Pure and Applied Functional Analysis
Volume 5, Number 6, 2020, 1377–1394

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

METHOD OF PARAMETER EXCLUSION. SOME RECOLLECTIONS AND SOME NEW RESULTS

MICHAEL JAKOBSON

Abstract. I start by some recollections about Sinai-Alexeev seminar in 60's and 70's and about several occasions when Sinai ideas influenced my work.

Then I review basic technique of parameter exclusion for families of one-dimensional maps with one critical point and with several critical points. In the last part some Hénon-like maps are discussed.

1. Some recollections

During my first year at the Moscow University V.M.Alexeev was my professor at the advanced calculus course. I was quite impressed by his style and asked him to be my adviser. That's how I joined Sinai-Alexeev seminar in Ergodic Theory and Dynamical Systems.

Alexeev suggested that I study classical works of Julia [21] and Fatou [8, 9] on iterations of rational functions from the point of view of modern dynamical systems.

During 60's and 70's dynamics became very popular thanks to the famous works of Kolmogorov, Rohlin, Sinai, Arnold, Anosov, and simultaneous works of Smale's school in USA.

Participants of the Sinai-Alexeev seminar met every week to present and discuss recent published works or preprints. Sinai, Alexeev, Anosov, Gurevich, Katok, Margulis, Oseledetz, Ratner, Stepin and many other speakers from Leningrad (now Saint Petersburg), Gorkii (now Nizhny Novgorod) and other cities and countries presented their new results many of which later became famous. One can find some recollections of that period in [22]. So I was lucky to be at the right place at the right time.

In particular topics in structural stability and technique of symbolic dynamics were introduced by Smale [31], Anosov [3], Anosov and Sinai [4], Sinai [30], Alexeev [2].

In my PhD I applied these new methods to study structural stability of polynomial and rational maps based on fundamental works of Julia [21] and Fatou [8, 9]. When studying works of Fatou I was amazed to see how his vision and approach were similar to our approach in 60's.

At some point Fatou clearly stated his famous conjecture and described main steps needed to check it. In modern language the question is:

2010 *Mathematics Subject Classification.* 37E05, 37D45, 37D35.
Key words and phrases. Parameter exclusion, He'non-like maps, SRB measures.

are rational maps $R(z)$ of given degree $d \geq 2$ (in particular polynomials $P(z)$) with hyperbolic dynamics, dense in the space of all complex rational maps (polynomials) ?

In spite of significant progress of the last 50 years, that problem remains unsolved even for quadratic polynomials.

Another topic which started to develop at that time was dynamics of real one-dimensional maps, where I proved density of hyperbolic dynamics in C^1 topology [12]. In full generality it was solved in [23].

In the space of one-dimensional maps with critical points the set of maps with non-hyperbolic dynamics was not very well understood. In few cases existence of smooth invariant measures with good stochastic properties was proved by Ulam and von Neumann [33], Bunimovich [7] and Ruelle [27].

I started to work on ergodic theory of maps similar to the Chebyshev map but in a more general C^2 context.

It happened that for several years during the Summer Sinai and I used the same railway station to go to our houses. Once we met in an overcrowded bus and I mentioned difficulties related to the critical point. Then Sinai suggested a new idea, he said: "use the first return map ". By that time expanding map with countably many pieces of continuity were studied in the works of Adler [1] and Walters [34]. It turned out that the first return maps induced by Chebyshev-like transformation generate expanding maps with countably many pieces and uniformly bounded distortions, and as a result the existence of absolutely continuous invariant measures with strong mixing properties was proved, see [16].

At some point after my presentation of several results of that type at the seminar Sinai asked what do I think about the measure of parameters such that respective systems have absolutely continuous invariant measures. My answer at that time was that it should be zero.

After the seminar Sinai, Bunimovich and I walked to the nearest metro station which took about 20-30 minutes. Then we continued our discussion for a while until it became too cold. They tried to convince me that the measure of stochastic parameters should be positive. They argued by analogy with the construction of infinite Markov partitions for Sinai billiards where new elements were constructed in decreasing neighborhoods of images of singularities. Later they suggested that similar ideas could be used to prove the positivity of the measure of parameters such that smooth perturbations of the stadium billiard system are ergodic. Actually that question about billiards is more difficult or similar to the famous unsolved Chirikov *standard map problem.*

I started to work on that problem for one-dimensional maps, developed what is now called a *parameter exclusion* method, and eventually proved that for families of maps close to $x \to ax(1-x)$ the measure of parameters with stochastic behavior is positive, see [13].

2. Dimension one, case of one critical point.

(1) Here we outline several features of a method based on [13, 14].

After some preliminary construction, which includes transition to a first return map and taking several iterates of that map we get a family of one-dimensional C^2 mappings F_t depending on the parameter $t \in \mathcal{T}_0 = [t_0, t_1]$ with the following properties.

For each t, F_t is piecewise continuous with a finite number of branches. The union of the domains of these branches is an interval I independent of t. The branches of F_t are of three types.

(a) There is a critical branch $h = h(t, x)$, whose domain is called *central domain* . Central domain $\delta_0(t)$ contains a single critical point O_1 of F_t. Without loss of generality one can assume that the critical point does not depend on t, for all t

$$h_x(t, O_1) = 0$$

(b) Monotone expanding branches which we also call *good branches*

$$f_i \colon \Delta_i \to I \tag{2.1}$$

satisfying for all t

$$|f_{ix}| > R_0 \tag{2.2}$$

where $R_0 > 1$ is a large constant.

(c) Branches g_i which map preimages of central domain δ_0 diffeomorphically onto δ_0

$$g_i \colon \delta_0^{-n_i} \to \delta_0 \tag{2.3}$$

satisfying for all t

$$|g_{ix}| > a_0 > 0 \tag{2.4}$$

(2) The above domains form a partition ξ_0 of I and we assume that the elements of that partition vary continuously with t . All new branches in the inductive process are constructed inside δ_0 and preimages $\delta_0^{-n_i}$. We assume the maps defined above satisfy the following properties for all $t \in \mathcal{T}_0$.

(a) Let $W(t) = h(t, O_1)$ be the critical value of h. We assume its speed is bounded away from zero by some $V_0 > 0$.

$$|W_t| > V_0 \tag{2.5}$$

When t varies in $\mathcal{T}_0$, the critical value $W(t)$ moves through the elements of some partition η_0. We assume that partition η_0 consists of domains Δ_i and $\delta_l^{-n_i}$ of the same type as elements of ξ_0. Elements of η_0 are mapped by some powers of F onto respective elements of ξ_0. It is convenient to consider ξ_0 as a partition on the x-axis and and η_0 as a partition on the y-axis.

(b) Let $f \colon \Delta f \to I$ or $g \colon \delta_0^{-n_i} \to \delta_0$ be maps defined on elements of η_0. We assume their domains are moving much slower than the critical value:

$$\frac{|f_t|}{|f_x|}, \frac{|g_t|}{|g_x|} < \epsilon_0 \ll V_0. \tag{2.6}$$

(c) The *distortion* $\Theta(f)$ of a diffeomorphism f defined on a domain Δf is the following supremum over $z \in \Delta f$

$$\Theta(f) = \sup \frac{|f_{xx}(z)|}{|f_x(z)|} |\Delta f| \tag{2.7}$$

We assume the maps f, g satisfy the above properties (2.2) and (2.4) and have uniformly bounded distortions. There exists $D_0 > 0$ such that all good maps $f\colon \Delta f \to I$ satisfy

$$\Theta(f) < D_0 \tag{2.8}$$

and there exists a small $\epsilon_0 > 0$ such that all maps $g\colon \delta_0^{-n_i} \to \delta_0$ satisfy

$$\Theta(g) < \epsilon_0 \tag{2.9}$$

(d) For all t the measure of the union of good branches in I is close to one

$$meas \bigcup \Delta f > 1 - \epsilon_0 \tag{2.10}$$

(e) We assume the variations of lengths of elements in ξ_0 and in η_0 are small

$$1 - \epsilon_0 < \frac{|\Delta(t_1)|}{|\Delta(t_2)|}, \frac{\left|\delta_0^{-n_i}(t_1)\right|}{\left|\delta_0^{-n_i}(t_2)\right|} < 1 + \epsilon_0 \tag{2.11}$$

for all $t_1, t_2 \in \mathcal{T}_0$.

Then we get the following theorem, [14].

Theorem 2.1. *There exist $\bar{R}_0, \bar{\epsilon}_0$ such that if the above conditions are satisfied with $R_0 > \bar{R}_0$ and $\epsilon_0 < \bar{\epsilon}_0$ and D_0 uniformly bounded, then there is a set of parameters of positive measure such that the respective maps F_t have SRB measures and the relative measure of such parameters tends to one when $\bar{R}_0 \to \infty$ and $\bar{\epsilon}_0 \to 0$.*

For families considered in [14] one can use a preliminary construction and get a family of maps F_t satisfying the above conditions, where ϵ_0 can be made arbitrary small, R_0 arbitrary large and other parameters uniformly bounded. That implies theorem (2.1). In applications one can vary the preliminary construction and use computer assisted estimates.

3. Dimension one, case of several critical points.

(1) Here we follow [15] and outline the construction in the case of several critical points.

There are m critical branches $h_l, l = 1, \dots m$, whose domains are called *central domains.* Each central domain δ_l contains a single critical point O_l of F_t. Without loss of generality one can assume that O_l do not depend on t and so for $l = 1, \dots m$ and for all t we have

$$h_{lx}(O_l) = 0$$

Let $W_l(t) = h_l(O_l, t)$ be the l-th critical value. We consider m parameter intervals $\mathcal{T}_l$ corresponding to the motion of $W_l(t)$ through elements of partitions $\eta_l = \{\Delta f, \delta_i^{-k}\}$ with the same properties as in the case of one critical point.

As an example one can include the third Chebyshev polynomial $T = 4x(x^2 - \frac{3}{4})$ in a one-parameter family $T_a = ax(x^2 - \frac{3}{4})$, and consider a close to 4. To make it more general one can consider a small C^2 perturbation of T.

As in the case of one critical point, when a approaches 4 critical values $W_l(t)$ cross consecutively intervals $I_{ln}(t)$ which accumulate to the respective repelling fixed points q_l. Let $\mathcal{T}_l = \cup T_{nl}$ be the union of parameter intervals corresponding to the motion of $W_l(t)$ through $I_{ln}(t)$. If $W_l(t)$ are moving independently we do not expect T_{nl_1} to coincide with T_{nl_2}. We need to consider parameter values such that the motion of all $W_l(t)$ is defined $\mathcal{T} = \cap_{l=1\ldots m}\mathcal{T}_l$.

However for the purpose of inductive estimates we want $W_l(t)$ to move through entire elements of η_l. In order to reconcile these contradictory requirements we first refine some elements of η_l. Then as discussed in the next subsection we can exclude some small proportion of parameters so that the remaining parameter intervals correspond to the motion of $W_l(t)$ through entire elements of η_l.

We assume our family satisfies assumptions similar to (2.5)-(2.11) from the previous section.

That allows to prove a theorem similar to (2.1) for families with several critical points, see [15].

(2) *Specifics of parameter exclusion in the presence of several critical points.*

For maps with several critical points the inductive construction in the phase space is similar to the one for *Unimodal Maps*, see [13, 14]. However in the parameter space there are some specifics which reappear in the case of Hénon-like maps, where the number of critical branches grow at consecutive steps of induction. Below we outline main ideas of the construction in the parameter space.

(a) At a given step of induction we define admissible and non-admissible domains in the phase space for each critical value $W_l(t)$. As the end-points of the elements of η_l move slower than $W_l(t)$ it follows that to each element $\Delta \in \eta_l(t)$ corresponds a parameter interval $\mathcal{D}$ such that $W_l(t) \in \Delta$ when $t \in \mathcal{D}$.

In order to keep distortions of maps $g : \delta_k^{-n_i} \to \delta_k$ small we position critical values outside of some enlargements $\tilde{\delta}_k^{-n_i} \supset \delta_k^{-n_i}$ and consider as inadmissible the locations of $W_l(t)$ inside $\tilde{\delta}_k^{-n_i}$. Then *admissible parameter values* $W_l(t)$ belong to the remaining $\mathcal{D}$. Let

$$T_l = \bigcup \mathcal{D} \tag{3.1}$$

be the union of l- *admissible* parameter intervals.

As the measure of the union of $\delta_k^{-n_i}$ is small one can choose relatively big enlargements which imply (2.9) and at the same time satisfy

$$\frac{|T_l|}{|\mathcal{T}_l|} > 1 - \epsilon_0, \tag{3.2}$$

where ϵ_0 is small.

Let us define

$$\mathcal{A}_0 = \bigcap_{l=1}^{m} \mathcal{T}_l. \tag{3.3}$$

Then $\mathcal{A}_0$ is the initial set of parameters that are admissible for all critical values simultaneously. If ϵ_0 is sufficiently small then the relative measure of $\mathcal{A}_0$ in each $\mathcal{T}_l$ satisfies

$$\frac{|\mathcal{A}_0 \cap \mathcal{T}_l|}{|\mathcal{T}_l|} > 1 - \epsilon_1 \tag{3.4}$$

which is arbitrary close to one if ϵ_0 is sufficiently small. Here ϵ_1 is another small constant.

(b) When we choose parameter values we require in particular that $W_l(t)$ do not belong to central domains δ_k. That means we delete a fixed proportion of parameters. If we do it at every step of induction then we end up with a set of admissible parameters of measure zero. Therefore at some step of induction we must construct new good elements inside each central domain δ_k and allow W_l to enter these elements.
We get a partition ξ_1^l of the central domain δ_l by considering the pull-back

$$\xi_1^l = h_l^{-1} \eta_l. \tag{3.5}$$

Consider $W^l(t)$ which is moving through an admissible domain Δ_1^l. Let $\mathcal{D}_1^l$ be the respective interval of parameters . For $t \in \mathcal{D}_1^l$ the new central domain, which contains the critical point O_l, is $\delta_1^l = h_l^{-1} \Delta_1^l$.
Note that differently from the partitions ξ_0 and η_l, which are defined and vary continuously for all t, the partitions ξ_1^l are defined and vary continuously only for $t \in \mathcal{D}_1^l$.

(c) In our construction in order to get consecutive refinements of the central domains δ_n^l at steps $n = 1, 2, \ldots$ we first pull back some partition onto a domain Δ_{n-1}^l, which contains the critical value $W^l(t)$, and after that we pull back that new partition from Δ_{n-1}^l onto δ_{n-1}^l by h_l^{-1}.
Let us denote by

$$\mathcal{I}_1 = \bigcap_{l=1}^{m} \mathcal{D}_1^l \tag{3.6}$$

one of the nonempty intersections of l-admissible parameter intervals at the first step of induction. We call it the *intersection of rank one*. By construction at each step of induction admissible parameter intervals $\mathcal{D}_{i_1 i_2 \ldots i_k}^l$ of rank k are partitioned into admissible intervals $\mathcal{D}_{i_1 i_2 \ldots i_k i_{k+1}}^l$ of the next rank $k+1$ and some inadmissible intervals. Then respective intersections

$$\mathcal{I}_{k+1} = \bigcap_{l=1}^{m} \mathcal{D}_{i_1 \ldots i_{k+1}}^l \tag{3.7}$$

are defined. By construction each $\mathcal{I}_{k+1}$ belongs to only one $\mathcal{I}_k$.

Let us consider an intersection of rank n_1

$$\mathcal{I}_{n_1} = \bigcap_{l=1}^{m} \mathcal{D}^l_{i_1 \ldots i_{n_1}} \tag{3.8}$$

and a respective intersection of rank one

$$\mathcal{I}_{n_1} \subset \mathcal{I}_1 = \bigcap_{l=1}^{m} \mathcal{D}^l_1, \tag{3.9}$$

where

$$\mathcal{D}^l_{i_1 \ldots i_{n_1}} \subset \mathcal{D}^l_1 \tag{3.10}$$

When we are doing parameter choice at step n_1 we want to use that the total measure of inadmissible elements in the phase space is small. Let us define a *union of rank* n_1 corresponding to the intersection (3.8) of rank n_1 by

$$\mathcal{U}_{n_1} = \bigcup_{l=1}^{m} \mathcal{D}^l_{i_1 \ldots i_{n_1}}. \tag{3.11}$$

As at step n_1 we pull back partitions of rank 1, we get that pull-backs are well defined if the *union of rank* n_1 lies inside the respective *intersection of rank* 1

$$\mathcal{U}_{n_1} \subset \mathcal{I}_1 \tag{3.12}$$

We delete Intersections $\mathcal{I}_{n_1}$ that do not satisfy (3.12).

Let us estimate the measure of the deleted parameter intervals. By construction (3.12) is not satisfied if and only if the following holds.

Exclusion Property.

One of the intervals $\mathcal{D}^l_{i_1 \ldots i_{n_1}}$ contains a boundary point of some $\mathcal{D}^k_1$.

Let N_1 be the number of intervals $\mathcal{D}$ of rank 1, thus $2N_1$ the number of their boundary points.

Let s_{n_1} be the maximum of the lengths of the intervals $\mathcal{D}^l_{i_1 \ldots i_{n_1}}$ of rank n_1. As the length of the union of intervals with a nonempty intersection does not exceed $2s_{n1}$ we get that the total measure deleted in order to satisfy exclusion property does not exceed

$$4N_1 s_{n_1} \tag{3.13}$$

By construction s_{n_1} decrease exponentially, thus (3.13) does the same.

(d) At a general step n of induction when doing parameter choice we pull back an earlier partition $\xi_{[nx_0]}$, where x_0 is a small constant and $[nx_0]$ is the integer part of nx_0. Then we show that for a sufficiently small x_0 the measure deleted based on exclusion property decays exponentially. For that purpose we use *uniform scaling* in the phase space, see [15]. By construction the domains of good branches at step n of the induction satisfy

$$c_1 b_1^n < |\Delta f_n| < c_2 a_1^n, \tag{3.14}$$

where

$$0 < b_1 < a_1 < 1.$$

Estimates of speeds imply that parameter intervals corresponding to the movement of $W_l(t)$ through Δf_n satisfy similar inequalities with another choice of constants:

$$c_1' b_1^n < |\mathcal{D}_n| < c_2' a_1^n. \tag{3.15}$$

In order to get well-defined partitions we need each n-union to be a subset of the respective $[nx_0]$-intersection

$$\mathcal{U}_n \subset \mathcal{I}_{[nx_0]} \tag{3.16}$$

We delete $\mathcal{I}_n$ which do not satisfy 3.16. Then as above at step n_1 we check that the measure of the deleted intervals is less than

$$C b_1^{-[nx_0]} a_1^n, \tag{3.17}$$

which is exponentially small for large n if

$$\frac{a_1}{b_1^{x_0}} < 1. \tag{3.18}$$

At first steps of induction we can pull back the same initial partition and therefore we can choose x_0 arbitrary small. So it is easy to satisfy (3.18), although one should note that at these special first steps of induction, when we pullback the same partition ξ_0 several times, we can loose a lot of measure in the parameter space.

4. Hénon-like maps

(1) *Some historical remarks and some open problems.*

As a part of a joint project with Sheldon Newhouse [19] we consider some Hénon-like families and prove that the measure of parameters with stochastic behavior is positive.

Here we make some historical remarks and outline some similarities with one-dimensional technique of parameter exclusion.

In 1976 Michel Hénon who was a French mathematician and astronomer and worked for a long time at the Nice Observatory studied a map with a "strange attractor"

$$x_{n+1} = 1 - ax_n^2 + y_n, \;\; y_{n+1} = bx_n$$

for parameter values $a = 1.4, b = 0.3$.

Later numerical estimates gave Hausdorff dimension of Hénon attractor close to 1.261.

The study of Hénon map generated a lot of activity in the area of Dynamical Systems, in particular remarkable works by Benedicks, Carleson, Young and others, see [5, 6, 36].

However in spite of all activity rigorous results about existence and properties of Hénon attractor were obtained only for unspecified small values of b and values of a close to 2.

A natural question is:

For a small neighborhood of Hénon values, say

$$\begin{aligned} 1.39 < a < 1.41 \\ .29 < b < .31 \end{aligned} \tag{4.1}$$

is there a set of parameters a, b of positive measure such that respective Hénon maps have a Sinai-Ruelle-Bowen measure?

A related problem for quadratic family

$$f_a : x \to ax(1-x)$$

can be formulated as follows.

Problem.
Develop an algorithm which estimates the measure of parameters a such that f_a has an SRB measure within a randomly chosen interval of parameter (a_1, a_2) .

One method for finding such a inside an interval not adjacent to the Chebyshev value $a = 4$ was developed in my work [14].

Based on that algorithm Yu-Ru Huang [11] proved that inside the interval $(3.99512, 3.99513)$ the relative measure of stochastic parameters is greater than 5.881582×10^{-15}. For the family

$$x \to x^2 - a$$

Luzzatto and Takahasi [24] proved that inside the interval $(2 - 10^{-4990}, 2)$ adjacent to the Chebyshev value $a = 2$ stochastic values of parameter a occupy more than 97%.

In the opposite direction Tucker and Wilczak [32] obtained an estimate for a lower bound of the measure of structurally stable parameter values in quadratic family.

Recently Golmakani, Koudjinan, Luzzatto and Pilarczyk [10] proved that parameter intervals where most of parameter values are stochastic, occupy more than 90% of the total measure of parameters in quadratic family.

(2) *Some new two-dimensional models.*

Here we study some piecewise smooth models which combine hyperbolic behavior with small determinant together with Hénon-like behavior with determinant $b \leq 1$.

Differently from [5] our technique does not use that the maps under consideration are small perturbations of one-dimensional maps. This approach combined with computer assisted estimates may be useful in the study of Hénon-like maps with not so small jacobian.

As an example we consider a piecewise smooth family of maps

$$f_t : \mathcal{D} \to \mathcal{D}$$

where $\mathcal{D} = \mathcal{Q} \cup \mathcal{B}$, $\mathcal{Q}$ and $\mathcal{B}$ are rectangles. The domain $\mathcal{Q} = [-A, A] \times [-1, 1]$ where the constant $A > 0$ is large, so $\mathcal{Q}$ is like a long strip.

Based on the range of parameters of the construction specified below we use the domain

$$\mathcal{B} = [A, A + \lambda k + 1] \times [-2, 2] \tag{4.2}$$

The strip $\mathcal{Q}$ is a union

$$\mathcal{Q} = \mathcal{D}_0 \cup \mathcal{D}_c \cup \mathcal{D}_1$$

The domains $\mathcal{D}_0$ and $\mathcal{D}_1$ are long strips and the domain

$$\mathcal{D}_c = [-k, k] \times [-1, 1] \tag{4.3}$$

is in the middle of $\mathcal{Q}$.

We assume that restricted to $\mathcal{D}_0$ and $\mathcal{D}_1$ f_t acts as an affine transformation which does not depend on t. On $\mathcal{D}_0$ f_t is defined by

$$\begin{aligned} f_1(x, y) &= \lambda x + S \\ f_2(x, y) &= \epsilon y + \sigma \end{aligned} \tag{4.4}$$

On $\mathcal{D}_1$ f_t is defined by

$$\begin{aligned} f_1(x, y) &= -\lambda x + S \\ f_2(x, y) &= -\epsilon y - \sigma \end{aligned} \tag{4.5}$$

Here $\epsilon > 0$ is small, λ is close to 2, the horizontal shift S is of the same order as A. Two vertical shifts $\pm\sigma$ are used to separate images of $\mathcal{D}_0$ and $\mathcal{D}_1$. We use $\sigma = 0.67$.

$\mathcal{D}_c$ is a union of its central piece $\mathcal{D}_{ct}$ and two adjacent pieces $\mathcal{D}_{lt}$ and $\mathcal{D}_{rt}$.

On $\mathcal{D}_{ct}$ the map f_t acts as a composition of a parabolic map with a constant right shift by S along the x axis. The parabolic map is given by Hénon formula with determinant $b \leq 1$ and $t \in [t_0, t_1]$.

$$\begin{aligned} X &= -x^2 + by + t \\ Y &= -x \end{aligned} \tag{4.6}$$

Respectively the map $f_t | D_{ct}$ is defined by

$$\begin{aligned} X &= -x^2 + by + t + S \\ Y &= -x \end{aligned} \tag{4.7}$$

The central domain $\mathcal{D}_{ct}$ is bounded by the lines $y = 1, y = -1$, and by two pieces of the parabola

$$by = -0.3 + x^2 - t \tag{4.8}$$

The right boundary of $\mathcal{D}_{lt}$ coincides with the left boundary of $\mathcal{D}_{ct}$ and the left boundary of $\mathcal{D}_{lt}$ is a subinterval of $x = -k$. Similarly $\mathcal{D}_{rt}$ is adjacent to $\mathcal{D}_{ct}$ on the right and bounded on the right by a subinterval of $x = k$. Inside the domains $\mathcal{D}_{lt}$ and $\mathcal{D}_{rt}$ we define smooth *bump* maps $G(x, y, t)$. On $\mathcal{D}_{lt}$ the bump map connects (4.7) to (4.4) and on $\mathcal{D}_{rt}$ the bump map connects (4.7) to (4.5).

All our maps are compositions of *local* maps with the constant right shift by S along the x axis. Therefore it is enough to define bump maps which connect (4.6) to the local affine maps.

$$f_{1l}(x,y) = \lambda x \qquad (4.9)$$
$$f_{2l}(x,y) = \epsilon y + \sigma$$

on $\mathcal{D}_0$ and

$$f_{1l}(x,y) = -\lambda x \qquad (4.10)$$
$$f_{2l}(x,y) = -\epsilon y - \sigma$$

on $\mathcal{D}_1$. Locally the image of $\mathcal{D}_c$ is contained in a rectangle $\mathcal{B}'$

$$\mathcal{B}' = [-\lambda k, 1] \times [-2, 2] \qquad (4.11)$$

where k is from (4.3) and λ is close to 2.

Hénon map (4.6) maps $\mathcal{D}_{ct}$ onto a parabolic region which we call a *hook* and denote it $\mathcal{H}_{ct}$. For all $t \in [t_0, t_1]$ the hook is a subset of $\mathcal{B}'$ bounded by two subintervals of the vertical line $X = -0.3$ and by two parabolas $X = -Y^2 + b + t$ and $X = -Y^2 - b + t$.

Then we extend smoothly Hénon map on $\mathcal{D}_{ct}$ by bump maps which map $\mathcal{D}_{lt}$ and $\mathcal{D}_{rt}$ onto two curvilinear rectangles which extend the hook's handles up to $X = -\lambda k$.

Images $S_0 = f(\mathcal{D}_0)$ and $S_1 = f(\mathcal{D}_1)$ are horizontal strips of height 2ϵ which have full width in $[-A, A]$. Local images of $S_i \cap \mathcal{D}_c$ are two hooks and their extensions located in $\mathcal{B}'$.

We choose connecting functions $G(x,y,t)$ in such a way that their restrictions to the left and right boundaries of $\mathcal{D}_c$ are two subintervals of $X = -\lambda k$ which do not depend on t. After translation by S extended hooks are attached to $x = A$ on the right.

The choice of $G(x,y,t)$ is flexible. The main restrictions are dictated by the relation between standard and *implicit* coordinates , see below subsection 9. These relations imply that it is enough to keep G_{1x} and Jacobian determinant of $G(x,y,t)$ uniformly bounded away from 0.

We denote $l = \sqrt{1 + by + t}$ and define

$$\eta_{lk}(x) = c_0^{-1} \int_l^x \exp\Big(-(s-l)^{-1} - (k-s)^{-1}\Big) \qquad (4.12)$$

where

$$c_0 = \int_l^k \exp\Big(-(s-l)^{-1} - (k-s)^{-1}\Big) \qquad (4.13)$$

Then for $\sqrt{1 + by + t} \leq x \leq k$ we define $G(x,y,t)$ by

$$G_1(x,y,t) = \Big(by + t - x^2\Big)\Big(1 - \eta_{lk}(x)\Big) - (\lambda x)\eta_{lk}(x) \qquad (4.14)$$
$$G_2(x,y,t) = (-x)\Big(1 - \eta_{lk}(x)\Big) - (\sigma + \epsilon y)\eta_{lk}(x)$$

For $-\sqrt{1+by+t} \geq x \geq -k$ we use symmetric functions $G_1(x,y,t)$ and $G_2(x,y,t)$. Finally the image of $\mathcal{D}_c$ under the Hénon map and $G(x,y,t)$ is translated to the right by S.

On the boundary $x = A$ horizontal strips S_i have the same heights as the handles of the hook. They are attached to the handles along the vertical intervals $[\sigma - \epsilon, \sigma + \epsilon]$ and $[-\sigma - \epsilon, -\sigma + \epsilon]$.

(3) *Parameters of the construction.*
A and S are large, λ is close to 2.

We assume $x = -A$ contains a fixed saddle point. That implies

$$\lambda(-A) + S = -A \tag{4.15}$$

or

$$S = (\lambda - 1)(A) \tag{4.16}$$

As the image of $x = -k$ is contained in $x = A$ we get

$$-\lambda k + S = A \tag{4.17}$$

or

$$S = A + \lambda k \tag{4.18}$$

We use $k = 4$. Then from (4.15), (4.18) we get

$$\lambda = \frac{2A}{A-4} \tag{4.19}$$

Estimates in local coordinates are independent of A and S.

Computations with $b = 0.3$, parameter values $0.149 \leq t \leq 0.151$ and various $\epsilon \leq 0.1$, resulted in G_{1x} uniformly bounded away from 0 by a constant independent of ϵ and Jacobian determinant uniformly bounded away from 0 by $c\epsilon$. Main restrictions on parameters of the construction are dictated by inductive arguments near $x = 0$.

(4) *Initial partition of* $\mathcal{Q}$.

Let us denote ξ_{00} the partition of $\mathcal{Q}$ into $\mathcal{D}_c$, $\mathcal{D}_0$ and $\mathcal{D}_1$. Let ξ_{0m} be the partition of $\mathcal{Q}$ obtained by m consecutive pullbacks of ξ_{00} using compositions of f^{-1}.

Elements of ξ_{0m} are of two types. First type are rectangles $E_{i_0 \ldots i_m} = E_{i_0} \cap f^{-1}E_{i_1} \cap f^{-m}E_{i_m}$ where $i_s = 0, 1$, $E_0 = \mathcal{D}_0, E_1 = \mathcal{D}_1$.

They are mapped by f^{m+1} linearly onto full width substrips of $\mathcal{Q}$ with expansion λ^{m+1} and contraction ϵ^{m+1}. We use m equal to the integer part of $\log_\lambda A$. Then the widths of $E_{i_0 \ldots i_m}$ are of the order 1.

Second type are preimages $\mathcal{D}_c^{-l}, l = 0, 1, \ldots m$ of the central zone, which are called *holes*.

The widths of holes decrease from $|\mathcal{D}_c|$ to $|\mathcal{D}_c^{-m}|$ which are of the order A^{-1}.

We assume the map $f_t|\mathcal{B}$ does not depend on t and maps $\mathcal{B}$ affinely with the same expansion λ and contraction ϵ onto a horizontal strip $\bar{S}_2 \subset E_{00\ldots 0}$ disjoint from S_0 and S_1. Then f^{m+1} maps $\bar{S}_2$ onto a full width substrip of

$\mathcal{Q}$.

The above model satisfies *starting conditions* which allow to prove the following theorem [19].

Theorem 4.1. *For a sufficiently large A there is a set of parameters of positive Lebesgue measure $\mathbf{M} \subset [0.149, 0.151]$ such that for $t \in \mathbf{M}$ the map f_t has an attractor with an ergodic SRB measure. When $A \to \infty$ the relative measure of $\mathbf{M}$ in $[0.149, 0.151]$ tends to* 1.

One can use a similar construction and get a C^∞ map $f(x, y, t)$ defined on $\mathcal{D}$. Then we get a similar theorem for respective C^∞ maps. Note that main restrictions are not related to the bump maps. They arize from inductive estimates for the Hénon map.

(5) *Geometric structure.*

At step n of induction the main structure constructed in $\mathcal{Q}$ is called *Horizontal Grid*, and it is denoted HG_n. HG_n is a union of finitely many horizontal strips $S = S_k^{(n)}$ with disjoint interiors.

Horizontal means that tangent vectors to the top and bottom are uniformly close to $(1, 0)$.

The heights of $S_k^{(n)}$ decrease exponentially with n.

Horizontal strips $S = S_k^{(n)}$ are partitioned into curvilinear rectangles with disjoint interiors E, Z_n, Z_i^{-k}, $i = 0, 1, \dots n$ which create *vertical structure.*

In the middle of horizontal strips are located central zones Z_n, which are rectangles with top and bottom close to horizontal and two parabolic sides making exponentially small angles with top and bottom.

They have exponentially decreasing widths, and their heights are exponentially small comparatively to their widths. Central zones are partitioned into small heights *squeezed rectangles* of the same types E and Z_i^{-k}.

Under Hénon maps vertical intervals located in the middle of Z_n are mapped onto horizontal intervals in the middle of the hooks. We call such horizontal intervals *tips of the hooks.*

Rectangles E are mapped onto full width strips by the maps which we denote $f : E \to S$.

Rectangles Z_i^{-k} are mapped onto central zones Z_i by the maps which we denote $g : Z_i^{-k} \to Z_i$.

Note that images under parabolic maps p_t of almost horizontal curves are curves close to parabolas.

Let $\mathcal{Z}_0 = \cup Z_0$. We call $p_t(HG_n \cap \mathcal{Z}_0) = PG_n$ the *Parabolic Grid.*

Horizontal structure in HG_n corresponds to parabolic structure in PG_n.

PG_n consists of hooks and *squeezed rectangles* with almost parabolic "horizontal boundaries" and almost vertical "vertical boundaries". By using parameter exclusion we put tips of the hooks far from holes.

(6) *Tools.*

We combine some new technique with various tools developed in several preceding works.

(a) The general method of *parameter exclusion* goes back to [13]. Some modifications suitable for numerical estimates were introduced in [14].

(b) The idea of doing distortion estimates in *local adapted* coordinates such that at the origin the axes are tangent to the stable and unstable manifolds and the differential takes the diagonal form was used in [17] and [18]. In the Hénon-like situation the angles between adapted axes converge to zero, when points are close to the local critical lines. Respective objects are curvilinear squeezed parallelograms.

(c) We use an adapted version of Palis-Yoccoz implicit coordinates where independent coordinates are y-coordinate in the domain and x-coordinate in the range, see [25] and [26]. In our approach compositions between *squeezed affine-like maps* substitute Palis-Yoccoz compositions between affine-like and parabolic maps.

(d) The existence of a positive measure set of parameters with SRB measures is based on a version of the general approach developed in [13] . The technique of dealing with a growing number of thin hooks is similar to one discussed in the previous section for maps with several critical points.

(e) In the course of inductive construction central zones and their preimages are filled with new rectangles E and new preimages Z_i^{-k}. At the end of induction central zones and their preimages disappear, and only E remain which are mapped onto full width strips.
That collection of E form a *pre-Markov partition.*
A transition $F : E_1 \to E_2$ is admissible if $S_1 = F(E_1)$ *properly* intersects E_2 and F has enough contraction. As contraction in our model is stronger than expansion we get more and more admissible transitions, which results in a construction of a Markov partition which consists of Cantor sets of positive measure. That was studied in [20].

(f) We get uniform distortion and parameter estimates for the power map on two types of two-dimensional rectangles:
Full height two-dimensional rectangles - preimages of rectangles E, and squeezed rectangles close to critical lines. Using full height rectangles helps to explore *contraction stronger than expansion* property and we get uniform distortion and parameter estimates on unstable curves in all rectangles E.

(g) Ergodic and statistical properties of the original map f can be studied by using the technique of [20, 28, 29, 36].

(7) *Structures in the parameter space.*

At step n of induction we consider different hooks which are moving through different full height rectangles W. Such W are mapped by power maps F onto full width strips S. We fix some small parameter of the construction $0 < s_2 < 1$ and consider an earlier partition $\xi_{[s_2 n]}$ restricted to S and pull it back into W.

Then we get inside W of step n a pullback of a partition of step $[s_2 n]$.

Let $J_i^{(n)}$ be respective parameter intervals. All elements of $\xi_{[s_2 n]}$ are defined simultaneously for

$$t \in \cap J_k^{([s_2 n])}$$

At the same time for the choice of parameter we need all pullbacks into various W to be defined simultaneously for

$$t \in \cup J_i^{(n)}$$

So as for one-dimensional maps with several critical points, we require an inclusion

$$\cup J_i^{(n)} \subset \cap J_i^{([s_2 n])}$$

In order to have such an inclusion we eliminate unions $\cup J_i^{(n)}$ of intervals $J_i^{(n)}$ satisfying two conditions.

1) They have nonempty intersection.
2) One of them intersects an endpoint of some earlier constructed interval $J_k^{([s_2 n])}$.

The number of $J_k^{([s_2 n])}$ can be controlled and the number of hooks at step n can be controlled, so we can control the measure of parameter values that we loose because of such restrictions.

(8) *Horizontal subdivisions and the number of hooks.*

As quadratic maps interchange axes a tip of some hook has width equal to the height of respective central zone.

At step n in order to fit tips of hooks into good rectangles where we can maintain the inductive process, we need respective heights to decrease fast.

That is done by *horizontal subdivisions.* In our construction each central domain Z is the image of some full height post P and horizontal subdivision of Z is pushed forward from the horizontal subdivision of P.

Horizontal subdivision of P is done by using the choice of parameter at earlier steps.

By using that contraction in our model is stronger than expansion we can control the number of horizontal subdivisions and get an estimate for the number of hooks at step n. We prove that this number does not exceed

$$(c\sqrt{A})5^{\epsilon_2 n}$$

where ϵ_2 is a small constant. Here 5 is the number of symbols corresponding to initial domains $\mathcal{D}_0$,$\mathcal{D}_1$, $\mathcal{B}$ and $+$, $-$, corresponding to two quadratic roots.

Then at step n the loss of parameter due to all parameter exclusions has an exponentially small measure.

(9) *Palis-Yoccoz implicit coordinates* .

The key technical ingredients are inductive estimates of distortions.

We use an adapted version of Palis-Yoccoz implicit coordinates, see [25] and [26].

Adapted version means that for some fixed point (x_0, y_0, t_0) we use local coordinate systems with diagonal differential at that point.

In our approach compositions between *squeezed affine-like maps* substitute Palis-Yoccoz compositions between affine-like and parabolic maps.

In general implicit coordinates approach which does not make difference between forward and backward iterates is compatible with our use of both images and preimages of pre-Markov rectangles.

By using implicit adapted coordinates we prove bounded distortions for two types of maps.

1. Maps from full height rectangles E onto full width strips S.
   and
2. Maps from squeezed rectangles E located in central zones δ_n onto full width strips S.

Then one proves that restricted to unstable leaves in the base of the attractor distortions of arbitrary compositions are uniformly bounded. That implies for the power map as in the classical SRB models existence of conditional measures with densities on unstable leaves given by

$$\lim_{n\to\infty} \frac{\prod_{s=1}^{n} \frac{1}{\mid D^u F(F^{-s}z) \mid}}{\int_{W^u} \prod_{s=1}^{n} \frac{1}{\mid D^u F(F^{-s}z) \mid}}$$

Then one constructs an F-invariant measure with such conditional measures. From there an f-invariant measure μ is obtained by a tower construction. In order to study ergodic and statistical properties of μ one constructs Markov partitions.

(10) *From pre-Markov to Markov partitions.*

Rectangles E that we construct make a pre-Markov partition for the power map F. To get a Markov partition we proceed as follows.

Consider some full height rectangle E_1 of partition ξ_{00}. Its image under F_1 is a full width strip S_1 in $\mathcal{Q}$ of height ϵ^m. It intersects properly many good rectangles E. Preimages $F_1^{-1}(S_1 \cap E)$ of proper intersections are full height subrectangles E_{1i} of E_1.

Admissible compositions $F \circ F_1$ have stronger contraction, so at the second step we can exclude smaller proportion of gaps from E_{1i}.

Similarly we consecutively exclude smaller and smaller proportions of inadmissible intersections and get uniform estimates on a Cantor sets C_1 of positive measure, which consists of full height stable manifolds W^s.

Inside the gaps excluded at the first step we can start a similar construction after two initial iterates. Then we get uniform estimates on a Cantor set of positive measure C_2. These estimates are worse than on C_1, but still uniform.

Then we repeat that argument and get a sequence of similar sets C_k. The union of C_k has full measure on the attractor. That implies absolute continuity of the stable foliation and as a result ergodicity of (F, μ).

Statistical properties of the original map f can be studied by using the technique of [20, 28, 29, 36].

In particular arguments of [36] imply the decay of correlations for the original system faster than any power.

Similarly to $1-d$ case our method does not imply exponential decay of correlation. One possibility to get exponential decay is to delete more parameters by using large deviation arguments in the spirit of [35].

Acknowledgement. The author is thankful to the referee for important corrections and to L.Bunimovich for important recollections.

References

[1] R. Adler, *F-expansions revisited*, Lecture Notes in Math. **318** (1973), 1–5.

[2] V. M. Alekseev, *Quasi-random dynamical systems, I*, Math. of the USSR, Sbornik **5** (1968), 73–128.

[3] D. V. Anosov, *Geodesic Flows on Closed Riemann Manifolds with Negative Curvature*, Proc. Steklov Inst. Mat., 90, 1967.

[4] D. V. Anosov and Ya. G. Sinai, *Some smooth ergodic systems*, Russian Math. Surveys **22** (1967), 103–167.

[5] M. Benedicks and L. Carleson, *The dynamics of the Hénon map*, Ann. Math. **133** (1991), 73–169.

[6] M. Benedicks and L.-S. Young, *Markov extensions and decay of correlations for certain Hénon maps*, Astérisque **261** (2000), 13–56.

[7] L. A.Bunimovich, *A transformation of the circle*, Mat. Zametki **8** (1970), 205–216.

[8] P. Fatou, *Sur les équations fonctionnelle, I*, Bull. Soc. Math. France **47** (1919), 161–271.

[9] P. Fatou, *Sur les équations fonctionnelle,II, III*, Bull. Soc. Math. France **48** (1920), 33–94, 208–314.

[10] A. Golmakani, C. E. Koudjinan, S. Luzzatto and P. Pilarczyk, *Rigorous numerics for the quadratic family*, Preprint, 2020.

[11] Yu-Ru Huang, *Estimating measure of stochastic parameters nonadjacent to the Chebyshev value*, PhD, University of MD, 2012.

[12] M. V. Jakobson, *On smooth mappings of the circle into itself*, Math. USSR-Sb. **85** (1971), 163–188.

[13] M. V. Jakobson, *Absolutely continuous invariant measures for one-parameter families of one-dimensional maps*, Communications Math. Phys. **81** (1981), 39–88.

[14] M. Jakobson, *Piecewise smooth maps with absolutely continuous invariant measures and uniformly scaled Markov partitions*, Proceedings in Symposia in Pure Math. **69** (2001), 825–881.

[15] M. Jakobson, *Parameter choice for families of maps with many critical points*, Modern dynamical systems and applications, Cambridge Univ. Press, Cambridge, 2004, pp. 359–364.

[16] M. Jakobson, *Topological and metric properties of one-dimensional endomorphisms*, Soviet Math. Dokl **19** (1978), 1452–1456.

[17] M. V. Jakobson and S. E. Newhouse, *A two-dimensional version of the Folklore Theorem*, American Math. Soc. translations, Ser. 2 **171** (1996), 89–105.

[18] M. V. Jakobson and S. E. Newhouse, *Asymptotic measures for hyperbolic piecewise smooth mappings of a rectangle*, Asterisque **261** (2000), 103–160.

[19] M. Jakobson and S. E. Newhouse, *SRB measures for certain Hénon-like families*, In preparation.

[20] M. Jakobson and L. Simonelli, *Countable Markov partitions suitable for thermodynamic formalism*, Journal of Modern Dynamics **13** (2018), 199–219.

[21] G. Julia, *Mémoire sur l'itération des fonctions rationnelle*, J. Math Pures et Appl. **1** (1918), 47–245.

[22] A. Katok, *Moscow dynamics seminars of the nineteen seventies and the early career of Yasha Pesin*, Discrete and continuous Dynamical systems **22** (2008), 1–22.

[23] O. Kozlovski, W. Shen and S. van Strien, *Density of hyperbolicity in dimension one*, Annals of Mathematics **166** (2007), 145–182.

[24] S. Luzzatto and H. Takahasi, *Computable conditions for the occurrence of non-uniform hyperbolicity in families of one-dimensional maps*, Nonlinearity **19** (2006), 1657–1695.

[25] J. Palis and J.-C. Yoccoz, *Implicit formalism for affine-like maps and parabolic composition*, Global Analysis of Dynamical Systems, 2001, pp. 67–87.

[26] J. Palis and J.-C. Yoccoz, *Non-uniformly hyperbolic horseshoes arising from bifurcations of Poincaré heteroclinic cycles*, Publ. Math. Inst. Hautes Études Sci. **110** (2009), 1–217.

[27] D. Ruelle, *Repellers for real analytic maps*, Ergodic Theory Dynam. Systems **2** (1982), 99–107.

[28] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Ergodic Theory Dynam. Systems **19** (1999), 1565–1593.

[29] O. Sarig, *Thermodynamic formalism for countable Markov shifts*, Proc. Symp. Pure Math. **89** (2015), 81–117.

[30] Ya.G. Sinai, *Markov partitions and C-diffeomorphisms*, Func. Anal. Prilozh. **2** (1968), 64–89.

[31] S. Smale, *Diffeomorphisms with many periodic points*, Differential and combinatorial Topology (A Symposium in Honor of Marstone Morse), Princeton University Press, 1965, pp. 63–80.

[32] W.Tucker and D. Wilczak, *A rigorous lower bound for the stability regions of the quadratic map*, Physica D: Nonlinear Phenomena **238** (2009), 1923–1936.

[33] S. Ulam and J. von Neumann, *On the combination of stochastic and deterministic processes*, Bull. Am. Math. Soc. vol 53 , no. 11, 1947.

[34] P. Walters, *Invariant measures and equilibrium states for some mappings which expand distances*, Trans. AMS **236** (1978), 121–153.

[35] J.-C. Yoccoz, *A Proof of Jakobson's Theorem* Lectures, College de France, 2015.

[36] Lai-Sang Young. Statistical properties of Dynamical Systems with Some Hyperbolicity. *Annals of Math*, 147, n. 3 :585–650, 1998.

Manuscript received December 28 2019
revised May 25 2020

Michael Jakobson
Department of Mathematics, University of Maryland, College Park, MD, USA
E-mail address: mvy@math.umd.edu