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WHAT MATHEMATICAL BILLIARDS TEACH US ABOUT STATISTICAL PHYSICS?

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ABSTRACT. We survey applications of the theory of hyperbolic (and to a lesser extent non hyperbolic) billiards to some fundamental problems of statistical physics and their mathematically rigorous derivations in the framework of classical Hamiltonian systems.

1. INTRODUCTION

The pursuit of mathematical rigour has arguably pervaded physical sciences from as early as Galileo's times. Statistical mechanics, which was founded in the second half of the 19$^{\text{th}}$ century by the likes of Rudolf Clausius (1822–1888), James Clerk Maxwell (1831–1879), Ludwig Boltzmann (1844–1906), and Josiah Willard Gibbs (1839–1903), has ever since remained at the forefront of this endeavour.

However, as Landau and Lifshitz wrote back in 1937–1939 [93, Preface to early Russian editions],

> It is a fairly widespread delusion among physicists that statistical physics is the least well-founded branch of theoretical physics. Reference is generally made to the point that some of its conclusions are not subject to rigorous mathematical proof; and it is overlooked that every other branch of theoretical physics contains just as much non-rigorous proofs, although these are not regarded as indicating an inadequate foundation for such branches.

The same authors were careful to further warn us that, in general, "mathematical rigour is not readily attainable in theoretical physics." And, indeed, statistical physics may have helped to shape their view.

To be sure, the foundations of statistical mechanics were, from the onset, the subject of intense debate. Foremost among the issues that came under scrutiny is the ergodic hypothesis [59], which was understood to be a preliminary to showing that gas molecules behave in a stochastic fashion. Roughly speaking, an ergodic system[1]

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[1]Herein, ergodicity is meant as a property of a probability measure, and, unless explicitly stated, not of an infinite one.

must be such that almost all initial conditions yield trajectories that fill evenly the subset of the phase space compatible with its conservation laws[2]. There was, however, compelling evidence that *Hamiltonian systems* are generically not ergodic, which was ultimately encapsulated in the work of Markus and Meyer [106]; see section 3 for further details. The consensus among physicists and mathematicians alike thus emerged that ergodic Hamiltonian systems are the exception rather than the rule, see e.g. [131, p 137] or [19, Section 4], and led some authors to question, paraphrasing Wigner [147], the "unreasonable effectiveness of statistical mechanics" [75, p 75]. The subject is indeed rife with controversy [56].

The theory of *hyperbolic billiards*, initiated by the seminal 1963 and 1970 works of Sinai [126, 127], provides a singular exception to this unpleasant state of affairs, which has been hailed as a milestone in the development of ergodic theory; see [124]. To this day, and apart from the gas of hard balls, only a few examples of ergodic Hamiltonian systems have been identified [51, 87, 91, 92]. Although generic Hamiltonian systems with a fixed number of degrees of freedom should indeed not be expected to be ergodic, it must be said that Boltzmann's ergodic hypothesis refers to the so-called thermodynamic limit, where the number of particles grows to infinity. It remains unclear whether the fulfilment of this form of the ergodic hypothesis does require the type of hard core interaction exhibited by billiards or is in fact applicable to Hamiltonian systems with smooth interactions. The conjectures drawn in Subsection 3.3 reflect similar expectations.

Although the singularities incurred by billiards lead to a number of conceptual and technical difficulties, Sinai's unique achievement lied primarily in creating the methods to handle them. Sinai's theory of hyperbolic billiards thus opened new perspectives for investigating the foundations of statistical physics. The main objective of this paper is to collect a few key problems of statistical physics to which billiard models can be successfully applied. When it comes to the derivation of fundamental laws in a Hamiltonian framework, billiard models—in spite of their singularities—are mathematically more easily tractable than other types of interactions.

The applications of billiard theory to statistical physics we wish to illustrate are the following:

(1) Boltzmann's ergodic hypothesis;
(2) Brownian motion;
(3) Fourier's law of heat conduction.

They will be addressed in separate sections.

Concerning item (2) above, we should remark that Einstein's 1905 proof [60]—however ingenious it was—was far from being rigorous[3]. The important and interesting feature we wish to illustrate is that the derivation of the macroscopic law

[2]Somewhat more precisely, ergodicity means that, while in equilibrium and over long periods of time, the time spent by a system in some region of the phase space of microstates with compatible energy is proportional to the volume of this region.

[3]Its conditions do not actually hold, but this fact, of course, does not diminish its utmost radical novelty and scientific significance; see the excellent review [55].

of Brownian motion[4] starts from the deterministic dynamics at the so-called microscopic level, which therefore entails accounting for breaking the time-reversal symmetry of the law of motion at the micro-scale to justify the irreversible diffusive behaviour at the macro-scale. Mathematically speaking, this derivation relies on the appropriate control of correlation decay.

We should also note that items (2) and (3) both deal with diffusive processes. However, in the former, the focus is on mass transport of a tagged particle, whereas, in the latter, energy is being transported. From a mathematical point of view, these problems are quite different; while the former can be modeled with a low-dimensional system, the latter ultimately results from the interactions of infinitely many particles. It should further be noted that the ideas discussed here are still under progress, even though, on a physical level, they are already rather well understood.

Beyond these three items, we also formulate a few open problems we consider interesting and draw up conjectures that we find compelling.

On the occasion of Sinai's award of the 2014 Abel prize, several surveys were dedicated to his numerous scientific achievements; see the volume [81], which includes three contributions devoted to the theory of hyperbolic billiards [27, 124, 137] and that cover a wide range of topics in detail. We will therefore refer to these surveys whenever suitable and go into further details only where we feel more recent developments so warrant. Moreover, we do not treat important problems such as quantum billiards, the Boltzmann-Grad limit (with a couple of exceptions), or the Boltzmann equation to name but a few.

We make one additional remark which pertains to the contents of our survey. There are mathematical or physical constructions that are isomorphic to billiard models. Just to name some of them, consider the Lorentz gas, the Rayleigh gas, or the wind–tree gas. If results related to them are obtained by the methods of hyperbolic billiards, we consider them relevant to the present survey. Otherwise our selection will be somewhat arbitrary; we mention topics that are connected to some interesting problem or phenomenon related to billiards.

About the structure of this paper: The models to be treated are introduced in Section 2, with relevant notations. In Sections 3, 4 and 5, the three problems mentioned above are respectively discussed. In Appendix, we recall two results related to two Conjectures presented in Subsection 3.3.

2. Models

We introduce below the different models that will be treated in the sequel. We start with the general definition of billiards in Subsection 2.1, and present several useful definitions. Lorentz processes and gases are subsequently defined in Subsections 2.2 and 2.3 respectively. For our purpose, the former refers to a single particle on a dispersing billiard table and the latter to a countable number of copies of it. In Subsection 2.4, examples of non-dispersing billiards are presented. They are generally referred to as wind–tree models. We then turn to higher-dimensional billiards,

[4] *Brownian motion* generically refers to the diffusive transport process of a tagged particle, which therefore implies the existence of the underlying transport coefficient, whose expression can be inferred through the Green-Kubo formula, or equivalent techniques; see, e.g., [67].

beginning with general hard ball systems in Subsection 2.5. A class of such systems with spatial ordering is presented in Subsection 2.6. We finish with the Rayleigh gas in Subsection 2.7.

2.1. **Billiards.** As far as notations go, we mainly follow [39] for planar billiards and [14] for multidimensional ones.

Billiards are defined in Euclidean domains bounded by a finite number of smooth boundary pieces. For our purpose a *billiard* is a dynamical system describing the motion of a point particle in a connected, compact domain $\mathcal{Q} \subset \mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. In general, the boundary $\partial\mathcal{Q}$ of the domain is assumed to be piecewise C^3-smooth, i.e. there are no corner points; if $0 < J < \infty$ is the number of such pieces, we can write $\partial\mathcal{Q} = \cup_{1\leq\alpha\leq J}\,\partial\mathcal{Q}_\alpha$. Connected components of $\mathbb{T}^d \setminus \mathcal{Q}$ are called *scatterers*. Motion is uniform inside $\mathcal{Q}$ and specular reflections take place at the boundary $\partial\mathcal{Q}$; in other words, a particle propagates freely until it collides with a scatterer, where it is reflected elastically, i.e. following the classical rule that the angle of incidence be equal to the angle of reflection.

Since the absolute value of the velocity is a first integral of motion, the phase space of our billiard is defined as the product of the set of spatial configurations by the $(d-1)$-sphere, $\mathcal{M} = \mathcal{Q} \times \mathbb{S}_{d-1}$, which is to say that every phase point $x \in \mathcal{M}$ is of the form $x = (q,\, v)$, with $q \in \mathcal{Q}$ and $v \in \mathbb{R}^d$ with norm $|v| = 1$. According to the reflection rule, $\mathcal{M}$ is subject to identification of incoming and outgoing phase points at the boundary $\partial\mathcal{M} = \partial\mathcal{Q} \times \mathbb{S}_{d-1}$. The billiard dynamics on $\mathcal{M}$ is called the *billiard flow* and denoted by $S^t : t \in (-\infty, \infty)$, where $S^t : \mathcal{M} \to \mathcal{M}$. The set of points defined by the trajectory going through $x \in \mathcal{M}$ is denoted $S^{\mathbb{R}}x$. The smooth, invariant probability measure of the billiard flow, μ on $\mathcal{M}$, also called the Liouville measure, is essentially the product of Lebesgue measures on the respective spaces, i.e. $\mathrm{d}\mu = \text{const.}\,\mathrm{d}q\,\mathrm{d}v$, where the constant is $(\mathrm{vol}\,\mathcal{Q}\ \mathrm{vol}\,\mathbb{S}_{d-1})^{-1}$.

For later reference we recall the time-discrete *collision map* (or billiard map, or Poincaré section map) of the billiard flow. Let us denote by $\partial\mathcal{M} = \partial\mathcal{Q} \times \mathbb{S}^+_{d-1}$ the set of all phase points with spatial coordinates at the boundary of a scatterer and velocities pointing outwards (here $\mathbb{S}^+_{d-1}$ refers to the corresponding hemisphere). For a point $x \in \mathcal{M}$, let $\tau(x) = \min\{s > 0 : S^s(x) \in \partial\mathcal{M}\}$ denote the first hitting time on the billiard boundary. Then for $(\xi,\, v) \in \partial\mathcal{M}$, τ is the first return time to the boundary and the collision map $T : \partial\mathcal{M} \to \partial\mathcal{M}$ is defined via $T(\xi,\, v) = S^\tau(\xi,\, v)$. The natural invariant measure for the collision map is then $\mathrm{d}\nu(\xi,\, v) = \text{const.}\,\langle n(\xi), v\rangle\,\mathrm{d}\xi\,\mathrm{d}v$ where $n(\xi)$ is the unit normal vector of the boundary $\partial\mathcal{Q}$ at $\xi \in \partial\mathcal{Q}$, directed inward $\mathcal{Q}$, $\langle\cdot,\cdot\rangle$ is the scalar product, and the constant is $(d-1)\,(\mathrm{vol}\,\partial\mathcal{Q}\ \mathrm{vol}\,\mathbb{S}_{d-2})^{-1}$.

We end with the following two sets of definitions.

Definition 2.1 (Dispersing and semi-dispersing Billiards)**.** We say that a billiard is *dispersing* (resp. *semi-dispersing*) if its smooth boundary pieces, i.e. the scatterers, are strictly convex (resp. convex) when viewed from inside $\mathcal{Q}$. Because of these *convexity properties*, semi-dispersing billiards, whose pre-eminent examples are hard ball systems in parallelepipeds or on tori, exhibit different degrees of *hyperbolicity*. In this paper, we generally refer to planar dispersing billiards as *Sinai billiards*;

see [127]. Sinai was also responsible for initiating the study of higher-dimensional dispersing and semi-dispersing billiards; see in particular [36, 129].

Definition 2.2 (Infinite and finite horizons)**.** Collision–free orbits are a distinctive feature of some billiards which are said to have infinite horizons.

(1) Denote by $\mathcal{M}_{\text{free}} \subset \mathcal{M}$ the subset of collision–free orbits, i.e.

$$\mathcal{M}_{\text{free}} = \{x \in \mathcal{M} \,:\, S^{\mathbb{R}}x \cap \partial\mathcal{M} = \emptyset\}\,.$$

(2) The billiard has *finite horizon* if $\mathcal{M}_{\text{free}} = \emptyset$. Otherwise it has *infinite horizon.*

This notion applies to the models defined in Subsections 2.2-2.4 below.

2.2. **Lorentz Process.** The Lorentz process was introduced in 1905 by H. A. Lorentz [99] for the study of a dilute electron gas in a metal.[5] While Lorentz considered the motion of a collection of independent pointlike particles moving uniformly among immovable metalic ions modeled by elastic spheres, we consider here the uniform motion of a single pointlike particle in a fixed array of spherical scatterers with which it interacts via elastic collisions[6]; see, however, Subsection 2.3 below for the extension of the process to many particles.

Thus defined, the *Lorentz process* is the billiard dynamics of a point particle on a billiard table $\mathcal{Q} = \mathbb{R}^d \setminus \cup_{\alpha=1}^{\infty} O_\alpha$, where the scatterers O_α, $1 \le \alpha \le \infty$, are strictly convex with C^3-smooth boundaries. Generally speaking, it could happen that $\mathcal{Q}$ has several connected components. For simplicity, however, we assume that the scatterers are disjoint and that $\mathcal{Q}$ is unbounded and connected. The phase space of this process is then given according to the above definition, namely $\mathcal{M} = \mathcal{Q} \times \mathbb{S}_{d-1}$.

It should finally be noted that, under this assumption, the Liouville measure $\mathrm{d}\mu = \mathrm{d}q\,\mathrm{d}v$, while invariant, is infinite. If, however, there exists a regular lattice of rank d for which we have that, for every point z of this lattice, $\mathcal{Q} + z = \mathcal{Q}$, then we say that the corresponding Lorentz process is *periodic.* In this case, the Liouville measure is finite (more exactly, its factor with respect to the lattice is finite).

2.3. **Lorentz Gas.** A closely related object is the *Lorentz gas*, by which me mean the joint motion of a countable number of completely independent Lorentz processes[7]. By keeping with our previous notation, the phase space of the Lorentz gas is thus $\mathcal{M}^\infty = \Pi_{j=1}^{\infty}\mathcal{M}_j$ where each $\mathcal{M}_j$ is a copy of $\mathcal{M}$, the phase space of the Lorentz process. Furthermore, we only consider points in $\mathcal{M}^\infty$ which are locally finite, that is, for every bounded $\mathcal{A} \subset \mathbb{R}^d$, $\sum_{j=1}^{\infty} \mathbb{1}_{q_j \in \mathcal{A}} < \infty$.

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[5] In fact, Drude [53] had introduced a similar model as early as 1900. While the models of Drude and Lorentz gave different ratios between the thermal and electrical conductivities, both were in accordance with the empirically observed Wiedemann–Franz law and, in that respect, provided decisive early contributions to the kinetic theory of gases [80].

[6] More generally, the model can be extended to strictly convex scatterers rather than spherical ones.

[7] In our nomenclature, we thus make a distinction between the dynamics of a single particle and that of countably many of them. The former case refers to the *process*, or a *flow*, defined in the previous Subsection, and the latter to a *gas.* This usage differs from that adopted by many authors who use the notions of gas and process interchangeably.

Now the smooth invariant measure of the dynamics is a Poissonian measure[8] in $\mathcal{M}^\infty$, with a uniform density; see [105] for more details.

We note here that, in principle, we could permit the velocity space of the Lorentz gas to be the whole of $\mathbb{R}^d$ rather than $\mathbb{S}_{d-1}$, as, in fact, did Lorentz who considered Maxwellian distributions of velocities. However, since the energies of the particles are individually conserved, this would be a trivial generalisation.

2.4. **Wind–Tree Models.** The wind–tree gas was initially proposed by P. and T. Ehrenfest in 1912, [59, Appendix to Section 5] as a "much simplified model" aiming to understand "what the position of the Stosszahlansatz is in the Maxwell-Boltzmann investigations." Since then, it has been extended and generalised in many ways.

Generally speaking, the *wind–tree process* (or *flow*) is analogous to the Lorentz process. Its distinctive feature, however, is that it is not dispersing and, in that sense, is a neutral version of it. The main difference with the Lorentz process is indeed that the scatterers of the wind–tree process are parallepipeds (rhomboids, cuboids, . . .), usually parallelly positioned[9]. Consequently no hyperbolicity is present.

By extension, the *wind–tree gas* consists of countably many independent copies of wind–tree processes. Notations analogous to those of Subsection 2.2-2.3 carry over.

We first mention two planar models with identical rectangular scatterers, whose sides are parallel to the coordinate axes.

2.4.1. *Aperiodic wind trees.* The first one of them, actually a family of models, was studied in [103]. The authors consider the set of unit square cells with *square* scatterers of sides $2\,r$, $\frac{1}{4} \leq r < \frac{1}{2}$, centered at points $(a, b) \in [0, 1]^2$, and such that the scatterers are contained within the unit cells. This set may thus be parametrised by

$$\mathcal{A} = \{(a,\, b) : r \leq a,\, b \leq 1 - r\}\,,$$

with the topology inherited from $\mathbb{R}^2$. On the plane, the parameter space is $\mathcal{A}^{\mathbb{Z}^2}$, with the product topology. Then each parameter value $g = \{(a_{i,j}, b_{i,j}) : (i, j) \in \mathbb{Z}^2\} \in \mathcal{A}^{\mathbb{Z}^2}$ defines a wind–tree billiard in the plane, with the collection of square scatterers $O_{i,j}$, each centered at point $g_{i,j} + (i,\, j)$, $(i,\, j) \in \mathbb{Z}^2$.

In this case the billiard table is $\mathcal{Q}_g = \mathbb{R}^2 \setminus \cup_{(i,j)\in\mathbb{Z}^2} O_{i,j}$ and the dynamics is $S^t_g : \mathcal{M}_g \to \mathcal{M}_g : -\infty < t < \infty$, with phase space $\mathcal{M}_g$ defined in Section 2.4.3 below. This billiard is called the wind–tree process.

2.4.2. *Periodic wind trees.* The second model is a periodic version of the wind–tree model, introduced in [76] and investigated more recently in [42, 63]. The scatterers are upright isomorphic *rectangles* $O_{i,j}$ with sides of lengths $0 < a, b < 1$ and centered at the lattice points of $\mathbb{Z}^2$.

[8]It is at the same time the Gibbsian measure.

[9]The Ehrenfests' wind–tree model thus consists of square obstacles positioned at random on the two-dimensional plane, with their diagonals parallel to the plane's axes.

In this case, the billiard table is $\mathcal{Q}_{a,b} = \mathbb{R}^2 \setminus \cup_{(i,j)\in\mathbb{Z}^2} O_{i,j}$ and the dynamics on this table, $S^t : \mathcal{M} \to \mathcal{M} : -\infty < t < \infty$, is a billiard, i.e. a wind–tree process. We now turn to the definition of phase space $\mathcal{M}$.

2.4.3. *Discrete velocity space.* A characteristic of wind–tree models is, in general, that the set of possible directions for the billiard flow is finite. The Ehrenfests themselves thus considered the space of four velocity directions $(\pm 1, 0)$, $(0, \pm 1)$.

If we allow for an arbitrary initial angle θ, $0 < \theta < 2\pi$, the vertical and horizontal reflections on the plane generate the set of four different possible directions $\{\pm\theta, \pm(\pi - \theta)\}$ the particle can take at any time. Letting $[p]$ denote the set of the corresponding unit velocity vectors,

$$[p] = \left\{ v \in \mathbb{S}_1 : \arctan \frac{v_2}{v_1} \in \{\pm\theta, \pm(\pi - \theta)\} \right\},$$

the phase space, in the case of the aperiodic wind–tree model, is then $\mathcal{M}_g = \mathcal{Q}_g \times [p]$ and, in the case of the periodic one, $\mathcal{M} = \mathcal{Q}_{a,b} \times [p]$.

The corresponding invariant measures of the flows S^t_g and S^t are in both cases the infinite measures $\mathrm{d}\mu \propto \mathrm{d}q$ (up to the counting measure for the velocity space).

2.4.4. *Higher-dimensional cases.* In Subsection 4.4 we will also recall results on a *random wind–tree model* in $\mathbb{R}^3$. There the scatterers form an array of randomly placed, identically oriented cubes with sides again parallel to the coordinate axes. As with the planar case, the velocities of the particle form a finite set, this time with eight elements. Indeed, following [100], fix a vector $p = (p_1, p_2, p_3)$ with $p_i > 0 \ \ \forall i$ and let $|p| = \sqrt{p_1^2 + p_2^2 + p_3^2}$ denote its norm. Then the set of possible velocities is

$$[p] = \left\{ v \in \mathbb{S}_2 : |v_i| = \frac{p_i}{|p|} \right\}.$$

2.5. **Hard Ball Systems.** Assume that, in general, a system of $N : N \geq 2$, identical (for simplicity) balls of unit masses and radii $r > 0$ are placed at non-overlapping positions in $\mathbb{T}^\nu = \mathbb{R}^\nu / \mathbb{Z}^\nu$, the ν-dimensional unit torus, $(\nu \geq 2)$, and given random velocities $v_i : 1 \leq i \leq N$. The dynamics corresponds to the uniform motion of the ball particles with elastic collisions when they get into contact.

Denote the phase point of the i^{th} ball by $(q_i, v_i) \in \mathbb{T}^\nu \times \mathbb{R}^\nu$. The configuration space $\mathcal{Q}$ of the N balls is a subset of $\mathbb{T}^{N\nu}$, obtained from $\mathbb{T}^{N\nu}$ by cutting out the $\binom{N}{2}$ cylindric scatterers,

$$C_{i,j} = \left\{ (q_1, \ldots, q_N) \in \mathbb{T}^{N\nu} : \mid q_i - q_j \mid < 2r \right\},$$

$1 \leq i < j \leq N$. That is, $\mathcal{Q} := \mathbb{T}^{N\nu} \setminus \cup_{1 \leq i < j \leq N} C_{i,j}$. The (kinetic) energy $K = \frac{1}{2} \sum_1^N v_i^2$ and the total momentum $P = \sum_1^N v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $K = \frac{1}{2}$, $P = 0$. (If $P \neq 0$, then the system has an additional conditionally periodic or periodic motion.) Now, for these values of K and P, we define our dynamical system.

The set $\mathcal{Q}$, a compact, flat Riemannian manifold with boundary, such as identified above, is the configuration space of our system. Its phase space is $\mathcal{M} := \mathcal{Q} \times \mathbb{S}_{N\nu - 1}$. The Liouville measure $\mathrm{d}\mu = \text{const.}\, \mathrm{d}q\, \mathrm{d}v$ is invariant with respect to the evolution $S^{\mathbb{R}} := \{S^t : \ t \in \mathbb{R}\}$ of our dynamical system defined by elastic collisions of the balls of unit masses and their uniform free motion. The dynamics can, indeed, be defined for $\mu-$a. e. phase point. This is a billiard which, after restricting to $P = 0$ and fixing the center of mass, is dispersing for $N = 2$ and semi-dispersing for all $N \geq 3$.

2.6. **Systems of Spatially Localised Hard Balls.** After the ergodicity of gases of two hard balls [29, 127, 129], then three [88], and four [89], had been established, Bunimovich *et al.* [28] observed that

> Unfortunately, new and serious technical problems, which require the development of some specific methods, appear at each step from N to $N+1$ balls.

To go around this difficulty, Bunimovich *et al.* [28] put forth a family of models that, as they write, "are intermediate ones between the gas of hard balls and the Lorentz gas model" and thus lend themselves to a systematic study of ergodicity for arbitrary number of particles. While these models are gases of hard balls (in the usual sense of the expression), they exhibit the distinctive feature that individual balls are trapped in their own cells and thereby retain a form of spatial order. Models in this class thus feature both the collisional dynamics of a gas and cristalline spatial structure of a solid.

As far as ergodicity is concerned, the fundamental advantage of these models over other systems of hard balls is that the collision sequences of the particles are much simpler and easier to control. Another important feature is that, while the models allow for heat transport through energy exchanges, they prevent mass transport. This raises the prospect, as emphasised by the authors of [28], that

> it would be interesting to investigate the kinetic properties of these models, e. g. diffusion of energy.

While the *raison d'être* of the models, i.e. ergodicity of a gas of any number of particles, has been superseded by the understanding of the inductive step alluded to above and the ultimate proof of the ergodic hypothesis for hard ball systems [124], the point above has been key to continued interest in these models and we will comeback to it in the sequel.

2.6.1. *Bunimovich–Liverani–Pellegrinotti–Suhov models.* Here we consider a specific planar version of the models, which easily lends itself to various generalisations in two and higher spatial dimensions.

The model is illustrated in Figure 1 and can be constructed according to the following few steps. Consider a regular honeycomb lattice on the plane. Unit cells alternate between upward and downward equilateral triangles of unit sides. Around each lattice point, discs of radii ρ_{f}, $0 < \rho_{\mathrm{f}} < \frac{1}{2}$, are removed from the plane, which serve as fixed scatterers. Inside every triangle, let there be one circular moving ball of unit mass and radius ρ_{m}, $\frac{1}{2} - \rho_{\mathrm{f}} < \rho_{\mathrm{m}} < \frac{1}{\sqrt{3}} - \rho_{\mathrm{f}}$. Their positions must be chosen so that moving balls do not overlap with any of the fixed scatterers, as well as among each other.

The lower bound $\rho_{\mathrm{f}} + \rho_{\mathrm{m}} > \frac{1}{2}$ ensures that each disc remains in the cell it starts from; see overlap among dotted circles in the figure. The upper bound $\rho_{\mathrm{f}} + \rho_{\mathrm{m}} < \frac{1}{\sqrt{3}}$ is so as to leave enough room to fit the moving ball in the remaining space.

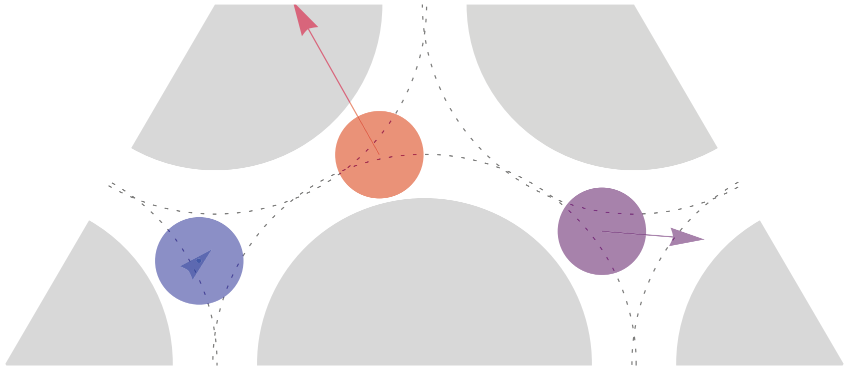

FIGURE 1. An illustration of the two-dimensional version of the BLPS model built on a honeycomb lattice. Fixed scatterers of radii ρ_{f} are coloured in gray and moving balls of radii ρ_{m} in different hues of blue and red. Differences in these colors, as well as in the velocity arrow sizes, reflect differences in the values of kinetic energy. The dotted circles have radii $\rho_{\mathrm{m}} + \rho_{\mathrm{f}}$.

To allow for collisions among moving balls, however, we must further assume that the balls are large enough, i.e.

$$\rho_{\mathrm{m}} > \rho_{\mathrm{c}} = \sqrt{(\rho_{\mathrm{f}} + \rho_{\mathrm{m}})^2 - \tfrac{1}{4}}\,.$$

Otherwise there would be no interaction and the model would be of little interest.

Given initial positions and velocities, the moving balls follow the billiard dynamics. That is, the balls move uniformly until an elastic collision event occurs, either among two moving balls, or against a scatterer.

The model introduced above is, of course, one with an infinite number of particles. However, it lends itself to different restrictions with a finite number of particles, such as a one-dimensional chain of alternating upward and downward cells as illustrated by the figure, or, in its simplest form, a gas of two moving balls, each trapped in their own cells. The corresponding billiard in this minimal case is a four-dimensional semi-dispersing one.

2.6.2. *Bálint–Gilbert–Nándori–Szász–Tóth model of pistons and balls.* A variant of the BLPS model was proposed by Bálint *et al.* [15]. Unlike the former model, which, as described above, consists of a collection of similar unit cells spanning the two-dimensional plane, the latter model consists of a hybrid juxtaposition of one-dimensional-like particles, called pistons, and two-dimensional ball particles. Energy exchanges in the system are therefore mediated by ball-piston interactions.

Here we consider linear chains such as shown in Figure 2. While point particles (balls in figure) move about their respective planar cells in the usual fashion of two-dimensional billiards, the motion of pistons is restricted to horizontal line segments (represented as rectangles in the figure), along which they move back and forth. Importantly, the lengths of these cells are large enough that piston particles, which have a fixed vertical span, penetrate into the two neighbouring ball cells

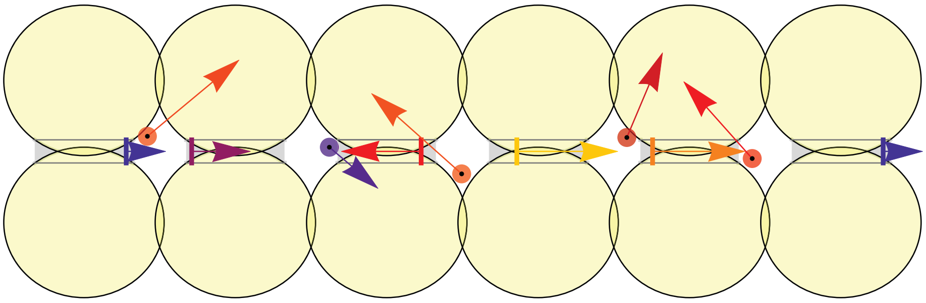

FIGURE 2. An illustration of the BGNST model alternating ball and piston cells along a one-dimensional lattice. In this specific example, periodic boundary conditions apply, so that the left– and right–most pistons are identical. The color-coding and arrow sizes are as described in the caption of Figure 1.

and may thus mediate energy exchanges between neighbouring balls through successive collisions. The system so-defined is a ball-piston gas and obeys the billiard dynamics.

An interesting property is that the minimal model consisting of a single pair of ball and piston can be viewed as a point particle moving about a three-dimensional cavity of cylindrical shape with a corner cut out, i.e. a three-dimensional semi-dispersing billiard; see [15] for further details. As far as controlling the decay of correlations between particles in neighbouring cells, this renders the BGNST model more easily tractable than the BLPS model.

2.7. **Rayleigh Gas.** In 1891 Lord Rayleigh [120] introduced a model system of binary mixture to study relaxation to equilibrium in the framework of the theory of gases. Originally posed as a one-dimensional model, this is a gas of two types of hard balls, one very small relatively to the other. It was in fact the first theoretical attempt towards the description of Brownian motion. As one of its limiting cases, it contains the Lorentz gas.

For our purpose, the *Rayleigh gas* is a binary gas of hard ball particles in $\mathbb{R}^d$: $d \geq 1$, whereby a single particle, called the *Rayleigh particle*, with mass $M > 0$ and radius $r > 0$, interacts with a collection of point particles of unit masses oblivious to one another. Having specified the positions and velocities of all particles, the gas thus evolves according to billiard dynamics, i.e. uniform motion until an elastic collision occurs when the Rayleigh particle and one of the point particles come into contact; see [132, 134] for further details.

So defined, the dynamics has, of course, no invariant probability measure. Several modifications have thus been considered, some of these are described below.

2.7.1. *Coordinate system fixed to the Rayleigh particle.* The dynamics with coordinate system fixed to the Rayleigh particle has been termed the *Münchhausen picture* [134][10]. This system admits an invariant probability measure in the form of

(1) the product of a Poissonian measure for the gas of point particles in the product space of positions outside the Rayleigh particle
(2) with Gaussian distribution of their velocities as well as that of the Rayleigh particle (with variance depending on its mass parameter M).

2.7.2. *Semi-permeable Rayleigh gas.* An alternative is to restrict the motion of the center of the Rayleigh particle to a finite region—say a sphere of radius R—acting as permeable for the point particles and reflecting for the Rayleigh particle. This dynamics has a natural invariant probability measure which is the product of

(1) a uniform spatial distribution for the center of the Rayleigh particle in the sphere of radius R,
(2) under the condition that the center of the Rayleigh particle is selected, the atoms are distributed according to a Poisson point distribution in the complement of the R-sphere of the Rayleigh particle, and
(3) as above, a Gaussian distribution of the velocities.

2.7.3. *Rayleigh gas on the half line.* On the positive half-line the distance of the Rayleigh particle has a probability distribution therefore the whole system has a time-invariant distribution. In fact, denote the positions of the particles by $0 < X < x_1 < x_2 < \dots$ where X is the position of the Rayleigh particle and $(x_1, x_2, \dots)$ are the positions of the point particles. The invariant probability measure is the product of

(1) a Poisson process on $\mathbb{R}_+$ and
(2) as above, a Gaussian distribution of the velocities.

3. Boltzmann's ergodic hypothesis

At its core, statistical mechanics aims at characterising the properties of matter in the bulk by accounting for the contributions of its constitutive molecules. From the point of view of dynamical systems, a central problem is to understand the emerging behaviour of systems whose states evolve according to a set of ordinary or partial differential equations, with an emphasis on Hamiltonian systems, or, alternatively, whose evolution is specified by an iterative system, with emphasis on symplectic maps.

The 1960's and 1970's saw spectacular progress in providing a mathematically rigorous basis for statistical mechanics. In particular, a new and rich arsenal of mathematical methods was discovered, which includes, to mention but a few examples, Kolmogorov–Arnold–Moser theory, renormalisation group methods, bifurcation theory, the theory of strange attractors, the hyperbolic theory of dynamical

[10] It is worth mentioning that the work of reference [134] was motivated by Sinai's Turán Memorial Lectures held in Budapest in 1985; see [130, 133]

systems, thermodynamic formalism, Lyapunov exponents and entropy theory, Ornstein's theory of isomorphisms[11].

Sinai's 1970 proof of the ergodicity of the gas of two hard discs on the two-dimensional torus and his 1963 hypothesis on the ergodicity of any number of hard ball particles, nowadays referred to as the Boltzmann–Sinai ergodic hypothesis [124, 135], was a substantial contribution to the aforementioned list of notable breakthroughs.

It must, however, be said that, in the list of stochastic properties of a system, ergodicity belongs to the weaker ones. There are several stronger such properties which we do not treat here; for instance, mixing, K-mixing, or Bernoulli. Nevertheless, in Section 4, we address another important qualitative property, that of *correlation decay* and its rate, which is most significant for the sake of assessing the diffusive properties of physical systems.

We recall below Boltzmann's ergodic hypothesis, Subsection 3.1, which deals with the limit of particle number increasing to infinity. Unfortunately, our understanding of this case remains very limited, to the point that formulating the appropriate statement remains a delicate issue. Before we attempt at this goal, we therefore treat separately and in more details the two cases of fixed and finite number of particles, Subsection 3.2, or infinite, Subsection 3.3. We then allow ourselves to provide some thoughts on the former case in Subsection 3.4.

3.1. **Boltzmann's Ergodic Hypothesis.**

The hypothesis is essentially contained in the following approximate

Statement (Boltzmann's Ergodic Hypothesis)**.** For large systems of interacting particles in equilibrium, time averages are close to ensemble averages.

Before delving into the specifics of the ergodic hypothesis, we open a parenthesis to make mention of its controversial implications for statistical physics. For instance, in discussing relaxation to equilibrium, M. Kardar [85, pp 61–62] writes :

> This brings us to the problem of ergodicity, which is whether it is justified to replace time averages with ensemble averages. In measuring the properties of any system, we deal with only one representative of the equilibrium ensemble. However, most macroscopic properties do not have instantaneous values and require some form of averaging. For example, the pressure P exerted by a gas results from the impact of particles on the walls of the container. The number and momenta of these particles varies at different times and different locations. The measured pressure reflects an average over many characteristic microscopic times. If over this time scale the representative point of the system moves around and uniformly samples the accessible points in phase space, we may replace the time average with the ensemble average.

[11]The volume [142], based on a 1983 school devoted to regular and chaotic motions in dynamical systems, summarises some of the main achievements at the time, with, in particular, a substantive and rich introductory paper by A. S. Wightman, [142, *Introduction to the Problems*], putting them in a historical perspective and thus providing an excellent snapshot on the early history of dynamical systems, starting from Newton, via Poincaré and Birkhoff.

One might infer from these words that the ergodic hypothesis is indeed fundamental for the definition of such an elementary notion as pressure. Yet Kardar goes on to write :

> For a few systems it is possible to prove an ergodic theorem, which states that the representative point comes arbitrarily close to all accessible points in phase space after a sufficiently long time. However, the proof usually works for time intervals that grow exponentially with the number of particles N, and thus exceed by far any reasonable time scale over which the pressure of a gas is typically measured. As such the proofs of the ergodic theorem have so far little to do with the reality of macroscopic equilibrium.

The latter sentence may be interpreted to say that the relevant time scale is that which controls the decay of correlations at the scale of molecular motion (Kardar's characteristic microscopic time). Apart from fluctuations which are essentially controlled by the underlying number of particles, physicists expect measurements of macroscopic observables such as pressure to yield consistent values provided their timescale is large enough with respect to the correlations of molecular motion [138].

Remark (Equilibrium averages)**.** In what follows, equilibrium averages always refer to the *microcanonical ensemble*, i.e. averages with respect to the Liouville equilibrium measure μ on the submanifold of the phase space specified by the trivial invariants of motion. More precisely, the ergodic hypothesis states that if f is an observable (i.e. a bounded measurable function on the phase space $\mathcal{M}$ of the system), then, as the size of the system (say the number N of particles) and observation time T both tend to infinity,

$$\frac{1}{T}\int_0^T f(S^t(x))\,\mathrm{d}t \to \int_{\mathcal{M}} f(x)\mathrm{d}\mu(x)$$

where $S^t(x)$ is the time evolution of the phase point $x \in \mathcal{M}$.

Boltzmann formulated his celebrated hypothesis in the sense reflected by the above remark, for a gas with an increasing number N of particles. In his subtle hypothesis neither the mathematical sense of the limits $N \to \infty$ or $T \to \infty$ nor their order were precisely given and it did not satisfy the demands of accuracy required by physics, much less by mathematics. While the ergodic theorems[12] of von Neumann [145] and of Birkhoff [21, 22] were instrumental mathematical achievements, it still remained completely open whether systems of interest to physics are ergodic or not.

In this sense, *Sinai's precise formulation of the ergodic hypothesis* for a physical system with a fixed number of degrees of freedom [126]—later called the *Boltzmann–Sinai ergodic hypothesis* [124, 135]—and, moreover, his *proof of the ergodicity of two colliding discs on the two-dimensional torus* caused a sensation. The sequence of events that took place between the works of Birkhoff and von Neumann [21, 145], on the one hand, and Sinai [126, 127], on the other hand, are treated in [135] and we refer to it for further details.

[12] Many historical aspects of ergodic theory were recently reviewed in [109] to commemorate these two ergodic theorems on the occasion of PNAS 100th Anniversary; see also the accompanying commentary [9].

We recall below the definition of ergodicity and otherwise refer to the surveys [124, 135] or the monograph [40].

Definition 3.1. Let $(\mathcal{M}, S^{\mathbb{R}}, \mu)$ (resp. $(\mathcal{M}, T^{\mathbb{Z}}, \mu)$) be a group of probability preserving maps. In the case of a flow (resp. map), a subset $A \subset \mathcal{M}$ is called invariant (mod 0) if for each $t \in \mathbb{R}$ one has $\mu(S^t A \,\Delta\, A) = 0$ for each $t \in \mathbb{R}$ (resp. $\mu(A \,\Delta\, T^{-1}A) = 0$). A flow (or a map) is ergodic if all invariant subsets are trivial, i.e. they have measure 0 or 1.

If, in general, we are given a dynamical system defined by a Hamiltonian H of $N \geq 2$ particles on $\mathbb{T}^\nu : \nu \geq 1$, then by fixing its invariants of motion, the time-evolution of the system provides a probability-preserving flow and the problem of ergodicity is actually about the ergodicity of *this flow*. In the present section we fix the Hamiltonian (and thus the dimension, too) and, as said above, we will separately—and in different depths—discuss in Subsections 3.2–3.4 the cases

(1) $N < \infty$,
(2) $N = \infty$,
(3) $N \to \infty$.

3.2. **Fixed Number N of Hard Balls.** This case was amply covered in the reviews [124,135] and our exposition will therefore be concise. The review [27] includes a discussion about stadium and related billiards, which we do not treat here.

In words, one has in mind a system of N particles on $\mathbb{T}^\nu : \nu \geq 1$ interacting via a smooth Hamiltonian H and moving on the submanifold of the phase space specified by the invariants of motion. Early results by Markus and Meyer [106] showed that in the space of smooth Hamiltonians both nonergodic systems and ergodic ones form dense open subsets[13].

3.2.1. *Boltzmann–Sinai Ergodic Hypothesis.* Consider the dynamical system of N identical, sufficiently small, elastic hard balls moving on $\mathbb{T}^\nu$. Take the submanifold of the phase space specified by fixing its invariants of motion (energy and momentum). Assume that the conserved momentum vector is 0, which also allows to fix the position of the center of mass. This submanifold has dimension $d = 2(N-1)\nu - 1$ (deduction of 1 for the energy and twice ν for the momentum and center of mass).

Statement (Boltzmann–Sinai Ergodic Hypothesis)**.** The system of $N : N \geq 2$ elastic hard balls on $\mathbb{T}^\nu : \nu \geq 2$ is, for sufficiently small radius of the balls, ergodic on the submanifold of the phase space specified by the invariants of motion.

Remark. Simple arguments lead to the following corollaries:

(1) If the hypothesis is true and the total momentum is not zero, then the motion is the product of an ergodic flow and a conditionally periodic motion;
(2) If the hypothesis is true, then there is a finite number of ergodic components in the case when the radius of the balls is not small.

Sinai's highly acclaimed 1970 paper [127] on the ergodicity (and even K-property) of what became known as two-dimensional Sinai billiards with finite horizon drew

[13]Of course, these subsets can both be very small so it may—and in fact does—occur that none of them represents a 'typical' behaviour.

on important prior works. On the one hand, N. S. Krylov [90], the great Russian theoretical physicist, observed in 1942 that the interaction of hard balls is hyperbolic in a sense similar to that which had appeared earlier in the proofs of Hedlund [77] and Hopf [83] of the ergodicity of geodesic motion in hyperbolic geometry. On the other hand, those were exactly the results of Hedlund and Hopf which motivated Anosov and Sinai to create a beautiful and far-reaching theory for smooth uniformly hyperbolic dynamical systems; see [6, 7, 125].

Sinai's 1970 work on two-dimensional billiards [127] was innovative in several respects. As a brief side-note, it must be said that Sinai's 1970 paper is also a gem of mathematical style. Because of the abundance of new and original ideas, Sinai had to find an appropriate balance between completeness and conciseness, providing sufficient amount of information for readers to follow his arguments while keeping the paper brief enough to be at all readable. Even so, his work was hard to understand and, apart from results by the Moscow school, for many years, developments of his theory were unsurprisingly few[14].

After a long history consisting of several breakthroughs, which is nicely recalled in [124], Simányi [123] eventually completed the proof of the Boltzmann–Sinai hypothesis for an arbitrary number of balls in any dimension[15].

3.3. **Infinite Number of Particles.** As said earlier, the $N \to \infty$ case—and the appropriate formulation of the ergodic hypothesis—is hard. The $N = \infty$ case is more easily amenable to study. Some of the interesting results are surveyed below and some conjectures are drawn.

Key features of the associated models we wish to emphasise are, on the one hand, the good spatial mixing property of the invariant (Gibbs) measure, and, on the other hand, the fact that such spatial mixing does indeed lead to nice ergodic properties. In Reference [73], this phenomenon is called *escape of local information to infinity.* It appears in all results we report below.

We also remark that, in some instances, spatial mixing may coexist with local mixing due to the interaction (more concretely its hyperbolicity). While some proofs make use of the latter, we expect the former will play a more prevailing role.

3.3.1. *The ideal gas in* $\mathbb{R}^d : d \geq 1$. Here the invariant measure is the Poissonian measure in the product space $\mathbb{R}^d \times B$ where $B \subset \mathbb{R}^d$ is arbitrary. The proof of ergodicity is due to Sinai and Volkovysskii [144].

3.3.2. *The Lorentz gas in* $\mathbb{R}^2$. Sinai [128] showed that, under general conditions for an otherwise arbitrary configuration of identical and fixed circular scatterers, the planar Lorentz gas is ergodic. The proof exploits the hyperbolicity of planar dispersing billiards. We expect that the result can also be generalised to higher dimensions and therefore propose the

[14] In 1974 Gallavotti [64] provided his own version of Sinai's proof, whereas Gallavotti and Ornstein [65] were able to use part of his results to go from the K-property of Sinai billiards to their Bernoulli property. We may further note that: (i) Gerhard Keller, in his MSc thesis [86], written under the guidance of Konrad Jacobs in Erlangen, reproduced Sinai's original proof; (ii) Vetier was among the first who could delve deep into Sinai's method, applying it to a new model: the Sinai billiard in a potential field; see [140, 141].

[15] Simányi's result also covers the case of arbitrary masses of the balls.

Conjecture 1. The *Lorentz gas* in $\mathbb{R}^d : d \geq 2$ with spherical scatterers is ergodic under general conditions for an otherwise arbitrary configuration of identical and fixed scatterers.

Here it would be much interesting to understand what is the minimal information that ensures ergodicity.

Remark. Proving the ergodicity of multidimensional dispersing billiards is based on the so called theorem on local ergodicity. In particular, this theorem makes the assumption that scatterers are algebraic [12], which is expected to be only a technical assumption. Nevertheless—beyond the self-sufficient appeal of the Conjecture—it would be interesting to prove it for a class of scatterers wider than merely algebraic ones.

3.3.3. *Rayleigh gas.* As explained in Subsection 2.7, one among the ways to obtain a dynamical system with an invariant probability is to consider a semi-permeable wall such that the Rayleigh particle be reflected by this barrier while the gas particles go through it unscathed. The ergodicity of the semi-permeable Rayleigh gas was shown for the case $d = 1$ in [74] and generalised to $d \geq 2$ in [62].

The alternative half-line version of the Rayleigh gas ($d = 1$) was also proven to be ergodic in [24].

The Rayleigh gas can be understood as an infinite-dimensional billiard where collisions with the Rayleigh particle correspond to collisions with a cylindrical scatterer. A very weak form of hyperbolicity is therefore present, but the aforementioned results do not use it explicitly.

3.3.4. *Wind tree gas.* The theory of wind–tree processes has recently been the subject of spectacular new and important results, partially thanks to the results on the Lyapunov exponents of Kontsevich-Zorich cocycles [149]. In the Appendix, we mention two different, relatively simple results of this theory pertaining to the models discussed in paragraphs 2.4.1 and 2.4.2, which allow us to formulate below two conjectures about the ergodicity of the corresponding wind–tree gases. Our goal is to emphasise that the wind–tree gas could be ergodic without need for local hyperbolicty of the dynamics!

Conjecture 2. The aperiodic Ehrenfest wind–tree gas whose scatterer configuration satisfies the conditions of the wind–tree flow of [103] is ergodic (cf. Theorem A.1 of Appendix A).

Conjecture 3. Under the conditions of [42] on the shape parameter of the rectangles, the periodic wind–tree gas is ergodic (cf. Theorem A.2 of Appendix A).

Remark. Let us emphasize again the difference between the wind–tree *process* (which describes a single particle) and the wind–tree *gas* (which describes infinitely many independent particles). In particular, by the results of reference [63], even after restriction to the subset of Subsection 2.4.3, the periodic wind–tree process is not ergodic for almost all parameters.

3.4. **Number of Particles Increasing to Infinity.** As mentioned before we know very little about this most fundamental problem whose goal would be to clarify the situation around Boltzmann's ergodic hypothesis. Its importance to both quantum and classical realms is well summarised by Penrose and Lebowitz [95]:

> This lack of knowledge is regrettable because only for an infinite system (by which term we mean the limit of a finite system as its size becomes infinite) can one expect to find strictly irreversible behavior in quantum mechanics. Moreover, the distinction between microscopic and macroscopic observables, which appears essential to any complete theory of irreversibility and kinetic equations, can only be formulated precisely for infinite systems.

We refer to the following two quotes. The first one is due to Wightman [142, *Introduction to the Problems*, p 20]:

> There are three traditional reasons often given for the legitimacy of the traditional methods, despite non–ergodicity of the flow.
>
> A (Thermodynamic limit) The results of classical statistical mechanics may be valid in the *thermodynamic* or *bulk limit* (number of degrees of freedom, $N \to \infty$; volume, $V \to \infty$; $\frac{N}{V} \to \rho < \infty$). This might happen because the relative phase volume of the non-ergodic portion of the flow approaches zero in the thermodynamic limit, so that observables become insensitive to the non-ergodic portion.
>
> B (Macroscopic observables) There might be a restricted class of observables, called *macroscopic*, to which classical statistical mechanics would apply in the thermodynamic limit. In that limit, the macroscopic observables might be insensitive to the non-ergodic portion of the flow even if its relative phase volume does not go to zero.
>
> C (Grain of Dust or Heat Bath) The idealization that the system under study is isolated should be made more realistic by the inclusion of coupling to outside systems. Then the extreme sensitivity of the non-ergodicity to initial conditions might result in an averaged behavior consistent with classical statistical mechanics.

The second quote we wish to refer to is due to Dobrushin [48] who, assessing the nonfullfilment of the ergodic hypothesis with respect to the foundations of statistical mechanics, offers that:

> For example, it is possible that even the Ergodic Hypothesis is not valid and there are several ergodic components and, for large N one of these components covers the main part of the phase space. Another variant seems more plausible. For large N there is a lot of small ergodic components which are mixed in a so complex way that using an observation in a fixed volume we almost can not distinguish between these components. It is difficult to formulate exactly such hypothesis and even more difficult to deduce its implications.

The reader will appreciate that Dobrushin's first hypothesis corresponds to Wightman's reason (A), and the second one to reason (B). There is unfortunately little progress to report on these questions.

4. Brownian motion

One of the fundamental cornerstones of contemporary science, the *atomic theory of matter*, did not win universal acceptance until the early 20$^{\text{th}}$ century. While a form of the atomic theory was taught as early the 4$^{\text{th}}$ and 5$^{\text{th}}$ centuries BC by Democritus [54], its ultimate approval was the result of

(1) Einstein's 1905 [60] ingenious characterisation of Avogadro's number through his derivation of the diffusion equation on the basis of atomic theory,
(2) followed by Perrin's experimental determination [117] of Avogadro's number based on Einstein's arguments, which later earned Perrin the 1926 Nobel Prize in physics "for his work on the discontinuous structure of matter, and especially for his discovery of sedimentation equilibrium."

It thus naturally became a paramount objective within the mathematical physics community to derive Brownian motion ab initio, i.e. starting from the framework of Hamiltonian mechanics. Many surveys of Einstein's work on Brownian motion and the many results it inspired are available in the literature. Here we limit our references to Duplantier's centenary survey [55] for general background and the two recent reviews [136, 137], which cover the approach to diffusion via billiard theory.

The emergence of Brownian motion for the Lorentz process relies on the decay of correlations. The most widely used approach in ergodic theory is via appropriate upper bounds on the correlation decay of some nice functions. In Subsection 4.1 below, we begin with some early results, which are rather based on random walks and where correlations are actually absent. We discuss some of their implications for limiting regimes of billiard dynamics which are widely used in the physics literature. As far as the mathematical theory of billiards is concerned, our treatment of two-dimensional models in Subsection 4.2 will be concise, as details are available in the aforementioned two recent reviews. The interesting and still open case of dimension $d \geq 3$ is the topic of Subsection 4.3. Finally, in Subsection 4.4, recent results applying to the random Lorentz and wind–tree processes are reviewed.

4.1. **Random Walks: Absence of Correlations.**

The simplest mathematical formalisation of Brownian motion is given by the Wiener process [146]. An elementary mathematical result for its derivation from underlying probabilistic laws was the classic claim of reference [61] (see also [52, 118]), namely that the Wiener process arises as the diffusive limit of a simple symmetric random walk (with discrete time step); see [84]. In some sense, the underlying probabilistic dynamics is here specified at a mesoscopic level of description, finer than the Wiener process, which defines the macroscopic evolution, but coarser than a hypothetical deterministic law at the microscopic scale.

In that respect, it is interesting to note that the baker's transformation[16] had been introduced earlier by Hopf [82, Paragraph 12], who established the isometry with the

[16] While Hopf does not make use of the name "baker's map" in his monograph, he does remark that "the repeated execution of which is reminiscent of the production of puff pastry."

Bernoulli process $B(\frac{1}{2}, \frac{1}{2})$; see also [8, Appendice 7]. The baker's transformation is both area-preserving and time-reversible, which confers it a special status as it may be conceived of as a caricature of a Hamiltonian system whose statistical properties are straightforward. While physicists focused mostly on its mixing properties and were careful to warn against giving it too much significance [95], it is interesting to note that the baker's transformation composed with translations on the lattice provides a simple and straightforward deterministic law which can be interpreted as a microscopic representation of the Wiener process [66].

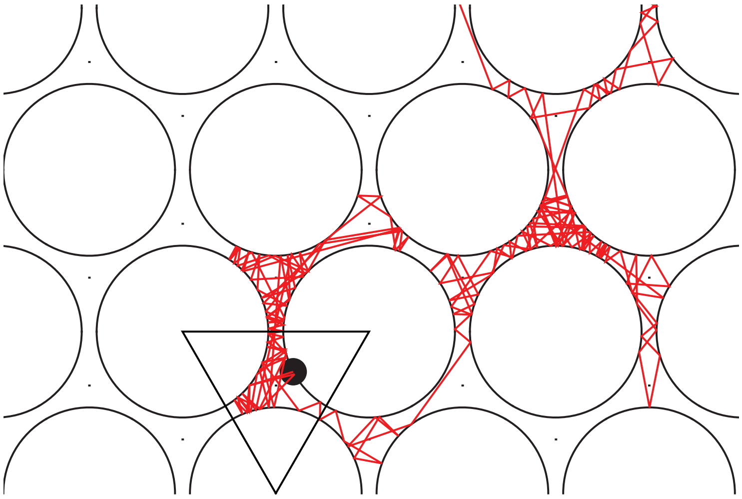

FIGURE 3. Diffusion in a finite-horizon Sinai billiard may be described by a random walk in the limit of vanishing spacing between obstacles. In physics literature, this is referred to as the Machta-Zwanzig regime [102].

In the case of the planar periodic Lorentz process with finite horizon, such as depicted in Figure 3, the picture is obviously complex. Suffice it to say, however, that convergence to the Wiener process follows essentially from weak bounds on correlation decay (these bounds are required to be at least summable); see Subsection 4.2.

A regime of particular interest is that of narrow spacing between circular obstacles, which, in the physics literature, is often referred to after Machta and Zwanzig [102]. If one considers the motion of a tracer particle between the triangular cells of the honeycomb lattice of Figure 3 and increases the diameter of the obstacles to near the lattice parameter value, the hopping events of the tracer particle between neighbouring cells become asymptotically rare, so that correlations between two such successive events are virtually absent. The distribution of hopping times, i. e. the return times to the boundary segments separating neighbouring cells, is then expected to follow an exponential law [79, 116]. At the level of lattice cells, the tracer's dynamics thus reduces, to a continuous-time Markov jump process. Moreover, and much like its discrete-time counterpart referred to above, this process has a straightforward diffusive scaling limit, which yields a Wiener process with diffusion coefficient given according to a dimensional estimate, that is, one fourth the lattice constant squared multiplied by the properly scaled average hopping rate. Furthermore, similar results can be derived in arbitrary dimensions (under the finite horizon condition and so long as correlations can be ignored—a very strong claim!).

4.2. **Markov Techniques for Sinai Billiards.** As said earlier, we limit our discussion to list the main techniques used in the study of *finite horizon Sinai billiards*[17]. They are:

(1) *Markov partitions* and stretched exponential correlation decay,
(2) *Markov sieves* and stretched exponential correlation decay,
(3) *Young towers* and exponential correlation decay,
(4) *Standard pairs*;

see references [30–32] for Markov partitions, [33, 34] for Markov sieves, [148] for Young towers, and [38] for standard pairs. The latter is a kind of approximate Markov partition whose great advantage is that it also lends itself to the description of a family of singular, hyperbolic dynamics obtained from one another by continuous perturbations.

Markov techniques are primarily used as tools to handle the correlation decay of physically or geometrically significant functions. One thus obtains probabilistic limit theorems for these functions that are central to the analysis of the relevant transport processes. The case of Brownian motion is discussed in some detail in the review article [137]. It is also interesting to note that, as an application to the momentum current, the same techniques were exploited in [35] to establish the existence of both shear and bulk viscosities of the solid phase of a periodic system of two hard balls on the two-dimensional torus, a model completely analogous to the Sinai billiard. We further refer to [143] where the extension of these results to the liquid phase on a hexagonal torus was investigated[18].

Before going into further details about billiards, we make the following

Remark. In probability theory there is a wealth of results about the most delicate and detailed properties of convergence to Wiener process of a random walk (or on the closeness of these two processes in various senses). Nowadays—in parallel with the aforementioned progress on the theory of Sinai billiards—one can observe a quite similar, rich and much promising development. Namely, what had earlier been known for sums of independent variables is becoming a goal to be ascertained for the Lorentz process, i.e. for sums of variables determined by a deterministic motion. Relevant references include [1, 49, 50, 104, 114, 115].

It is our opinion that the aforementioned advances have been strongly consolidated in the last decades by the publication of the monograph by Chernov and

[17]We refer to [137] and references therein for a review of infinite horizon Sinai billiards in the plane, and to [44, 110] for the multidimensional case.

[18]In contrast to the square geometry where the liquid phase is associated with an infinite horizon regime, the hexagonal geometry exhibits a liquid phase which spans both finite and infinite horizon regimes.

Markarian [39], which has tremendously helped in promoting research in this highly technical subject. Similarly, interest in the alternative functional analytic method of *transfer operators* greatly benefited from the monograph by Baladi [11]. A recent and notable breakthrough in the application of this method is the result of Baladi, Demers and Liverani [13], establishing exponential decay of correlations for two-dimensional finite-horizon Sinai billiard flows without corner points.

4.3. **The Complexity Hypothesis for Higher-Dimensional ($d \geq 3$) Billiards.** The contents of the previous subsection are unfortunately restricted to the planar case. It would be utmost important to understand the multidimensional situation, which is much different from the planar one.

As to *correlation decay* for multidimensional billiards, we know of only one result, due to Bálint and Tóth [17]. Assuming finite horizon and smooth boundary pieces, it makes the following

Statement. Under the complexity hypothesis[19] (to be formulated below) the planar theory extends to the multidimensional case. That is to say, for the map, correlations of smooth functions decay exponentially.

Let us turn now to the complexity hypothesis, which is formulated for the time-discretised process, the so-called Poincaré section map; see Subsection 2.1. Billiards are singular dynamical systems, so if one applies the billiard map T to some nice, smooth convex (in other words expanding) submanifold $\Sigma \subset \partial M$, then, under the iterations of T, the images $T^n\Sigma$ of Σ will typically be chopped up into several smooth pieces (tangent collisions or collisions at corner points of the boundary of the billiard table indeed break the images of Σ).

Let us denote by K_n the upper bound on the number of smooth pieces of $T^n\Sigma$. The precise form of the so-called subexponential complexity hypothesis is borrowed here from [17, Subsection 2.4.2]:

Conjecture 4 (Complexity hypothesis)**.** For typical[20] three-dimensional billiards with smooth scatterers and finite horizon, there exists $\lambda > 1$, which is strictly less than the smallest expansion rate on the unstable cones of the billiard, such that $K_n = o(\lambda^n)$.

We note here that Bálint and Tóth [18] have also found an example where the subexponential complexity hypothesis does not hold. This example, however, is quite special. Because of the utmost importance of settling the complexity hypothesis in general, the authors offer a bottle of Unicum Riserva[21] for the construction of a billiard which exemplifies the complexity hypothesis[22].

[19] After Sinai's original work [127], the complexity hypothesis has been used in every work dealing with hyperbolic two-dimensional billiards. Roughly speaking it says that hyperbolicity wins over the effect of singularities. We note that, in the planar case, hyperbolicity is exponential in nature, whereas the chopping up effect caused by singularities is only algebraic (in fact quadratic).

[20] In any reasonable sense of typicality, for instance, with respect to the C^r topology ($r \geq 3$) on the scatterers.

[21] Unicum Riserva is a popular beverage specialty of the celebrated Hungarian firm Zwack Unicum PLC founded in 1790.

[22] While it is known that finite-horizon periodic billiards with non-overlapping spheres exist in any dimension [26, 78], their explicit construction is far from straightforward, even in three

Let us make an additional remark in closing. Namely, there is an interesting cultural difference between ergodic theory and other branches of mathematics. For instance, in number theory, it is common to prove conditional results, most notably assuming the validity of Riemann hypothesis. So far as we know, this is not common in the theory of dynamical systems. To take the example of the theory of multi-dimensional Sinai billiards (or perhaps of semi-dispersing billiards as well), new developments might be achievable under the complexity hypothesis. A convincing computational evidence of its validity is also desirable.

4.4. **Some Results for the Random Lorentz and Wind–Tree processes.** Below we recall two recent closely connected results due to Lutsko and Tóth [100, 101]. They apply to two different processes in $\mathbb{R}^3$, the random Lorentz process, on the one hand, and the random wind–tree process, on the other hand. The authors prove that, in the Boltzmann-Grad limit, both converge to Wiener processes. Our goal in mentioning these results is to provide support to our conjectures formulated in Subsection 3.3.4, since, here too, one draws a close analogy between the Lorentz and the wind–tree processes.

Remark. It is important to emphasize that this analogy applies to the random Lorentz and wind tree processes. For the periodic processes, details of the local dynamics are more relevant. In particular, the planar periodic wind–tree process has anomalous diffusive properties; see [42]. Another interesting model recently studied in [10] is a planar periodic wind–tree process with rounded off scatterers. This process regains local hyperbolicity and it is expected to limit to Brownian motion, but potentially with a new type of scaling.

Let us introduce notations. Depending on the nature of the process, the fixed scatterers are respectively *balls* of radii r or upright *cubes* of sides r. In both cases, the centers of the scatterers are fixed according to Poisson point processes of positive density, $\rho > 0$. Initially, the point particle is placed at the origin. In the Boltzmann-Grad limit, we have $r \to 0$ and $\rho \to \infty$ while (say) $\rho\, r^2 \to \pi$. In this limit the Lorentz process $S_r^t : 0 \leq t < \infty$ (resp. the wind–tree process $S_{r,[\mathrm{p}]}^t : 0 \leq t < \infty$) is a Markov jump process with a probability tending to 1. For more details, see [100, 101].

The authors investigate in both papers the so-called *averaged-quenched limit*, such that both the initial velocity directions of the point particle and the configuration of the scatterers are random and the limit is taken while averaging with respect to these random configurations.

In the forthcoming theorems, T denotes a scale such that[23], as $r \to 0$, $T \to \infty$. We also let $\rho = \pi\, r^{-2}$.

dimensions. In particular, no such configuration can be obtained with lattice packings of spheres, that is, when congruent spherical scatterers are placed at the vertices of a regular lattice. The horizons of such billiards are infinite unless the scatterers are allowed to overlap [121], which breaks the assumption of smooth scatterers.

[23]The difference in the conditions on the increase of T as $r \to 0$ in the two theorems is certainly a technical one.

Theorem 4.1 (Random Lorentz process [101])**.** *Assume, in addition, that* $\lim_{r\to 0} r^2 \,|\log r|^2\, T = 0$. *Then, as* $r \to 0$*, and in the aforementioned averaged-quenched sense,*

$$\frac{S_r^{tT}}{\sqrt{T}} \Longrightarrow W(t) \qquad t \in [0,\infty)\,,$$

where $W(t)$ *is the standard Wiener process in* $\mathbb{R}^3$ *and the convergence is weak convergence in* $C[0,\infty)$.

Theorem 4.2 (Random wind–tree process [100])**.** *Assume, in addition, that* $\lim_{r\to 0} r^2\, T = 0$. *Then, as* $r \to 0$*, and in the aforementioned averaged-quenched sense,*

$$\frac{S_{r,[p]}^{tT}}{\sqrt{T}} \Longrightarrow W(t) \qquad t \in [0,\infty)\,,$$

where $W(t)$ *is the Wiener process in* $\mathbb{R}^3$*, with covariance matrix* $\Sigma = \mathrm{diag}(v_1^2, v_2^2, v_3^2)$*, defined in terms of the possible velocity directions, and the convergence is weak convergence in* $C[0,\infty)$.

Before concluding this section, we mention a novel approach to random Lorentz processes by Aimino and Liverani [1] via deterministic walks in random environments.

5. Fourier's law of heat conduction

Ever since the formulation by Fourier of his *théorie analytique de la chaleur*, heat conduction has intrigued both engineers and scientists, physicists and mathematicians alike, as has, in fact, much of Fourier's scientific legacy [20]. Besides the articles [23, 43, 47, 108], which were published in the aforementioned volume, many excellent additional surveys and volumes have been published, recounting progress in this area; in particular and among the more recent ones, see [25,45,46,96,97,111].

Among these reviews, Bonetto *et al.* [25] has had considerable influence in shaping current understanding of the topic and focus research on key issues, depicting the state of affairs at the turn of the millennium in a style accessible to both mathematics and physics communities, covering both stochastic and deterministic models. A more recent survey on the derivation of Fourier's law starting from a Hamiltonian description, which is closely relevant to our discussion, can be found in [98]. For the derivation of large scale dynamics from stochastic models, a basic reference has been and remains [132].

While the pioneering work of B. J. Adler and T. E. Wainwright [2–4] established billiard models in the form of hard spheres as potent tools for the numerical study of nonequilibrium transport processes (as well as critical phenomena for that matter), much of the theoretical effort on the study of transport phenomena such as heat conduction has focused mostly on weakly anharmonic chains which bear no connection to billiards; see, e.g., [41]. This situation changed in the last two decades with the emergence of new and promising models, which we turn to below. In Subsection 5.1, we discuss the nonequilibrium Lorentz gas and an interacting version of it, which, as we explain, has features of billiard dynamics, but is not an actual billiard. The next Subsection 5.2 deals with proper billiard models of the type described in

Subsection 2.6. In a regime of rare interactions, these billiards lend themselves to a two-step strategy to deriving Fourier's law of heat conduction. The main ideas are recalled and perspectives are given.

5.1. **Nonequilibrium Lorentz Gases.** Before we go into our main discussion, we wish to mention an interesting model of heat conduction thought to obey Fourier's law, namely a form of periodic Lorentz gas whose circular scatterers freely rotate and exchange energy with point particles via "perfectly rough collisions", thereby mediating interactions among the gas particles [94, 107]. While this model is not quite a billiard, it has some of its main features.

To motivate the model, let us mention that a two-dimensional slab (i.e. an array of cells with finite horizontal and vertical lengths) of the periodic Lorentz gas may be driven out of equilibrium by putting, say, its vertical boundaries in contact with external reservoirs or baths. These reservoirs could be chemostats, acting like absorbing boundaries for particles that collide with them, and randomly injecting particles into the system at specified rates. Or they may be thermal baths which reflect colliding particles while randomly changing their velocities according to set temperatures. And a combination of both chemostats and thermal baths is also possible.

To give a concrete example, we mention that a numerical study of such a Lorentz gas driven out of equilibrium by two thermal baths at different temperatures was reported in [5], where, under reasonable assumptions, it was found that temperature profiles are consistent with macroscopic laws. In such a situation, the distributions of the gas particles exhibit near local equilibrium properties. That is, the velocity distributions are "close" to Maxwellians with a position-dependent temperature (how close depends on the value of the local temperature gradient). This is sometimes referred to as *local thermal equilibrium.* Of course, in this system, particles do not interact, so that each one of them retains the same energy between successive interactions with the thermal baths. Local thermal equilibrium is therefore merely a reflection of the superposition of independent gas particles. Loosely speaking, "warm" particles stay warm, just as "cold" particles stay cold; the answer to the question of where and in what proportions they mix determines the local temperature. One may be critical of the lack of actual equilibration among the particles sharing the same location (meaning lattice cell), but that is the essence of the model and one cannot have it both ways.

The Lorentz gas with rotating scatterers referred to earlier is precisely designed to achieve an exchange of energy between particles sharing the same location that the usual Lorentz gas lacks. Thanks to the mechanism of perfectly rough collisions proposed by the authors of [94] (see, however, [119] for a similar proposition), a particle which enters a cell will effectively leave the cell with an energy different from that with which it entered, which, in a sense, behaves like a random variable similar to that generated by a thermal bath at the local temperature of the cell (which is therefore a dynamically evolving quantity itself). It must be acknowledged that our description of the phenomenology of this model is a bit of an idealisation. Reality is indeed more intricate, but such details are not relevant here.

Thus why is this model not a billiard? This has to do with the mechanism of perfectly rough collisions alluded to above. Circular scatterers, whose centers are affixed to the vertices of the lattice possess a rotational degree of freedom. Whenever a point particle of the gas hits a scatterer, the perfectly rough collision rules are such that the radial component of the particle's momentum changes sign (in the same way that it does in an elastic collision). At the same time, the tangential component of its momentum and the angular momentum of the rotating scatterer undergo a linear interaction parametrised by the equivalent of a friction coefficient, but such that the total energy is fixed. This is therefore different from an elastic collision, which, irrespective of the rotational degree of freedom of the scatterers, would leave the energy of the particle unchanged[24].

Regardless of the fact that its dynamics is not Hamiltonian, the model [94] gave rise to a number of interesting contributions; see in particular [58].

5.2. **Heat Conduction in Gases of Locally Confined Hard Balls.** Even if the models described above were Hamiltonian, as is the variant with rotating rods numerically studied in [57], a systematic derivation of Fourier's law remains far out of reach. The control of correlations is indeed very difficult to achieve due to the high-dimensionality of the models.

In that respect, the billiard models studied in [68] belong to the class of BLPS models [28] described in Section 2.6.1. While these models are high-dimensional billiards with many of the properties of hard ball gases, the trapping mechanism of the individual balls induces a spatial order which considerably simplifies the physical picture in the sense that mass transport is prohibited and only energy transport takes place. This has opened new perspectives for a systematic derivation of Fourier's law, especially in a regime of rare interactions.

5.2.1. *Rarely interacting BLPS billiards.* To explain the notion of rare interactions, we note that BLPS models have two readily identifiable timescales. The first one has to do with the trapping mechanism and denotes the average time that separates successive collisions between a moving ball and the walls of its cell (assuming no interactions with a neighbouring moving ball takes place in that time interval). Call τ_{w} this time, which trivially scales with the inverse square root of the kinetic energy of the ball and otherwise depends only on the geometry of the cell through the sum $\rho_{\mathrm{m}} + \rho_{\mathrm{f}}$ of the radii of fixed scatterers ρ_{f} and moving balls ρ_{m}; see [37]. The second timescale is that which, in average, separates successive collisions between two neighbouring moving balls (under the assumption that no interactions involving other balls take place in that time interval). Call τ_{b} this time, which trivially scales with the inverse square root of the sum of the kinetic energies of the two balls. Importantly, its dependence on the geometry of the cells involves a parameter, that which measures how close the radius ρ_{m} of the moving balls is to the critical radius ρ_{c} defined in Subsection 2.6.2.

[24] It is worthwhile noting here that a different model with similar properties was considered in [57], where circular scatterers were replaced by rods (termed "needles" by the authors because they have zero thickness). The energy exchange mechanism between the rotating rods and point particles is defined through conservation of linear and angular momenta. Contrary to the perfectly rough collisions described above, such collisions are in fact consistent with billiard dynamics [72].

Crucially, one can vary $\rho_{\rm m}$ and $\rho_{\rm f}$ while keeping their sum, and hence $\tau_{\rm w}$, fixed, even as $\rho_{\rm m} \to \rho_{\rm c}$. Assuming $\epsilon := \rho_{\rm m} - \rho_{\rm c} \,:\, 0 < \epsilon \ll 1$, it is not difficult to show that $\tau_{\rm b} \propto \epsilon^{-1}$ [69]. We therefore have the following separation between the two timescales,

$$\tau_{\rm b} \gg \tau_{\rm w} \,.$$

This result embodies the rare interaction regime and implies a typical relaxation to local equilibrium of the internal degrees of freedom (averaging of the positions and velocity directions) *before* a collision among moving balls takes place. It is key to reducing the dynamics of energy exchanges among moving balls to a Markov jump process; see below.

This discussion naturally brings us back to an important aspect of the rarely interacting BLPS billiards, which concerns the distinction between microscopic and macroscopic observables alluded to in Subsection 3.4. Much like with diffusion in the Machta-Zwanzig regime of finite-horizon periodic Lorentz gases, the lattice coordinates provide the natural basis for inferring a set of macroscopic positions with attached local energies given in terms of the velocities squared of the trapped particles. These local energies can furthermore be interpreted as temperatures, which are indeed defined without ambiguity under the assumption of relaxation to local equilibria, expected in the regime of rare interactions. Their counterparts, i.e. the microscopic variables, which represent the internal degrees of freedom (positions of the balls inside the cells and their velocity angles) are rapidly averaged over on the timescales of energy exchanges and do not bear further influence on the process.

We remark that, as is the case with the Machta-Zwanzig regime of diffusion discussed in Subsection 4.1, the dimensionality of the dynamics has little relevance to the assumptions that drive the local relaxation process described above. Billiard models of both higher and lower dimensionalities have in fact been studied elsewhere [15, 70], exhibiting similar properties.

5.2.2. *Reduction to a Markov jump process.* A minimal model to study the process of energy exchanges in the rare interaction regime is a billiard table of only two cells, each with its own moving ball trapped inside it, and sharing a boundary through which collisions between the two balls are possible. At the value $\rho_{\rm m} = \rho_{\rm c}$, i.e. $\epsilon = 0$ in the notation introduced above, the two balls are effectively isolated and cannot collide; the model is then akin to an insulator. This four-dimensional billiard is therefore the product of two Sinai-billiards whose cells have triangular shapes such as outlined by the dotted circles in Figure 1.

One would like to study the $0 < \epsilon \ll 1$ energy-conducting regime of rare interactions as a perturbation of this insulating regime[25]. Theoretically, this would amount to generalising the method of standard pairs [38] to this situation. A somewhat similar strategy was actually described in [113] to study the diffusive limit of a gas of two hard balls moving in a finite-horizon dispersing planar billiard table.

As explained in [112], however,

[25]It is indeed an interesting feature of these models that a normally-conducting regime is expected to emerge out of an insulating one. The situation is very different with weakly anharmonic oscillators, whose harmonic limits have infinite conductivity.

> At the present state of dynamical methods [this problem] unfortunately defies a rigorous approach. So far the apparently strongest method: that of standard pairs elaborated in detail in [38] does not permit extension to truly higher-dimensional models, like this one. (The main technical reason is that the conservation of the standard pair structure after one collision can not be controlled for some particular configurations of the colliding disks.)

With respect to the model under consideration, it implies, for instance, that there is no way of bounding the correlation decay of nice functions that would be necessary to justify the first step of the Gaspard-Gilbert two-step approach outlined in [68].

It was with this obstacle in mind that a simpler variant of the BLPS class of models was designed, as described in Subsection 2.6.2, which, in its minimal configuration is a three-dimensional semi-dispersing billiard. In [15], the reduction to a Markov jump process was discussed on phenomenological grounds and a derivation of the associated transition kernel was given in terms of conditional mean free times. Still, progress towards applying the method of [38] to this model so far remains too limited to make any serious claim. Nevertheless, the authors of [16] have announced a result on this issue and have, in fact, already clarified one important step toward a rigorous proof.

Along these lines, we formulate below a conjecture which experts will have anticipated, but which gives us the opportunity to emphasize a result of fundamental importance[26]:

Conjecture 5. The main result of [13] on the exponential decay of correlations of planar Sinai billiard flows is also true for billiards with corner points (assuming that the angle of the intersecting boundary pieces at these corners is not zero).

To be more specific, we recall that, as said in Subsection 4.2, a strong advantage of the method of standard pairs is that it is also suitable for the joint treatment of perturbations of a dynamical system, a situation which, by its very nature, arises in the rare interaction regime of the models under consideration. Building upon the breakthrough results of [13], the authors of [16] thus showed that, if one considers a dispersing billiard flow on the two-dimensional torus whose initial measure is concentrated on the highly singular measure determined by a standard pair, then one still has exponential convergence to the absolutely continuous equilibrium Liouville measure.

5.2.3. *Hydrodynamic limit.* Even if the strategy outlined above for the reduction of the energy exchange dynamics to a Markov jump process were fully justified, the problem of deriving Fourier's law of heat conduction (and determining the heat conductivity) is a separate challenge and, at that, rather more complicated than obtaining the diffusive limits of simple symmetric random walks such as discussed in Subsection 4.1.

[26]We should, in particular, note that the content of this conjecture is a necessary step towards controlling the decay of correlations between neighbouring particles in the model of heat conduction presented in section 2.6.2. The explanation is simple: the minimal three-dimensional billiard model isomorphic to a single ball-piston pair has corner points, a case not covered by reference [13] (or by reference [16] for that matter).

In order to implement the second step of the Gaspard-Gilbert approach [68] for calculating the heat conductivity of the BGNST model of [15] (as well as that of other models[27]), one ought to take the hydrodynamic limit of the associated Markov jump process. At present the most hopeful approach is to apply the variational method of Varadhan [139]; see also [132] for a general discussion in the framework of stochastic lattice gases.

An important contribution in that direction is a result by Sasada [122], who obtained a lower bound on the spectral gap of the Gaspard-Gilbert model [68], namely const. N^{-2} (with a strictly positive constant), where N denotes the size of the system. As far as the BGNST model is concerned, the problem is slightly more complicated due to the hybrid nature of ball-piston cells. We transpose Sasada's result to the following

Conjecture 6 (Spectral gap for the BGNST model of [15])**.** The spectral gap of the piston model is bounded below by const. N^{-2} (with a strictly positive constant), where $N \in \mathbb{Z}^+$ denotes the number of ball-piston pairs (aligned along a one-dimensional chain).

Although Varadhan's method was formulated for models different from ours, it is expected to work here as well. There is, however, a caveat, namely that Varadhan's conditions on the coefficients of his models are perhaps too restrictive for the BGNST model. Consequently, more work is needed so as to relax these conditions and make them applicable.

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Appendix A. Some results on wind–tree models

Below we recall two theorems which we refer to in Subsection 3.3. Both models were introduced in Subsection 2.4. The first theorem claims (infinite measure) ergodicity about an aperiodic model with square scatterers.

Theorem A.1 (Málaga-Sabogal–Troubetzkoy [103])**.** *Using the notations of Section 2.4.1, we claim: There is a dense subset $\mathcal{G} \subset \mathcal{A}^{\mathbb{Z}^2}$ of parameters such that for*

[27] A computational method which relies on Varadhan's variational formula and the calculation of exact bounds was implemented in [71] and applied to a specific model, yielding a precise estimate of heat conductivity with a (negative) contribution from dynamical correlations estimated to be less than four parts in ten thousands of the static contribution—the published figure is in fact much more precise.

each $g \in \mathcal{G}$ there is a dense $G_\delta-$subset of directions $\mathcal{H} \subset \mathbb{S}_1$ of full measure such that the billiard flow on Q_g in the direction θ is ergodic for every $\theta \in \mathcal{H}$.

For simplicity we recall only part of a theorem of [42] that reflects well its flavour. It claims that the wind–tree process with periodic rectangular scatterers of size $a \times b$ makes large excursions (cf. escape of local information to infinity in reference [73]).

Theorem A.2 (Delecroix-Hubert-Lelièvre [42]). *For Lebesgue-almost all $(a, b) \in [0, 1)^2$, for Lebesgue-almost all $\theta \in \mathcal{H}$ and every point in $Q = Q_{a,b}$ (with infinite forward orbit)*

$$\limsup_{T\to\infty} \frac{\log d(p, S_\theta^T(p))}{\log T} = \frac{2}{3}\,.$$

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