)} \int_{\mathbb{R}^r} e^{-c^{(n)} \mathbf{u}^2} q_n(\mathbf{x} - \mathbf{u}, T) q_n(\mathbf{x} + \mathbf{u}, T)\, d\mathbf{u}. \tag{2.9}$$

Also, by (1.3),

$$q_0(\mathbf{x},T)=\frac{1}{Z_0(T)}\exp\left\{(c_0A_0-T)\frac{|\mathbf{x}|^2}{2}-\kappa T^2\frac{|\mathbf{x}|^4}{4}\right\}. \tag{2.10}$$

The norming constants $Z_n(T)$ in the previous formulas are determined by the condition that

$$\int_{\mathbb{R}^r} q_n(\mathbf{x},T)\,d\mathbf{x}=1.$$

Thus, the functions $q_n(\mathbf{x},T)$ are defined recursively by formulas (2.9) and (2.10). Our goal is to derive an asymptotics of the functions $q_n(\mathbf{x},T)$ as $n\to\infty$. Then the asymptotics of the functions $p_n(\mathbf{x},T)$ can be found by means of formula (2.7). The advantage of the functions $q_n(\mathbf{x},T)$ is that their recursive equation (2.9) does not depend on T.

The method of paper [5] can be adapted in the study of the low temperature approximation. We shall follow this approach. Due to the rotational symmetry of the Hamiltonian (1.1), the function $q_n(\mathbf{x},T)$ depends only on $|\mathbf{x}|$. Define the function $\bar q_n(t,T)$, $t\in\mathbb{R}^1$, $n=0,1,2,\dots$, such that

$$q_n(\mathbf{x},T)=C_n(T)\,\bar q_n(|\mathbf{x}|,T), \tag{2.11}$$

with a norming constant $C_n(T)$ such that

$$\int_0^\infty \bar q_n(t,T)\,dt=1.$$

We will define

$$\bar q_n(t,T)=0 \quad \text{for} \quad t<0.$$

Put also

$$M_n(T)=\int_0^\infty t\,\bar q_n(t,T)\,dt, \quad n=0,1,\dots, \tag{2.12}$$

and define the rescaled probability density functions

$$f_n(t,T)=\frac{1}{c^{(n)}}\bar q_n\left(M_n(T)+\frac{t}{c^{(n)}},T\right), \quad t\in\mathbb{R}^1, \quad n=0,1,\dots. \tag{2.13}$$

Then

$$\bar q_n(t,T)=c^{(n)}f_n\left(c^{(n)}(t-M_n(T)),T\right), \tag{2.14}$$

and

$$\int_{-\infty}^\infty f_n(t,T)\,dt=1, \quad \int_{-\infty}^\infty tf_n(t,T)\,dt=0.$$

The order parameter $M_n(T)$ in (2.12) is very convenient for the asymptotic recursive analysis. Later we will relate it to the parameters $\bar M_n(T)$ and $\hat M_n(T)$ introduced in formulae (1.9) and (1.15), respectively.

A low temperature approximation can be applied in the case when $M_n(T)$ is relatively large, comparing with the size of the neighborhood of $M_n(T)$ in which the function $f_n(t,T)$ is essentially concentrated. In this case we follow the behaviour of the pair $(f_n(t,T),M_n(T))$. To describe this procedure introduce the notation

$\mathbf{c} = \{c^{(n)},\ n = 0, 1, \dots\}$. The rotational invariance of the function $q_n(\cdot, T)$ suggests the definition of the operator

$$\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M} f(t) = \int_{u \in \mathbb{R}^1,\, \mathbf{v} \in \mathbb{R}^{r-1}} \exp\left\{ -\frac{u^2}{c^{(n)}} - \mathbf{v}^2 \right\}$$
$$\times f\left(c^{(n)} \left(\sqrt{\left(M + \frac{t}{c^{(n+1)}} + \frac{u}{c^{(n)}} \right)^2 + \frac{\mathbf{v}^2}{c^{(n)}}} - M \right) \right)$$
$$\times f\left(c^{(n)} \left(\sqrt{\left(M + \frac{t}{c^{(n+1)}} - \frac{u}{c^{(n)}} \right)^2 + \frac{\mathbf{v}^2}{c^{(n)}}} - M \right) \right) du\, d\mathbf{v}.$$

Formula (2.9) together with the definition of the function $f_n(t, T)$ yields that

$$\bar{q}_{n+1}\left(M_n(T) + \frac{t}{c^{(n+1)}}, T \right) = \frac{c^{(n+1)}}{Z_n(T)} \bar{\mathbf{Q}}^{\mathbf{c}}_{n, M_n(T)} f_n(t, T)$$

with

$$Z_n(T) = \int_{-c^{(n+1)} M_n(T)}^{\infty} \bar{\mathbf{Q}}^{\mathbf{c}}_{n, M_n(T)} f_n(t, T)\, dt.$$

The norming constant $Z_n(T)$ is determined by the condition

$$\int_0^\infty \bar{q}_{n+1}(t, T)\, dt = 1.$$

Define also

$$m_n(T) = m_n(f_n(t, T)) = \frac{1}{Z_n(T)} \int_{-c^{(n+1)} M_n(T)}^{\infty} t\, \bar{\mathbf{Q}}^{\mathbf{c}}_{n, M_n(T)} f_n(t, T)\, dt \tag{2.15}$$

and

$$\mathbf{Q}^{\mathbf{c}}_{n, M_n(T)} f_n(t, T) = \frac{1}{Z_n(T)} \bar{\mathbf{Q}}^{\mathbf{c}}_{n, M_n(T)} f_n(t + m_n(T), T).$$

Then

$$f_{n+1}(t, T) = \mathbf{Q}^{\mathbf{c}}_{n, M_n(T)} f_n(t, T) \quad \text{and} \quad M_{n+1}(T) = M_n(T) + \frac{m_n(T)}{c^{(n+1)}}. \tag{2.16}$$

To formulate a good approximation of the operator $\mathbf{Q}^{\mathbf{c}}_{n, M_n(T)}$, let us introduce the numbers

$$\bar{c}_n = \frac{c^{(n)}}{c^{(n-1)}} = \frac{(1 + A_n) l_n}{(1 + A_{n-1}) l_{n-1}}, \quad n = 1, 2, \dots. \tag{2.17}$$

The arguments of the function f in the definition of the operator $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M}$,

$$\ell^{\mathbf{c},\pm}_{n,M}(t, u, \mathbf{v}) = c^{(n)} \left(\sqrt{\left(M + \frac{t}{c^{(n+1)}} \pm \frac{u}{c^{(n)}} \right)^2 + \frac{\mathbf{v}^2}{c^{(n)}}} - M \right), \tag{2.18}$$

can be well approximated by a simpler expression because of the estimate

$$\left| \ell^{\mathbf{c},\pm}_{n,M}(t, u, \mathbf{v}) - \left(\frac{t}{\bar{c}_{n+1}} \pm u + \frac{\mathbf{v}^2}{2M} \right) \right| \le 100 \left(\frac{|\mathbf{v}|^4}{c^{(n)} M^3} + \frac{t^2 + u^2}{c^{(n)} M} \right)$$

which holds for $|t| < \frac{1}{4}c^{(n+1)}M,\ |u| < \frac{1}{4}c^{(n)}M$ and $\mathbf{v}^2 < c^{(n)}M^2$. This estimate suggests that for low temperatures T, when $M_n(T)$ is not small, the operator $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}$ can be well approximated by the operator $\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}$ defined as

$$\begin{aligned}\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}f(t,T) = \int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} & e^{-\mathbf{v}^2} f\left(\frac{t}{\bar{c}_{n+1}}+u+\frac{\mathbf{v}^2}{2M_n(T)},T\right) \\ & \times f\left(\frac{t}{\bar{c}_{n+1}}-u+\frac{\mathbf{v}^2}{2M_n(T)},T\right)du\,d\mathbf{v}.\end{aligned} \tag{2.19}$$

The elaboration of the above indicated method will be called the *low temperature approximation.* It works well when $M_n(T)$ is much larger than the range where the function $f_n(t,T)$ is essentially concentrated. For $n=0$ the starting value $M_0(T)$ at very low temperatures $T>0$ is very large. In this case the low temperature expansion can be applied. As we shall see later, the approximation of $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}$ by $\mathbf{T}^{\bar{c}}_{n,M_n(T)}$ yields that

$$M_{n+1}(T) \sim M_n(T) - \frac{r-1}{4c^{(n)}M_n(T)}, \tag{2.20}$$

which, in turn, implies that

$$M^2_{n+1}(T) \sim M^2_n(T) - \frac{r-1}{2c^{(n)}}. \tag{2.21}$$

It follows from (2.3) and (1.5) that

$$2 \le A_n \le 2.03, \qquad \lim_{n\to\infty} A_n = 2, \tag{2.22}$$

hence if Condition 1 is satisfied, then not only $\lim\limits_{n\to\infty} c_n = 1$, but also $\lim\limits_{n\to\infty} \bar{c}_n = 1$, and by (2.8),

$$3 \le \frac{c^{(n)}}{l_n} \le 3.03, \qquad \lim_{n\to\infty} \frac{c^{(n)}}{l_n} = 3. \tag{2.23}$$

This allows us to rewrite (2.21) as

$$M^2_{n+1}(T) \sim M^2_n(T) - \frac{r-1}{6l_n}. \tag{2.24}$$

This formula underlines the importance of the Dyson condition (1.2).
Namely, if the series

$$B = \sum_{n=1}^{\infty} l_n^{-1} \tag{2.25}$$

converges then $M_n(T)$ remains large for all n if $T>0$ is small. Indeed, assume that $T < c_0A_0/2$. Then it follows from (2.10) that $M_0^2(T) > C(\kappa T^2)^{-1}$, hence by (2.24), neglecting the error term,

$$M_n^2(T) \ge M_0^2(T) - \frac{r-1}{6}\sum_{n=0}^{\infty} l_n^{-1} \ge C(\kappa T^2)^{-1} - C_1 \gg 1$$

for all n if $T > 0$ is small, which was stated. On the other hand, if the series (2.25) diverges, then for some n, $M_n(T)$ becomes small, and the approximation (2.20) becomes inapplicable.

The low temperature approximation can be applied when $M_n(T)$ is not small. When $M_n(T)$ is small a different approximation is natural. If the function $q_n(\mathbf{x}, T)$ is essentially concentrated in a ball whose radius is much less than $\left(c^{(n)}\right)^{-1/2}$, then a small error is committed if the kernel function $e^{-c^{(n)}\mathbf{u}^2}$ in formula (2.9) is omitted. This means that the formula expressing $q_{n+1}(\mathbf{x})$ by $q_n(\mathbf{x})$ can be well approximated through the convolution $q_{n+1}(\mathbf{x}) = q_n * q_n(\mathbf{x})$. This approximation will be called the *high temperature approximation.* If the high temperature approximation can be applied for $q_n(\mathbf{x}, T)$, then the function $q_{n+1}(\mathbf{x}, T)$ is even more strongly concentrated around zero. Hence, as a detailed analysis will show, if at a temperature T it can be applied for a certain n_0, then it can be applied for all $n \geq n_0$.

Finally, there are such pairs (n, T) for which the function $q_n(\mathbf{x}, T)$ can be studied neither by the low nor by the high temperature approximation. We call the set of such pairs an *intermediate region.* We shall prove that if the sequence $c^{(n)}$ sufficiently slowly tends to infinity and the function $q_n(\mathbf{x}, T)$ is out of the region where the low temperature approximation is applicable, then the density function $q_{n+1}(\mathbf{x}, T)$ will be more strongly concentrated around zero than the function $q_n(\mathbf{x}, T)$. Moreover, in *finitely many* steps the function $q_{n+k}(\mathbf{x}, T)$ will be so strongly concentrated around zero that after this step the high temperature approximation is applicable. It is important that the number of steps k needed to get into the high temperature region can be bounded independently of the parameter T.

The main part of the paper consists of an elaboration of the above heuristic argument.

3. Formulation of auxiliary theorems

To describe the region where the low temperature approximation will be applied we define some sequences $\beta_n(T)$ which depend on the temperature T. Define recursively,

$$\begin{aligned} \beta_N(T) &= \frac{\left(c^{(N)}\right)^2}{2^N}, \\ \beta_{n+1}(T) &= \left(\frac{\bar{c}_{n+1}^2}{2} + \sqrt{\frac{\beta_n(T)}{c^{(n)}}}\right)\beta_n(T) + \frac{10}{M_n^2(T)} \qquad \text{for } n \geq N, \end{aligned} \tag{3.1}$$

where the number N is defined in (1.12), $\bar{c}_n$ in (2.17) and $M_n(T)$ in (2.12). As it will be seen later, these numbers measure how strongly the functions $f_n(x, T)$ are concentrated around zero. We define the low temperature region, where low temperature approximation will be applied.

Definition of the low temperature region. *A pair (n, T) is in the low temperature region if the following properties (1) and (2) hold.*

(1) $0 < T \leq c_0 A_0/2$, *where A_0 was defined in (2.3).*
(2) *Either $0 \leq n \leq N$ with the number N introduced in (1.12) or $n > N$ and $\frac{\beta_{n-1}(T)}{c^{(n-1)}} \leq \eta$ with the number η appearing also in (1.12).*

The temperature T is in the low temperature region if the pair (n,T) is in the low temperature region for all numbers n. Let us remark that by (2.4) and (1.5)

$$1 \le l_n = \prod_{j=1}^{n} c_j \le 1.01^n,$$

hence by (2.23),

$$3 \le c^{(n)} \le 3.03 \cdot 1.01^n. \tag{3.2}$$

Therefore, by (3.1),

$$\frac{\beta_N(T)}{c^{(N)}} = \frac{c^{(N)}}{2^N} \le \frac{1}{c^{(N)}} \le \eta \tag{3.3}$$

hence the pair $(N+1,T)$ is in the low temperature region if $T \le c_0 A_0/2$. Since $\beta_{n+1}(T) \ge \frac{10}{M_n^2(T)}$ the pair (n,T) can get out of the low temperature region only if $M_n(T)$ becomes very small.

To define the high temperature region introduce the notations

$$\begin{aligned} h_n(\mathbf{x},T) &= \left(c^{(n)}\right)^{-r/2} q_n\left(\frac{\mathbf{x}}{\sqrt{c^{(n)}}},T\right), \\ D_n^2(T) &= \int_{\mathbb{R}^r} \mathbf{x}^2 h_n(\mathbf{x},T)\,d\mathbf{x}. \end{aligned} \tag{3.4}$$

where the function $q_n(\mathbf{x},T)$ is defined in (2.7). Let us also introduce the probability measure $H_{n,T}$,

$$H_{n,T}(\mathbf{A}) = \int_{\mathbf{A}} h_n(\mathbf{x},T)\,d\mathbf{x}, \quad \mathbf{A} \subset \mathbb{R}^r, \tag{3.5}$$

on $\mathbb{R}^r$.

Definition of the high temperature region. *A pair (n,T) is in the high temperature region if $D_n^2(T) < e^{-1/\eta^2}$ with the number η in formula (1.12), where $D_n^2(T)$ is defined in (3.4). The temperature T is in the high temperature region if there exists a threshold index $n_0(T)$ such that (n,T) is in the high temperature region for all $n \ge n_0(T)$.*

It may happen that a pair (n,T) belongs neither to the low nor to the high temperature region. Then we say that (n,T) belongs to the *intermediate region.* Let us remark that we introduced two numbers N and η in formula (1.12), and in the formulation of the subsequent results N and η will denote these numbers. The following result is very important for us.

Theorem 3.1. *There exists a number $\kappa_0 = \kappa_0(N)$ such that for all $0 < \kappa < \kappa_0$ (where κ appears in formula (1.3)) and $0 < \bar\eta < \eta$ there is a number $L = L(\bar\eta,\kappa)$ for which the following is true. Assume that Conditions 1 and 5 (with $\bar\eta$ and this number $L = L(\bar\eta,\kappa)$) hold. We consider such temperatures T for which there are numbers n such that the pair (n,T) does not belong to the low temperature region. Let $\bar n(T) \ge 0$ be the smallest number n with this property.*

If the pair $(\bar{n}(T), T)$ does not belong to the high temperature region (which means that $(\bar{n}(T), T)$ is in the intermediate region), then there exist some numbers $K = K(\bar{\eta}, \kappa) > 0$, $\tilde{\eta} = \tilde{\eta}(\bar{\eta}, \kappa) > 0$, and $k = k(\bar{\eta}, \kappa) \in \mathbb{N}$ such that

$$D^2_{\bar{n}(T)}(T) < K, \qquad \tilde{\eta} < D^2_{\bar{n}(T)+k}(T) < e^{-1/\eta^2}.$$

This implies in particular that the pair $(\bar{n}(T) + k, T)$ with this index k belongs to the high temperature region.

We shall also prove the following corollary of Theorem 3.1. (See the Remark after the proof of Lemma 6.1.)

Corollary. *Under the conditions of Theorem 3.1 all temperatures $T > 0$ belong either to the low or to the high temperature region. If the Dyson condition (1.2) holds, then all sufficiently low temperatures belong to the low and all sufficiently high temperatures to the high temperature region. If the Dyson condition (1.2) is violated, then all temperatures $T > 0$ belong to the high temperature region.*

The next theorem concerns the *low temperature region.*

Theorem 3.2. *There exists a number $\kappa_0 = \kappa_0(N)$ such that for all $0 < \kappa < \kappa_0$ the following is true. Assume that the Dyson condition (1.2) and Conditions 1 and 2 hold. Assume that the temperature T is in the low temperature region. Then the numbers $M_n(T)$ defined in (2.12) have a limit,*

$$\lim_{n\to\infty} M_n(T) = M_\infty(T), \tag{3.6}$$

and

$$\lim_{n\to\infty} \frac{M_n^2(T) - M_\infty^2(T)}{\frac{r-1}{2} \sum\limits_{k=n}^{\infty} \frac{1}{c^{(k)}}} = 1. \tag{3.7}$$

In addition,

$$\lim_{n\to\infty} \left\| \frac{1}{M_n(T)} f_n\left(\frac{t}{M_n(T)}, T\right) - g(t) \right\| = 0, \tag{3.8}$$

where

$$\|f(t)\| = \sum_{j=0}^{2} \sup_{t \ge -c^{(n)} M_n(T)} e^{|t|} \left| \frac{d^j f(t)}{d\,t^j} \right|, \tag{3.9}$$

$f_n(t, T)$ is introduced in (2.13), and the probability density $g(t)$ is defined as a solution of the fixed point equation (1.18).

Part (3) of Theorem 1.5, with the exception of estimate (1.35), follows from Theorem 3.2 and the additional relation $M_\infty(T) > 0$ if $T < T_c$ which follows from the results in Theorem 3.4 formulated at the end of this section. Indeed, we can express the function $p_n(\mathbf{x}, T)$ in terms of $f_n(t, T)$. Namely, by (2.7), (2.11), and

(2.14)

$$p_n(\mathbf{x},T) = L_n^{-1}(T)\,\exp\left(-\frac{A_n l_n|\mathbf{x}|^2}{2T}\right) \tag{3.10}$$
$$\times f_n\left(\frac{c^{(n)}}{\sqrt{T}}\left(|\mathbf{x}|-\sqrt{T}\,M_n(T)\right),T\right)$$

with an approriate norming constant $L_n(T)$. Let us write that $|\mathbf{x}|^2 = (\sqrt{T}\,M_n(T)+|\mathbf{x}|-\sqrt{T}\,M_n(T))^2$, hence

$$\exp\left(-\frac{A_n l_n|\mathbf{x}|^2}{2T}\right) = \exp\left\{-\frac{A_n l_n}{2T}\left[TM_n^2(T)\right.\right.$$
$$\left.\left.+2\sqrt{T}\,M_n(T)(|\mathbf{x}|-\sqrt{T}\,M_n(T))+(|\mathbf{x}|-\sqrt{T}\,M_n(T))^2\right]\right\},$$

and substitute it into (3.10). This leads to the equation

$$p_n(\mathbf{x},T) = \tilde{L}_n^{-1}(T)\tilde{f}_n\left(\frac{|\mathbf{x}|-\tilde{M}_n(T)}{\tilde{V}_n(T)},\,T\right) \tag{3.11}$$

with an appropriate norming constant $\tilde{L}_n(T)$, where

$$\begin{aligned}
\tilde{M}_n(T) &= \sqrt{T}\,M_n(T), \qquad \tilde{V}_n(T) = \frac{\sqrt{T}}{c^{(n)}M_n(T)},\\
\tilde{f}_n(t,T) &= f_n\left(\frac{t}{M_n(T)},T\right)\exp\left(-\frac{A_n l_n t}{c^{(n)}}-\varepsilon_n(t,T)\right),\\
\varepsilon_n(t,T) &= \frac{A_n l_n t^2}{2(c^{(n)})^2 M_n^2(T)}.
\end{aligned}$$

Observe that by (2.22) and (2.23)

$$\lim_{n\to\infty}\frac{A_n l_n}{c^{(n)}} = \frac{2}{3}, \qquad \lim_{n\to\infty}\frac{A_n l_n}{2(c^{(n)})^2 M_n^2(T)} = 0,$$

hence (3.8) implies that there is some $C_0>0$ such that

$$\lim_{n\to\infty}\left\|\frac{1}{M_n(T)}\tilde{f}_n(t,T)-C_0 g\,(t)\,e^{-2t/3}\right\|' = 0,$$

where

$$\|f(t)\|' = \sum_{j=0}^{2}\sup_{t\ge -c^{(n)}M_n^2(T)} e^{|t|/3}\left|\frac{d^j f(t)}{d\,t^j}\right|.$$

This also implies that there exist some real number a and $C'>0$ such that

$$\lim_{n\to\infty}\left\|\frac{1}{M_n(T)}\tilde{f}_n(t-a,T)-C' g\,(t-a)\,e^{-2t/3}\right\|' = 0, \tag{3.12}$$

with such numbers a and $C'>0$ for which the relations

$$\int_\infty^\infty C' g(t-a)e^{-2t/3}\,dt = 1 \text{ and } \int_\infty^\infty C' t g(t-a)e^{-2t/3}\,dt = 0$$

hold.

Let us define for all $b > 0$ the function $\pi(t|b) = C'bg(bt-a)e^{-2bt/3}\,dt$. These functions satisy the relations

$$\int_{\infty}^{\infty} C'b\pi(t|b)\,dt = 1 \text{ and } \int_{\infty}^{\infty} C'bt\pi(t|b)\,dt = 0.$$

Moreover, the number $b > 0$ can be chosen in such a way that the identity

$$\begin{aligned}\int_{\infty}^{\infty} C'bt^2\pi(t|b)\,dt &= b^{-2}\int_{\infty}^{\infty} C'b(bt)^2\pi(t|b)\,dt \\ &= b^{-2}\int_{\infty}^{\infty} C't^2 g(t-a)e^{2t/3}\,dt = 1\end{aligned}$$

also holds. Let us define the function $\pi(t) = \pi(t|b)$ with this parameter b. In such a way we constructed a function $\pi(t)$ that satisfies relations (1.21) and (1.22). Moreover, if we define the functions

$$\tilde{\pi}_n(t,T) = C_n(T)\tilde{f}_n(bt-a,T) \tag{3.13}$$

with these numbers a and b and with such a norming constant $C_n(T)$ for which

$$\int_{-\infty}^{\infty} \tilde{\pi}_n(t,T)\,dt = 1,$$

then these functions satisfy the relation

$$\lim_{n\to\infty} \|\tilde{\pi}_n(t) - \pi(t)\|' = 0 \tag{3.14}$$

because of relations (3.12). Because of (3.14) we also have $\int_{-\infty}^{\infty} \tilde{\pi}_n(t,T)\,dt = 1$, $\lim\limits_{n\to\infty}\int_{-\infty}^{\infty} t\tilde{\pi}_n(t,T)\,dt = 0$, $\lim\limits_{n\to\infty}\int_{-\infty}^{\infty} t^2\tilde{\pi}_n(t,T)\,dt = 1$, and because of (3.11) and (3.13)

$$\begin{aligned}\tilde{\pi}_n(t,T) &= C_n'(T)^{-1}\bar{p}_n((bt-a)\tilde{V}_N(T) + \tilde{M}_n(T),T) \\ &= b\tilde{V}_n(T)\bar{p}_n(bt-a)\tilde{V}_N(T) + \tilde{M}_n(T),T).\end{aligned}$$

(The normalization constant in the second identity of the last formula is determined by the fact that both $\bar{p}_n(t,T)$ and $\tilde{\pi}_n t,T)$ are probability density functions).

Hence

$$\begin{aligned}\int_{-\infty}^{\infty} t\tilde{\pi}_n(t,T)\,dt &= b\tilde{V}_n(T)\int_{-\infty}^{\infty} t\bar{p}_n(bt-a)\tilde{V}_N(T) + \tilde{M}_n(T),T)\,dt \\ &= \int_{-\infty}^{\infty} \frac{t-\tilde{M}_n(T)+a\tilde{V}_n(T)}{b\tilde{V}_n(T)}\bar{p}_n(t,T)\,dt \\ &= \frac{\bar{M}_n(T)-\tilde{M}_n(T)+a\tilde{V}_n(T)}{b\tilde{V}_n(T)} \to 0 \quad \text{as } n\to\infty,\end{aligned} \tag{3.15}$$

and

$$\begin{aligned}&\int_{-\infty}^{\infty} t^2\tilde{\pi}_n(t,T)\,dt = b\tilde{V}_n(T)\int_{-\infty}^{\infty} t^2\bar{p}_n(bt-a)\tilde{V}_N(T) + \tilde{M}_n(T),T)\,dt \\ &= \int_{-\infty}^{\infty}\left(\frac{(t-\bar{M}_n(T)) + (\bar{M}_n(T)-\tilde{M}_n(T)+a\tilde{V}_n(T))}{b\tilde{V}_n(T)}\right)^2 \bar{p}_n(t,T)\,dt\end{aligned}$$

$$= \frac{V_n^2(T) + \left((\bar{M}_n(T) - \tilde{M}_n(T)) + a\tilde{V}_n(T)\right)^2}{b^2\tilde{V}_n^2(T)} \to 1 \quad \text{as } n \to \infty. \tag{3.16}$$

Relations (3.15) and (3.16) together with Theorem 3.2, the inequqailty $M_\infty(T) > 0$ and the definition of the quantities $\tilde{M}_n(T)$ and $\tilde{V}_n(T)$ imply relations (1.34) and (1.36). Indeed, by Theorem 3.2 $\lim\limits_{n\to\infty} \tilde{M}_n(T) = M(T)$ with $M(T) = \sqrt{T} M_\infty(T)$, and since $\tilde{V}_n(T) \to 0$ as $n \to \infty$ relation (3.15) implies that $\lim\limits_{n\to\infty} (\bar{M}_n(T) - \tilde{M}_n(T)) = 0$. Formula (1.34) follows from these relations with the above defined number $M(T)$. Relations (3.15) and (3.16) together imply that $\lim\limits_{n\to\infty} \frac{V_n(T)}{\tilde{V}_n(T)} = b$. On the other hand, $\lim\limits_{n\to\infty} l_n \tilde{V}_n(T) \lim\limits_{n\to\infty} \frac{c^{(n)}}{3} \frac{T}{c^{(n)}\sqrt{T} M_n(T)} = \frac{T}{3M(T)}$. These relations imply (1.36).

Finally to prove relation (1.37) let us observe how the functions $\pi_n(t,T)$ and $\tilde{\pi}_n(t,T)$ can be expressed with the help of the function $\bar{p}_n(t,T)$. Besides, both are probability density functions, and the integrals $\int_{-\infty}^{\infty} t\tilde{\pi}_n(t,T)\,dt$ and $\int_{-\infty}^{\infty} t^2\tilde{\pi}_n(t,T)\,dt$ tend to zero and 1 as $n \to \infty$, while the correponding integrals for $\pi_n(t,T)$ are equal exactly to these limit values for all parameters n. This implies that the identity $\pi_n(t,T) = (1+\varepsilon_n)\tilde{\pi}_n((1+\varepsilon_n)t+\delta_n, T)$ with such numbers $\varepsilon_n = \varepsilon_n(T)$ and $\delta_n = \delta_n(T)$ for which $\lim\limits_{n\to\infty} \varepsilon_n = 0$, and $\lim\limits_{n\to\infty} \delta_n = 0$. It can be proved with the help of this observation that relation (3.14) remains valid if we replace the functions $\tilde{\pi}_n(t,T)$ by $\pi_n(t,T)$ in it, and this means that formula (1.37) is valid.

Now we formulate a theorem about the *high temperature region.* Put

$$\tilde{h}_n(\mathbf{x}, T) = 2^{-rn/2} q_n\left(2^{-n/2}\mathbf{x}, T\right) = \left(\frac{c^{(n)}}{2^n}\right)^{r/2} h_n\left(\sqrt{\frac{c^{(n)}}{2^n}}\mathbf{x}, T\right), \tag{3.17}$$

and define the probability measures

$$\tilde{H}_{n,T}(\mathbf{A}) = \int_{\mathbf{A}} \tilde{h}_n(\mathbf{x}, T)\,d\mathbf{x}, \quad \mathbf{A} \subset \mathbb{R}^r \tag{3.18}$$

on $\mathbb{R}^r$.

Theorem 3.3. *There exists a number $\kappa_0 = \kappa_0(N)$ such that for all $0 < \kappa < \kappa_0$ and $0 < \bar{\eta} < \eta$ there exists a number $L = L(\bar{\eta}, \kappa)$ such that the following is true. Assume that Conditions 1 and 5 (with $\bar{\eta}$ and this $L = L(\bar{\eta}.\kappa)$) hold, and T is in the high temperature region. Then the measures $\tilde{H}_{n,T}$ defined in (3.18) converge weakly to the normal distribution on $\mathbb{R}^r$ with expectation zero and covariance matrix $\sigma^2(T)\mathbf{I}$ with some $\sigma^2(T) > 0$, where $\mathbf{I}$ denotes the identity matrix.*

If T belongs to the high temperature region, but the pair $n = (0, T)$ does not belong to it, (i.e. the temperature T is not too high), then the inequality

$$C_1 \frac{2^{\bar{n}(T)}}{c^{(\bar{n}(T))}} \le \sigma^2(T) \le C_2 \frac{2^{\bar{n}(T)}}{c^{(\bar{n}(T))}} \tag{3.19}$$

also holds with some $C_2 > C_1 > 0$, where $\bar{n}(T)$ is defined in Theorem 3.1.

Remark. Not only the convergence of the measures $\tilde{H}_{n,T}$ but also the convergence of their density functions $\tilde{h}_n(\mathbf{x}, T)$ could be proved. But the proof of the convergence of the distribution is simpler, and it is also sufficient for our purposes.

Corollary. *Let $\bar{H}_{n,T}$ denote the probability measure on $\mathbb{R}^r$ with the density function*

$$2^{-rn/2}T^r p_n(2^{-n/2}\sqrt{T}\mathbf{x}, T).$$

Under the conditions of Theorem 3.3 the measures $\bar{H}_{n,T}$ have the same Gaussian limit as the measures $\tilde{H}_{n,T}$ defined in Theorem 3.3 as $n \to \infty$.

Our last theorem concerns the *critical point*. We want to show that there is a critical temperature T_c such that above it all temperatures belong to the high and below it all temperatures belong to the low temperature region. We also want to describe the situation in the neighborhood of the critical temperature in more detail. In Theorem 3.4 we state such a result.

Theorem 3.4. *There exists a number $\kappa_0 = \kappa_0(N)$ such that for all $0 < \kappa < \kappa_0$ there exists a number $L = L(\bar{\eta}, \kappa)$ such that the following is true. Assume that Conditions 1–4 are satisfied. Then for a fixed n the set of temperatures T for which (n, T) belongs to the low temperature region forms an interval $(0, T_n]$, and the sequence T_n, $n = 1, 2, \dots$, is monotone decreasing in n. Define the critical temperature T_c as the limit, $T_c = \lim\limits_{n\to\infty} T_n$. Then $c_0A_0/4 > T_c > 0$. The function $M_\infty(T) = \lim\limits_{n\to\infty} M_n(T)$ exists in the interval $(0, T_c]$, and for a fixed n the function $M_n(T)$ is strictly decreasing on the interval $(0, T_n]$. The relation $M_\infty(T_c) = 0$ holds. If $T_c + \varepsilon > T > T_c$ with some $\varepsilon > 0$, then the inequality*

$$C_1 \sum_{k=\bar{n}(T)}^{\infty} \frac{1}{c^{(k)}} < T - T_c < C_2 \sum_{k=\bar{n}(T)}^{\infty} \frac{1}{c^{(k)}} \tag{3.20}$$

holds with some appropriate numbers $C_2 > C_1 > 0$, where $\bar{n}(T)$ is defined in Theorem 3.1. If $T_c - \varepsilon < T < T_c$ with a sufficiently small $\varepsilon > 0$, then

$$C_1(T_c - T)^{1/2} < M_\infty(T) < C_2(T_c - T)^{1/2}. \tag{3.21}$$

4. Basic estimates in the low temperature region

In this section we give some basic estimates on the function $f_n(x, T)$ and its derivatives (with respect to the variable x) if the pair (n, T) is in the low temperature region. These estimates state in particular, that in the definition of the functions $f_n(x, T)$ the right scaling was chosen. With the scaling in formula (2.13) the function $f_n(x, T)$ is essentially concentrated in a finite interval whose size depends only on $M_n(T)$. Both the results and proofs are closely related to those of Sections 3—6 in paper [5].

First we consider the case of *small* indices $0 \le n \le N$, where the number N defined in (1.12) (cf. Section 4 in [5]), and we begin with $n = 0$. Assume that $T < c_0A_0/2$ and $\kappa > 0$ is small (exact conditions on the smallness of κ will be given later). In this case the function $\bar{q}_0(x, T)$ has its maximum in the points $\bar{M}_0(T)$ (see (2.10)), where

$$\bar{M}_0(T) = \left(\frac{A_0c_0 - T}{\kappa T^2}\right)^{1/2} \tag{4.1}$$

is a large number. From (2.10) we obtain that

$$\frac{1}{c^{(0)}}\bar{q}_0\left(\bar{M}_0(T)+\frac{x}{c^{(0)}},T\right) = \frac{1}{Z_0(T)}\exp\left\{-\left(A_0c_0-T\right)\left(\frac{x}{c^{(0)}}\right)^2\left(1+\frac{x}{2c^{(0)}\bar{M}_0(T)}\right)^2\right\}, \tag{4.2}$$

where

$$Z_0(T)=\int_{-\bar{M}_0(T)}^{\infty}\exp\left\{-(A_0c_0-T)\left(\frac{x}{c^{(0)}}\right)^2\left(1+\frac{x}{2c^{(0)}\hat{M}_0(T)}\right)^2\right\}dx. \tag{4.3}$$

It can be proved by means of the identity

$$c^{(0)}(M_0(T)-\bar{M}_0(T)) = \frac{\int_{-c^{(0)}\bar{M}_0(T)}^{\infty} x\exp\left\{-(c_0A_0-T)\left(\frac{x}{c^{(0)}}\right)^2\left(1+\frac{x}{2c^{(0)}\bar{M}_0(T)}\right)^2\right\}dx}{\int_{-c^{(0)}\bar{M}_0(T)}^{\infty}\exp\left\{-(A_0c_0-T)\left(\frac{x}{c^{(0)}}\right)^2\left(1+\frac{x}{2c^{(0)}\bar{M}_0(T)}\right)^2\right\}dx} \tag{4.4}$$

that

$$\left|M_0(T)-\bar{M}_0(T)\right|\le\frac{\text{const.}}{M_0(T)}\le\ \text{const.}\sqrt{\kappa}\,T, \tag{4.5}$$

where $M_0(T)$ is defined (2.12). This shows that $\bar{M}_0(T)$ is a very good approximation to $M_0(T)$. Straightforward calculation yields with the help of formulas (4.1) and (4.3) that

$$\left|Z_0(T)-\frac{c^{(0)}\sqrt{\pi}}{\sqrt{(A_0c_0-T)}}\right|\le\ \text{const.}\sqrt{\kappa}\,T, \tag{4.6}$$

and from (4.1)–(4.6) we obtain that

$$\left|\frac{\partial^j}{\partial x^j}\left(f_0(x,T)-\frac{\sqrt{A_0c_0-T}}{c^{(0)}\sqrt{\pi}}\exp\left\{-(A_0c_0-T)\left(\frac{x}{c^0}\right)^2\right\}\right)\right| \le\ \text{const.}\,\kappa^{1/4}e^{-2|x|/c^{(0)}}\quad\text{if } |x|<\log\kappa^{-1},\quad j=0,1,2, \tag{4.7}$$

and

$$\left|\frac{\partial^j}{\partial x^j}f_0(x,T)\right|\le C\exp\left\{-\frac{(A_0c_0-T)}{4c^{(0)}}\left|2x+\frac{x^2}{c^{(0)}M_0^2(T)}\right|\right\} \quad\text{for } x\ge -c^{(0)}M_0(T),\quad j=0,1,2. \tag{4.8}$$

A relatively small error is committed if M_n is very large and the arguments $\ell^{\pm}_{n,M_n}(x,u,v)$ (defined in formula (2.18)) of the function f_n in the operator $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M}f_n$ are replaced by $x\pm u$. Exploiting this fact one can prove, using a natural adaptation of the proof of Proposition 1 of paper [5], the following

Proposition 4.1. *There exists a number $\kappa_0=\kappa_0(N)$ such that if*

(i) $0<\kappa<\kappa_0$, *and*
(ii) $0<T\le c_0A_0/2$,

then the relations

$$\left|\frac{\partial^j}{\partial x^j}\left(f_n(x,T)-\frac{\sqrt{A_0c_0-T}}{\sqrt{\pi}}\frac{2^n}{c^{(n)}}\exp\left\{-2^n(A_0c_0-T)\left(\frac{x}{c^{(n)}}\right)^2\right\}\right)\right|$$
$$\le B(n)\kappa^{1/4}e^{-2^{n+1}|x|/c^{(n)}},\quad \text{if } |x|<2^{-n}\log\kappa^{-1},\quad j=0,1,2,$$
$$\left|\frac{\partial^j}{\partial x^j}f_n(x,T)\right|\le B(n)\exp\left\{-\frac{(A_0c_0-T)}{4}\frac{2^n}{c^{(n)}}\left|2x+\frac{x^2}{c^{(n)}M_n^2(T)}\right|\right\}$$
$$\text{for } x\ge -c^{(n)}M_n(T),\quad j=0,1,2,$$

and

$$|M_n(T)-\bar M_0(T)|\le B(n)\sqrt{\kappa}\,T \tag{4.9}$$

hold for all $0\le n\le N$ *with the function* $\bar M_0(T)$ *defined in (4.1) and a function* $B(n)$ *which depends neither on* T *nor on* κ.

We formulate and prove, similarly to paper [5], certain inductive hypotheses about the behaviour of the functions $f_n(x,T)$ for $n\ge N$ if the pair (n,T) is in the low temperature region. In the formulation of these hypotheses we apply the sequence $\beta_n(T)$ defined in (3.1) and the sequence $\alpha_n(T)$ defined as

$$\begin{aligned}\alpha_N(T)&=\tfrac{1}{200}\tfrac{(c^{(N)})^2}{2^N},\\ \alpha_{n+1}(T)&=\left(\tfrac{\bar c_{n+1}^2}{2}-\sqrt{\tfrac{\beta_n(T)}{c^{(n)}}}\right)\alpha_n(T)+\tfrac{10^{-12}}{M_n^2(T)}\qquad\text{for } n\ge N.\end{aligned}\tag{4.10}$$

To formulate the inductive hypotheses we also introduce a regularization of the functions $f_n(x,T)$.

Definition of the regularization of the functions $f_n(x,T)$**.** *Let us fix a* C^∞-*function* $\varphi(x)$, $-\infty<x<\infty$, *such that* $\varphi(x)=1$ *for* $|x|\le 1$, $0\le\varphi(x)\le 1$ *if* $1\le x\le 2$ *and* $\varphi(x)=0$ *for* $|x|\ge 2$. *Then the regularization of the function* $f_n(x,T)$ *is*

$$\varphi_n(f_n(x,T))=A_n\varphi\left(\frac{x+B_n}{100\sqrt{c^{(n)}}}\right)f_n(x+B_n,T),$$

with norming constants A_n *and* B_n *such that*

$$\int_{-\infty}^{\infty}\varphi_n(f_n(x,T))\,dx=1,\qquad \int_{-\infty}^{\infty}x\varphi_n(f_n(x,T))\,dx=0.$$

Now we formulate the inductive hypotheses.

Hypothesis $I(n)$**.**

$$\left|\frac{\partial^j f_n(x,T)}{\partial x^j}\right|\le\frac{C}{\beta_n(T)^{(j+1)/2}}\exp\left\{-\frac{1}{\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n)}M_n(T)}\right|\right\}$$
$$\text{for } j=0,1,2,\quad x\ge -c^{(n)}M_n(T),$$

with a universal constant $C>0$. *One could choose, e.g.,* $C=10^{20}$.

Hypothesis $J(n)$**.**

$$|\tilde{\varphi}_n f_n(t+is,T)| \le \frac{e^{\beta_n(T)s^2}}{1+\alpha_n(T)t^2} \qquad \textit{if} \qquad |s| \le \frac{2}{\sqrt{\beta_{n+1}(T)}},$$

where

$$\tilde{\varphi}_n f_n(t+is,T) = \int e^{i(t+is)x}\varphi(f_n(x,T))\,dx.$$

Corollary of Proposition 4.1. *Under the conditions of Proposition 4.1, the inductive hypotheses $I(n)$ and $J(n)$ hold for $n = N$ with a universal constant $C > 0$ in hypothesis $I(n)$. (For instance, one can choose $C = 10^5$.)*

Before formulating the main result of this Section, we introduce the operators $\mathbf{T}_n$. They are appropriate scaling of the operators $\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}$ defined in formula (2.19), but these operators will be applied only for the regularization of the functions $f_n(x,T)$ and not for the functions $f_n(x,T)$ themselves. Put

$$\begin{aligned}\mathbf{T}_n\varphi_n(f_n(x,T)) = {} & \frac{2}{\bar{c}_{n+1}\pi^{\frac{r-1}{2}}}\int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} e^{-\mathbf{v}^2} \\ & \times \varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}} - \frac{r-1}{4M_n(T)} + u + \frac{\mathbf{v}^2}{2M_n(T)}, T\right)\right) \\ & \times \varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}} - \frac{r-1}{4M_n(T)} - u + \frac{\mathbf{v}^2}{2M_n(T)}, T\right)\right)\,du\,d\mathbf{v},\end{aligned} \tag{4.11}$$

with the constants $\bar{c}_n$ defined in (2.17) and the number $V(S^{r-2})$ introduced in Proposition 1.4. The main result of this section is the following

Proposition 4.2. *There exists $\kappa_0 = \kappa_0(N) > 0$ such that if*

(i) *the inductive hypotheses $I(n)$ and $J(n)$ hold for the function $f_n(x,T)$,*
(ii) $0 < \kappa < \kappa_0$, *(κ appears in formula (1.3)), and*
(iii) *the pairs (m,T) belong to the low temperature region for all* $0 \le m \le n$,

then the inductive hypotheses $I(n+1)$ and $J(n+1)$ hold for the function $f_{n+1}(x,T)$. Also there exist universal constants C_1, K_1, K_2 and K_3 such that the following estimates hold:

(a)

$$\begin{gathered}M_{n+1}(T) = M_n(T) - \frac{r-1}{4c^{(n)}M_n(T)} + \frac{\gamma_n(T)}{c^{(n)}}, \\ \textit{where } \ |\gamma_n(T)| \le C_1\frac{\beta_{n+1}(T)}{c^{(n+1)}}\sqrt{\beta_{n+1}(T)}\end{gathered} \tag{4.12}$$

(b)

$$1 \le \frac{\beta_{n+1}(T)}{\alpha_{n+1}(T)} \le K_1, \tag{4.13}$$

(c) *For* $x > -c^{(n+1)} M_{n+1}(T)$ *and* $j = 0, 1, 2,$

$$
\begin{aligned}
\left| \frac{\partial^j}{\partial x^j} \left[f_{n+1}(x,T) - \mathbf{T}_n \varphi_n(f_n(x,T)) \right] \right| &\leq \frac{K_2 C^4}{\beta_{n+1}^{(j+1)/2}(T)} \frac{\beta_n(T)}{c^{(n)}} \\
\times \Bigg[\exp \left\{ -\frac{1}{\sqrt{\beta_{n+1}(T)}} \left| 2x + \frac{x^2}{c^{(n+1)} M_{n+1}(T)} \right| \right\} \\
+ \exp \left\{ -\frac{2|x|}{\sqrt{\beta_{n+1}(T)}} \right\} \Bigg]. \qquad (4.14)
\end{aligned}
$$

(d) *For* $x \in \mathbb{R}^1$ *and* $j = 0, 1, 2, 3, 4,$

$$
\left| \frac{\partial^j}{\partial x^j} \mathbf{T}_n \varphi_n(f_n(x,T)) \right| \leq \frac{K_3 C^2}{\beta_{n+1}^{(j+1)/2}(T)} \exp \left\{ -\frac{2|x|}{\sqrt{\beta_{n+1}(T)}} \right\}. \qquad (4.15)
$$

The proof of Proposition 4.2 is based on the observation that the operator $\mathbf{T}_n$ approximates the operator $\mathbf{Q}^{\mathbf{c}}_{n,M_n(T)}$ very well, and it has a relatively simple structure. Namely, it can be written by writing the vectors $\mathbf{v} \in \mathbb{R}^{r-1}$ in formula (4.11) in spherical coordinates in the form

$$
\begin{aligned}
\mathbf{T}_n \varphi_n(f_n(x,T)) &= \frac{4}{\bar{c}_{n+1} \Gamma(\frac{r-1}{2})} \int_0^\infty w^{r-2} e^{-w^2} \\
&\quad \varphi_n(f_n) * \varphi_n(f_n) \left(\frac{2x}{\bar{c}_{n+1}} + \frac{w^2}{M_n(T)} - \frac{r-1}{2M_n(T)}, T \right) dw,
\end{aligned}
$$

where $w = |\mathbf{v}|$. Then we get with the substitution $\frac{w^2}{M_n(T)} = u$ that

$$
\begin{aligned}
\mathbf{T}_n \varphi_n(f_n(x,T)) &= \frac{2}{\bar{c}_{n+1} \Gamma(\frac{r-1}{2})} \int_0^\infty M_n(T) (M_n(T)u)^{(r-3)/2} e^{-M_n(T)u} \\
&\quad \varphi_n(f_n) * \varphi_n(f_n) \left(\frac{2x}{\bar{c}_{n+1}} + u - \frac{r-1}{2M_n(T)}, T \right) du \\
&= \frac{2}{\bar{c}_{n+1}} \varphi_n(f_n) * \varphi_n(f_n) * k^-_{M_n(T)} \left(\frac{2x}{\bar{c}_{n+1}} - \frac{r-1}{2M_n(T)} \right), \qquad (4.16)
\end{aligned}
$$

where $*$ denotes convolution, $k^-_{M_n(T)}(x) = k_{M_n(T)}(-x)$, and $k_{M_n(T)}(x) = M_n(T) k\left(M_n(T)x\right)$ with $k(x) = \frac{1}{\sqrt{\pi x}} e^{-x}$ $k(x) = \frac{x^{(r-3)/2} e^{-x}}{\Gamma(\frac{r-1}{2})}$ for $x > 0$, and $k(x) = 0$ for $x \leq 0$.

The operator $\mathbf{T}_n$ has a certain contraction property which can be expressed in the Fourier space. the Fourier transform of $\tilde{\mathbf{T}}_n \tilde{\varphi}_n(f_n(\xi,T))$ can be expressed with the help of formula (4.16). One gets that

$$
\begin{aligned}
&\tilde{\mathbf{T}}_n \tilde{\varphi}_n(f_n(\xi,T)) \\
&\quad = \exp \left\{ i \frac{(r-1)\bar{c}_{n+1}}{4M_n(T)} \xi \right\} \tilde{k} \left(-\frac{\bar{c}_{n+1} \xi}{2M_n(T)} \right) \left[\tilde{\varphi}_n \left(f_n \left(\frac{\bar{c}_{n+1}}{2} \xi, T \right) \right) \right]^2,
\end{aligned}
$$

$$(4.17)\qquad = \frac{\exp\left\{i\frac{(r-1)\bar c_{n+1}}{4M_n(T)}\xi\right\}}{\left(1+i\frac{\bar c_{n+1}\xi}{2M_n(T)}\right)^{(r-1)/2}}\left[\tilde\varphi_n\left(f_n\left(\frac{\bar c_{n+1}}{2}\xi,T\right)\right)\right]^2.$$

In this calculation we have exploited that $k(x)$ is the density function of the gamma distribution with parameter $\frac{r-1}{2}$, whose characteristic function equals $(1-i\xi)^{-(r-1)/2}$. It follows from formula (4.17) that $\mathbf{T}_n\varphi_n(f_n(x,T))$ is the density function of a random variable with expectation zero.

The proof of Proposition 4.2 is a natural adaptation of the proof of the corresponding result (of Proposition 3) in paper [5]. Hence we only explain the main points and the necessary modifications.

Because of the inductive property $I(n)$ $f_n(x,T)$ is essentially concentrated in a neighbourhood of the origin of size $\sqrt{\beta_n(T)}$, and if (n,T) is in the low temperature domain and $\eta>0$ is chosen sufficiently small, then $\frac{|x|}{100\sqrt{c^{(n)}}}\le\frac{\eta}{10}$ for $|x|\le\sqrt{\beta_n(T)}$, and the function $f_n(x,T)$ (disregarding the scaling with the numbers A_n and B_n) is not changing in the typical region by the regularization of the function $f_n(x,T)$. This is the reason why such a regularization works well.

The proof of Proposition 4.2 contains several estimates. First we list those results whose proof apply the bound on $f_n(x,T)$ formulated in the Inductive hypothesis $I(n)$. One can bound the differences

$$\frac{\partial^j}{\partial x^j}(\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}f_n(x,T)-\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}\varphi_n(f_n(x,T)))\quad\text{(Lemma 4 in [5])},$$

$$\frac{\partial^j}{\partial x^j}(\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}\varphi_n(f_n(x,T))-\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}\varphi_n(f_n(x,T)))\quad\text{(Lemma 5 in [5])},$$

with the help of Property $I(n)$ similarly to paper [5]. The absolute value of these expressions can be bounded for all $\varepsilon>0$ by

$$\frac{\beta_n}{c^{(n)}}\frac{C_1(\varepsilon)C^2}{\beta_n^{(j+1)/2}(T)}\exp\left\{-\frac{2(1-\varepsilon)}{\bar c_{n+1}\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n+1)}M_n(T)}\right|\right\}$$

with some appropriate constant $C_1(\varepsilon)>0$ if $f_n(x,T)$ satisfies Condition $I(n)$.

The main difference between these estimates and the analogous results in paper [5] is that the upper bounds given for the above expressions contain a small multiplying factor $\frac{\beta_n(T)}{c^{(n)}}$. In paper [5] the multiplying factors 2^{-n} and $1/c^{(n)}$ appear instead of this term. In the proof of this paper we had to make some modifications, because while in paper [5] only very low temperatures were considered when $M_n(T)$ is strongly separated from zero, now we want to give an upper bound under the weaker condition formulated in the definition of the low temperature region. The proofs are very similar. The only essential difference is that in the present case the typical region, where a good asymptotic approximation must be given is chosen as the interval $|x|<10\sqrt{c^{(n)}}$, i.e. it does not depend on the value of $M_n(T)$.

Also the expression $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}f_n(x,T)$ can be bounded together with their first two derivatives with the help of Property $I(n)$ in the same way as in Lemma 3 of paper [5]. But this estimate is useful only for large x. It can be proved, similarly to the proof of the corresponding result in paper [5] (lemma 7) that the scaling constants which appear in the formulas expressing $\mathbf{Q}^{\mathbf{c}}_{n,M_n(T)}$ through $\bar{\mathbf{Q}}^{\mathbf{c}}_{n,M_n(T)}$ and $\mathbf{T}_n$ through $\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}$ are very close to each other. Here again the multiplying factor $\frac{\beta_n(T)}{c^{(n)}}$ appears in the error term instead of the multiplying factor $1/c^{(n)}$ in paper [5]. This Lemma 7 in [5] is a technical result which expresses the difference of the functions $\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}F_1(x)$ and $\bar{\mathbf{T}}^{\mathbf{c}}_{n,M_n(T)}F_2(x)$ together with its derivatives if we have a control on the difference of the original functions $F_1(x)$ and $F_2(x)$. We gain such kind of information from the inductive hypothesis $I(n)$. They give a good control on the difference $f_{n+1}(x,T) - \mathbf{T}_n\varphi_n(f_n(x,t))$. The consequences of these results are formulated in Proposition 2 in paper [5]. These results also imply an estimate on the Fourier transforms $\tilde{\varphi}_{n+1}(f_{n+1}(\xi,T)) - \tilde{\mathbf{T}}_n\tilde{\varphi}_n(f_n(\xi,T)))$ and $\tilde{\mathbf{T}}_n\tilde{\varphi}_n(f_n(\xi,T))$ and also on their analytic continuation. This is done in lemma 8 in paper [[5]. Now again the analogous result holds under the conditions of the present paper with the difference that the term c^{-n} must be replaced $\frac{\beta_n(T)}{c^{(n)}}$. The estimate obtained for $\tilde{\mathbf{T}}_n\tilde{\varphi}_n(f_n(\xi,T))$ in such a way is relatively weak, it is useful only for large ξ.

The above results are not sufficient to prove Proposition 4.2. In particular, they do not explain why the right scaling was chosen in the definition of the function $f_n(x,T)$. Their role is to bound the error which is committed when $\mathbf{Q}^{\mathbf{c}}_{n,M_n(T)}f_n(x,T)$ is replaced by $\mathbf{T}_n\varphi(f_n(x,T))$. The function $\mathbf{T}_n\varphi_n(f_n(x,T))$ together with its derivatives and Fourier transform can be well bounded by means of formula (4.17) and the inverse Fourier transform. In the estimations leading to such bounds the inductive hypothesis $J(n)$ plays a crucial role. The proof of Lemma 9 in paper [5] can be adapted to the present case without any essential difficulty. But, the parameters α_n, β_n and c must be replaced by $\alpha_n(T)$, $\beta_n(T)$ and $\bar{c}_{n+1}$ in the present case.

Proposition 4.2 can be proved similarly to its analog, Proposition 3 in paper [5]. The notation must be adapted to the notation of the present paper. Besides, the small coefficient $c^{-n/2}$ appearing in the proof of Proposition 3 in [5] must be replaced by $\sqrt{\frac{\beta_n(T)}{c^{(n)}}}$. There is one point where a really new argument is needed in the proof. This argument requires a more detailed discussion. It is the proof of relation (4.13), i.e. of the fact that $\alpha_n(T)$ and $\beta_n(T)$ have the same order of magnitude. Their ratio must be bounded by a number independent of η. The proof of the analogous result in paper [5] exploited the fact that in the model of that paper the sequence $c^{(n)}$ tended to infinity exponentially fast. In the present case this property does not hold any longer, hence a different argument is needed. The validity of relation (4.13) has a different cause for relatively small and large indices n.

For large n it can be shown that both $\beta_n(T)$ and $\alpha_n(T)$ have the same order of magnitude as $M_n^{-2}(T)$, and for large n these relations imply (4.13). If n is relatively small and $M_0(T)$ is large, then $M_n^{-2}(T)$ is much less than $\alpha_n(T)$ and $\beta_n(T)$. In this case the above indicated argument does not work, but it can be proved that for such indices n the numbers $\beta_n(T)$ are decreasing exponentially fast, and the proof of relation (4.13) for such n is based on this fact.

To distinguish between small and large indices n define the number

$$N_1(T) \quad = \quad \left\{\min n\colon n \ge N, \text{ and } \beta_{n+1}(T) \le \frac{100}{M_n^2(T)}\right\},$$

$$(N_1(T) = \infty \text{ if there is no such } n). \tag{4.18}$$

where the number N was defined in formula (1.12). We shall later see that $N_1(T) < \infty$ for all $0 < T \le c_0 A_0/2$.

First we prove relation (4.13) under the additional condition $n \le N_1(T)$. In this case $\beta_{m+1}(T) \le \frac{c_{m+1}^2}{2}\beta_m(T) + \frac{\beta_m(T)}{10}$ for $m \le n$, and because of Condition 1

$$\beta_{m+1}(T) \le \frac{2}{3}\beta_m(T) \quad \text{if } m \le N_1(T) \tag{4.19}$$

for all $N \le m \le n$.

Hence $\sqrt{\frac{\beta_{m+1}(T)}{c^{(m+1)}}} \le \frac{5}{6}\sqrt{\frac{\beta_m(T)}{c^{(m)}}}$, $\sqrt{\frac{\beta_m(T)}{c^{(m)}}} \le \left(\frac{5}{6}\right)^{m-N}\sqrt{\frac{\beta_N(T)}{c^{(N)}}}$,

$$\begin{aligned} 1 \le \frac{\beta_{m+1}(T)}{\alpha_{m+1}(T)} &\le \max\left(\frac{\frac{c_{m+1}^2}{2} + \sqrt{\frac{\beta_m(T)}{c^{(m)}}}}{\frac{c_{m+1}^2}{2} - \sqrt{\frac{\beta_m(T)}{c^{(m)}}}} \cdot \frac{\beta_m(T)}{\alpha_m(T)},\, 10^{13}\right) \\ &\le \max\left(\exp\left\{5\sqrt{\frac{\beta_m(T)}{c^{(m)}}}\right\} \cdot \frac{\beta_m(T)}{\alpha_m(T)},\, 10^{13}\right) \end{aligned}$$

for $N \le m \le n$, and

$$\frac{\beta_{n+1}(T)}{\alpha_{n+1}(T)} \le \max\left(\frac{\beta_N(T)}{\alpha_N(T)},\, 10^{13}\right)\exp\left\{5\sum_{m=N}^{n}\sqrt{\frac{\beta_m(T)}{c^m(T)}}\right\} \le K.$$

The above argument together with the observation that $\beta_N(T) \gg M_N^{-2}(T)$ if the parameter $t > 0$ in (1.3) is sufficiently small and $T \le c_0 A_0/2$ imply that $N < N_1(T)$, and the pair (n, T) is in the low temperature region for all $n \le N_1(T)$. The latter property follows from the fact that by formula (4.19) the sequence $\frac{\beta_n(T)}{c^{(n)}}$ is monotone decreasing for $N \le n \le N_1(T)$.

In the case $n > N_1(T)$ we can prove by induction with respect to n together with the inductive proof of Proposition 4.2 that

$$\begin{aligned} \beta_{n+1}(T) \quad &\le \quad \frac{100}{M_n^2(T)} \quad \text{if } n \ge N_1(T). \\ &\text{and } (n,T) \text{ is in the low temperature region.} \end{aligned} \tag{4.20}$$

By applying formula (4.20) for $n-1$ and the fact that (n, T) is in the low temperature region we get that the term $\gamma_{n-1}(T)$ in formula (4.12) can be bounded as

$$|\gamma_{n-1}(T)| \le \frac{\beta_n(T)}{c^{(n)}}\sqrt{\beta_n(T)} \le \eta\frac{10}{M_{n-1}(T)} \le \frac{1}{8C_1 M_{n-1}(T)} \tag{4.21}$$

with the same number C_1 which appears in (4.12) if the number $\eta > 0$ was chosen sufficiently small. Then formula (4.12) implies that $M_n(T) \le M_{n+1}(T)$. Hence we get by applying again formula (4.20) with $n-1$ that $M_n(T) < M_{n-1}(T)$, and

$$\beta_{n+1}(T) \le \frac{2}{3}\beta_n(T) + \frac{10}{M_n^2(T)} \le \frac{200}{3M_{n-1}^2(T)} + \frac{10}{M_n^2(T)} \le \frac{100}{M_n^2(T)}.$$

This means that formula (4.20) also holds for n. Relation (4.20) together with the definition of the sequence $\alpha_n(T)$ imply that for $n \ge N_1(T)$

$$\alpha_{n+1}(T) \ge \frac{10^{-12}}{M_n^2(T)} \ge 10^{-14}\beta_n(T),$$

i.e. formula (4.13) is also valid for $n > N_1(T)$ if (n,T) is in the low temperature domain. With the help of this argument Proposition 4.2 can be proved by an adaptation of the proof of the corresponding result in [5].

We formulate and prove a lemma which describes some properties of the numbers $\beta_n(T)$ in the cases when $n \le N_1(T)$ or $n \ge N_1(T)$. Several parts of it were already proved in the previous arguments.

Lemma 4.3. *Let $0 < T \le c_0 A_0/2$. If the parameter $\kappa > 0$ in formula (1.3) is sufficiently small, then the following statements are valid:*

(1) *The number $N_1(T)$ defined in (4.18) is finite, and $N_1(T) > N$.*
(2) *The pair $(N_1(T), T)$ is in the low temperature region.*
(3) *The relations (4.19), (4.20) hold.*
(4) *If $n \ge N_1(T)$ and (n,T) is in the low temperature region then*

$$M_n(T) - \frac{3}{8c^{(n)}M_n(T)} \le M_{n+1}(T) \le M_n(T) - \frac{1}{8c^{(n)}M_n(T)}. \tag{4.22}$$

(5) *If $N \le n \le N_1(T)$ then*

$$\begin{aligned} M_n(T) - \frac{1}{4c^{(n)}M_n(T)} - \eta\left(\frac{2}{3}\right)^{(n-N)/2} &\le M_{n+1}(T) \\ \le M_n(T) - \frac{1}{4c^{(n)}M_n(T)} + \eta\left(\frac{2}{3}\right)^{(n-N)/2}&. \end{aligned} \tag{4.23}$$

(6) *We have that*

$$N_1(T) - N \le 10\log(1/\kappa T^2). \tag{4.24}$$

(7) *If $M_n(T) < 10$ then $n \ge N_1(T)$.*

Proof of Lemma 4.3. Formulas (4.19) and (4.20) were already proved in the previous argument, and since (N,T) is in the low temperature region, i.e. $\beta_N(T) \ge \eta c^N$, relation (4.19) implies that (n,T) is in the low temperature region for all $N \le n \le N_1(T)$. Formula (4.22) follows from formula (4.21) with the replacement of $n-1$ by n and formula (4.12). By relation (4.19) $\beta_n(T) \le \left(\frac{2}{3}\right)^{n-N}$ if $N \le n \le N_1(T)$. Hence it follows from (4.12) that

$$M_{n+1}(T) \le M_n(T) + \frac{\beta_{n+1}(T)}{c^{(n)}}\frac{\sqrt{\beta_{n+1}(T)}}{c^{(n)}} \le M_n(T) + \eta\left(\frac{2}{3}\right)^{(n-N)/2}, \tag{4.25}$$

and even relation (4.23) holds in this case.

Relation (4.25) and the estimate obtained for $\beta_n(T)$ imply that $M_n^2(T) \le (M_N(T)+1)^2 \le 2M_N^2(T)$ and $\beta_{n+1}(T)M_n^2(T) \le 2M_N^2(T)\left(\frac{2}{3}\right)^{n-N}$ if $n \le N_1(T)$. This relation together with the definition of the index $N_1(T)$ defined in (4.18) imply that $2M_N^2(T)\left(\frac{2}{3}\right)^{n-N} \ge 100$ if $n < N_1(T)$. Applying the last formula for

$n = N_1(T) - 1$ we get that $(N_1(T) - 1 - N)) \log \frac{3}{2} \le \log \frac{M_N^2(T)}{50}$. Since $M_N^2(T) \sim$ const. $\frac{1}{\kappa T^2}$ this relation implies that $N_1(T)$ is finite, and moreover it satisfies (4.24). Finally, if the inequalities $M_n(T) \le 10$ and $n < N_1(T)$ held simultaneously, then the inequality $M_n^2(T)\beta_{n+1}(T) \le 100 \left(\frac{2}{3}\right)^{n-N} \le 100$ would also hold. This relation contradicts to the assumption $n < N_1(T)$. Hence also the last statement of Lemma 4.3 holds. □

The previous results enable us to describe the different behaviour of the model in the cases when the Dyson condition (1.2) is satisfied and when it is not. This will be done in Lemma 4.4. It shows that if (1.2) *is not satisfied* then for all T there is a pair (n, T) which does not belong to the low temperature region, while if (1.2) *is satisfied*, then all sufficiently low temperatures T belong to the low temperature region. In the latter case the asymptotic behaviour of the spontaneous magnetization $M_n(T)$ can be described for large n. The description of the behaviour of the function $q_n(x, T)$ in the case when T does not belong to the low temperature region needs further investigation, and this will be done in Sections 5 and 6. A more detailed investigation of the case when T belongs to the low temperature region will be done in Section 7. We finish this section with the proof of a result about the behaviour of the magnetization $M_n(T)$ at low temperatures $T > 0$ which will be useful in the subsequent part of the paper.

Lemma 4.4. *Let $0 < T \le c_0 A_0/2$, and let the parameter $\kappa > 0$ be sufficiently small. If the Dyson condition (1.2) is not satisfied, then for all $T > 0$ there is some $n = n(T)$ for which (n, T) does not belong to the low temperature region. If, on the other hand, condition (1.2) is satisfied, then T belongs to the low temperature region for sufficiently small $T > 0$. In this case relation (3.6) and, under the additional Condition 2, also relation (3.7) hold.*

Proof of Lemma 4.4. It follows from formulas (4.22) and (4.23) that

$$-\frac{1}{c^{(n)}} \le M_{n+1}^2(T) - M_n^2(T) \le -\frac{1}{8c^{(n)}} \tag{4.26}$$

if $n \ge N_1(T)$ and the pair (n, T) is in the low temperature region, and

$$\begin{aligned} &-\frac{1}{2c^{(n)}} - 10\left(\frac{2}{3}\right)^{n-N}(M_N(T)+1) \le M_{n+1}^2(T) - M_n^2(T) \\ &\qquad \le -\frac{1}{2c^{(n)}} + 10\left(\frac{2}{3}\right)^{n-N}(M_N(T)+1) \end{aligned} \tag{4.27}$$

if $N \le n \le N_1(T)$. Formula (4.26) can be obtained by taking square in formula (4.22) and observing that $c^{(n)} M_n(T)^2 > 10\eta^{-1}$. Formula (4.27) can be deduced similarly from (4.23) by observing first that the right-hand side of (4.23) implies that $M_n(T) \le M_N(T) + 1$ for $N \le n \le N_1(T)$.

Formulas (4.26) and (4.27) imply that if a temperature $T > 0$ is in the low temperature region, then

$$\sum_{k=N}^{n} \frac{1}{c^{(n)}} \le 8(M_N^2(T) - M_n^2(T)) + 30(M_N(T)+1) \le 8M_N^2(T) + 30(M_N(T)+1)$$

for all $n \geq N$, where the number N is defined in (1.12). Since the right-hand side of the last formula does not depend on n, this implies that (1.2) holds.

In the other direction, if (1.2) holds, then since by Proposition 4.1

$$\lim_{T\to\infty} M_0(T) = \lim_{T\to\infty} M_N(T) = \infty,$$

there is some number $\bar{T} \leq c_0 A_0/2$ such that for all temperatures $0 < T \leq \bar{T}$ $M_N^2(T) > 8 \sum\limits_{n=N}^{\infty} \frac{1}{c^{(n)}} + 30 M_n(T) + 31$. If $T > 0$ satisfies the above inequality, then the left-hand side of the inequalities (4.26) and (4.27) imply that if the pair (n, T) is in the low temperature domain and $n \geq N_1(T)$, then

$$M_n^2(T) > M_N^2(T) - 8 \sum_{n=N}^{n} \frac{1}{c^{(n)}} 30(M_n(T) + 1)) \geq 1.$$

Hence $M_n^2(T) > 1$ for all n, and T is in the low temperature region.

Let $T > 0$ be in the low temperature region. If $n > m > N_1(T)$, then by (4.26)

$$\left| M_n^2(T) - M_m^2(T) \right| \leq \sum_{k=m}^{n} \frac{1}{c^{(k)}}.$$

Since in this case Condition 1 holds, the last relation implies that $M_n^2(T)$, $n = 1, 2, \dots$, is a Cauchy sequence, and relation (3.6) holds. We claim that if Condition 2 also holds, then for any $\varepsilon > 0$

$$-\frac{r - 1 + \varepsilon}{2c^{(n)}} \leq M_{n+1}^2(T) - M_n^2(T) \leq -\frac{r - 1 - \varepsilon}{2c^{(n)}} \tag{4.28}$$

if $n \geq n(\varepsilon)$. Relation (3.7) is a consequence of (4.28). Relation (4.28) can be deduced from (4.12) and (4.20) if we show that for any temperature $T > 0$ in the low temperature region

$$\lim_{n\to\infty} \frac{\beta_n(T)}{c^{(n)}} = 0. \tag{4.29}$$

Relation (4.29) holds under Condition 2, since by (4.26) in this case for all $n > N_1(T)$,

$$M_n^2(T) \geq \lim_{k\to\infty} \left(M_n^2(T) - M_k^2(T) \right) \geq \frac{1}{8} \sum_{k=n}^{\infty} \frac{1}{c^{(k)}},$$

and

$$\frac{\beta_n(T)}{c^{(n)}} \leq \frac{100}{M_{n-1}^2(T) c^{(n)}} \leq 800 \left(c^{(n)} \sum_{k=n-1}^{\infty} \frac{1}{c^{(k)}} \right)^{-1}.$$

Under Condition 2 the last expression tends to zero as $n \to \infty$. This implies formula (4.28). Lemma 4.4 is proved. □

5. Estimates in the intermediate region. The proof of theorem 3.1

In this section we give some estimates on $q_n(\mathbf{x},T)$ when the pair (n,T) belongs neither to the low nor to the high temperature region and prove Theorem 3.1 with their help.

Let us consider the number $\bar n=\bar n(T)$ introduced in the formulation of Theorem 3.1, namely

$$\bar n(T)=\min\{n\colon\ D_n^2(T)<e^{-1/\eta^2}\}.$$

In Lemmas 5.1 and 5.2 we shall prove some estimates about a scaled version of the function $q_{\bar n(T)}(\mathbf{x},T)$, where $q_n(\mathbf{x},T)$ was defined in (2.7). In Lemma 5.1 the case $T\le c_0A_0$, and in Lemma 5.2 the case $T\ge c_0A_0$ will be considered. Lemmas 5.1 and 5.2 yield some estimates on the tail-behaviour of a scaled version of the function $q_{\bar n(T)}(\mathbf{x},T)$. This will be needed to start an inductive procedure for all $n\ge\bar n(T)$ which state that the functions $q_n(\mathbf{x},T)$ become more and more strongly concentrated around zero as the index n is increasing. This procedure is based on Lemmas 5.3 and 5.4. The role of Lemma 5.3 is to give an appropriate lower bound for the norming constant $Z_n(T)$ in the definition of the function $q_n(\mathbf{x},T)$. Then in Lemma 5.4 we prove some contraction property of the operator which maps an appropriate scaled version of the distribution function with density function $\text{const.}\,\bar q_{n-1}(|\mathbf{x}|,T)$ to an appropriate scaled version of the distribution function with density $\text{const.}\,\bar q_n(|\mathbf{x}|,T)$, $\mathbf{x}\in\mathbb{R}^r$. The proof of Lemma 5.4 will exploit the rotation symmetry of the model. Theorem 3.1 will be proved by means of these lemmas.

To formulate these results we introduce some notations. Let us introduce the functions

$$\hat h_n(\mathbf{x},T)=\left(c^{(\bar n(T))}\right)^{-r/2}q_n\left(\frac{\mathbf{x}}{\sqrt{c^{\bar n(T)}}},T\right),\quad \mathbf{x}\in\mathbb{R}^r, \tag{5.1}$$

and measures

$$\hat H_{n,T}(\mathbf{A})=\int_{\mathbf{A}}\hat h_n(\mathbf{x},T)\,d\mathbf{x},\quad \mathbf{A}\subset\mathbb{R}^r, \tag{5.2}$$

in the space $\mathbb{R}^r$. Define also the function

$$\hat H_{n,T}(R)=\hat H_{n,T}(\{\mathbf{x}\colon |\mathbf{x}|\ge R\})\quad \text{for } R\ge 0. \tag{5.3}$$

The functions $\hat h_{n,T}$ and measures $\hat H_{n,T}$ are similar to the functions $h_{n,T}$ and measures $H_{n,T}$ defined in (3.4) and (3.5). The only difference is that the scaling of $q_n(\mathbf{x},T)$ in (5.2) and (5.3) is made by means of $c^{(\bar n(T))}$ instead of $c^{(n)}$. If Condition 5 is satisfied with a sufficiently small $\bar\eta$ and sufficiently large $L(\bar\eta,T)$, and $n-\bar n(T)$ is not too large, then the approximation of $c^{(n)}$ by $c^{(\bar n(T))}$ is sufficiently good for our purposes. Hence it will be enough to have a good control on the measure $\hat H_{n,T}$. In Lemma 5.3 we give a bound on it for large $|\mathbf{x}|$ and in Lemma 5.4 we prove an estimate which enables to bound $\hat H_{n,T}(R)$ for small R too.

With the help of these results we can prove that starting from $\bar n=\bar n(T)$ after finitely many steps k the pair $(\bar n+k,T)$ is in the high temperature region. Moreover, this number k can be bounded from above independently of the temperature T. First we formulate Lemma 5.1.

Lemma 5.1. *Under the conditions of Proposition 4.2, the function* $h_{\bar{n}(T)}(\mathbf{x},T)$ *defined in (3.4) satisfies the inequality*

$$h_{\bar{n}(T)}(\mathbf{x},T) \le \exp\left\{\frac{K}{\eta} - \frac{|\mathbf{x}|^2}{10}\right\} \quad \text{if} \quad T \le c_0 A_0/2 \tag{5.4}$$

with an appropriate $K > 0$*. For* $T \le c_0 A_0/2$ *the pair* $(\bar{n}(T), T)$ *does not belong to the high temperature region, and there exists some* $\tilde{\eta} = \tilde{\eta}(\eta)$ *such that the function* $\hat{H}_{n,T}(\cdot)$ *defined in (5.3) satisfies the inequality*

$$\hat{H}_{\bar{n}(T),T}\left(\tilde{\eta}^{-1}\right) \le 1/2, \quad \text{if} \quad T \le c_0 A_0/2, \tag{5.5}$$

i.e. for $T \le c_0 A_0/2$ *there is a ball with its center in the origin whose radius depends only on* η*, and whose* $\hat{H}_{\bar{n}(T),T}$ *measure is greater than* $1/2$*.*

Proof of Lemma 5.1. Let us introduce the function

$$\bar{h}_n(x,T) = \frac{1}{\sqrt{c^{(n)}}}\bar{q}_n\left(\frac{x}{\sqrt{c^{(n)}}}, T\right), \quad x \ge 0$$

with the function $\bar{q}_n$ introduced in (2.11). Observe that

$$\int_0^\infty \bar{h}_n(x,T)\,dx = 1.$$

Let us apply Proposition 4.2 with the choice $n = \bar{n}(T) - 1$. Since hypothesis $I(n)$ holds for $n = \bar{n}$, we obtain that

$$\begin{aligned} f_{\bar{n}(T)}(x,T) \quad &\le \quad \frac{K}{\beta^{1/2}_{\bar{n}(T)-1}(T)} \exp\left\{-\frac{1}{\sqrt{\beta_{\bar{n}(T)}(T)}}\left|2x + \frac{x^2}{c^{(\bar{n}(T))}M_{\bar{n}(T)}(T)}\right|\right\} \\ &\qquad \text{if } x > -c^{(\bar{n}(T))}M_{\bar{n}(T)}(T) \end{aligned}$$

with some universal constant $K > 0$. It follows from this relation that the function $\bar{h}_{\bar{n}(T)}(x,T) = \sqrt{c^{(\bar{n}(T))}} f_{\bar{n}(T)}\left(\sqrt{c^{(\bar{n}(T))}}x - c^{(\bar{n}(T))}M_n(T)), T\right)$ satisfies the inequality

$$\begin{aligned} \bar{h}_{\bar{n}(T)}(x,T) \quad &\le \quad K\left(\frac{c^{(\bar{n}(T))}}{\beta_{\bar{n}(T)-1}(T)}\right)^{1/2} \\ &\qquad \exp\left\{\frac{1}{\sqrt{\beta_{\bar{n}(T)}(T)}}\left(c^{(\bar{n}(T))}M_{\bar{n}(T)}(T) - \frac{x^2}{M_{\bar{n}(T)}(T)}\right)\right\}. \end{aligned}$$

The inequalities $\beta_{\bar{n}(T)}(T) > \eta c^{(\bar{n}(T))}$ and $\beta_{\bar{n}(T)-1}(T) \le \eta c^{(\bar{n}(T)-1)}$ hold.

Lemma 4.3 implies that the fractions $\frac{\beta_{\bar{n}(T)}(T)}{\beta_{\bar{n}(T)-1}(T)}$, $\frac{M_{\bar{n}(T)}(T)}{M_{\bar{n}(T)-1}(T)}$ and $\beta_{\bar{n}(T)}(T)M_{\bar{n}(T)}(T)^2$ are separated both from zero and infinity, hence

$$\frac{c^{(\bar{n}(T))}}{\beta_{\bar{n}(T)-1}(T)} \le \frac{\text{const.}}{\eta},$$

$$\frac{c^{(\bar{n}(T))}M_{\bar{n}(T)}(T)}{\sqrt{\beta_{\bar{n}(T)}(T)}} \le \frac{\text{const.}}{\eta},$$

and

$$\frac{1}{M_{\bar n(T)}(T)\sqrt{\beta_{\bar n(T)}(T)}} \ge \frac{1}{20}.$$

These inequalities together with the last relation imply that

$$\bar h_{\bar n(T)}(x,T) \le e^{\bar K/\eta} e^{-x^2/20} \quad x \ge 0, \tag{5.6}$$

with an appropriate $\bar K > 0$. Since

$$h_{\bar n(T)}(\mathbf{x},T) = C(T)\bar h_{\bar n(T)}(|\mathbf{x}|,T), \quad \mathbf{x} \in \mathbb{R}^r, \tag{5.7}$$

with an appropriate number $C(T) > 0$, estimate (5.4) can be deduced from (5.6) if we give a good upper bound for the constant $C(T)$ in (5.7). Observe that because of (5.7)

$$C(T)^{-1} = \int_{\mathbb{R}^r} \bar h_{\bar n(T)}(|\mathbf{x}|,T)\,d\mathbf{x} = \mathrm{Vol}(S^{r-1})\int_0^\infty x^{r-1}\bar h_{\bar n(T)}(x,T)\,dx, \tag{5.8}$$

hence

$$\begin{aligned} C(T)^{-1} &= \mathrm{Vol}(S^{r-1})\int_0^\infty x^{r-1}\bar h_{\bar n(T)}(x,T)\,dx \\ &\ge \mathrm{Vol}(S^{r-1})R^{r-1}\left(1-\int_0^R \bar h_{\bar n(T)}(x,T)\,dx\right) \end{aligned}$$

for any $R > 0$. On the other hand, by formula (5.6)

$$\int_0^R \bar h_{\bar n(T)}(x,T)\,dx \le \frac{1}{2}$$

if $0 < T \le c_0A_0/2$ and $R \le e^{-K/\eta}$ with a sufficiently large $K > 0$. Hence $C(T)^{-1} \ge \frac{1}{2}\mathrm{Vol}(S^{r-1})e^{-K(r-1)}$. This means that $C(T) \le e^{Kr/\eta}$ in (5.7), and inequality (5.4) follows from (5.6).

We shall prove that $\bar n(T)$ does not belong to the high temperature region with the help of the following estimate on $D^2_{\bar n(T)}(T)$. In its proof we shall apply formula (5.8).

$$\begin{aligned} D^2_{\bar n(T)}(T) &= \int_{\mathbb{R}^r} |\mathbf{x}|^2 h_{\bar n(T)}(\mathbf{x},T)\,dx = \mathrm{Vol}(S^{r-1})\int_0^\infty x^{r+1}h_{\bar n(T)}(x,T)\,dx \\ &= C(T)\mathrm{Vol}(S^{r-1})\int_0^\infty x^{r+1}\bar h_{\bar n(T)}(x,T)\,dx \\ &\ge C(T)\mathrm{Vol}(S^{r-1})\left(\int_0^\infty x^{r-1}\bar h_{\bar n(T)}(x,T)\,dx\right)^{(r+1/(r-1)} \\ &= \left(\int_0^\infty x^{r-1}\bar h_{\bar n(T)}(x,T)\,dx\right)^{2/(r-1)} \\ &\ge \left(\int_0^\infty x\bar h_{\bar n(T)}(x,T)\,dx\right)^2 = \left(M_{\bar n(T)}\sqrt{c^{(\bar n(T))}}\right)^2. \end{aligned}$$

Since $M_n^2 \ge \frac{10}{\beta_{n+1}}$ if $n \ge N+1$, $N+1$ is in the low temperature region if $T \le c_0A_0/2$, (see (3.3) and the subsequent sentence in our discussion), and $\frac{M_{\bar n(T)}}{M_{\bar n(T)-1}} \le$

const., hence

$$D^2_{\bar n(T)}(T) \ge M^2_{\bar n(T)} c^{(\bar n(T))} \ge \text{const.} \frac{c^{(\bar n(T))}}{\beta_{\bar n(T)}} \ge \frac{\text{const.}}{\eta}.$$

This implies that $\bar n(t)$ is not in the high temperature region. □

In the next Lemma 5.2 we shall formulate some properties of the function $h_n(\mathbf{x},T)$ defined in (3.4) in the case $n=0$. For the sake of a better discussion we define the function $\bar h_n(x,T)$, $x \ge 0$, by the formula $\bar h_n(x,T) = h_n(|\mathbf{x}|,T)$, and from now on $\bar h_n(x,T)$ means this function. (It differs slightly from the function $\bar h_n(x,T)$ applied in the proof of Lemma 5.1 where a different norming constant was applied.)

If $T \ge c_0A_0/2$, then $\bar n(T)=0$, and

$$\bar h_{\bar n(T)}(|\mathbf{x}|,T) = \text{const.}\, q_0\left((c^{(0)})^{-1/2}\mathbf{x}, T\right),$$

where $q_0(\mathbf{x},T)$ is defined in (2.10). Hence

$$\bar h_{\bar n(T)}(x,T) = \frac{1}{Z_0(T)} \exp\left\{\left(\frac{A_0c_0 - T}{c^{(0)}}\right)\frac{x^2}{2} - \kappa T^2 \frac{x^4}{4(c^{(0)})^2}\right\} \quad \text{if } T \ge c_0A_0/2 \tag{5.9}$$

with the norming constant (for the function $h_{\bar n(T)}(x,T)$)

$$Z_0(T) = \text{Vol}(S^{r-1}) \int_0^\infty x^{r-1} \exp\left\{\left(\frac{A_0c_0 - T}{c^{(0)}}\right)\frac{x^2}{2} - \kappa T^2 \frac{x^4}{4(c^{(0)})^2}\right\} dx. \tag{5.10}$$

Using formulas (5.9) and (5.10), we will prove the following

Lemma 5.2. *There is a constant $\kappa_0 = \kappa_0(N) > 0$ such that if $0 < \kappa < \kappa_0$, Condition 1 is satisfied, and $T \ge c_0A_0/2$, then $\bar n(T) = 0$, and*

$$\bar h_{\bar n(T)}(x,T) \le \textit{const.}\, T^{r/2} \exp\left\{\frac{1}{\kappa} - \frac{T}{2c^{(0)}}x^2\right\} \quad \textit{if } T \ge \frac{c_0A_0}{2}. \tag{5.11}$$

$$\bar h_{\bar n(T)}(x,T) \le \textit{const.}\, T^{r/2} \exp\left\{\frac{Tx^2}{2c^{(0)}}\right\} \quad \textit{if } T \ge \frac{c_0A_0}{2}. \tag{5.12}$$

$$\bar h_{\bar n(T)}(x,T) \le \textit{const.}\, e^{-Tx^2/4} \quad \textit{if } T \ge 10A_0, \textit{ and } x \ge T^{-1/3}. \tag{5.13}$$

The const. in formulas (5.11)—(5.13) depend only on the dimension r of the model.

The pair $(\bar n(T),T)$ belongs to the high temperature region if T is very large, e.g. if $T \ge e^{-1/\eta^9}$, and it does not belong to it if $T>0$ is relatively small, e.g. if $T \le \eta^{-100}$. If $(\bar n(T),T)$ does not belong to the high temperature region, then the function $h_{\bar n(T)}(\mathbf{x},T)$ defined in formula (3.4) satisfies the inequality

$$h_{\bar n(T)}(\mathbf{x},T) \le \exp\{K(\eta,\kappa) - \alpha|\mathbf{x}|^2\} \tag{5.14}$$

with a constant $\alpha = \alpha(\eta) > 0$ and an appropriate number $K(\eta,\kappa)$ depending only on κ and η. In this case there is a constant $B = B(\eta,\kappa) > 0$ in such a way that the quantity $\hat H_{\bar n(T),T}(\cdot)$ defined in (5.3) satisfies the inequality

$$\hat H_{\bar n(T),T}(B) \le \frac{1}{2}. \tag{5.15}$$

This means that if the pair $(\bar n(T),T)$ *is not in the high temperature region (and* $T\ge c_0A_0/2$*), then there is a radius* $B=B(\eta,\kappa)$ *such that the* $\hat H_{\bar n(T),T}$ *measure of the ball* $\{x\colon |x|\le B(\eta,\kappa)\}$ *is bigger than* $1/2$*.*

If $(\bar n(T),T)=(0,T)$ *is in the high temperature region, then*

$$\hat H_{\bar n(T),T}(x)\le K_1e^{-K_2\eta^2x^2}\quad \textit{for all } x>0 \tag{5.16}$$

with some universal constants $K_1>0$ *and* $K_2>0$*.*

Proof of Lemma 5.2. First we estimate the norming factor $Z_0(T)$ from below. Let us observe that

$$\frac{A_0c_0-T}{c^{(0)}}\frac{x^2}{2}-\kappa T^2\frac{x^4}{4(c^{(0)})^2}\ge -Tx^2\left(\frac{1}{2c^{(0)}}+\frac{1}{4(c^{(0)})^2}\right)\ge -10Tx^2,$$

if $\kappa Tx^2\le 1$ and Condition 1 holds. Hence

$$\begin{aligned}Z_0(T)\ &\ge\ \mathrm{Vol}\,(S^{r-1})\int_0^{1/\sqrt{\kappa T}}x^{r-1}e^{-10Tx^2}\,dx\\ &=\ \mathrm{Vol}\,(S^{r-1})\int_0^{1/\sqrt{\kappa}}\frac{x^{r-1}e^{-10x^2}}{T^{r/2}}\,dx\ge \mathrm{const.}T^{-r/2}.\end{aligned}\tag{5.17}$$

Now, if $T\ge c_0A_0/2$, then

$$\begin{aligned}\frac{A_0c_0-T}{c^{(0)}}\frac{x^2}{2}-\kappa T^2\frac{x^4}{4(c^{(0)})^2}\ &\le\ -\frac{1}{2}\frac{Tx^2}{c^{(0)}}+\max_{x\colon\, x\ge 0}\left(\frac{Tx^2}{c^{(0)}}-\frac{\kappa}{4}\left(\frac{Tx^2}{c^{(0)}}\right)^2\right)\\ &=\ -\frac{T}{2c^{(0)}}x^2+\frac{1}{\kappa},\end{aligned}$$

and combining this with (5.17), we obtain (5.11).

The estimate $\frac{A_0c_0-T}{c^{(0)}}\frac{x^2}{2}-\kappa T^2\frac{x^4}{4(c^{(0)})^2}\le\frac{Tx^2}{2c^{(0)}}$ yields (5.12).

If $T\ge 10A_0$, then

$$\frac{A_0c_0-T}{c^{(0)}}\frac{x^2}{2}-\kappa T^2\frac{x^4}{4(c^{(0)})^2}\le -\frac{T}{3}x^2\le -\frac{T}{4}x^2-\frac{T^{1/3}}{12},\quad \text{for } |x|\ge T^{-1/3},$$

which together with (5.17) imply inequality (5.13).

Furthermore, (5.13) implies that if $T>e^{-1/\eta^9}$, then the pair $(0,T)$ belongs to the high temperature region. Indeed, if we estimate the integral expressing $D_0^2(T)$, then by this relation the contribution of the domain $\{\mathbf{x}\colon\ |\mathbf{x}|\ge T^{-1/3}\}$ to this integral is very small. On the other hand, the contribution of the domain $\{\mathbf{x}\colon\ |\mathbf{x}|\le T^{-1/3}\}$ is less than $T^{-2/3}$ which is also very small in this case. To see that for $T<\eta^{-100}$ the pair $(0,T)$ does not belong to the high temperature domain it is enough to observe that in this case by formula (5.12) the $H_{0,T}$ measure of the ball $\{\mathbf{x}\colon |\mathbf{x}|\le\eta^{100}\}$ is less than const. $T^{r/2}\eta^{100r}\le 1/2$. Hence in this case $D_0^2(T)\ge\frac12\eta^{200}$. (We get this estimate by restricting the integral expressing $D_0^2(T)$ to the domain $\{\mathbf{x}\colon |\mathbf{x}|\ge\eta^{100}\}$). This means that T is not in the high temperature region.

Inequality (5.11) together with the fact that if the pair $(0,T)$ does not belong to the high temperature region then $T\le e^{-1/\eta^9}$ imply relations (5.14) and (5.15).

Since $T>\eta^{-100}$ if the pair $(0,T)$ is in the high temperature region, relation (5.13) implies relation (5.16). Lemma 5.2 is proved. $\square$

To formulate Lemmas 5.3 and 5.4 we rewrite formula (2.9) for the functions $\hat{h}_n(x,T)$ defined in (5.1). It has the form

$$\hat{h}_{n+1}(\mathbf{x},T)=\frac{1}{Z_n(T)}\int_{\mathbb{R}^r}\exp\left\{-\frac{c^{(n)}}{c^{(\bar{n}(T))}}\mathbf{u}^2\right\}\hat{h}_n(\mathbf{x}-\mathbf{u},T)\hat{h}_n(\mathbf{x}+\mathbf{u},T)\,d\mathbf{u} \tag{5.18}$$

with

$$Z_n(T)=\int_{\mathbb{R}^r\times\mathbb{R}^r}\exp\left\{-\frac{c^{(n)}}{c^{(\bar{n}(T))}}\mathbf{u}^2\right\}\hat{h}_n(\mathbf{x}-\mathbf{u},T)\hat{h}_n(\mathbf{x}+\mathbf{u},T)\,d\mathbf{u}\,d\mathbf{x} \tag{5.19}$$

for all $n\ge\bar{n}(T)$.

Let us also introduce the moment generating function of the measures $\hat{H}_{n,T}$, defined in (5.2):

$$\varphi_{n,T}(\mathbf{u})=\int_{\mathbb{R}^r}e^{\mathbf{ux}}\hat{h}_{n,T}(\mathbf{x})\,d\mathbf{x},\quad \mathbf{u}\in\mathbb{R}^r,$$

where $\mathbf{ux}$ denotes scalar product. By studying the properties of the moment generating function $\varphi_{n,T}(\mathbf{u})$, we get an upper bound for the function $\hat{H}_{n,T}(R)$ for large values R. Namely, we have the following result:

Lemma 5.3. *There exists some $\kappa_0=\kappa_0(N)$ such that for all $0<\kappa<\kappa_0$ and $0<\bar{\eta}<\eta$ (with the numbers N and η defined in (1.12)) the following relations hold. If we have such a positive integer L for which Conditions 1 and 5 (with $\bar{\eta}$ and this number L) are satisfied, then for all such temperatures $T>0$ for which the number $\bar{n}(T)$ exists, and the pair$(\bar{n}(T),T)$ does not belong to the high temperature region, the inequality*

$$\hat{H}_{\bar{n}(T)+l,T}(R)\le e^{-2^l\alpha R^2/5r}\quad \text{if } R\ge D \text{ and } 0\le l\le L \tag{5.20}$$

holds with appropriate constants $\alpha>0$ and $D>0$, and also the norming factor $Z_n(T)$ in (5.19) can be estimated as

$$Z_{\bar{n}(T)+l}(T)\ge D_1\quad \text{for } 0\le l\le L \tag{5.21}$$

with some constant $D_1>0$. These constants can be chosen as some functions of κ and $\bar{\eta}$, i.e. $\alpha=\alpha(\kappa,\bar{\eta})$, $D=D(\kappa,\bar{\eta})>0$ and $D_1=D_1(\kappa,\bar{\eta})>0$. This means in particular that they do not depend on the temperature T.

Proof of Lemma 5.3. It follows from formulas (5.4) and (5.14) that

$$\varphi_{\bar{n}(T),T}(\mathbf{u})\le\exp\left\{K_0+\frac{\mathbf{u}^2}{\alpha}\right\}\quad\text{for all } \mathbf{u}\in\mathbb{R}^r$$

with some $K_0=K_0(\eta,\kappa)>100$ and $\alpha=\alpha(\eta)>0$. It can be seen by induction with respect to l that

$$\varphi_{\bar{n}(T)+l,T}(\mathbf{u})\le\exp\left\{2^lK_l+\frac{\mathbf{u}^2}{2^l\alpha}\right\}\quad\text{for all } 0\le l\le L \text{ and } \mathbf{u}\in\mathbb{R}^r \tag{5.22}$$

with some $K_0>0$ and

$$K_l=K_{l-1}-\frac{\log Z_{\bar{n}(T)+l-1,T}}{2^l},\quad 1\le l\le L. \tag{5.23}$$

Indeed, the function $\hat h_{\bar n(T)+l+1,T}(\mathbf{x},T)$ is increased if the kernel term $\exp\left\{-\frac{c^{(n)}}{c^{(\bar n(T))}}\mathbf{u}^2\right\}$ is omitted from the integral in (5.18), and the integral turns into the convolution $2\hat h_{\bar n(T)+l,T}*\hat h_{\bar n(T)+l,T}(2\mathbf{x})$ after this change. By computing this convolution with the help of the inductive hypothesis and dividing it by $Z_{\bar n(T)+l+1}$ we get an upper bound for $\varphi_{\bar n(T)+l+1,T}(\mathbf{u})$. Formulas (5.22) and (5.23) follow from these calculations. We will prove formulas (5.20) and (5.21) from these relations by induction for l together with the inductive hypothesis that

$$K_l \le B \quad \text{for all } 0\le l\le L \tag{5.24}$$

with some constants $B>10$ depending only on κ and $\bar\eta$.

Observe that formula (5.22) with the choice of vectors of the form $(u,0)$, $u\in\mathbb{R}^1$, $u>0$, $0\in\mathbb{R}^{r-1}$. implies that the function $\hat H_{\bar n(T)+l,T}(R)$ defined in formulas (5.2) and (5.3) satisfies the inequality

$$\begin{aligned}\hat H_{\bar n(T)+l,T}(R) &\le r\hat H_{\bar n(T)+l,T}\left(\{\mathbf{x}=(x_1,\mathbf{x}_2)\in\mathbb{R}^r,\ x_1>\frac{R}{\sqrt r}\right)\\ &\le r\exp\left\{-\frac{uR}{\sqrt r}+2^lK_l+\frac{u^2}{2^l\alpha}\right\}\end{aligned}$$

for all real numbers u. In particular,

$$\hat H_{\bar n(T)+l,T}(R)\le r\exp\left\{2^l\left(K_l-\frac{R^2\alpha}{4r}\right)\right\} \tag{5.25}$$

with the choice $u=\frac{2^lR\alpha}{2\sqrt r}$. Hence

$$\hat H_{\bar n(T)+l,T}\left(\sqrt{\frac{4r(r+1)B}{\alpha}}\right)\le re^{-2^lrB}\le\frac12 \tag{5.26}$$

with the number $B>0$ appearing in (5.24). Formula (5.26) implies that

$$\hat H_{\bar n(T)+l,T}\left(\left\{\mathbf{x}\colon \mathbf{x}\in\mathbb{R}^r,\ |\mathbf{x}|\le\sqrt{\frac{4r(r+1)B}{\alpha}}\right\}\right)\ge\frac12.$$

For $\mathbf{z}\in\mathbb{R}^r$ and $u>0$ let $K(\mathbf{z},u)=\{\mathbf{x}\colon \mathbf{x}\in\mathbb{R}^r, |\mathbf{x}-\mathbf{z}|\le u\}$ denote the ball with center $\mathbf{z}$ and radius u. Since the ball $\left\{\mathbf{x}\colon\ \mathbf{x}\in\mathbb{R}^r,\ |x|\le\sqrt{\frac{4r(r+1)B}{\alpha}}\right\}$ can be covered by $C(r)B(\alpha\bar\eta)^{-1}$ balls of radius $\sqrt{\bar\eta}$, where $C(r)>0$ depends only on r, there is a ball $K(\mathbf{z},\sqrt{\bar\eta})$ of radius $\sqrt{\bar\eta}$ whose $\hat H_{n,T}$ measure (this measure was defined in (5.2)) is greater than $\frac{\alpha\bar\eta}{C(r)B}$. Hence

$$\hat H_{\bar n(T)+l,T}\times\hat H_{\bar n(T)+l,T}\left(K(\mathbf{z},\sqrt{\bar\eta})\times K(\mathbf{z},\sqrt{\bar\eta})\right)\ge\frac{\alpha^2\bar\eta^2}{(C(r)B)^2},$$

and because of Condition 5 the expression $Z_n(T)$ defined in (5.19) can be estimated for $n=\bar n(T)+l$ as follows:

$$\begin{aligned}Z_{\bar n(T)+l}(T)&\ge 2^{-r}\int_{\mathbf{x}\in K(\mathbf{z},\sqrt{\bar\eta}),\,\mathbf{u}\in K(\mathbf{z},\sqrt{\bar\eta})}\exp\left\{-\frac{c^{(\bar n(T)+l)}}{c^{\bar n(T)}}\frac{(\mathbf{x}-\mathbf{u})^2}{4}\right\}\\&\quad\times\hat h_{\bar n(T)+l,T}(\mathbf{x})\hat h_{\bar n(T)+l,T}(\mathbf{u})\,d\mathbf{x}\,d\mathbf{u}\\&\ge e^{-5}\hat H_{\bar n(T)+l,T}\times\hat H_{\bar n(T)+l,T}\left(K\left(\mathbf{z},\sqrt{\bar\eta}\right)\times K\left(\mathbf{z},\sqrt{\bar\eta}\right)\right)\\&\ge e^{-5}\frac{\alpha^2\bar\eta^2}{(C(r)B)^2}.\end{aligned}$$

In the above estimation we have exploited that because of Condition 5 and Condition 1 $\frac{c^{(\bar n(T)+l)}}{c^{\bar n(T)}}\le\frac{5}{\bar\eta}$, hence $\frac{c^{(\bar n(T)+l)}}{c^{\bar n(T)}}\frac{(\mathbf{x}-\mathbf{u})^2}{4}\le 5$ if $\mathbf{x}\in K\left(\mathbf{z},\sqrt{\bar\eta}\right)$ and $\mathbf{u}\in K\left(\mathbf{z},\sqrt{\bar\eta}\right)$.

The last relation implies (5.21) with $D_1=\frac{e^{-5}\alpha^2\bar\eta^2}{(C(r)B)^2}$, although we still have to show that the number D_1 (depending on B) can be bounded by a number which does not depend on the parameter T. We show with the help of (5.21), (5.23) and the inductive hypothesis (5.24) that $K_l\le(1-2^{-(l+1)})B$ if the number B is chosen as $B=\max(2K_0,K^*)$, where K^* is the larger solution of the equation $x=2\log\frac{x^2}{\bar D_1}$ with $\bar D_1=\frac{e^{-5}\alpha^2\bar\eta^2}{C(r)^2}$. This means that B in (5.24) can be chosen as a number not depending on T.

Indeed, this relation holds for $l=0$, and if it holds for $l-1$, then

$$K_l\le(1-2^{-(l+1)})B-2^{-(l+1}\left(B-2\log\frac{B^2}{\bar D_1}\right)\le(1-2^{-(l+1)})B$$

if $B\ge K^*$.

This implies (5.21) (with the constant D_1 not depending on T and the validity of the inductive hypothesis (5.24) for $0\le l\le L$. Finally, relation (5.20) follows from (5.24) and (5.25). Lemma 5.3 is proved. □

Formulas (5.18) and (5.19) can be rewritten for the function $\hat H_{n,T}(R)$ defined in (5.3) as

$$\begin{aligned}\hat H_{n+1,T}(R)&=\frac{2^r}{Z_n(T)}\int_{|\mathbf{x}|\ge R}\int_{\mathbf{u}\in\mathbb{R}^r}\exp\left\{-\frac{c^{(n)}}{c^{(\bar n(T))}}\mathbf{u}^2\right\}\\&\quad\times\hat h_n(\mathbf{x}-\mathbf{u},T)\hat h_n(\mathbf{x}+\mathbf{u},T)\,d\mathbf{u}\,d\mathbf{x}\\&=\frac{1}{Z_n(T)}\int_{\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|\ge R}\int_{\mathbf{u}\in\mathbb{R}^r}\exp\left\{-\frac{c^{(n)}}{c^{(\bar n(T))}}\frac{(\mathbf{x}-\mathbf{u})^2}{4}\right\}\\&\quad\times\hat H_{n,T}(\,d\mathbf{x})\hat H_{n,T}(\,d\mathbf{u})\end{aligned}\tag{5.27}$$

with

$$Z_n(T)=\int_{\mathbf{x}\in\mathbb{R}^r}\int_{\mathbf{u}\in\mathbb{R}^r}\exp\left\{-\frac{c^{(n)}}{c^{(\bar n(T))}}\frac{(\mathbf{x}-\mathbf{u})^2}{4}\right\}\hat H_{n,T}(\,d\mathbf{x})\hat H_{n,T}(\,d\mathbf{u}).$$

for all $R\ge0$. We apply these formulas in the proof of the following Lemma 5.4. The proof of Lemma 5.4 also exploits the rotational invariance of the measure $\hat H_{n,T}$.

Lemma 5.4. *Let the conditions of Lemma 5.3 hold. Then there exist some numbers $\delta = \delta(\bar\eta, D_1) > 0$ and $M = M(\bar\eta, D_1) > 0$ depending only on the numbers D_1 in formula (5.21) and $\bar\eta$ in Condition 5 in such a way that*

$$\hat H_{\bar n(T)+l+1,T}((1-\delta)R) \le \frac{1}{2}\hat H_{\bar n(T)+l,T}((1-\delta)R) + M\hat H_{\bar n(T)+l,T}(R)) \tag{5.28}$$
$$\textit{for all } R>0 \textit{ and } 0\le l\le L.$$

Proof of Lemma 5.4. Observe that

$$\left\{\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right| \ge (1-\delta)R\right\} \subset \{|\mathbf{x}|\ge R\}\cup\{|\mathbf{u}|\ge R\}$$
$$\cup\{|\mathbf{x}|\ge(1-\delta)R,\ \arg(\mathbf{x},\mathbf{u})\le\alpha\}\cup\{|\mathbf{u}|\le(1-\delta)R,\ \arg(\mathbf{x},\mathbf{u})\ge\alpha\}$$

for all $R>0$ and $0<\delta<1$ with $\alpha = 2\arccos(1-\delta)$. Indeed, if $\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right| \ge (1-\delta)R$, then either $|\mathbf{x}|>R$ or $|\mathbf{u}|>R$ or both $|\mathbf{x}|$ and $|\mathbf{u}|$ is less than R, but in this case either $|\mathbf{x}|>(1-\delta)R$ or $|\mathbf{u}|>(1-\delta)R$, and the angle between the vectors $\mathbf{x}$ and $\mathbf{u}$ must be small. On the other hand, because of the rotational invariance of the measure $\hat H_{n,T}$

$$\hat H_{\bar n(T)+l,T}\times\hat H_{\bar n(T)+l,T}\left(\{(\mathbf{x},\mathbf{y})\colon |\mathbf{x}|\ge(1-\delta)R,\ \arg(\mathbf{x},\mathbf{u})\le\alpha\}\right)$$
$$\le\frac{\alpha}{\pi}\hat H_{\bar n(T)+l,T}(\{\mathbf{x}\colon |\mathbf{x}|\ge(1-\delta)R\}) = \frac{\alpha}{\pi}\hat H_{\bar n(T)+l,T}((1-\delta)R).$$

The last two relations together with (5.27) and the inequality $\frac{\alpha}{\pi}\le\sqrt{\delta}$ imply that

$$\hat H_{\bar n(T)+l+1,T}((1-\delta)R) \le \frac{1}{Z_n(T)}\left(2\sqrt{\delta}\hat H_{\bar n(T)+l,T}((1-\delta)R) + 2\hat H_{\bar n(T)+l,T}(R)\right). \tag{5.29}$$

Relation (5.28) follows from (5.29) and (5.21) if we choose $\delta>0$ so small that the inequality $\frac{2\sqrt{\delta}}{D_1}\le\frac12$ holds. Lemma 5.4 is proved. □

Next we prove Theorem 3.1 with the help of the previous results.

Proof of Theorem 3.1. First we give a good estimate on $H_{\bar n(T)+l}(R)$ if the conditions of Lemma 5.3 hold with a sufficiently large $L = L(\kappa,\bar\eta)$ and $l\le L$ is sufficiently large. For this goal we introduce the following quantities.

Put $P(j,l) = P(j,l,T) = \hat H_{\bar n(T)+l}((1-\delta)^j D)$, $j=0,1,\dots,$ $0\le l\le L$ with the number D appearing in (5.20) and δ in Lemma 5.4. Clearly, $P(j,l)\le 1$ for all j and l. By Lemma 5.4

$$P(j,l+1)\le\frac12 P(j,l) + MP(j-1,l),\quad j\ge1, \tag{5.30}$$

and by relation (5.20) $P(0,l)\le e^{-\alpha 2^l D^2/5r}$ if $l\le L$. Hence there is a constant $k_0>0$ in such a way that $P(0,k_0+l)\le\left(\frac23\right)^l$ if $k_0+l\le L$. Because of this relation, the inequality $P(j,l)\le1$ and formula (5.30) there is a constant $k_1\ge k_0$ in such a way that $P(1,k_1+l)\le\frac{1}{3M}\left(\frac23\right)^l$ and $P(1,k_1+l)\le\left(\frac23\right)^l$ if $k_1+l\le L$. Similarly, there is a constant k_2 such that $P(2,k_2+l)\le\frac{1}{3M}\left(\frac23\right)^l$, and $P(2,k_2+l)\le\left(\frac23\right)^l$ if $k_2+l\le L$. This procedure can be continued, and we get a sequence $k_0\le k_1\le k_2\le\cdots$ in such a way that the inequality $P(p, k_p + l) \le \left(\frac{2}{3}\right)^l$ holds if $k_p + l \le L$. The numbers k_p depend only on the parameter κ in (1.3) and the number $\bar\eta$ in Condition 5. The above procedure can be continued till $k_p \le L$. In such a way we have proved that for all fixed $j \ge 0$

$$\hat H_{\bar n(T)+l}\left((1-\delta)^p D\right) \le C(p)\left(\frac{2}{3}\right)^l,$$

if $0 \le l \le L$. The above relation together with formula (5.20) imply that if Condition 5 holds with a sufficiently large constant $L = L(\bar\eta, t)$, then an integer $k > 0$ can be chosen independently of the parameter T in such a way that

$$\hat H_{\bar n(T)+l,T}(R) \le 2\exp\left\{-\frac{e^{1/\eta^3}R^2}{\bar\eta/5}\right\} \quad \text{for all } R > 0 \text{ and } k \le l \le L(\bar\eta, t). \tag{5.31}$$

Indeed, by (5.20) relation (5.31) holds for $R \ge D$ if $l \ge k_0'$ with some $k_0' > 0$ and by the previous inequality for all $j = 1, 2, \dots$ it also holds for $(1-\delta)^j D \le R < (1-j)^{j-1}D$ if $l \ge k'j$ with a sufficiently large $k'j \ge$. On the other hand, it is enough to demand this inequality for finitely many indices j, since relation (5.31) automatically holds if $\frac{e^{1/\eta^3}R^2}{\bar\eta/5} \le \log 2$.

Since the measure $H_{n,T}$ defined in (3.5) satisfies the relation

$$H_{\bar n(T)+l,T}\{\mathbf{x}\colon |\mathbf{x}| > R\} = \hat H_{\bar n(T)+l,T}\left(\sqrt{\frac{c^{(\bar n(T))}}{c^{(\bar n(T)+l)}}}R\right) \le \hat H_{\bar n(T)+l,T}\left(\sqrt{\frac{\bar\eta}{5}}R\right)$$

relation (5.31) implies that

$$H_{\bar n(T)+l,T}(R) \le 2\exp\left\{-e^{1/\eta^3}R^2\right\} \quad \text{for all } R > 0, \text{ and } l^* \le l \le L \tag{5.32}$$

with some appropriate $l^* \ge 0$. Relation (5.32) implies in particular that $D^2_{\bar n(T)+l}(T) < e^{-1/\eta^2}$, i.e. $(\bar n(T)) + l, T)$ is in the high temperature region if $l^* \le l \le L$. The inequality $D^2_{\bar n(T)}(T) < K$ follows from (5.20) with $l = 0$.

To complete the proof of Theorem 3.1 we have to give a lower bound for $D^2_{\bar n(T)+k}(T)$. Let us introduce the following notation: Given two positive numbers $R_2 > R_1 > 0$ let $\mathbf{K}(R_1, R_2) = \{\mathbf{x}\colon \mathbf{x} \in \mathbb{R}^r,\ R_1 \le |x| \le R_2\}$ denote the annulus between the concentrical balls with center in the origin and radii R_1 and R_2. We claim that for any $0 \le l \le L$ there exist some positive numbers $R_1(l) = R_1(l, \bar\eta, \kappa)$, $R_2(l) = R_2(l, \bar\eta, \kappa)$ and $A(l) = A(l, \bar\eta, \kappa) > 0$ such that the measure of the annulus determined by these numbers satisfies the inequality

$$\hat H_{\bar n(T)+l,T}(\mathbf{K}(R_1(l), R_2(l)) \ge A(l), \quad 0 \le l \le L \tag{5.33}$$

if the pair $(\bar n(T), T)$ does not belong to the high temperature region. Observe that the relation between the measures $\hat H_{\bar n(T)+l,T}$ and $H_{\bar n(T)+l,T}$ implies that relation (5.33) also holds for $H_{\bar n(T)+l,T}(\mathbf{K}(R_1(k), R_2(k))$ (i.e. the function $\hat H(\cdot)$ can be replaced by $H(\cdot)$ in formula (5.33)) if the radii $R_2(k)$ and $R_1(k) > 0$ are multiplied with an appropriate number. This implies that the variance $D^2_{\bar n(T)+k,T}$ can be bounded from below by a positive number which depends only on k and $\bar\eta$. Hence we can choose e.g. $k = L = L(\kappa, \bar\eta)$ as the number k satisfying the properties demanded in Theorem 3.1.

We shall prove a slightly stronger statement than relation (5.33) which will be useful in later applications. We shall prove that

$$\hat H_{\bar n(T)+l,T}\left(\mathbf{K}\left(\frac{1}{2^l}R_1, \left(\frac{\sqrt3}{2}\right)^l R_2\right)\right) \ge A(l), \quad 0 \le l \le L, \tag{5.34}$$

with some numbers $R_2 > R_1 > 0$ and $A(l) > 0$ if the pair $(0, T)$ does not belong to the high temperature region. The numbers R_j can be chosen in such a way that $R_j = R_j(\eta, \kappa)$, $j = 1, 2$.

Let us first observe that relation (5.34) holds for $l = 0$ if $\bar n(T)$ is not in the high temperature region. This follows from relations (5.4) and (5.5) in the case $T \le c_0A_0/2$ and from (5.11) and (5.15) if $T \ge c_0A_0/2$, but $(\bar n(T), T)$ does not belong to the high temperature region. Indeed, formulas (5.5) and (5.15) make possible to choose the number R_2 in such a way that the $H_{\bar n(T),T}$ measure of the ball with center in the origin and radius $R_2 = R_2(\eta)$ is greater than $1/2$. By formulas (5.4) and (5.11) we can choose the number $R_1 = R_1(\eta)$ in such a way that by cutting out from this ball the ball with radius R_2 and center in the origin the remaining annulus $\mathbf{K}(R_1, R_2)$ has a measure greater than $1/4$. In the case $T \ge c_0A/2$ we also exploited in the above argument that T cannot be very large if $(\bar n(T), T)$ is not in the high temperature region. By Lemma 5.2 $T \le e^{-1/\eta^9}$ in this case.

We claim that

$$\hat H_{\bar n(T)+l+1,T}\left(\mathbf{K}\left(\frac{\bar R_1}{2}, \frac{\sqrt3}{2}\bar R_2\right)\right) \ge B(\bar R_1, \bar R_2, \bar\eta)\hat H_{\bar n(T)+l,T}(\mathbf{K}(\bar R_1, \bar R_2))^2 \tag{5.35}$$

for all $0 \le l \le L$ and $\bar R_2 > \bar R_1 > 0$ and an appropriate constant

$$B(\bar R_1, \bar R_2, \bar\eta) > 0.$$

Relation (5.34) follows from (5.35) and the fact that it holds for $l = 0$.

In the proof of relation (5.35) we exploit the relation

$$\left\{(\mathbf{u},\mathbf{x})\colon \mathbf{u}\in\mathbb{R}^r, \mathbf{x}\in\mathbb{R}^r, \frac{\bar R_1}{2} \le \left|\frac{\mathbf{x}+\mathbf{u}}{2}\right| \le \frac{\sqrt3}{2}\bar R_2,\ \frac{\pi}{3} \le \arg(\mathbf{x},\mathbf{u}) \le \frac{\pi}{2}\right\}$$
$$\supset \left\{(\mathbf{u},\mathbf{x})\colon \mathbf{u}\in\mathbb{R}^r, \mathbf{x}\in\mathbb{R}^r,\ \bar R_1 \le |\mathbf{x}|, |\mathbf{u}| \le \bar R_2,\ \frac{\pi}{3} \le \arg(\mathbf{x},\mathbf{u}) \le \frac{\pi}{2}\right\}.$$

It follows from relation (5.19) that $Z_{\bar n(T)+l+1}(T) \le 1$, since we get an upper bound for it by omitting the kernel term $\exp\left\{-\frac{c^{(n)}}{c^{(\bar n(T))}}\mathbf{u}^2\right\}$ from the integral in (5.19). Hence the previous relation together with (5.27) and the rotational invariance of the measure $\hat H_{\bar n(T)+l,T}$ yield that

$$\hat H_{\bar n(T)+l+1,T}\left(\mathbf{K}\left(\frac{\bar R_1}{2}, \frac{\sqrt3}{2}\bar R_2\right)\right) = \frac{1}{Z_{\bar n(T)+l+1}(T)}\int\int_{\frac{\sqrt3}{2}\bar R_2 \ge \left|\frac{\mathbf{x}+\mathbf{u}}{2}\right| \ge \frac{\bar R_1}{2},\, \mathbf{x},\mathbf{u}\in\mathbb{R}^r}$$
$$\exp\left\{-\frac{c^{(\bar n(T)+l)}}{\bar c^{(\bar n(T))}}\frac{(\mathbf{x}-\mathbf{u})^2}{4}\right\}\hat H_{\bar n(T)+l,T}(\,d\mathbf{x})\hat H_{\bar n(T)+l,T}(\,d\mathbf{u})$$

$$\geq e^{-\bar{R}_2^2/\bar{\eta}} \int\int_{\substack{\frac{\sqrt{3}}{2}\bar{R}_2 \geq \left|\frac{\mathbf{x}+\mathbf{u}}{2}\right| \geq \frac{\bar{R}_1}{2},\ \mathbf{x},\mathbf{u}\in\mathbb{R}^2,\\ \frac{\pi}{3}\leq \arg(\mathbf{x},\mathbf{u})\leq\frac{\pi}{2}}} \hat{H}_{\bar{n}(T)+l,T}(\,dx)\hat{H}_{\bar{n}(T)+l,T}(\,d\mathbf{u})$$

$$\geq e^{-\bar{R}_2^2/\bar{\eta}} \int\int_{\bar{R}_2\geq|\mathbf{x}|,|\mathbf{u}|\geq\bar{R}_1,\frac{\pi}{3}\leq\arg(\mathbf{x},\mathbf{u})\leq\frac{\pi}{2}} \hat{H}_{\bar{n}(T)+l,T}(\,d\mathbf{x})\hat{H}_{\bar{n}(T)+l,T}(\,d\mathbf{u})$$

$$= C(r)e^{-\bar{R}_2^2/\bar{\eta}}\hat{H}_{\bar{n}(T)+l,T}(\mathbf{K}(\bar{R}_1,\bar{R}_2))^2$$

with an appropriate constant $C(r) > 0$. The last estimate implies relation (5.35) with $B(\bar{R}_1,\bar{R}_2,\bar{\eta}) = C(r)e^{-\bar{R}_2^2/\bar{\eta}}$. Theorem 3.1 is proved. □

6. Estimates in the high temperature region. The proof of theorem 3.3

To study the behaviour of the function $f_n(x,T)$ in the high temperature region we need a starting index $n = \tilde{n}(T)$ for which a good estimate is known about the tail behaviour of the measure $H_{\tilde{n}(T),T}$. We also need a lower bound for the variance $D_n^2(T)$ defined in (3.4) for $n \geq \tilde{n}(T)$. This requirement will be also taken into consideration in the definition of $\tilde{n}(T)$. Let us first define the number

$$l_0 = l_0(T) = \min\left\{ l\colon \left(\frac{\sqrt{3}}{2}\right)^l R_2 \leq \frac{\bar{\eta}}{10}\right\} \tag{6.1}$$

if the pair $(0,T)$ is not in the high temperature region, where $\bar{\eta}$ appeared in Condition 5, and the number R_2 was introduced in formula (5.34). Now define

$$\tilde{n}(T) = \begin{cases} 0 \quad \text{if } (0,T) \text{ is in the high temperature region} \\ \bar{n}(T)+l \text{ with the smallest } l \text{ satisfying both (5.32) and the} \\ \qquad\qquad \text{inequality } l\geq l_0 \text{ with } l_0 \text{ defined in (6.1)} \\ \qquad\qquad \text{if } (0,T) \text{ is not in the high temperature region.}\end{cases} \tag{6.2}$$

It follows from the results of the previous section that for a temperature T which is not in the low temperature region the inequality $0 \leq \tilde{n}(T) - \bar{n}(T) \leq L(\bar{\eta},t)$ holds if the number L in Condition 5 is chosen sufficiently large. The measure $H_{\tilde{n}(T),T}$ introduced in formula (3.5) is strongly concentrated around the origin. Indeed, formulas (5.16) and (5.32) give a good estimate for the $H_{\tilde{n}(T),T}$ measure of the sets $\{\mathbf{x}\colon |\mathbf{x}| \leq R\}$ for all $R \geq 0$.

Let us introduce the moments of the functions $h_{\tilde{n}(T)+l}(\mathbf{x},T)$ defined in (3.4).

$$M_k(l,T) = \int_{\mathbb{R}^r} |\mathbf{x}|^k h_{\tilde{n}(T)+l}(\mathbf{x},T)\,d\mathbf{x} \quad l\geq 0,\ k\geq 1.$$

We shall estimate the moments $M_2(l,T)$ and $M_4(l,T)$. It follows from relations (5.16) and (5.32) that

$$M_2(0,T) \leq \eta^* \quad \text{and} \quad M_4(0,T)\leq \eta^* \quad \text{with } \eta^* = e^{-1/\eta^2} \tag{6.3}$$

for all $T > 0$ which is not in the low temperature region. To get lower bounds for the second moments $M_2(l,T)$ let us introduce the truncated second moments

$$M_{2,\text{tr.}}(l,T) = M_{2,\text{tr.}}\left(\frac{1}{10},l,T\right) = \int_{|\mathbf{x}|\leq\frac{1}{10}} |\mathbf{x}|^2 h_{\tilde{n}(T)+l}(\mathbf{x},T)\,d\mathbf{x}.$$

It follows from (5.13) if $(0,T)$ is in the high temperature region and from (5.34) and the definition of $\tilde n(T)$ if $(0,T)$ is not in the high temperature region that

$$\begin{aligned} &M_{2,\mathrm{tr.}}(0,T)>0, && \text{for all } T\ge c_0A_0/2\\ &M_{2,\mathrm{tr.}}(0,T)>\tilde\eta, && \text{if } T\ge c_0A_0/2 \text{ and } (0,T) \text{ is not in the}\\ & && \text{high temperature region} \end{aligned} \tag{6.4}$$

with some $\tilde\eta=\tilde\eta(\bar\eta,\kappa)>0$. First we shall bound $M_2(l,T)$ and $M_4(l,T)$ from above in Lemma (6.1) for all $l\ge0$. Then the second moment $M_2(l,T)$ will be bounded from below in Lemma 6.2. These estimates enable us to prove the central limit theorem for $g_{\tilde n(T)+l}(x,T)$ by means of the characteristic function technique.

Simple calculation yields that

$$\begin{aligned} M_k(l+1,T) &= \frac{2^r}{Z_l(T)}\int e^{-\mathbf{u}^2}|\mathbf{x}|^k h_{\tilde n(T)+l}\left(\frac{\mathbf{x}}{\sqrt{\bar c_{\tilde n(T)+l+1}}}-\mathbf{u},T\right)\\ &\qquad h_{\tilde n(T)+l}\left(\frac{\mathbf{x}}{\sqrt{\bar c_{\tilde n(T)+l+1}}}+\mathbf{u},T\right)d\mathbf{x}\,d\mathbf{u} \end{aligned} \tag{6.5}$$

for all $l\ge0$ and $k\ge1$ with

$$\begin{aligned} Z_l(T) &= 2^r\int e^{-\mathbf{u}^2} h_{\tilde n(T)+l}\left(\frac{\mathbf{x}}{\sqrt{\bar c_{\tilde n(T)+l+1}}}-\mathbf{u},T\right)\\ &\qquad h_{\tilde n(T)+l}\left(\frac{\mathbf{x}}{\sqrt{\bar c_{\tilde n(T)+l+1}}}+\mathbf{u},T\right)d\mathbf{x}\,d\mathbf{u} \end{aligned} \tag{6.6}$$

with the constants $\bar c_n$, $n=1,2,\dots$ defined in (2.17). These formulas will be used in the proof of the following

Lemma 6.1. *Under the conditions of Theorem 3.3 the inequalities*

$$M_2(l,T) \le \eta^*\left(\frac23\right)^l, \tag{6.7}$$

$$Z_l(T) \ge \left(\bar c_{\tilde n(T)+l}\right)^{r/2}\left(1-8\sqrt{\eta^*}\left(\frac56\right)^l\right), \tag{6.8}$$

$$M_2(l+1,T) \le \frac{\bar c_{\tilde n(T)+l+1}}{2}\left(1+10\sqrt{\eta^*}\left(\frac56\right)^l\right)M_2(l,T), \tag{6.9}$$

$$M_2(l,T)\le 2\cdot 2^{-l}\frac{c^{(\tilde n(T)+l)}}{c^{(\tilde n(T))}}\eta^* \quad \textit{and} \quad M_4(l,T)\le 5\cdot 4^{-l}\left(\frac{c^{(\tilde n(T)+l)}}{c^{(\tilde n(T))}}\right)^2\eta^* \tag{6.10}$$

hold for all $l\ge0$ with the same number η^* which appears in (6.3).

Proof of Lemma 6.1. Relation (6.7) holds for $l=0$ by relation (6.3). We shall prove that if relation (6.7) holds for an integer l, then relations (6.8) and (6.9) also hold for this l. Then we prove that if relations (6.7) and (6.9) hold for some l, then relation (6.7) holds also for $l+1$. These statements imply relations (6.7), (6.8) and (6.9). We prove them with the help of the following calculations.

It follows from formulas (6.5) and (6.6) that

$$
\begin{aligned}
(6.11)\qquad M_k(l+1,T) &= \frac{\left(\bar c_{\tilde n(T)+l+1}\right)^{r/2}}{Z_l(T)}\int \exp\left\{-\frac{(\mathbf{x}-\mathbf{u})^2}{4}\right\}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|^k \\
&\qquad \left(\bar c_{\tilde n(T)+l+1}\right)^{k/2} h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\,(u,T)\; d\mathbf{x}\,d\mathbf{u} \\
&\le \frac{\left(\bar c_{\tilde n(T)+l+1}\right)^{(k+r)/2}}{2^k Z_l(T)}\int |\mathbf{x}+\mathbf{u}|^k h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\,(\mathbf{u},T)\; d\mathbf{x}\,d\mathbf{u}
\end{aligned}
$$

for all $l\ge 0$ and $k\ge 1$, and

$$
\begin{aligned}
Z_l(T) &= (\bar c_{\tilde n(T)+l+1})^{r/2}\int \exp\left\{-\frac{(\mathbf{x}-\mathbf{u})^2}{2}\right\} \\
&\qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\,(\mathbf{u},T)\; d\mathbf{x}\,d\mathbf{u} \\
&\ge (\bar c_{\tilde n(T)+l+1})^{r/2}e^{-4\sqrt{M_2(l,T)}}\int_{|\mathbf{x}|\le M_2(l,T)^{1/4},\,|\mathbf{u}|\le M_2(l,T)^{1/4}} \\
&\qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\,(\mathbf{u},T)\; d\mathbf{x}\,d\mathbf{u} \\
&\ge (\bar c_{\tilde n(T)+l+1})^{r/2}e^{-4\sqrt{M_2(l,T)}}\left(1-2\int_{|\mathbf{x}|\ge M_2(l,T)^{1/4}} h_{\tilde n(T)+l}(\mathbf{x},T)d\mathbf{x}\right) \\
&\ge (\bar c_{\tilde n(T)+l+1})^{r/2}e^{-4\sqrt{M_2(l,T)}} \\
&\qquad \left(1-\frac{2}{\sqrt{M_2(l,T}}\int_{|\mathbf{x}|\ge M_2(l,T)^{1/4}} x^2 h_{\tilde n(T)+l}(\mathbf{x},T)\,d\mathbf{x}\right).
\end{aligned}
$$

Hence

$$
Z_l(T)\ge (\bar c_{\tilde n(T)+l+1})^{r/2}e^{-4\sqrt{M_2(l,T)}}\left(1-2\sqrt{M_2(l,T)}\right).
$$

The last relation and formula (6.7) for l together imply that

$$
\begin{aligned}
Z_l(T) &\ge (\bar c_{\tilde n(T)+l+1})^{r/2}\left(1-5\sqrt{M_2(l,T)}\right)\left(1-2\sqrt{M_2(l,T)}\right) \\
&\ge (\bar c_{\tilde n(T)+l+1})^{r/2}\left(1-8\sqrt{M_2(l,T)}\right) \\
&\ge (\bar c_{\tilde n(T)+l+1})^{r/2}\left(1-8\sqrt{\eta^*}\left(\frac{5}{6}\right)^l\right),
\end{aligned}
$$

and this is relation (6.8) for the number l. Relation (6.11) for $k=2$ and formula (6.8) for l together yield that

$$
\begin{aligned}
M_2(l+1,T) &\le \frac{\left(\bar c_{\tilde n(T)+l+1}\right)^{(2+r)/2}}{2Z_l(T)}M_2(l,T) \\
&\le \frac{\bar c_{\tilde n(T)+l+1}}{2}\left(1+10\sqrt{\eta^*}\left(\frac{5}{6}\right)^l\right)M_2(l,T),
\end{aligned}
$$

and this is formula (6.9) for l. Finally, if η is chosen sufficiently small, then formulas (6.7) and (6.9) for l imply (6.7) for $l+1$. Thus formulas (6.7) — (6.9) are proved.

The first relation in (6.10) follows from the first relation in (6.3) and (6.9). Formula (6.11) with the choice $k=4$, (6.8) and the first formula in (6.10) imply that

$$\begin{aligned}
M_4(l+1,T) &\le \frac{(\bar c_{\tilde n(T)+l+1})^{(r+4)/2}}{8Z_l(T)}\left(3M_2(l,T)^2+M_4(l,T)\right)\\
&\le \frac{1}{8}\bar c^2_{\tilde n(T)+l+1}\left(1+10\sqrt{\eta^*}\left(\frac{5}{6}\right)^l\right)\left(3M_2(l,T)^2+M_4(l,T)\right)\\
&\le 2\cdot 4^{-l}\left(\frac{\bar c^{(\tilde n(T)+l+1)}}{c^{(\tilde n(T))}}\right)^2\eta^{*2}+\frac{M_4(l,T)}{6}.
\end{aligned}$$

The second relation in (6.10) follows by induction from the last inequality and the second inequality in (6.3). Lemma 6.1 is proved. □

Remark. The Corollary formulated after Theorem 3.1 follows from Theorem 3.1, formula (6.10) and Lemma 4.4. Indeed, if T is not in the low temperature, then by Theorem 3.1 the pair $(\tilde n(T),T)$ with the definition of $\tilde n(T)$ given in (6.1) is in the high temperature domain. By formula (6.10) all pairs (n,T), $n\ge\tilde n(T)$, are in the high temperature region, i.e. if $T>0$ is not in the low temperature region, then it is in the high temperature region. The remaining statements of the Corollary are contained in Lemma 4.4.

In the next lemma we prove an estimate from below for $M_2(l,T)$.

Lemma 6.2. *Put*

$$\sigma^2(l,T)=2^l\frac{c^{(\tilde n(T))}}{c^{(\tilde n(T)+l)}}M_2(l,T),\qquad l\ge 0.$$

Under the conditions of Theorem 3.3 the limit

$$\bar\sigma^2(T)=\lim_{l\to\infty}\sigma^2(l,T)>0$$

exists, and it is positive for all $T>0$. If $\tilde n(T)\ne 0$, i.e. if $(0,T)$ is not in the high temperature region, then there exist two constants $C_2>C_1>0$ depending only on the parameter $\tilde\eta$ in formula (6.4) in such a way that the inequalities

$$C_1\le\bar\sigma^2(T)\le C_2 \tag{6.12}$$

hold. The upper bound in (6.12) holds for all $T>0$ which is not in the low temperature region.

Proof of Lemma 6.2. The hard part of the proof is to show that $\sigma^2(l,T)$ has a nonnegative lim inf. It follows simply from formula (6.6) that $Z_l(T)\le(\bar c_{\tilde n(T)+l+1})^{r/2}$. A natural lower bound for $M_2(l,T)$ can be obtained in the following way. By formula (6.5) and the upper bound for $Z_l(T)$

$$\begin{aligned}
M_2(l+1,T) &\ge \bar c_{\tilde n(T)+l+1}\int e^{-(\mathbf{x}-\mathbf{u})^2/4}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|^2\\
&\qquad\qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}(\mathbf{u},T)\,d\mathbf{x}\,d\mathbf{u}\\
&= \frac{\bar c_{\tilde n(T)+l+1}}{4}\left(2M_2(l,T)-\int|\mathbf{x}+\mathbf{u}|^2\left(1-e^{-(\mathbf{x}-\mathbf{y})^2/4}\right)\right.
\end{aligned}$$

$$
\begin{aligned}
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u}\Big) \\
\geq\;& \frac{\bar c_{\tilde n(T)+l+1}}{2}\Big(M_2(l,T)-\int\frac14|\mathbf{x}+\mathbf{u}|^2|\mathbf{x}-\mathbf{u}|^2 \\
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u}\Big) \\
\geq\;& \frac{\bar c_{\tilde n(T)+l+1}}{2}\Big(M_2(l,T)-\int\frac12(|\mathbf{x}|^4+|\mathbf{u}|^4) \\
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u}\Big) \\
=\;& \frac{\bar c_{\tilde n(T)+l+1}}{2}\left(M_2(l,T)-M_4(l,T)\right). \qquad (6.13)
\end{aligned}
$$

However, this estimate is useful only if we know that the right-hand side in it is non-negative. We do not know such an estimate for small l, hence in this case we apply a different argument. Clearly

$$M_2(l,T)\geq M_{2,\mathrm{tr.}}(l,T),$$

where $M_{2,\mathrm{tr.}}(l,T)$ is the truncated moment. On the other hand, we get by using an argument similar to the previous calculation and making the observation

$$
\begin{aligned}
&\left\{(\mathbf{x},\mathbf{u})\colon \mathbf{x}\in\mathbb{R}^r,\mathbf{u}\in\mathbb{R}^r,\bar c_{\tilde n(T)+l+1}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|\leq\frac{1}{10}\right\} \\
&\qquad\supset\left\{(\mathbf{x},\mathbf{u})\colon \mathbf{x}\in\mathbb{R}^r,\mathbf{u}\in\mathbb{R}^2,|\mathbf{x}|\leq\frac{1}{10},|\mathbf{u}|\leq\frac{1}{10},\arg\left(\mathbf{x},\mathbf{u}\right)\subset I\right\}
\end{aligned}
$$

with $I=\left(\frac{\pi}{50},\frac{49\pi}{50}\right)\cup\left(\frac{51\pi}{50},\frac{99\pi}{50}\right)$ that

$$
\begin{aligned}
M_{2,\mathrm{tr.}}(l+1,T)\;\geq\;& \bar c_{\tilde n(T)+l+1}\int_{\bar c_{\tilde n(T)+l+1}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|\leq\frac{1}{10}} e^{-(\mathbf{x}-\mathbf{u})^2/4}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|^2 \\
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u} \\
\geq\;& \bar c_{\tilde n(T)+l+1}e^{-1/100}\int_{|\mathbf{x}|\leq\frac{1}{10},|\mathbf{u}|\leq\frac{1}{10},\arg(\mathbf{x},\mathbf{u})\subset I}\left|\frac{\mathbf{x}+\mathbf{u}}{2}\right|^2 \\
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u} \\
=\;& \frac{\bar c_{\tilde n(T)+l+1}}{4}e^{-1/100}\int_{|\mathbf{x}|\leq\frac{1}{10},|\mathbf{u}|\leq\frac{1}{10},\arg(\mathbf{x},\mathbf{u})\subset I}(\mathbf{x}^2+\mathbf{u}^2) \\
& \qquad h_{\tilde n(T)+l}(\mathbf{x},T)h_{\tilde n(T)+l}\left(\mathbf{u},T\right)\,d\mathbf{x}\,d\mathbf{u} \\
=\;& \bar c_{\tilde n(T)+l+1}e^{-1/100}\frac{12}{25}M_{2,\mathrm{tr.}}(l,T)\geq\frac13\bar c_{\tilde n(T)+l+1}M_{2,\mathrm{tr.}}(l,T).
\end{aligned}
$$

The last estimate implies that
(6.14)

$$\sigma^2(l,T)=2^l\frac{c^{(\tilde n(T))}}{c^{(\tilde n(T)+l)}}M_2(l,T)\geq 2^l\frac{c^{(\tilde n(T))}}{c^{(\tilde n(T)+l)}}M_{2,\mathrm{tr.}}(l,T)\geq\left(\frac23\right)^l M_{2,\mathrm{tr.}}(0,T).$$

On the other hand, it follows from (6.13) and the second inequality in (6.10) that

$$\begin{aligned} \sigma^2(l+1,T) &\ge \sigma_2(l,T) - 2^{l+1}\frac{c^{(\tilde n(T))}}{c^{\tilde n(T)+l+1}} M_4(l,T) \\ &\ge \sigma^2(l,T) - \frac{5\eta^*}{2^l c_{\tilde n(T)+l+1}} \frac{c^{(\tilde n(T)+l)}}{c^{(\tilde n(T))}} \ge \sigma^2(l,T) - 50\eta^* \left(\frac{3}{4}\right)^l . \end{aligned} \tag{6.15}$$

Because of (6.14) and (6.4) an index $\bar l \ge 0$ can be chosen in such a way that

$$\sigma^2(\bar l, T) \ge 1000\eta^* \left(\frac{3}{4}\right)^{\bar l},$$

and if the pair $(0,T)$ is not in the high temperature region, then we may choose $\bar l$ so that $\bar l \le K(\bar\eta,\kappa)$ with some appropriate $K(\bar\eta,\kappa)$. Hence relation (6.15) implies that

$$\frac{\sigma^2(\bar l+l+1,T)}{\sigma^2(\bar l,T)} \ge \frac{\sigma^2(\bar l+l,T)}{\sigma^2(\bar l,T)} - \frac{1}{20}\left(\frac{3}{4}\right)^l .$$

This relation and the bound on $\sigma^2(\bar l,T)$ imply that $\liminf\limits_{l\to\infty} \sigma^2(l,T) > 0$, and this $\liminf$ can be bounded by a positive number which depends only on $\bar\eta$ and κ if $(0,T)$ is not in the high temperature region. The analogous result for $\limsup$ follows from (6.9). To complete the proof it is enough to show that the $\liminf$ is actually $\lim$. To prove this let us observe that for any $\varepsilon > 0$ and $N > 0$ there is some $m > N$ such that $\sigma^2(m,T) < \liminf\limits_{n\to\infty} \sigma^2(n,T) + \varepsilon$. Then by formula (6.9)

$$\begin{aligned} \sigma^2(n,T) &\le \sigma^2(m,T) \prod_{l=m}^{n} \left(1 + 10\sqrt{\eta^*}\left(\frac{5}{6}\right)^l\right) \\ &\le \liminf_{n\to\infty} \sigma^2(n,T) + 2\varepsilon, \quad n > m \end{aligned}$$

for any $\varepsilon > 0$ if $N = N(\varepsilon)$ is chosen sufficiently large. Lemma 6.2 is proven. □

To prove Theorem 3.3 let us introduce the characteristic functions

$$\varphi_n(s,T) = \int_{\mathbb{R}^r} e^{is\mathbf{x}} \tilde h_n(\mathbf{x},T)\, d\mathbf{x}, \quad s \in \mathbb{R}^r$$

and moments

$$\tilde M_k(n,T) = \int_{\mathbb{R}^r} |\mathbf{x}|^k \tilde h_n(\mathbf{x},T)\, d\mathbf{x},$$

where the function $\tilde h(\mathbf{x},T)$ was defined in (3.17). Clearly,

$$\tilde M_k(n,T) = \left(\frac{2^n}{c^{(n)}}\right)^{k/2} M_k(n-\tilde n(T),T) \quad \text{if } n \ge \tilde n(T).$$

In particular, $\tilde M_2(n,T) = \frac{2^{\tilde n(T)}}{c^{(\tilde n(T))}} \sigma^2(n-\tilde n(T),T)$. We shall prove Theorem 3.3 by means of the usual characteristic function technique. The following lemma plays a crucial role in the proof.

Lemma 6.3. *Under the conditions of Theorem 3.3 the relation*

$$\lim_{n\to\infty} \frac{c^{(\tilde n(T))}}{2^{\tilde n(T)}} \tilde M_2(n,T) = \bar\sigma^2(T) \tag{6.16}$$

holds with the constant $\bar{\sigma}^2(T)$ appearing in Lemma 6.2, and

$$\lim_{n\to\infty} \sup_{|\mathbf{s}|\le A} \left| \log \varphi_n(\mathbf{s},T) + \frac{2^{\tilde{n}(T)}}{c^{(\tilde{n}(T))}} \bar{\sigma}^2(T) \frac{\mathbf{s}^2}{2} \right| \to 0 \tag{6.17}$$

for all $A > 0$.

Proof of Lemma 6.3. Relation (6.16) follows from Lemma 6.2, and it follows from the second relation in (6.10) that $\tilde{M}_4(n,T) \le 5 \left(\frac{2^{\tilde{n}(T)}}{c^{(\tilde{n}(T))}}\right)^2 \eta^*$. Hence the characteristic function φ can be estimated as

$$\left| \varphi_n(\mathbf{s},T) - \left(1 - \tilde{M}_2(n,T)\frac{\mathbf{s}^2}{2}\right) \right| \le \left(\frac{2^{\tilde{n}(T)}}{c^{(\tilde{n}(T))}}\right)^2 \eta^* |\mathbf{s}|^4 \tag{6.18}$$

$$\text{for } n \ge \tilde{n}(T) \text{ and } \mathbf{s} \in \mathbb{R}^r.$$

In the proof of formula (6.18) we exploit that $\int (\mathbf{s},\mathbf{x}) \tilde{h}_n(\mathbf{x},T)\, d\mathbf{x} = 0$ and $\int (\mathbf{s},\mathbf{x})^3 \tilde{h}_n(\mathbf{x},T)\, d\mathbf{x} = 0$ because of the rotational symmetry of $\tilde{h}_n(\mathbf{x},T)$. The coefficient of $|\mathbf{s}|^4$ in (6.18) is bounded by a constant (depending on T), and the coefficient at $|\mathbf{s}|^2$ converges to the positive constant $\frac{2^{\tilde{n}(T)}}{c^{(\tilde{n}(T))}} \bar{\sigma}^2(T)$. Hence formula (6.18) implies that for any $\varepsilon > 0$,

$$\left| \log \varphi_n(\mathbf{s},T) + \frac{2^{\tilde{n}(T)}}{c^{(\tilde{n}(T))}} \bar{\sigma}^2(T) \frac{\mathbf{s}^2}{2} \right| \le \varepsilon \quad \text{if } n > n_1 \text{ and } |\mathbf{s}| \le \delta \tag{6.19}$$

with some $n_1 = n_1(\varepsilon, T)$ and $\delta = \delta(\varepsilon, T)$. By a rescaled version of the recursive formula (2.9) we can write

$$\begin{aligned}
\varphi_{n+1}(\sqrt{2}\mathbf{s},T) &= \frac{1}{Z_n(T)} \int \exp\left\{ i\mathbf{s}(\mathbf{x}+\mathbf{u}) - \frac{c^{(n)}(\mathbf{x}-\mathbf{u})^2}{4\cdot 2^n} \right\} \\
&\qquad \tilde{h}_n(\mathbf{x},T)\tilde{h}_n(\mathbf{u},T)\, d\mathbf{x}\, d\mathbf{u} \\
&= \frac{1}{Z_n(T)} \Bigg[\varphi_n(\mathbf{s},T)^2 - \int e^{i\mathbf{s}(\mathbf{x}+\mathbf{u})} \left(1 - \exp\left\{ -\frac{c^{(n)}(\mathbf{x}-\mathbf{u})^2}{4\cdot 2^n} \right\}\right) \\
&\qquad \tilde{h}_n(\mathbf{x},T)\tilde{h}_n(\mathbf{u},T)\, d\mathbf{x}\, d\mathbf{u} \Bigg]
\end{aligned}$$

with

$$Z_n(T) = \int \exp\left\{ -\frac{c^{(n)}(\mathbf{x}-\mathbf{u})^2}{4\cdot 2^n} \right\} \tilde{h}_n(\mathbf{x},T)\tilde{h}_n(\mathbf{u},T)\, d\mathbf{x}\, d\mathbf{u}.$$

The estimates

$$\begin{aligned}
&\left| \int e^{i\mathbf{s}(\mathbf{x}+\mathbf{u})} \left(1 - \exp\left\{ -\frac{c^{(n)}(\mathbf{x}-\mathbf{u})^2}{4\cdot 2^n} \right\}\right) \tilde{h}_n(\mathbf{x},T)\tilde{h}_n(\mathbf{u},T)\, d\mathbf{x}\, d\mathbf{u} \right| \\
&\qquad \le \int \frac{c^{(n)}(\mathbf{x}-\mathbf{u})^2}{4\cdot 2^n} \tilde{h}_n(\mathbf{x},T)\tilde{h}_n(\mathbf{u},T)\, d\mathbf{x}\, d\mathbf{u} = \frac{c^{(n)}}{2\cdot 2^n} \tilde{M}_2(n,T)
\end{aligned}$$

and similarly

$$1 \ge Z_n(T) \ge 1 - \frac{c^{(n)}}{2\cdot 2^n} \tilde{M}_2(n,T)$$

hold. Hence

$$\varphi_n^2(\mathbf{s},T)-\frac{c^{(n)}}{2\cdot 2^n}\tilde M_2(n,T)\le\varphi_{n+1}(\sqrt2\mathbf{s},T)\le\frac{\varphi_n^2(\mathbf{s},T)+\frac{c^{(n)}}{2\cdot 2^n}\tilde M_2(n,T)}{1-\frac{c^{(n)}}{2\cdot 2^n}\tilde M_2(n,T)}.$$

The term $\frac{c^{(n)}}{2^n}\tilde M_2(n,T)$ is much less than $\left(\frac23\right)^n$ for large n. If we have a positive lower bound on $\varphi_n(\mathbf{s})$ then we get by fixing some $K>0$ and taking logarithm in the last relation that

$$\left|\log\varphi_{n+1}\left(\sqrt2\mathbf{s},T\right)-2\log\varphi_n(\mathbf{s},T)\right|\le\left(\frac23\right)^n\quad\text{if } n>n_2 \text{ and } \varphi_n(\mathbf{s},T)\ge\frac1K \tag{6.20}$$

with some $n_2=n_2(K,T)$. Formula (6.17) can be deduced from (6.19) and (6.20). Indeed, define an index k by the relation $A\le\delta 2^{k/2}<\sqrt2 A$ with the numbers A and δ in (6.17) and (6.18). Put $K=2e^{-2^{\tilde n(T)}\bar\sigma^2(T)A^2/c^{(\tilde n(T))}}$ and let $\varepsilon\le\frac{1}{8K}$. Choose a number n_3 such that $\left(\frac23\right)^{n_3}\le\varepsilon$, and let us consider such indices n for which $n\ge\max(n_1(\varepsilon,T),n_2(K,T),n_3)$. Then simple induction yields that

$$\left|\varphi_{n+j}(\mathbf{s},T)+\frac{2^{\tilde n(T)}}{c^{(\tilde n(T))}}\bar\sigma^2(T)\frac{\mathbf{s}^2}{2}\right|\le\varepsilon+3\left(\frac23\right)^n\left(1-\left(\frac23\right)^{j+1}\right)\le4\varepsilon$$
$$\text{and } |\varphi_{n+j}(\mathbf{s},T)|\ge\frac1K$$

for $j\le k$ and $|\mathbf{s}|\le\delta2^{j/2}$. Since ε can be chosen arbitrary small in the last relation, it implies (with $j=k$) relation (6.17). Lemma 6.3 is proved. □

Theorem 3.3 follows from Lemmas 6.2 and 6.3. Indeed, Lemma 6.3 implies that the measures $\tilde H_{n,T}$ converge in distribution to the normal law with expectation zero and covariance $\frac{2^{\tilde n(T)}}{c^{(\tilde n(T))}}\bar\sigma^2(T)\mathbf{I}$. The bounds obtained for the variance follow from Lemma 6.2 and the observation that the difference $\tilde n(T)-\bar n(T)$ can be bounded by a number depending only on $\bar\eta$ and κ.

Let us finally show that Corollary to Theorem 3.3 follows from Theorem 3.3. By formulas (2.7), and (3.17) we can write

$$2^{-n}p_n(2^{-n/2}\sqrt T\mathbf{x},T)=C(n)\exp\left\{-\frac{l_nA_n}{2^{n+1}}\mathbf{x}^2\right\}\tilde h_n(\mathbf{x},T) \tag{6.21}$$

with an appropriate norming constant $C(n)$. Observe that the expressions at both sides of this identity are density functions, the measures with density function $\tilde h_n(\mathbf{x},T)$ have a limit as $n\to\infty$, the term $\left\{-\frac{l_nA_n}{2^{n+1}}\mathbf{x}^2\right\}$ is bounded, and it tends to 1 uniformly in any compact set as $n\to\infty$. These facts imply that $C(n)\to1$ in (6.21), and the measures with density functions $2^{-n}p_n(2^{-n/2}\sqrt T\mathbf{x},T)$ have the same limit as the measures with density functions $\tilde h_n(\mathbf{x},T)$. Hence the Corollary of Theorem 3.3 holds. □

7. Estimates in the low temperature region. The proof of theorem 3.2

The proof of Theorem 3.2 heavily exploits the results of Section 4. These results show that the replacement of the operator $\mathbf{Q}_n$ whose application makes possible to compute the function $f_{n+1}(x,T)$ by its linearization $\mathbf{T}_n$ causes only a negligible error. Formula (4.17) enables one to investigate the operator $\mathbf{T}_n$ in the Fourier space. In such a way good estimates can be obtained for the Fourier transform of a regularized version of the function $f_{n+1}(x,T)$. The results of Theorem 3.2 can be proved by means of these estimates with the help of inverse Fourier transformation.

It is simpler to work with an appropriately scaled version of the functions $f_n(x,T)$. Put

$$\bar{f}_n(x,T)=\frac{1}{M_n(T)}f_n\left(\frac{x}{M_n(T)},T\right)$$

and

$$\bar{\varphi}_n(f_n(x,T))=\frac{1}{M_n(T)}\varphi_n\left(f_n\left(\frac{x}{M_n(T)},T\right)\right).$$

We defined the function $\bar{\varphi}_n(f_n(x,T))$ by means of the definition of the regularization of the function $f_n(x,T)$ introduced in Section 4.

Let us also introduce the functions

$$\psi_{n+1}(f_n(x,T))=\frac{1}{M_n(T)}\mathbf{T}_n\varphi_n\left(f_n\left(\frac{x}{M_n(T)},T\right)\right).$$

The estimates of Proposition 4.2 and relation (4.17) can be rewritten for these new functions. We shall rewrite formulas (4.14) and (4.15) only in the case when $n>N_1(T)$ with the number $N_1(T)$ defined in formula (4.18), i.e. in the case when $\beta_n(T)$ and $M_n^{-1}(T)$ have the same order of magnitude. In this case $M_n(T)\sqrt{\beta_{n+1}(T)}\le 10$,

$$\begin{aligned}(7.1)\qquad &\left|\frac{\partial^j}{\partial x^j}\left(\bar{f}_{n+1}(x,T)-\psi_{n+1}(f_n(x,T))\right)\right|\\ &\le K_1\frac{\beta_n(T)}{c^{(n)}}\left[\exp\left\{-\frac{1}{10}\left|2x+\frac{x^2}{c^{(n+1)}}\right|\right\}+\exp\left\{-\frac{|x|}{5}\right\}\right]\\ &\le K_2\frac{\beta_n(T)}{c^{(n)}}e^{-|x|/10},\qquad x>-c^{(n+1)}M_{n+1}^2(T),\quad j=0,1,2,\end{aligned}$$

and

$$(7.2)\qquad \left|\frac{\partial^j}{\partial x^j}\psi_{n+1}(f_n(x,T))\right|\le K_3e^{-|x|/5},\quad x\in\mathbb{R}^1,\quad j=0,1,2,3,4,$$

with some universal constants K_1, K_2 and K_3. Formula (4.17) can be rewritten as

$$\begin{aligned}(7.3)\qquad \tilde{\psi}_{n+1}(f_n(\xi,T))&=\tilde{\mathbf{T}}_n\tilde{\varphi}_n(f_n(M_n(T)\xi,T))\\ &=\frac{\exp\left\{i\frac{(r-1)\bar{c}_{n+1}}{4}\xi\right\}}{\left(1+i\frac{\bar{c}_{n+1}}{2}\xi\right)^{(r-1)/2}}\,\tilde{\bar{\varphi}}_n^2\left(f_n\left(\frac{\bar{c}_{n+1}}{2}\xi,T\right)\right).\end{aligned}$$

We claim that under the conditions of Theorem 3.2,

$$(7.4)\qquad \lim_{n\to\infty}\sup_{x\ge -c^{(n)}M_n^2(T)}\left|\frac{\partial^j}{\partial x^j}\left(\bar{f}_n(x,T)-\bar{\varphi}(f_n(x,T))\right)\right|e^{|x|/20}=0,\qquad j=0,1,2.$$

Indeed, by relations (7.1) and (7.2)

$$(7.5)\qquad \left|\frac{\partial^j}{\partial x^j}\bar{f}_n(x,T)\right|\le e^{-|x|/10},\quad j=0,1,2,\quad \text{if }\ x\ge -c^{(n)}M_n^2(T),$$

and $\bar\varphi_n(f_n(x,T))$ is the appropriate scaling of the function

$$\varphi\left(\frac{x}{\sqrt{c^{(n)}}M_n(T)}\right) f_n\left(\frac{x}{\sqrt{c^{(n)}}M_n(T)}\right).$$

Under the conditions of Theorem 3.2, formula (4.29) holds, which implies that

$$\lim_{n\to\infty}\sqrt{c^{(n)}}M_n(T)=\infty.$$

This fact together with (7.5) allow us to give a good bound on the difference between the functions $\bar\varphi_n(f_n(x,T))$ and $\varphi\left(\frac{x}{\sqrt{c^{(n)}}M_n(T)}\right) f_n\left(\frac{x}{\sqrt{c^{(n)}}M_n(T)}\right)$.
Relation (7.4) can be deduced from this bound and formula (7.5).

It follows from Lemma 4.4 that $\lim\limits_{n\to\infty}\frac{M_{n+1}(T)}{M_n(T)}=1$. Relations (7.1) and (7.4) together with this fact imply that

$$\lim_{n\to\infty}\sup_{|x|<\infty}\left|\frac{\partial^j}{\partial x^j}(\psi_n(f_{n-1}(x,T))-\bar\varphi_n(f_n(x,T)))\right|e^{|x|/20}=0, \tag{7.6}$$
$$j=0,1,2.$$

Now we prove, using an adaptation of the proof of Lemmas 14 and 15 in [5], that the Fourier transforms of the functions $\psi_{n+1}(f_n(x,T))$ converge to the Fourier transform of the function $g(x)$, and this convergence is uniform in all compact domains. First we prove a modified version of this statement, where ψ_n is replaced with $\bar\varphi_n$ in a small neighbourhood of the origin. We want to work with the functions $\log\tilde{\bar\varphi}_n(f_n(\xi,T)))$. To do this, observe first that for $n>N_1(T)$ there is some constant $A>0$ such that all functions $\tilde{\bar\varphi}_{n+1}(f_n(\xi,T)))$ are separated from zero in the interval $|\xi|\le A$. Indeed,

$$\begin{aligned}|1-\tilde{\bar\varphi}_n(f_n(\xi,T)))| &\le \int|e^{ix\xi}-1|\bar\varphi_n(f_n(x,T)))\,dx\\ &\le \int|\xi||x|\bar\varphi_n(f_n(x,T)))\,dx\le \text{ const.}\,|\xi|.\end{aligned}$$

Similarly,

$$\left|\frac{\partial^j}{\partial\xi^j}\tilde{\bar\varphi}_n(f_n(\xi,T))\right|\le C(j)\quad\text{for all } j\ge 0 \text{ and } n\ge N_1(T).$$

Hence a constant $A>0$ can be chosen in such a way that

$$\sup_{|\xi|\le 2A}\max\left(|1-\tilde g(\xi)|,\sup_{n\ge N_1(T)}|1-\tilde{\bar\varphi}_n(f_n(\xi,T)))|\right)\le\frac12.$$

These estimates imply that

$$\sup\sup_{|\xi|\le A}\left|\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_n(f_n(\xi,T)))\right|\le C(T), \tag{7.7}$$

with a constant $C(T)<\infty$ independent of n. We claim that

$$\sup_{|\xi|\le A}\left|\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_n(f_n(\xi,T)))-\frac{d^2}{d^2\xi}\log\tilde g(\xi)\right|\to 0\quad\text{as } n\to\infty. \tag{7.8}$$

To prove (7.8) let us first observe that $\lim\limits_{n\to\infty}\bar c_n=1$ by Condition 1. By (7.6),

$$\lim_{n\to\infty}\sup_{|\xi|\le A}\left|\frac{\partial^2}{\partial\xi^2}\Big(\log\tilde\psi_{n+1}(f_n(\xi,T))-\log\tilde{\bar\varphi}_{n+1}(f_{n+1}(\xi,T))\Big)\right|=0,$$

and because of the estimates obtained for the derivatives of $\tilde{\bar\varphi}_n(\xi,T)$

$$\left|\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_n(f_n(\xi_1,T)))-\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_n(f_n(\xi_2,T)))\right|\le\ \text{const.}\,|\xi_1-\xi_2|$$
$$\text{if } |\xi_1|\le A \text{ and } |\xi_2|\le A$$

for all large indices n with a constant independent of n. Taking logarithm and then differentiating twice in formulas (7.3) and (1.20) we get with the help of the above observations that

$$\begin{aligned}&\sup_{|\xi|\le A}\left|\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_{n+1}(f_{n+1}(\xi,T)))-\frac{d^2}{d^2\xi}\log\tilde g(\xi)\right|\\&\qquad\le\frac12\sup_{|\xi|\le A}\left|\frac{\partial^2}{\partial\xi^2}\log\tilde{\bar\varphi}_n(f_n(\xi,T)))-\frac{d^2}{d^2\xi}\log\tilde g(\xi)\right|+\delta_n(T)\end{aligned}$$

with some sequence $\lim\limits_{n\to\infty}\delta_n(T)=0$. This relation together with (7.7) imply (7.8). Since

$$\frac{\partial}{\partial\xi}\log\tilde{\bar\varphi}_n(f_n(\xi,T)))\bigg|_{\xi=0}=\frac{d}{d\xi}\log\tilde g(\xi)\bigg|_{\xi=0}=0$$
$$\text{and }\ \log\tilde{\bar\varphi}_n(f_n(0,T)))=\log\tilde g(0)=0,$$

relation (7.8) also implies that

$$\lim_{n\to\infty}\sup_{|\xi|\le A}|\tilde{\bar\varphi}_n(f_n(\xi,T)))-\tilde g(\xi)|=0. \tag{7.9}$$

Moreover, relation (7.9) holds for all $A>0$. This can be proved similarly to the argument of Lemma 15 in [5]. One has to observe that because of the structure of formulas (7.3) and (1.20), the relation $\bar c_n\to1$ as $n\to\infty$, the continuity of the function $\tilde g(\xi)$ and the relation

$$\lim_{n\to\infty}\sup_{|\xi|<\infty}\left|\tilde\psi_{n+1}(f_n(\xi,T))-\tilde{\bar\varphi}_{n+1}(f_{n+1}(\xi,T)))\right|=0,$$

the validity of relation (7.9) in an interval $|\xi|\le A$ also implies its validity in the interval $|\xi|\le(2-\varepsilon)A$ for any $\varepsilon>0$. In relation (7.9) the function $\tilde{\bar\varphi}_n(f_n(\xi,T)))$ can be replaced by $\tilde\psi_{n+1}(f_n(\xi,T))$, i.e. the relation

$$\lim_{n\to\infty}\sup_{|\xi|\le A}\left|\tilde\psi_{n+1}(f_n(\xi,T)))-\tilde g(\xi)\right|=0 \tag{7.10}$$

holds for all $A>0$. It can be proved from (7.10), by means of inverse Fourier transformation, that

$$\lim_{n\to\infty}\sup_{|x|<\infty}\left|\frac{\partial^j}{\partial x^j}\psi_{n+1}(f_n(x,T)))-\frac{d^j}{dx^j}g(x)\right|=0,\quad j=0,1,2. \tag{7.11}$$

To prove (7.11) we need, besides the estimate (7.10), some bound about the decrease of the functions $\tilde{g}(\xi)$ and $\tilde{\psi}_{n+1}(f_n(\xi,T)))$ as $\xi \to \pm\infty$. The estimate (1.19) gives a good bound for the Fourier transform of the function $g(x)$. We can get a good estimate for the Fourier transform of the function $\psi_{n+1}(f_n(x,T))$ with the help of the inductive hypothesis $J(n)$ in Section 4 and relation (7.3). Rewriting the inductive hypothesis $J(n)$ for the function $\bar{\varphi}_n(f_n(x,T))$ we get with the help of some standard calculation that the Fourier transform $\tilde{\psi}_{n+1}(f_n(\xi,T))$ decreases at infinity faster than $|\xi|^{-4}$. These estimates are sufficient for the proof of (7.11). Relations (7.11) and (7.1) give an estimate on the function $\bar{f}_n(x,T)$ and its derivatives, which is equivalent to (3.8). Theorem 3.2 is proved. $\square$

8. Estimates near the critical point. The proof of theorems 3.4, 1.3, and 1.5

Our previous results suggest that $M^2_{n+1}(T) \sim M^2_n(T) - \frac{r-1}{2c^{(n)}}$, hence the derivative $\frac{dM^2_n(T)}{dT}$, as a function of n, changes very little if the pair (n,T) is in the low domain region (observe that $c^{(n)}$ does not depend on T). Therefore, it is natural to expect that $\frac{dM^2_\infty(T)}{dT}$ is of constant order below the critical value T_c, and $M^2_\infty(T) - M^2_\infty(T_c) \sim \text{const.}\,(T_c - T)$ for $T < T_c$. If T_n denotes the smallest T for which the pair (n,T) leaves the low temperature region at the n-th step, then the following heuristic argument may suggest the magnitude of $T_n - T_{n+1}$ for large n. Since both $c^{(n)}M^2_n(T_n) \sim \eta^{-1}$ and $c^{(n+1)}M^2_{n+1}(T_{n+1}) \sim \eta^{-1}$, besides this $M^2_n(T_{n+1}) - M^2_{n+1}(T_{n+1}) \sim \frac{r-1}{2c^{(n)}}$, $M^2_n(T_{n+1}) - M^2_n(T_n) \sim \frac{r-1}{2c^{(n)}}$. On the other hand, $M^2_n(T_{n+1}) - M^2_n(T_n) \sim T_{n+1} - T_n$. This argument suggests that $T_{n+1} - T_n \sim \frac{r-1}{2c^{(n)}}$ and $T_n - T_c \sim \sum\limits_{k=n}^{\infty} \frac{r-1}{2c^{(k)}}$. In this section we justify these heuristic arguments. The proofs are based on the following result:

Theorem 8.1. *There exists $\kappa_0 = \kappa_0(N)$ such that if (i) $0 < \kappa < \kappa_0$ in formula (1.7), (ii) Conditions 1—4 are satisfied, (iii) $0 < \bar{T} < c_0 A_0/2$, and (iv) the integer $n \ge 1$ has the property that the pair $(n,\bar{T})$ belongs to the low temperature region, then for all $0 < T < \bar{T}$ the pair (n,T) also belongs to the low temperature region, and the following inequalities hold for $T \le \bar{T}$:*

a.) If $0 \le n \le N$, then

$$\frac{C_1}{\sqrt{\kappa}T^2} < -\frac{dM_{n+1}(T)}{dT} < \frac{C_2}{\sqrt{\kappa}T^2} \quad \text{with some } \infty > C_2 > C_1 > 0.$$

b.) If $n \ge N$, then

$$\frac{dM_{n+1}(T)}{dT} = \frac{dM_n(T)}{dT}\left(1 + \frac{r-1}{4c^{(n)}M^2_n(T)} + \frac{\delta_n(T)}{c^{(n)}}\right),$$

$$\text{where } |\delta_n(T)| \le C\frac{\beta_{n+1}(T)}{c^{(n+1)}}\beta_n(T) \text{ with some appropriate } C > 0.$$

We will prove Theorem 8.1 in Appendix A below with the help of Proposition A which is proved also there. This result can be interpreted in an informal way as the "differentiation" of the asymptotic identity (4.12). The main difficulty in the proof of Proposition A is to bound the error caused by the linear approximation of the operator $\mathbf{Q}_n$ by $\mathbf{T}_n$ when differentiating with respect to T. To overcome this difficulty we need a good control not only on the functions $f_n(x,T)$ but also on their derivatives $\frac{\partial}{\partial T} f_n(x,T)$. Hence we have to work out the estimation of these derivatives. In particular, we have to find the inductive hypotheses describing their behaviour. These are the analogs of the inductive hypotheses $I(n)$ and $J(n)$ formulated in Section 4. It demands fairly much work to work out the details, but after the formulation and proof of these inductive hypotheses the proof of Proposition A is simple.

Proof of Theorem 3.4. We prove with the help of Proposition A that if the conditions of Theorem 3.4 hold, $0<\bar T<c_0A_0/2$ and the pair $(n-1,\bar T)$ belongs to the low temperature region, then there exist some constants $0<C_1<C_2$ independent of T such that

$$\frac{C_1}{\kappa T^3} < -\frac{dM_n^2(T)}{dT} < \frac{C_2}{\kappa T^3}. \tag{8.1}$$

for all $0<T\le \bar T$.

For $0\le n\le N$ (n,T) is in the low temperature region for $0<T<c_0A_0/2$. In this case Properties $K_1(n)$—$K_4(n)$ hold by Proposition A, and the validity of (8.1) for $n=0$ follows from relations (4.1), (4.5) and (A.1). Its validity for $0\le n\le N$ can be proved by induction with the help of Properties $K_1(n)$ and $K_3(n)$, $1\le n\le N$.

To prove formula (8.1) for $n>N$ first we show that

$$\begin{aligned} -\frac{dM_n^2(T)}{dT}\exp\left\{-K\left(\frac{\beta_{n+1}(T)}{c^{(n)}}\right)^2\right\} &\le -\frac{dM_{n+1}^2(T)}{dT} \\ &\le -\frac{dM_n^2(T)}{dT}\exp\left\{K\left(\frac{\beta_{n+1}(T)}{c^{(n)}}\right)^2\right\} \end{aligned} \tag{8.2}$$

for all $T<\bar T$ and $n\ge N$ with an appropriate $K>0$. Relation (8.2) is a consequence of Part b) of Proposition A, formula (4.12), the inequality $\beta_{n+1}(T)M_n^2(T)\ge 10$ and the relation $\frac{\beta_{n+1}(T)}{c^{(n)}}\le\eta$ if (n,T) is in the low temperature domain. Indeed, since

$$-\frac{dM_{n+1}^2(T)}{dT} = -\left(1-\frac{m_n(T)}{c^{(n+1)}M_n(T)}\right)\left(1-\frac{\frac{dm_n(T)}{dT}}{c^{(n+1)}\frac{dM_n(T)}{dT}}\right)\frac{dM_n^2(T)}{dT},$$

these relations imply that

$$\begin{aligned} -&\left(1-\frac{r-1}{4c^{(n+1)}M_n^2(T)}-C_1\frac{\beta_{n+1}^{3/2}(T)}{{c^{(n+1)}}^2M_n(T)}\right) \\ &\qquad\left(1+\frac{r-1}{4c^{(n+1)}M_n^2(T)}-C\frac{\beta_{n+1}^2(T)}{(c^{(n+1)})^2}\right)\frac{dM_n^2(T)}{dT} \\ &\le -\frac{dM_{n+1}^2(T)}{dT} \end{aligned}$$

$$\le -\left(1 - \frac{r-1}{4c^{(n+1)}M_n^2(T)} + C_1\frac{\beta_{n+1}^{3/2}(T)}{{c^{(n+1)}}^2 M_n(T)}\right)$$
$$\left(1 + \frac{r-1}{4c^{(n+1)}M_n^2(T)} + C\frac{\beta_{n+1}^2(T)}{{c^{(n+1)}}^2}\right)\frac{dM_n^2(T)}{dT}.$$

In this calculation we have exploited that $c^{(n+1)}M_n^2(T) \ge \frac{\beta_{n+1}(T)}{\eta}M_n^2(T) \gg 1$.

The left and right-hand side of this inequality can be bounded by

$$-\left(1 \pm K\frac{\beta_{n+1}^2(T)}{{c^{(n)}}^2}\right)\frac{dM_n^2(T)}{dT},$$

and formula (8.2) can be deduced from these relations.

For $N \le n \le N_1(T)$ with the number $N_1(T)$ defined in relation (4.18) relation (8.1) follows from (8.2) and (4.19). Since by (4.20) $\beta_{n+1}M_n^2(T) \le 100$ if $n \ge N_1(T)$ and the pair (n,T) is in the low temperature domain, to prove formula (8.1) with the help of (8.2) for $n > N_1(T)$ it is enough to show that

$$\sum_{k=N_1(T)}^{n} \frac{1}{\left(c^{(k)}M_k^2(T)\right)^2} \le L \text{ if } n \ge N_1(T)$$
$$\text{and } (n,T) \text{ is in the low temperature domain}$$

with a constant $L > 0$ independent of T and n. Since $M_n^2(T) \ge \frac{1}{10\beta_{n+1}(T)} \ge \frac{1}{10\eta c^{(n)}}$ and $M_k^2(T) = M_n^2(T) + (M_k^2(T) - M_n^2(T)) \ge \frac{1}{10\eta c^{(n)}} + \sum\limits_{j=k}^{n-1}\frac{1}{8c^{(j)}}$,

$$\sum_{k=N_1(T)}^{n} \frac{1}{\left(c^{(k)}M_k^2(T)\right)^2} \le \text{ const.} \sum_{k=N_1(T)}^{n} \frac{1}{\left(c^{(k)}\sum\limits_{j=k}^{n}\frac{1}{c^{(j)}}\right)^2} \le L$$

because of Condition 3. Hence formula (8.1) holds.

It follows from (1.2), Condition 4, and the results of Section 4 that all $T > c_0A_0/4$ belong to the high temperature region. Indeed, it follows from formulas (4.26), (4.27), (4.1), (4.5) and (4.9), that if $T > 0$ is in the low temperature region, then

$$0 \le M_n^2(T) \le M_N^2(T) - 30(M_N(T)+1) - \sum_{n=1}^{\infty}\frac{1}{8c^{(n)}} \le \frac{3}{\kappa T^2} - \sum_{n=1}^{\infty}\frac{1}{8c^{(n)}}$$

for all $n \ge N$, and $T \le \left(\sum\limits_{n=1}^{\infty}\frac{\kappa}{24c^{(n)}}\right)^{-1/2}$. Hence Condition 4 implies that $T \le c_0A_0/4$.

It follows from (8.2) that for a fixed n the function $M_n^2(T)$ is strictly monotone decreasing. Hence a simple induction with respect to n yields that the function $\beta_n(T)$ is a monotone increasing, continuous function of T for all $n > N$. Put

$$T_n = \sup\{T\colon (T,n) \text{ is in the low temperature region}\}. \tag{8.3}$$

The sequence T_n is monotone decreasing, hence the limit $T_c = \lim_{n\to\infty} T_n$ exists, and by Lemma 4.4 $T_c > 0$ under Dyson's condition (1.2). We want to show that

$$C_1 \sum_{k=n}^{\infty} \frac{1}{c^{(k)}} \le T_n - T_c \le C_2 \sum_{k=n}^{\infty} \frac{1}{c^{(k)}}. \tag{8.4}$$

Since we can handle the sequence $M_n(T)$ better than the sequence $\beta_n(T)$ we also introduce besides the sequence T_n defined in (8.3) the sequence $T(n)$

$$T(n) = \sup \left\{ T \colon M_n^2(T) \ge \frac{100}{c^{(n)}\eta} \right\}.$$

We will show that

$$T_{n+K} \le T(n) \le T_n \tag{8.5}$$

for all sufficiently large n with an appropriate $K > 0$, and

$$\frac{C_1}{c^{(n)}} \le T(n) - T(n+1) \le \frac{C_2}{c^{(n)}} \tag{8.6}$$

with some appropriate $C_2 > C_1 > 0$ for all sufficiently large n. Because of Condition 5 relation (8.4) follows from (8.5) and (8.6) together with the relation $\lim_{n\to\infty} T_n = T_c$.

If $T \le T(n)$, and $m \le n$ then either $m \le N_1(T)$ with the number $N_1(T)$ defined in (4.18) or $\beta_{m+1}(T) \le \frac{100}{M_m^2(T)} \le \frac{100}{M_n^2(T)} \le c^{(n)}\eta$. This implies that for $T \le T(n)$ the pair (m, T) is in the low temperature region for all $m \le n$, and $T(n) \le T_n$. This is the right-hand side of relation (8.5).

To prove its left-hand side observe that because of Condition 5 there is some K such that

$$\sum_{k=n}^{n+K-1} \frac{1}{8c^{(n)}} > \frac{100}{c^{(n)}\eta}$$

for all sufficiently large n with appropriate $K > 0$. We claim that for $T \ge T(n)$ the pair $(n+K, T)$ is not in the low temperature region. This relation implies the left-hand side of (8.5). If $(n+K, T)$ were in the low temperature region, then we would get with the help of formula (4.26) that

$$M_{n+K}^2(T) \le M_n^2(T) - \sum_{k=n}^{n+K-1} \frac{1}{8c^{(n)}} < \frac{100}{c^{(n)}\eta} - \sum_{k=n}^{n+K} \frac{1}{8c^{(n)}} < \frac{100}{c^{(n)}\eta} - \frac{100}{c^{(n)}\eta} = 0,$$

and this is a contradiction.

To prove formula (8.6) let us first observe that because of the continuity and strict monotonicity of the function $M_n^2(T)$, $M_n^2(T(n)) = \frac{100}{c^{(n)}\eta}$. It follows from the last statement of Lemma 4.3 and formula (8.1) that $N_1(T) \le n$ for all $T(n) - \varepsilon < T < T(n)$ with an appropriately small $\varepsilon > 0$. (The number $N_1(T)$ was defined in (4.18).). Hence we get with the help of formula (8.1) that for sufficiently large n and $T(n) - \varepsilon < T < T(n)$

$$\frac{100}{c^{(n)}\eta} - \frac{2}{c^{(n)}} + \bar{C}_1(T(n) - T) \le \frac{100}{c^{(n+1)}\eta} - \frac{1}{c^{(n)}} + \bar{C}_1(T(n) - T)$$

$$\begin{aligned}
&\le M_{n+1}^2(T)\\
&\le \frac{100}{c^{(n)}\eta} - \frac{1}{8c^{(n)}} + \bar C_2(T(n)-T)\\
&\le \frac{100}{\eta c^{(n+1)}} - \frac{1}{9c^{(n)}} + \bar C_2(T(n)-T)
\end{aligned}$$

with some appropriate constants $\bar C_2 > \bar C_1 > 0$. Hence the solution of the equation $M_{n+1}^2(T) = \frac{100}{c^{(n+1)}\eta}$ satisfies the inequality $K_1 < c^{(n)}(T - T(n)) < K_2$ with appropriate constants $K_2 > K_1 > 0$. Since the solution of this equation is $T(n+1)$, this fact implies relation (8.6).

It is not difficult to see that T_c is in the low temperature region. Since the inequality $M_n^2(T_c) = M_n^2(T(n)) + (M_n^2(T_c) - M_n^2(T(N))) \le \frac{100}{c^{(n)}\eta} +$ const. $(T(n) - T_c)$ holds for all large n because of (8.1), $\lim\limits_{n\to\infty} M_n(T_c) = 0$. Then relation (8.1) implies that

$$C_1\,(T_c - T)) \le M_n^2(T_c) - M_n^2(T) \le C_2\,(T_c - T)$$

with some positive constants $C_2 > C_1 > 0$ if $T_c \ge T \ge T_c - \varepsilon$. Letting n tend to infinity in the last relation we get formula (3.21). Since formula (8.4) is equivalent to (3.20) Theorem 3.4 is proved. □

Proof of Theorem 1.3. By Corollary of Theorem 3.1, if the Dyson condition (1.2) is violated then all temperatures $T > 0$ belong to the high temperature region. By Corollary of Theorem 3.3 relation (1.13) holds, and the measures $\tilde\nu_{n,T}(dx)$ tends to the standard normal distribution as $n \to \infty$. Theorem 1.3 is proved. □

Proof of Theorem 1.5.

Part 1). The convergence of $\tilde\nu_{n,T}(dx)$ to the r-dimensional standard Gaussian distribution and relation (1.26) follow from Corollary of Theorem 3.3. The asymptotics (1.27) follows from (3.19) and (3.20).

Part 2). Formula (1.28) follows from (3.7), and the convergence of $\tilde\nu_{n,T_c}(dx)$ to the uniform distribution on the sphere follows from Theorems 3.2 and 3.4. Namely, Theorem 3.4 tells us that the critical temperature T_c belongs to the *low temperature region*. Then formula (3.8) proves that the probability distribution $\tilde\nu_{n,T_c}(dx)$ converges to the uniform distribution on the sphere. As a matter of fact, (3.8) proves much more: it proves the convergence at $T = T_c$ of the distribution of normalized fluctuations of the mean spin along the radius to a limit. Indeed, by (3.8),

$$\lim_{n\to\infty} \left\| \frac{1}{M_n(T_c)} f_n\left(\frac{t}{M_n(T_c)}, T_c\right) - g\,(t) \right\| = 0,$$

where

$$\|f(t)\| = \sum_{j=0}^{2} \sup_{t \ge -c^{(n)} M_n^2(T)} e^{|t|} \left| \frac{d^j f(t)}{d\,t^j} \right|$$

and the probability density $g(t)$ is defined as a solution of the fixed point equation (1.18). By (2.13),

$$f_n(t, T_c) = \frac{1}{c^{(n)}} \bar q_n \left(M_n(T_c) + \frac{t}{c^{(n)}}, T_c \right),$$

hence

$$\lim_{n\to\infty}\left\|\frac{\bar{q}_n\left(M_n(T_c)+\frac{t}{c^{(n)}M_n(T_c)},T_c\right)}{c^{(n)}M_n(T_c)}-g\,(t)\right\|=0.$$

To obtain a scaling limit of q_n near $M_n(T_c)$, let us rewrite the latter formula as

$$\lim_{n\to\infty}\left\|\frac{\bar{q}_n\left(M_n(T_c)\left(1+\frac{t}{c^{(n)}M_n^2(T_c)}\right),T_c\right)}{c^{(n)}M_n(T_c)}-g\,(t)\right\|=0. \tag{8.7}$$

Let us evaluate the asymptotics of $c^{(n)}M_n^2(T_c)$ as $n\to\infty$. By (3.7),

$$\lim_{n\to\infty}\frac{M_n^2(T_c)}{\sum_{k=n}^{\infty}\frac{1}{c^{(k)}}}=\frac{r-1}{2}, \tag{8.8}$$

since $M_\infty(T_c)=0$. Define

$$\lambda_n=l_n\sum_{k=n}^{\infty}\frac{1}{l_k}. \tag{8.9}$$

By Condition 2 on the sequence $\{l_n\}$,

$$\lim_{n\to\infty}\lambda_n=\infty. \tag{8.10}$$

By (2.23)

$$\lim_{n\to\infty}\frac{c^{(n)}}{l_n}=3,$$

hence

$$\lim_{n\to\infty}\frac{\sum_{k=n}^{\infty}\frac{1}{c^{(k)}}}{\sum_{k=n}^{\infty}\frac{1}{l_k}}=\frac{1}{3},$$

and by (8.8), (8.9),

$$\lim_{n\to\infty}\frac{M_n^2(T_c)}{\sum_{k=n}^{\infty}\frac{1}{l_k}}=\lim_{n\to\infty}\frac{M_n^2(T_c)l_n}{\lambda_n}=\frac{r-1}{6}, \tag{8.11}$$

which relation is equivalent to (1.28). Therefore,

$$\lim_{n\to\infty}\frac{c^{(n)}M_n^2(T_c)}{\lambda_n}=\frac{r-1}{2}.$$

Substituting this limit into (8.7), we obtain that

$$\lim_{n\to\infty}\left\|\frac{2M_n(T_c)}{(r-1)\lambda_n}\bar{q}_n\left(M_n(T_c)\left(1+\frac{t}{\frac{r-1}{2}\,\lambda_n}\right),T_c\right)-g\,(t)\right\|=0. \tag{8.12}$$

This implies that the probability density $\bar{Z}_n(T_c)^{-1}\,\bar{q}_n(M_n(T_c)x,T_c)$ is localized in a neighborhood of order λ_n^{-1} of the point 1, and after the proper scaling it converges to $g\,(t)$ as $n\to\infty$.

Let us consider now the scaling limit of the probability density $\bar{p}_n(x,T_c)$. By equations (2.7) and (2.11),

$$\bar{p}_n(x,T_c)=Z_n(T_c)^{-1}\exp\left(-\frac{A_nl_nT_c^{-1}x^2}{2}\right)\bar{q}_n(T_c^{-1/2}x,T_c),\quad x\ge 0,$$

where $Z_n(T_c)^{-1}$ is a norming factor, hence

$$\bar{p}_n(T_c^{1/2}M_n(T_c)x,T_c) = Z_n(T_c)^{-1}\exp\left(-\frac{A_n l_n M_n^2(T_c)x^2}{2}\right)\bar{q}_n(M_n(T_c)x,T_c).$$

Applying the same scaling as in (8.12), we obtain that

$$\begin{aligned}\bar{p}_n&\left(T_c^{1/2}M_n(T_c)\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right),T_c\right)\\ &= Z_n(T_c)^{-1}\exp\left(-\frac{A_n l_n M_n^2(T_c)}{2}\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right)^2\right)\\ &\qquad\times\bar{q}_n\left(M_n(T_c)\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right),T_c\right).\end{aligned}\tag{8.13}$$

Consider the expression in the exponent,

$$\begin{aligned}\frac{A_n l_n M_n^2(T_c)}{2}\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right)^2 &= \frac{A_n l_n M_n^2(T_c)}{2}+\frac{2A_n l_n M_n^2(T_c)t}{(r-1)\lambda_n}\\ &\quad+\frac{2A_n l_n M_n^2(T_c)t^2}{(r-1)^2\lambda_n^2}.\end{aligned}\tag{8.14}$$

By (8.11), (2.22) and (8.10)

$$\lim_{n\to\infty}\frac{2A_n l_n M_n^2(T_c)}{(r-1)\lambda_n}=\frac{2}{3},\qquad \lim_{n\to\infty}\frac{2A_n l_n M_n^2(T_c)}{(r-1)^2\lambda_n^2}=0.\tag{8.15}$$

The constant term in (8.14) is not important, because it changes in (8.13) the norming constant only. Therefore, from (8.12)-(8.15) we obtain that

$$\lim_{n\to\infty}\left\|Z_n'^{-1}e^{(2/3)t}\bar{p}_n\left(T_c^{1/2}M_n(T_c)\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right),T_c\right)-g\,(t)\right\|=0.\tag{8.16}$$

This implies that

$$\begin{aligned}\lim_{n\to\infty}\bigg\|\frac{2T_c^{1/2}M_n(T_c)}{(r-1)\lambda_n}\,\bar{p}_n&\left(T_c^{1/2}M_n(T_c)\left(1+\frac{t}{\frac{r-1}{2}\lambda_n}\right),T_c\right)\\ &-Z^{-1}e^{-(2/3)t}g\,(t)\bigg\|'=0,\end{aligned}\tag{8.17}$$

where $Z^{-1}e^{-(2/3)t}g\,(t)$ is a probability density (this determines the constant Z), and

$$\|f(t)\|'=\sum_{j=0}^{2}\sup_{t\geq-\frac{r-1}{2}\lambda_n}e^{|t|/3}\left|\frac{d^j f(t)}{d\,t^j}\right|.\tag{8.18}$$

Substituting t for $(bt-a)$ in (8.17), we obtain that

$$\lim_{n\to\infty}\left\|\frac{A_n}{d_n}\bar{p}_n\left(A_n\left(1+\frac{t}{d_n}\right)\right)-\pi(t)\right\|'=0,\tag{8.19}$$

where $A_n > 0$ is some constant. Since

$$\int_{-\infty}^{\infty} t\,\bar p_n\left(\hat M_n(T_c)\left(1+\frac{t}{d_n}\right)\right)=0, \qquad \int_{-\infty}^{\infty} t\,\pi(t)\,dt=0,$$

we can replace A_n by $\hat M_n(T_c)$ in (8.19). This proves equation (1.32).

Part 3). The results of Part 3) of Theorem 3 were already proved with the exception of relation (1.35) in the discussion after the formulation of Theorem 3.2. But relation (1.35) is a direct consequence of relation (3.21) proved in Theorem 3.4 and the identity $M(T)=\sqrt{T}M_\infty(T)$ for $T<T_c$ which was also proved in the above discussion. Theorem 1.5 is proved. □

Appendix A. The proof of theorem 8.1

To prove Theorem 8.1 we need good estimates on the partial derivatives

$$g_n(x,T)=\frac{\partial}{\partial T}f_n(x,T),$$

of a scaled version of the functions $q_n(x,T)$. This can be done similarly to the estimation of the functions $f_n(x,T)$, done in Section 4. First we give estimates for the starting function $g_0(x,T)$, then prove that similar estimates hold for small indices n, more explicitly for $n\le N$ with the index N defined in (1.12). Then inductive hypotheses can be formulated and proved for the functions $g_n(x,T)$. In Section 4 we have introduced certain operators $\bar{\mathbf{Q}}_n$, their normalization $\mathbf{Q}_n$ and the linearization of these operators denoted by $\bar{\mathbf{T}}_n$ and $\mathbf{T}_n$. The inductive hypotheses formulated there were closely related to the properties of these operators. Now we want to work similarly. To do this we have to introduce some new operators. We introduce certain operators $\bar{\mathbf{R}}_n$ and $\mathbf{R}_n$ which are the derivatives of the operators $\bar{\mathbf{Q}}_n$ and $\mathbf{Q}_n$ with respect to the variable T. We also need their linear approximation which we shall denote by $\bar{\mathbf{U}}_n$ and $\mathbf{U}_n$. We have to study the action of these operators on the functions $g_n(x,T)=\frac{\partial}{\partial T}f_n(x,T)$ and their Fourier transform.

An appropriate description of the asymptotic behaviour of the starting functions $f_0(x,T)$ and numbers $M_0(T)$ were already given in formulas (4.2) — (4.8). Some more calculation yields, with the help of some formulas in Section 4, the following estimates for the derivatives of the magnetization $M_0(T)$ and the norming constant $Z_0(T)$ if $T<c_0A_0/2$.

$$\left|\frac{d}{dT}\left(M_0(T)-\bar M_0(T)\right)\right|\le \text{ const.}\sqrt{\kappa}.$$

$$\frac{C_1}{\sqrt{\kappa}\,T^2}<-\frac{dM_0(T)}{dT}<\frac{C_2}{\sqrt{\kappa}\,T^2}\quad\text{with some }\infty>C_2>C_1>0, \tag{A.1}$$

and

$$\left|\frac{dZ_0(T)}{dT}-\frac{\sqrt{\pi}}{2(A_0-T)^{3/2}}\right|\le \text{ const.}\sqrt{\kappa}.$$

The derivatives of the functions $\bar q_0(x,T)$ and $f_0(x,T)$ satisfy the inequalities

$$\left|\frac{\partial \bar q_0(x+M_0(T),T)}{\partial T}-\frac{\sqrt{A_0-T}}{\sqrt{\pi}}\left(x^2-\frac{1}{2(A_0-T)}\right)e^{-(A_0-T)x^2}\right|$$

$$\leq \text{ const.}\, \kappa^{1/4}, \quad \text{if } |x| < \log \kappa^{-1},$$

and

$$\left| \frac{\partial \bar{q}_0(x + M_0(T), T)}{\partial T} \right| \leq C \exp\left\{ -\frac{(A_0 - T)}{4} \left| 2x + \frac{x^2}{M_0^2(T)} \right| \right\} \quad \text{for } x \geq -M_0(T). \tag{A.2}$$

We shall apply the notation

$$g_n(x, T) = \frac{\partial f_n(x, T)}{\partial T}, \quad n = 0, 1, \dots. \tag{A.3}$$

Since $f_0(x, T) = q_0(x + M_0(T), T)$ the previous estimates together with the results of Section 4 yield a sufficiently good control on $g_0(x, T)$. The functions $g_n(x, T)$, $n = 1, 2, \dots$, can be estimated inductively with respect to the parameter n.

Put

$$\bar{\mathbf{R}}_n f_n(x, T) = \frac{\partial}{\partial T} \bar{\mathbf{Q}}^{\mathbf{c}}_{n, M_n(T)} f_n(x, T)$$

and

$$\mathbf{R}_n f_n(x, T) = \frac{\partial}{\partial T} \mathbf{Q}^{\mathbf{c}}_{n, M_n(T)} f_n(x, T) = g_{n+1}(x, T).$$

Then

$$\bar{\mathbf{R}}_n f_n(x, T) = \bar{\mathbf{R}}_n^{(1)} f_n(x, T) + \bar{\mathbf{R}}_n^{(2)} f_n(x, T) \tag{A.4}$$

with

$$\begin{aligned} \bar{\mathbf{R}}_n^{(1)} f_n(x, T) &= 2 \int_{u \in \mathbb{R}^1,\, \mathbf{v} \in \mathbb{R}^{r-1}} \exp\left\{ -\frac{u^2}{c^{(n)}} - \mathbf{v}^2 \right\} f_n(\ell^{\mathbf{c},+}_{n, M_n(T)}(x, u, \mathbf{v}), T) \\ &\qquad g_n(\ell^{\mathbf{c},-}_{n, M_n(T)}(x, u, \mathbf{v}), T)\, du\, d\mathbf{v}, \end{aligned}$$

where the functions $g_n(x, T)$ and $\ell^{\mathbf{c},\pm}_{n, M_n(T)}(x, u, \mathbf{v}), T)$ were defined in (A.3) and (2.18), and

$$\begin{aligned} \bar{\mathbf{R}}_n^{(2)} f_n(x, T) &= -2 \int_{u \in \mathbb{R}^1,\, \mathbf{v} \in \mathbb{R}^{r-1}} \exp\left\{ -\frac{u^2}{c^{(n)}} - \mathbf{v}^2 \right\} f_n(\ell^{\mathbf{c},+}_{n, M_n(T)}(x, u, \mathbf{v}), T) \\ &\qquad h_n(x, u, \mathbf{v}, T) \frac{\partial}{\partial x} f_n(\ell^{\mathbf{c},-}_{n, M_n(T)}(x, u, \mathbf{v}), T)\, du\, d\mathbf{v} \end{aligned}$$

with

$$\begin{aligned} h_n(x, u, \mathbf{v}, T) &= -\frac{\partial \ell^{\mathbf{c},-}_{n, M_n(T)}(x, u, \mathbf{v})}{\partial T} \\ &= \frac{M_n'(T) \mathbf{v}^2}{\sqrt{\left(M_n(T) + \frac{x}{c^{(n+1)}} - \frac{u}{c^{(n)}} \right)^2 + \frac{\mathbf{v}^2}{c^{(n)}}}} \\ &\qquad \frac{1}{\sqrt{\left(M_n(T) + \frac{x}{c^{(n+1)}} - \frac{u}{c^{(n)}} \right)^2 + \frac{\mathbf{v}^2}{c^{(n)}}} + \left(M_n(T) + \frac{x}{c^{(n+1)}} - \frac{u}{c^{(n)}} \right)}. \end{aligned}$$

The function $g_{n+1}(x, T)$ can be expressed as

$$g_{n+1}(x, T) = \mathbf{R}_n f_n(x, T)$$

$$
\begin{aligned}
&= \frac{\bar{\mathbf{R}}_n f_n(x+m_n(T),T)}{Z_n(T)} + \frac{\frac{\partial}{\partial x}\bar{\mathbf{Q}}_n f_n(x+m_n(T),T)}{Z_n(T)}\frac{dm_n(T)}{dT} \\
&\qquad - \frac{\bar{\mathbf{Q}}_n f_n(x+m_n(T),T)}{Z_n^2(T)}\frac{dZ_n(T)}{dT} \qquad \text{(A.5)}
\end{aligned}
$$

with

$$
Z_n(T) = \int_{-c^{(n)}M_n(T)}^{\infty} \bar{\mathbf{Q}}_n f_n(x,T)\,dx.
$$

If the parameter $\kappa > 0$ in formula (2.10) is sufficiently small, then $\bar{q}_0(x,T)$ and the functions $g_n(x,T)$, $n \le N$, can be estimated similarly to the proof of Proposition 4.1 or Proposition 1 in [5]. Relation (A.7) formulated below can be deduced from formula (A.2) similarly to the proof of Lemma 1 of that paper. Then an argument similar to the proof of Lemma 2 in [5] enables one to prove formula (A.6) formulated below. In this argument one can observe that a negligible error is committed if in the integrals appearing in the definition of $\bar{\mathbf{R}}_n f_n(x,T)$ the arguments $\ell^{\mathbf{c},\pm}_{n,M_n(T)}(x,u,\mathbf{v})$ defined in formula (2.18) are replaced by $\frac{x}{c_{n+1}} \pm u$. Some calculation also shows that we commit a negligible error by replacing $\mathbf{R}_n f_n(x,T)$ with $\frac{\bar{\mathbf{R}}_n^{(1)} f_n(x,T)}{Z_n(T)}$. In such a way we get that

$$
\begin{aligned}
&\Bigg| g_n(x,T) - \frac{\sqrt{A_0-T}}{\sqrt{\pi}}\frac{2^{n/2}}{c^{(n)}}\left(x^2 - \frac{1}{2(A_0-T)}\frac{{c^{(n)}}^2}{2^n}\right) \\
&\qquad \exp\left\{-(A_0-T)\frac{2^n x^2}{{c^{(n)}}^2}\right\}\Bigg| \\
&\quad \le C(n)\kappa^{1/4}\exp\left\{-\frac{(A_0-T)}{4}\frac{2^n}{c^{(n)}}\left|2x + \frac{x^2}{M_n^2(T)}\right|\right\} \\
&\qquad \text{if } |x| < 2^{-n}\log\kappa^{-1}, \qquad \text{(A.6)}
\end{aligned}
$$

$$
\text{(A.7)} \quad |g_n(x,T)| \le C(n)\exp\left\{-\frac{(A_0-T)}{4}\frac{2^n}{c^{(n)}}\left|2x + \frac{x^2}{M_n^2(T)}\right|\right\} \quad \text{for } x \ge -M_n(T),
$$

$$
|M_n(T) - M_0(T)| \le C(n)\kappa^{1/2}, \qquad \left|Z_n(t) - \frac{\sqrt{\pi}}{\sqrt{A_0-t}}\right| \le C(n)\kappa^{1/2}
$$

with some constant $C(n)$ which may depend on n but not on the parameter κ of the model.

The previous results are sufficient to handle the functions $g_n(x,T)$ for small indices $n \le N$. To work with indices $n \ge N$ we have to introduce, similarly to the argument in Section 4, the regularization of the functions $g_n(x,T)$, the linearization $\bar{\mathbf{U}}_n$ and $\mathbf{U}_n$ of the operators $\bar{\mathbf{R}}_n$ and $\mathbf{R}_n$ and to describe their action in the Fourier space.

Define the regularization of the function $g_n(x,T)$ as

$$
\varphi_n(g_n(x,T)) = \frac{\partial\varphi_n(f_n(x,T))}{\partial T}. \qquad \text{(A.8)}
$$

We want to approximate the operator $\mathbf{R}_n$ with a simpler operator $\mathbf{U}_n$ in analogy with the approximation of $\mathbf{Q}_n$ by $\mathbf{T}_n$. Then we formulate and prove some inductive hypothesis about the behaviour of the operators $\mathbf{R}_n$ and $\mathbf{U}_n$.

A natural approximation of the operators $\bar{\mathbf{R}}_n$ and $\mathbf{R}_n$ by some operators $\bar{\mathbf{U}}_n$ and $\mathbf{U}_n$ can be obtained by differentiating $\bar{\mathbf{T}}_n\varphi(f_n(x,T))$ and $\mathbf{T}_n\varphi_n(f_n(x,T))$ with respect to the variable T. These considerations suggest the definition of the operators

$$
\begin{aligned}
\bar{\mathbf{U}}_n\varphi_n(f_n(x,T)) = 2\int_{u\in\mathbb{R}^1,\mathbf{v}\in\mathbb{R}^{r-1}} & e^{-\mathbf{v}^2}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}+u+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\\
\Bigg\{\varphi_n\left(g_n\left(\frac{x}{\bar{c}_{n+1}}-u+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)&\\
-\mathbf{v}^2\frac{M_n'(T)}{2M_n(T)^2}\frac{\partial}{\partial x}\varphi_n & \left(f_n\left(\frac{x}{\bar{c}_{n+1}}-u+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\Bigg\}\,du\,d\mathbf{v}
\end{aligned}
$$

with the functions $g_n(x,T)$ and $\varphi_n(g_n(x,T))$ defined in (A.3) and (A.8) and

$$
\mathbf{U}_n\varphi_n(f_n(x,T)) = \mathbf{U}_n^{(1)}\varphi_n(f_n(x,T)) + \mathbf{U}_n^{(2)}\varphi_n(f_n(x,T))
$$

with

$$
\begin{aligned}
\mathbf{U}_n^{(1)}\varphi_n(f_n(x,T)) &= \frac{8}{\bar{c}_{n+1}\Gamma(\frac{r-1}{2})V(S^{r-2})}\\
&\int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} e^{-\mathbf{v}^2}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}+u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\\
&\qquad\varphi_n\left(g_n\left(\frac{x}{\bar{c}_{n+1}}-u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\,du\,d\mathbf{v},
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbf{U}_n^{(2)}\varphi_n(f_n(x,T)) &= \frac{8}{\bar{c}_{n+1}\Gamma(\frac{r-1}{2})V(S^{r-2})}\\
&\int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} e^{-\mathbf{v}^2}\left(\frac{(r-1)M_n'(T)}{4M_n^2(T)}-\mathbf{v}^2\frac{M_n'(T)}{2M_n(T)^2}\right)\\
&\qquad\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}+u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\\
&\qquad\frac{\partial}{\partial x}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}-u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)\,du\,d\mathbf{v}.
\end{aligned}
$$

We can calculate the Fourier transform of the functions $\bar{\mathbf{U}}_n\varphi_n(f_n(x,T))$, $\mathbf{U}_n^{(1)}\varphi_n(f_n(x,T))$ and $\mathbf{U}_n^{(2)}\varphi_n(f_n(x,T))$ by expressing them with the help of convolutions. This is similar to the proof of formula (4.17). In the calculations we exploit the following identity. As simple integration by parts shows $\widetilde{\frac{\partial}{\partial x}\varphi_n}(f_n(\xi)) = \int e^{i\xi x}\frac{\partial}{\partial x}\varphi_n(f_n(x))\,dx = -i\xi\tilde{\varphi}(f_n(\xi))$. Hence we get that

$$
\tilde{\bar{\mathbf{U}}}_n\tilde{\varphi}_n(f_n(\xi,T))
$$

$$= \frac{\bar{c}_{n+1}\Gamma(\frac{r-1}{2})V(S^{r-2})}{2} \frac{\tilde{\varphi}_n\left(f_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right)}{\left(1 + i\frac{\bar{c}_{n+1}}{2M_n(T)}\xi\right)^{(r-1)/2}} \tilde{\varphi}_n\left(g_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right)$$
$$+ i\frac{\bar{c}_{n+1}^2\Gamma(\frac{r+1}{2})V(S^{r-2})}{8} \frac{M_n'(T)}{M_n(T)^2}\xi \frac{\tilde{\varphi}_n^2\left(f_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right)}{\left(1 + i\frac{\bar{c}_{n+1}}{2M_n(T)}\xi\right)^{(r+1)/2}},$$

$$\tilde{\mathbf{U}}_n^{(1)}\tilde{\varphi}_n(f_n(\xi, T)) = 2\frac{\exp\left\{i\frac{(r-1)\bar{c}_{n+1}\xi}{4M_n(T)}\right\}}{\left(1 + i\frac{\bar{c}_{n+1}}{2M_n(T)}\xi\right)^{(r-1)/2}} \tilde{\varphi}_n\left(f_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right) \tilde{\varphi}_n\left(g_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right), \tag{A.9}$$

and

$$\tilde{\mathbf{U}}_n^{(2)}\tilde{\varphi}_n(f_n(\xi, T)) = -i\frac{\bar{c}_{n+1}(r-1)M_n'(T)}{2M_n(T)^2} \frac{\exp\left\{i\frac{(r-1)\bar{c}_{n+1}\xi}{4M_n(T)}\right\}}{\left(1 + i\frac{\bar{c}_{n+1}}{2M_n(T)}\xi\right)^{(r-1)/2}}\xi$$
$$\tilde{\varphi}_n^2\left(f_n\left(\frac{\bar{c}_{n+1}}{2}\xi, T\right)\right)\left(1 - \frac{1}{1 + i\frac{\bar{c}_{n+1}}{2M_n(T)}\xi}\right). \tag{A.10}$$

The above relation can also be extended to a larger set of the variables ξ in the complex plane by means of analytic continuation.

Now we formulate the inductive hypotheses we want to prove in the Appendix.

Property $K_1(n)$.

$$-\frac{dM_n(T)}{dT} > 0.$$

Property $K_2(n)$.

$$\begin{aligned} |g_n(x, T)| &= \left|\frac{\partial}{\partial T} f_n(x, T)\right| \\ &< K\left|\frac{dM_n(T)}{dT}\right| \exp\left\{-\frac{1}{\sqrt{\beta_n(T)}}\left|2x + \frac{x^2}{c^{(n)}M_n(T)}\right|\right\} \\ &\qquad \text{if } x > -c^{(n)}M_n(T) \end{aligned}$$

with a universal constant K.

Property $K_3(n)$.

$$\begin{aligned} &|g_n(x, T) - \mathbf{U}_{n-1}\varphi_{n-1}(f_{n-1}(x, T))| \\ &\quad < K\left|\frac{dM_n(T)}{dT}\right| \frac{\beta_n(T)}{c^{(n)}} \exp\left\{-\frac{1.4}{\sqrt{\beta_n(T)}}\left|2x + \frac{x^2}{c^{(n)}M_n(T)}\right|\right\} \\ &\qquad \text{if } x > -c^{(n)}M_n(T) \end{aligned}$$

with a universal constant K. The inequality remains valid if the function $g_n(x, T)$ is replaced by its regularization $\varphi_n(g_n(x, T))$.

The following property $K_4(n)$ which gives a bound on the Fourier transform of $\varphi_n(g_n(x,T))$ is an analog of Property $J(n)$.

Property $K_4(n)$.

$$|\tilde\varphi_n(g_n(-is,T)| \;=\; \left|\int e^{sx}\varphi_n(g_n(x,T)\,dx\right| \le \beta_n^{3/2}(T)s^2\left|\frac{dM_n(T)}{dT}\right| e^{\beta_n(T)s^2}$$
$$\text{if } |s| < \frac{2}{\sqrt{\beta_{n+1}(T)}}.$$

In Property $K_4(n)$ we formulated a weaker estimate than in $J(n)$. It is enough to have a good bound on the moment generating function, i.e. on the analytic continuation of the Fourier transform to the imaginary axis together with the trivial estimate $|\tilde\varphi_n(g_n(-is+t,T)| \le \tilde\varphi_n(g_n(-is,T)$ for all t.

The main result of the Appendix is the following Proposition A.

Proposition A. *Let the properties $K_1(m)$, $K_2(m)$, $K_3(m)$ and $K_4(m)$ hold in a neighbourhood of a parameter T together with the property $\frac{\beta_m(T)}{c^m} \le \eta$ (with the same small number $\eta > 0$ which appeared in the proof of Propositions 4.1 and 4.2) for all $N \le m \le n$, and let also the inductive hypotheses $I(n)$ and $J(n)$ be also satisfied. Then the properties $K_1(n+1)$, $K_2(n+1)$, $K_3(n+1)$ and $K_4(n+1)$ also hold for this parameter T. The expression*

$$\delta_n(T) = \frac{d}{dT}\left(m_n(T) - \frac{r-1}{4M_n(T)}\right) = \frac{dm_n(T)}{dT} + \frac{r-1}{4M_n^2(T)}\frac{dM_n(T)}{dT}$$

satisfies the inequality

$$|\delta_n(T)| \le C\left|\frac{dM_n(T)}{dT}\right| \frac{\beta_{n+1}(T)}{c^{(n+1)}}\beta_{n+1}(T) \tag{A.11}$$

with an appropriate $C > 0$, where $m_n(T)$ was defined in (2.15).

If we want to apply Proposition A, then first we have to show that properties $K_1(n)$, $K_2(n)$, $K_3(n)$ and $K_4(n)$ hold for $n = N$ if $T < c_0A_0/2$. This can be done with the help of an argument similar to the proof in the Corollary of Lemma 1 in [5]. Property $K_1(N)$ holds since $\frac{dM_N(T)}{dT}$ hardly differs from $\frac{dM_0(T)}{dT}$. Property $K_2(N)$ can be proved by means of relations (A.6) and (A.7). In the proof of Property $K_3(N)$ still the following additional observation is needed. Relation (A.6) remains valid if the function $g_N(x,T) = \mathbf{R}_N f_{N-1}(x,T)$ is replaced by $\mathbf{U}_N\varphi_{N-1}(f_{N-1}(x,T))$ in this formula. (The term $\frac{dM_n(T)}{dT}$ on the right-hand side of the inductive hypotheses do not play an important role for $n = N$. It is strongly separated from zero if $T \le c_0A_0/2$.)

Relation $K_4(N)$ can be proved again with the help of formulas (A.6), (A.7) and the relations

$$\int \varphi_n(g_n(x,T))\,dx = \int x\varphi_n(g_n(x,T))\,dx = 0.$$

These relations imply that the value of the function $\tilde\varphi_n(g_n(s,T)$ and of its first derivative is zero in the point $s = 0$. Hence it is enough to give a good estimate of the second derivative of $\tilde\varphi_n(g_n(s,T)$.

Let us formulate the following Corollary of Proposition A.

Corollary. *Under the Conditions of Theorem 3.4 the set of the points T for which (n,T) is in the low temperature region is an interval $(0,T_n)$ for all $n \geq 0$. The inductive hypotheses $K_1(n)$, $K_2(n)$, $K_3(n)$ and $K_4(n)$ hold for all $T \in (0,T_n)$.*

Proof of the Corollary. The Corollary simply follows from Proposition A by means of induction with respect to n. In this induction we assume the statement of the Corollary for a fixed n together with the assumption that $\beta_n(T)$ is monotone increasing in the variable T for $0 < T < T_n$. The Corollary and the additional assumption hold for $n = N$ with $T_N = c_0 A_0/2$. If properties $K_1(n)$, $K_2(n)$, $K_3(n)$ and $K_4(n)$ hold for n, then because of Property $K_1(n)$ the function $M_n(T)$ is monotone decreasing and $\beta_{n+1}(T)$ is monotone increasing in the variable T. Then $T_{n+1} = \min(T_n, \max(T\colon \beta_{n+1}(T) < \eta))$, and by Proposition A the statements of the Corollary hold for $n+1$. □

Before turning to the proof of Proposition A we prove Theorem 8.1 with its help.

Proof of Theorem 8.1. The proof of Part a.) is contained in the previous estimates of the Appendix. Part b.) can be obtained by differentiating the second formula in (2.16), and applying formula (A.11). □

Proof of Proposition A. Some calculation yields that because of properties $K_4(n)$, $J(n)$ relations (A.9) and (A.10) the Fourier transforms

$$\tilde{\mathbf{U}}_n^{(1)}\tilde{\varphi}_n(f_n(\xi,T)), \quad \tilde{\mathbf{U}}_n^{(2)}\tilde{\varphi}_n(f_n(\xi,T))$$

satisfy the inequalities

$$\begin{aligned}&\left|\tilde{\mathbf{U}}_n^{(1)}\tilde{\varphi}_n(f_n(t+is,T))\right| \\ &\quad\leq 2\left|\frac{dM_n(T)}{dT}\right|\left(\frac{\bar{c}_{n+1}}{2}s\right)^2\beta_n(T)^{3/2} \\ &\qquad \exp\left\{\left(\frac{\bar{c}_{n+1}^2\beta_n(T)}{2}+\frac{1}{M_n^2(T)}\right)s^2\right\}\frac{1}{1+\alpha_n(T)t^2}\end{aligned}$$

and

$$\begin{aligned}&\left|\tilde{\mathbf{U}}_n^{(2)}\tilde{\varphi}_n(f_n(t+is,T))\right| \\ &\quad\leq \frac{\bar{c}_{n+1}^2|M_n'(T)|}{8M_n(T)^3}(s^2+t^2)\exp\left\{\left(\frac{\bar{c}_{n+1}^2\beta_n(T)}{2}+\frac{1}{M_n^2(T)}\right)s^2\right\} \\ &\qquad \frac{1}{(1+\alpha_n(T)t^2)^2\left(1-\frac{\bar{c}_{n+1}}{2M_n(T)}s\right)}\end{aligned}$$

for $|s| < \frac{4}{\bar{c}_{n+1}\sqrt{\beta_{n+1}(T)}}$.

The function $\varphi_n(g_n(x,T))$ can be computed by means of the application of the inverse Fourier transformation and by replacement of the domain of integration from the real line to the line

$$\left\{z = i\,\mathrm{sign}\,x\,\frac{2}{\sqrt{\beta_{n+1}(T)}}+t,\ t\in\mathbb{R}^1\right\}.$$

We get, by applying the above estimates for the Fourier transforms $\tilde{\mathbf{U}}_n^{(1)}$ and $\tilde{\mathbf{U}}_n^{(2)}$ and exploiting the relation $\frac{M_n'(T)}{2M_n(T)^3} \le \frac{1}{200}\beta_{n+1}(T)^2 \frac{dM_n(T)^2}{dT}$ together with the fact that the constants $\alpha_n(T)$ and $\beta_n(T)$ introduced in the definition of Properties $I(n)$ and $J(n)$ have the same order of magnitude that

$$\begin{aligned}(\text{A.12}) \quad |\mathbf{U}_n\varphi_n(f_n(x,T))| &\le -K_1\frac{dM_n(T)}{dT}e^{-2|x|\beta_{n+1}(T)^{-1/2}} \\ &\le -K_2\frac{dM_n(T)}{dT}\exp\left\{-\frac{1}{\sqrt{\beta_{n+1}(T)}}\left|2x+\frac{x^2}{c^{(n+1)}M_{n+1}(T)}\right|\right\}.\end{aligned}$$

The estimates obtained for $\tilde{\mathbf{U}}_n^{(1)}$ and $\tilde{\mathbf{U}}_n^{(2)}$ yield, with the choice $t=0$ and some calculation that

$$\begin{aligned}\|\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(-is,T))| \quad &\le \quad -\frac{9}{10}\frac{dM_n(T)}{dT}\beta_{n+1}(T)^{3/2}s^2e^{\beta_{n+1}(T)s^2} \\ (\text{A.13}) \qquad\qquad & \qquad\qquad \text{if } |s|<\frac{2}{\sqrt{\beta_{n+2}(T)}}.\end{aligned}$$

(In the proof of Property $K_4(n+1)$ it will be important that the right-hand side of (A.13) is less than the expression at the right-hand side of the formula which defines Property $K_4(n+1)$.)

We need a good estimate on the difference of $\mathbf{R}_nf_n(x,T)-\mathbf{U}_n\varphi_n(f_n(x,T))$ and its Fourier transform. These expressions can be bounded similarly to the proof of the corresponding inequalities in the proof of Proposition 3 in paper [5]. One has to compare the difference of the corresponding terms in the expressions $\mathbf{Q}_n\varphi_n(f_n(x,T))$ and $\mathbf{R}_n\varphi_n(f_n(x,T))$. Some calculation yields that

$$(\text{A.14}) \qquad \left|Z_n(T)-\frac{\bar{c}_{n+1}\sqrt{\pi}}{2}\right| \le \frac{\beta_n(T)}{c^{(n)}}, \qquad \left|m_n(T)+\frac{1}{4M_n(T)}\right| \le \frac{\beta_n(T)}{c^{(n)}}\sqrt{\beta_n(T)},$$

$$\begin{aligned}(\text{A.15}) \qquad \left|\frac{dZ_n(T)}{dT}\right| \quad &\le \quad -K\frac{\beta_n(T)}{c^{(n)}}\beta_n^{1/2}(T)\frac{dM_n(T)}{dT}, \\ \left|\frac{d}{dT}\left(m_n(T)+\frac{1}{4M_n(T)}\right)\right| \quad &\le \quad -K\frac{\beta_{n+1}(T)}{c^{(n+1)}}\beta_{n+1}(T)\frac{dM_n(T)}{dT}.\end{aligned}$$

Relation (A.11) is a consequence of (A.15). Property $K_1(n+1)$ can be deduced from the above inequalities, since

$$\begin{aligned}-\frac{dM_{n+1}(T)}{dT} \quad &= \quad -\frac{dM_n(T)}{dT}+\frac{1}{c^{(n+1)}}\frac{dm_n(T)}{dT} \\ &\ge \quad -\frac{dM_n(T)}{dT}\left(1-\frac{1}{c^{(n+1)}}\left(\frac{1}{4M_n^2(T)}+K\frac{\beta_n^2(T)}{c^{(n)}}\right)\right) \\ &\ge \quad -\frac{1}{2}\frac{dM_n(T)}{dT}.\end{aligned}$$

Now we turn to the proof of Property $K_3(n+1)$. We do it by estimating the errors we make by replacing the terms in the sum at the right-hand side of (A.5) by their natural approximation if we replace $\mathbf{R}_nf(x,T)$ by $\mathbf{U}_n\varphi_n(f_n(x,T))$. (We also use formula (A.4) in that calculation.) We get, by applying again inequalities

(A.14) and (A.15) together with the estimates obtained for $f_n(x,T)$, similarly to the proof of the estimates in the lemmas needed for the proof of Lemma 3 in [3] that

$$\left|\frac{\bar{\mathbf{Q}}_n f_n(x+m_n(T),T)}{Z_n^2(T)}\frac{dZ_n(T)}{dT}\right|$$
$$\le K\frac{\beta_n(T)}{c^{(n)}}\left|\frac{dM_n(T)}{dT}\right|\exp\left\{\frac{-1.5}{\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n)}M_n(T)}\right|\right\}$$
$$\text{if } x \ge -c^{(n+1)}M_{n+1}(T),$$

$$\left|\frac{\frac{\partial}{\partial x}\bar{\mathbf{Q}}_n f_n(x+m_n(T),T)}{Z_n(T)}\frac{dm_n(T)}{dT} - \frac{8(r-1)}{\bar{c}_{n+1}\Gamma(\frac{r-1}{2})V(S^{r-2})}\frac{M_n'(T)}{4M_n^2(T)}\times\right.$$
$$\int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} e^{-\mathbf{v}^2}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}+u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)$$
$$\left.\frac{\partial}{\partial x}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}-u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)du\,d\mathbf{v}\right|$$
$$\le K\frac{\beta_n(T)}{c^{(n)}}\left|\frac{dM_n(T)}{dT}\right|\exp\left\{\frac{-1.5}{\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n)}M_n(T)}\right|\right\}$$
$$\text{if } x \ge -c^{(n+1)}M_{n+1}(T),$$

and

$$\left|\frac{\bar{\mathbf{R}}_n^{(2)} f_n(x+m_n(T),T)}{Z_n(T)} + \frac{8}{\bar{c}_{n+1}\Gamma(\frac{r-1}{2})V(S^{r-2})}\frac{M_n'(T)}{2M_n^2(T)}\times\right.$$
$$\int_{u\in\mathbb{R}^1,\,\mathbf{v}\in\mathbb{R}^{r-1}} \mathbf{v}^2 e^{-\mathbf{v}^2}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}+u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)$$
$$\left.\frac{\partial}{\partial x}\varphi_n\left(f_n\left(\frac{x}{\bar{c}_{n+1}}-u-\frac{r-1}{4M_n(T)}+\frac{\mathbf{v}^2}{2M_n(T)},T\right)\right)du\,d\mathbf{v}\right|$$
$$\le K\frac{\beta_n(T)}{c^{(n)}}\left|\frac{dM_n(T)}{dT}\right|\exp\left\{-\frac{1.5}{\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n)}M_n(T)}\right|\right\}$$
$$\text{if } x \ge -c^{(n+1)}M_{n+1}(T).$$

To prove Property $K_3(n+1)$ we still need an estimate which compares the terms

$$\frac{\bar{\mathbf{R}}_n^{(1)} f_n(x+m_n(T),T)}{Z_n(T)} \quad \text{and} \quad \mathbf{U}_n^{(1)}\varphi_n(f_n(x,T)).$$

We claim that

$$\left|\frac{\bar{\mathbf{R}}_n^{(1)} f_n(x+m_n(T),T)}{Z_n(T)} - \mathbf{U}_n^{(1)}\varphi_n(f_n(x,T))\right| \le K\frac{\beta_n(T)}{c^{(n)}}\left|\frac{dM_n(T)}{dT}\right|$$

$$\exp\left\{-\frac{1.5}{\sqrt{\beta_n(T)}}\left|2x+\frac{x^2}{c^{(n)}M_n(T)}\right|\right\} \quad \text{if } x \geq -c^{(n+1)}M_{n+1}(T).$$

This estimate can be proved by means of Property $K_3(n)$. With the help of this relation it can be shown that a negligible error is committed if in the integrals defining $\bar{\mathbf{R}}_n^{(1)} f_n(x+m_n(T),T)$ and $\mathbf{U}_n^1\varphi_n(f_n(x,T))$ the functions g_n and $\varphi_n(g_n)$ are replaced by the function $\mathbf{U}_n\varphi_{n-1}(f_{n-1})$. After this replacement the proof of Theorem 3.2 can be adapted, since we can bound not only the function $\mathbf{U}_n\varphi_{n-1}(f_{n-1})$, but also its partial derivative with respect to the variable x.

These estimates together imply Property $K_3(n+1)$, and some calculation shows that a version of Property $K_3(n+1)$, where the function $g_{n+1}(x,T)$ is replaced by its regularization $\varphi_{n+1}(g_{n+1}(x,T))$ is also valid. Since we gave a good estimate on $\mathbf{U}_n\varphi_n(f_n(x,T))$ in (A.12), some calculation yields the proof of Property $K_2(n+1)$. It remained to prove Property $K_4(n+1)$.

Because of (A.13) and (A.15) (The latter formula together with (2.15) and (2.16) imply that formula (A.13) remain valid with a slightly bigger coefficient if the term $\frac{dM_n(T)}{dT}$ is replaced by $\frac{dM_{n+1}(T)}{dT}$ in it), it is enough to give a good bound on the difference $\tilde{\varphi}_{n+1}(g_{n+1}(-is))-\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(-is))$ to prove property $K_4(n+1)$. This can be done in the following way:

By applying the modified property of $K_3(n+1)$, where the function $g_{n+1}(x)$ is replaced by $\varphi_{n+1}g_{n+1}(x)$ we get that

$$\begin{aligned}
&\left|\frac{\partial^2}{\partial s^2}\left[\tilde{\varphi}_{n+1}(g_{n+1}(-is,T))-\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(-is,T))\right]\right| \\
&\qquad \leq -\int x^2\frac{dM_n}{dT}\frac{\beta_{n+1}(T)}{c^{(n+1)}}\exp\left\{\left(|t|-\frac{2.8}{\sqrt{\beta_{n+1}(T)}}\right)x\right\}\,dx \\
&\qquad \leq K\frac{\beta_{n+1}^{5/2}(T)}{c^{(n+1)}}\frac{dM_n^2}{dT}
\end{aligned}$$

if $|s| \leq \frac{2}{\sqrt{\beta_{n+2}(T)}}$.

Since

$$\begin{aligned}
&\tilde{\varphi}_{n+1}(g_{n+1}(0,T))-\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(0,T)) \\
&\qquad = \frac{\partial}{\partial s}\left(\tilde{\varphi}_{n+1}(\tilde{g}_{n+1}(-is,T)-\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(-is,T))\right)\Big|_{s=0} = 0,
\end{aligned}$$

the last relation implies that

$$\left|\tilde{\varphi}_{n+1}(\tilde{g}_{n+1}(-is,T)-\tilde{\mathbf{U}}_n\tilde{\varphi}_n(f_n(-is,T)\right| \leq -K\frac{\beta_{n+1}(T)}{c^{(n+1)}}\beta_{n+1}^{3/2}\frac{dM_n(T)}{dT}s^2$$

if $|s| \leq \frac{2}{\sqrt{\beta_{n+2}(T)}}$. This estimate together with relation (A.13) imply Property $K_4(n+1)$ if the number η which is an upper bound for $\beta_{n+1}(T)/c^{(n+1)}$ is chosen sufficiently small. Proposition A is proved. □

Appendix B. The proof of proposition 1.2

Condition 1. We have that for $n \geq 1$,

$$1 < c_n = \left(\frac{1+an}{1+a(n-1)}\right)^{\lambda}.$$

Observe that c_n is decreasing and

$$\lim_{n\to\infty} c_n = 1, \qquad c_n \leq c_1 = (1+a)^{\lambda}.$$

This implies Condition 1.

Condition 2. We have that

$$(1+an)^{\lambda} \sum_{j=n}^{n+K} (1+aj)^{-\lambda} \geq \frac{K(1+an)^{\lambda}}{(1+a(n+K))^{\lambda}} \to K$$

as $n \to \infty$. This implies Condition 2.

Condition 3. For $k \leq n/2$ we estimate

$$l_k \sum_{j=k}^{n} l_j^{-1} = (1+ak)^{\lambda} \sum_{j=k}^{n} (1+aj)^{-\lambda} \geq C(1+ak)^{\lambda}(1+ak)^{-\lambda+1} = C(1+ak)^{-1}$$

and for $k > n/2$ and $n \geq j \geq k$ we estimate

$$l_k l_j^{-1} \geq C_0 > 0$$

hence

$$l_k \sum_{j=k}^{n} l_j^{-1} \geq C_0(n-k+1).$$

Thus,

$$\sum_{k=1}^{n} \left(l_k \sum_{j=k}^{n} l_j^{-1}\right)^{-2} \leq C^{-2} \sum_{k=1}^{n/2} (1+ak)^{-2} + C_0^{-2} \sum_{k=n/2}^{n} (n-k+1)^{-2} \leq C_1.$$

Condition 3 is checked.

Conditions 4 and 5 are obvious. □

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