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A GENERALIZED SEQUENTIAL FORMULA FOR SUBDIFFERENTIAL OF MULTI-COMPOSED FUNCTIONS DEFINED ON BANACH SPACES AND APPLICATIONS

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Abstract. In this paper, a sequential formula is obtained for the subdifferential of multi-composed convex functions in Banach spaces. We present two applications illustrating the main result of this study.

1. Introduction

Recently, a new class of optimization problems has been introduced in [4, 12], where the objective function is written as a composition of more than two functions. This type of problems is called multi-composed optimization problems. In fact, multi-composed optimization problems can be employed for modelling many practical problems in connection, for example, to facility location problems [5, 13, 14], fractional and entropy optimization [4]. By considering a convex multi-composed optimization problem with geometric and cone constraints, Wanka and Wilfer [12] have obtained the optimality conditions by using the conjugate duality approach (precisely, Lagrange duality) under a class of regularity conditions guaranteeing strong duality. In most cases, to derive necessary and sufficient optimality conditions for a general constrained convex optimization problem, a qualification condition is needed (for instance, generalized Slater condition). However, the qualification condition does not always hold even for very simple optimization problems. To overcome this difficulty, many authors investigate sequential optimality conditions characterizing optimal solution in terms of nets (or sequences) in exact subdifferentials at some nearby points without considering any qualification condition (see [1], [6], [7], [8], [9], [10]).

The aim of this work is twofold. First, by applying an interesting result of Fitzpatrick and Simons [3], we establish a sequential formula for the subdifferential of the sums of m $(m \geq 2)$ proper, convex and lower semicontinuous functions, without any qualification condition. Second, we formulate the sequential subdifferential of the multi-composed convex function $f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p$, by using an approach which enables us to reduce the subdifferential calculus of the multi-composed convex function to that of the sums of $p + 4$ convex functions $(p \geq 2)$. We present two applications illustrating the main result of this study.

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The rest of the work is written as follows. In Section 2, we introduce some definitions, notations from convex analysis and we present some important results used in what follows. In Section 3, we state a formula describing the sequential subdifferential of the sums of m ($m \geq 2$) proper, convex and lower semicontinuous functions, without any qualification condition. In Section 4, we provide sequential formula for the subdifferential of the multi-composed convex function. Finally, in Section 5, we derive sequential optimality conditions for a general multi-composed optimization problem with geometric and cone constraints, without considering any qualification condition. Moreover, we give an example dealing with facility location problems.

2. Notations, definitions and preliminaries

In this section we describe the notations used throughout this paper and present some preliminary results. Let X and Y be two Hausdorff locally convex spaces paired in duality by $\langle .,. \rangle$ where their topological duals X^* and Y^* are endowed respectively with the weak-star topology $w(X^*,X)$ and $w(Y^*,Y)$. For a subset $C \subseteq X^*$, we denote by $\overline{C}^{w(X^*,X)}$ the closure of C with respect to the weak-star topology in X^*. Consider a nonempty convex cone $K \subseteq Y$. We define by

$$K^* := \{y^* \in Y^* : \langle y^*, y \rangle \geq 0, \quad \forall y \in K\}$$

the dual cone of K. Further, on Y we consider the partial order " $\leqq_K$ " induced by K defined by

$$y_1, y_2 \in Y, \qquad y_1 \leqq_K y_2 \iff y_2 - y_1 \in K.$$

With respect to " $\leqq_K$ " the augmented set $Y \cup \{+\infty_Y\}$ is considered where $+\infty_Y$ is an abstract element verifying the following operations and conventions

$$y \leqq_K +\infty_Y, \quad y + (+\infty_Y) := (+\infty_Y) + y := +\infty_Y, \quad \forall y \in Y \cup \{+\infty_Y\},$$

$$\langle y^*, (+\infty_Y) \rangle := +\infty, \quad \alpha.(+\infty_Y) := +\infty_Y, \quad \forall y^* \in K^*, \quad \forall \alpha \geq 0.$$

Let us mention that throughout this paper all cones we consider contain the origin.

Let us now recall some well known concepts from convex analysis. For a given function $f : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, f is said to be proper if its effective domain $\text{dom} f := \{x \in X : f(x) \in \mathbb{R}\} \neq \emptyset$ and $f(x) > -\infty$ for all $x \in X$, and it is called convex if $f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_2)$ for all $x_1, x_2 \in X$ and all $\lambda \in [0,1]$. Moreover, a function $f : X \to \overline{\mathbb{R}}$ is called lower semicontinuous if $f(\overline{x}) \leq \liminf_{x \to \overline{x}} f(x)$ for all $\overline{x} \in X$. The conjugate function of $f : X \to \overline{\mathbb{R}}$ is defined by

$$\begin{array}{rccl} f^* : & X^* & \longrightarrow & \overline{\mathbb{R}} \\ & x^* & \longmapsto & \sup\limits_{x \in X} \{\langle x^*, x \rangle - f(x)\}. \end{array}$$

We have the so called Young-Fenchel inequality

$$f^*(x^*) + f(x) \geq \langle x^*, x \rangle, \quad \forall (x, x^*) \in X \times X^*.$$

Let $f : X \to \overline{\mathbb{R}}$ be a function and $\overline{x} \in \text{dom} f$, then the ε-subdifferential of f at $\overline{x}$, where $\varepsilon \geq 0$, and the subdifferential of f at $\overline{x}$ are defined respectively by

$$\partial_\varepsilon f(\overline{x}) := \{x^* \in X^* : f(x) \geq f(\overline{x}) + \langle x^*, x - \overline{x} \rangle - \varepsilon, \quad \forall x \in X\}$$

and

$$\partial f(\overline{x}) := \{x^* \in X^* : f(x) \geq f(\overline{x}) + \langle x^*, x - \overline{x}\rangle, \quad \forall x \in X\}.$$

We point out the relation between the subdifferential and the conjugate function

$$\partial f(\overline{x}) = \{x^* \in X^* : f^*(x^*) + f(\overline{x}) = \langle x^*, \overline{x}\rangle\}.$$

For a nonempty subset $C \subseteq X$, its topological interior is denoted by $int\, C$ and its indicator function $\delta_C : X \longrightarrow \overline{\mathbb{R}}$ is

$$\delta_C(x) := \begin{cases} 0, \text{ if } x \in C, \\ +\infty, \text{ otherwise.} \end{cases}$$

The normal cone $N_C(\overline{x})$ of C at $\overline{x} \in C$ is defined as the subdifferential of δ_C at $\overline{x}$, i.e.

$$N_C(\overline{x}) := \{x^* \in X^* : \langle x^*, x - \overline{x}\rangle \leq 0, \quad \forall x \in C\}.$$

Let Z be another Hausdorff locally convex space partially ordered by the convex cone $Q \subseteq Z$ and Z^* its topological dual space endowed with the weak-star topology $w(Z^*, Z)$. For a given vector mapping $g : Y \to Z \cup \{+\infty_Z\}$, g is called proper if its effective domain $\text{dom}g := \{y \in Y : g(y) \in Z\} \neq \emptyset$, and Q-epi closed if its epigraph

$$\text{epi}g := \{(y, z) \in Y \times Z : g(y) \leqq_Q z\}$$

is a closed subset of $Y \times Z$. The mapping g is said to be Q-convex if

$$g(\lambda y_1 + (1 - \lambda)y_2) \leqq_Q \lambda g(y_1) + (1 - \lambda)g(y_2),$$

for all $y_1, y_2 \in Y$ and all $\lambda \in [0, 1]$. Furthermore, the mapping $g : Y \to Z \cup \{+\infty_Z\}$ is called (K, Q)-nondecreasing on $\text{dom}g$ if for all $y_1, y_2 \in \text{dom}g$

$$y_1 \leqq_K y_2 \implies g(y_1) \leqq_Q g(y_2).$$

Let $h : X \longrightarrow Y \cup \{+\infty_Y\}$ be a given mapping, then the composed mapping $g \circ h : X \longrightarrow Z \cup \{+\infty_Z\}$ is defined by

$$(g \circ h)(x) := \begin{cases} g(h(x)), \text{ if } x \in \text{dom}h, \\ +\infty_Z, \text{ otherwise.} \end{cases}$$

It is easy to see that if $g : Y \to Z \cup \{+\infty_Z\}$ is (K, Q)-nondecreasing on $\text{dom}g$ and Q-convex, and h is K-convex with $h(\text{dom}h) \subseteq \text{dom}g$, then $g \circ h$ is Q-convex.

The following version of the Brønsted-Rockafellar Theorem was proved in [10] and will be used for computing the subdifferential of the sums of m $(m \geq 2)$ proper, convex and lower semicontinuous functions, without any qualification condition.

Theorem 2.1 ([10]). *Let $(X, \|.\|_X)$ be a Banach space and $f : X \to \overline{\mathbb{R}}$ be a proper, convex and lower semicontinuous function. Assume that $\overline{x} \in \text{dom}f$, then for any real number $\varepsilon > 0$ and any $x^* \in \partial_\varepsilon f(\overline{x})$, there exists $(x_\varepsilon, x^*_\varepsilon) \in X \times X^*$ such that*

$$x^*_\varepsilon \in \partial f(x_\varepsilon),\ \|x_\varepsilon - \overline{x}\|_X \leq \sqrt{\varepsilon},\ \|x^*_\varepsilon - x^*\|_{X^*} \leq \sqrt{\varepsilon}$$

and

$$|f(x_\varepsilon) - f(\overline{x}) - \langle x^*_\varepsilon, x_\varepsilon - \overline{x}\rangle| \leq 2\varepsilon.$$

3. Sequential subdifferential calculus for the sums of m functions ($m \geq 2$)

Let $(X, \|.\|_X)$ be a Banach space and $(X^*, w(X^*, X))$ its topological dual space paired in duality by $\langle ., . \rangle$. We write $x_j \xrightarrow[j\in J]{\|\cdot\|_X} 0$ (resp. $x_j^* \xrightarrow[j\in J]{w(X^*,X)} 0$) for the case when the net $\{x_j\}_{j\in J}$ converges to 0 in $(X, \|.\|_X)$ (resp. $\{x_j^*\}_{j\in J}$ converges to 0 in $(X^*, w(X^*, X))$).

The aim of this section is to give sequential formula for the subdifferential of the sums of m ($m \geq 2$) proper, convex and lower semicontinuous functions $f_1, ..., f_m : X \to \overline{\mathbb{R}}$. To this end, we use a result due to Fitzpatrick and Simons stated in the setting of locally convex space.

Lemma 3.1 ([3]). *Let $f_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., m$, be a proper, convex and lower semicontinuous function. If $\overline{x} \in \cap_{i=1}^m \mathrm{dom} f_i$, then*

$$\partial(f_1 + f_2 + ... + f_m)(\overline{x}) = \bigcap_{\eta>0} \overline{\partial_\eta f_1(\overline{x}) + \partial_\eta f_2(\overline{x}) + ... + \partial_\eta f_m(\overline{x})}^{w(X^*,X)}.$$

Theorem 3.2. *Let $(X, \|.\|_X)$ be a Banach space and $f_i : X \to \overline{\mathbb{R}}$, $i = 1, ..., m$, be a proper, convex and lower semicontinuous function. Assume that $\overline{x} \in \cap_{i=1}^m \mathrm{dom} f_i$. Then the following statements are equivalent*
(a) $x^* \in \partial(f_1 + f_2 + ... + f_m)(\overline{x})$;
(b) there exist nets $\{x_{i,j}\}_{j\in J} \subseteq \mathrm{dom} f_i$ and $\{x_{i,j}^*\}_{j\in J} \subseteq X^*$, $i \in \{1, ..., m\}$ such that

$$x_{i,j}^* \in \partial f_i(x_{i,j}) \ (j \in J), \ x_{i,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x}, \ \sum_{i=1}^m x_{i,j}^* \xrightarrow[j\in J]{w(X^*,X)} x^*$$

and

$$f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x} \rangle \xrightarrow[j\in J]{} 0.$$

Proof. (a)⇒(b) Suppose that $x^* \in \partial(f_1 + f_2 + ... + f_m)(\overline{x})$ and let $\mathcal{N}_0^*$ the set of all weak-star neighbourhoods of zero in X^*. Clearly, The order $\supset$ on $\mathcal{N}_0^*$ is directed. By Lemma 3.1

$$x^* \in \bigcap_{n\in\mathbb{N}^*} \overline{\partial_{\frac{1}{n}} f_1(\overline{x}) + \partial_{\frac{1}{n}} f_2(\overline{x}) + ... + \partial_{\frac{1}{n}} f_m(\overline{x})}^{w(X^*,X)}$$

and hence for each $n \in \mathbb{N}^*$ and $V \in \mathcal{N}_0^*$ there exist $y_{i,n,V}^* \in \partial_{\frac{1}{n}} f_i(\overline{x})$, $i = 1, ..., m$, such that

$$\sum_{i=1}^m y_{i,n,V}^* \in x^* + V. \tag{3.1}$$

By using Theorem 2.1, there exist $x_{i,n,V} \in \mathrm{dom} f_i$ and $x_{i,n,V}^* \in X^*$ such that

$$x_{i,n,V}^* \in \partial f_i(x_{i,n,V}), \tag{3.2}$$

$$\|x_{i,n,V} - \overline{x}\|_X \leq \frac{1}{\sqrt{n}}, \tag{3.3}$$

$$\|x^*_{i,n,V} - y^*_{i,n,V}\|_{X^*} \le \frac{1}{\sqrt{n}} \tag{3.4}$$

and

$$\left|f_i(x_{i,n,V}) - f_i(\overline{x}) - \langle x^*_{i,n,V}, x_{i,n,V} - \overline{x}\rangle\right| \le \frac{2}{n}, \tag{3.5}$$

with $i \in \{1, ..., m\}$, $V \in \mathcal{N}_0^*$ and $n \in \mathbb{N}^*$. Now, we consider the product partial ordering on $J := \mathbb{N}^* \times \mathcal{N}_0^*$ that we denote by $\prec$ and defined as follows: for any $(n, U) \in J$ and $(q, V) \in J$

$$(n, U) \prec (q, V) \quad \Longleftrightarrow \quad n \le q \text{ and } V \subset U.$$

From (3.1), one can see easily that for each $W \in \mathcal{N}_0^*$, there exists $(n_0, V_0) = (1, W) \in J$ such that for any $(n, V) \in J$

$$(n_0, V_0) \prec (n, V) \implies \sum_{i=1}^m y^*_{i,n,V} \in x^* + V \subset x^* + W$$

which means that

$$\sum_{i=1}^m y^*_{i,n,V} \xrightarrow[(n,V)\in J]{w(X^*,X)} x^*. \tag{3.6}$$

From (3.4) it follows that

$$\sum_{i=1}^m (x^*_{i,n,V} - y^*_{i,n,V}) \xrightarrow[(n,V)\in J]{\|\cdot\|_{X^*}} 0$$

and thus

$$\sum_{i=1}^m (x^*_{i,n,V} - y^*_{i,n,V}) \xrightarrow[(n,V)\in J]{w(X^*,X)} 0. \tag{3.7}$$

Therefore, from (3.6) and (3.7) we deduce that

$$\sum_{i=1}^m x^*_{i,n,V} \xrightarrow[(n,V)\in J]{w(X^*,X)} x^*. \tag{3.8}$$

By setting $x^*_{i,j} := x^*_{i,n,V}$ and $x_{i,j} := x_{i,n,V}$ with $j = (n, V) \in J$, we can see from (3.2), (3.3), (3.5) and (3.8) that

$$x^*_{i,j} \in \partial f_i(x_{i,j})\ (j \in J),\ x_{i,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ \sum_{i=1}^m x^*_{i,j} \xrightarrow[j\in J]{w(X^*,X)} x^*$$

and

$$f_i(x_{i,j}) - f_i(\overline{x}) - \langle x^*_{i,j}, x_{i,j} - \overline{x}\rangle \xrightarrow[j\in J]{} 0,$$

with $i \in \{1, ..., m\}$.

(b)$\Rightarrow$(a) Assume that for any $i \in \{1, ..., m\}$ there exist nets $\{x_{i,j}\}_{j\in J} \subseteq \mathrm{dom} f_i$ and $\{x^*_{i,j}\}_{j\in J} \subseteq X^*$ such that

$$x^*_{i,j} \in \partial f_i(x_{i,j})\ (j \in J),\ x_{i,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ \sum_{i=1}^m x^*_{i,j} \xrightarrow[j\in J]{w(X^*,X)} x^*$$

and

$$f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x}\rangle \underset{j\in J}{\longrightarrow} 0.$$

Since $x_{i,j}^* \in \partial f_i(x_{i,j})$, we have for all $x \in X$

$$f_i(x) \geq f_i(x_{i,j}) + \langle x_{i,j}^*, x - x_{i,j}\rangle,$$

and hence

$$f_i(x) \geq f_i(\overline{x}) + [f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x}\rangle] + \langle x_{i,j}^*, x - \overline{x}\rangle.$$

Thus by summing over i, we have

$$\sum_{i=1}^{m} f_i(x) \geq \sum_{i=1}^{m} f_i(\overline{x}) + \sum_{i=1}^{m}[f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x}\rangle] + \langle \sum_{i=1}^{m} x_{i,j}^*, x - \overline{x}\rangle, \quad \forall x \in X$$

and since

$$\sum_{i=1}^{m} x_{i,j}^* \xrightarrow[j\in J]{w(X^*,X)} x^* \text{ and } f_i(x_{i,j}) - f_i(\overline{x}) - \langle x_{i,j}^*, x_{i,j} - \overline{x}\rangle \underset{j\in J}{\longrightarrow} 0$$

we obtain

$$\sum_{i=1}^{m} f_i(x) \geq \sum_{i=1}^{m} f_i(\overline{x}) + \langle x^*, x - \overline{x}\rangle, \quad \forall x \in X$$

i.e.

$$x^* \in \partial(f_1 + f_2 + \ldots + f_m)(\overline{x}).$$

□

As a consequence, we recapture the formula for the case of two proper, convex and lower semicontinuous functions established in [9].

Corollary 3.3. *Let $(X, \|.\|_X)$ be a Banach space and $f_1, f_2 : X \to \overline{\mathbb{R}}$ be two proper, convex and lower semicontinuous functions. Assume that $\overline{x} \in \text{dom} f_2 \cap \text{dom} f_2$. Then, $x^* \in \partial(f_1 + f_2)(\overline{x})$ if and only if there exist nets $\{x_{1,j}\}_{j\in J} \subseteq \text{dom} f_1$, $\{x_{2,j}\}_{j\in J} \subseteq \text{dom} f_2$, $\{x_{1,j}^*\}_{j\in J} \subseteq X^*$ and $\{x_{2,j}^*\}_{j\in J} \subseteq X^*$ such that*

$$\begin{cases} x_{1,j}^* \in \partial f_1(x_{1,j}),\ x_{2,j}^* \in \partial f_2(x_{2,j})\ (j \in J), \\ x_{1,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ x_{2,j} \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ x_{1,j}^* + x_{2,j}^* \xrightarrow[j\in J]{w(X^*,X)} x^*, \\ f_1(x_{1,j}) - f_1(\overline{x}) - \langle x_{1,j}^*, x_{1,j} - \overline{x}\rangle \underset{j\in J}{\longrightarrow} 0, \\ f_2(x_{2,j}) - f_2(\overline{x}) - \langle x_{2,j}^*, x_{2,j} - \overline{x}\rangle \underset{j\in J}{\longrightarrow} 0. \end{cases}$$

4. Sequential subdifferential formula for multi-composed convex functions in Banach spaces

In what follows $(X, \|.\|_X)$, $(Y_i, \|.\|_{Y_i}), i = 0, ..., p$ $(p \geq 2)$ are Banach spaces paired in duality by $\langle ., . \rangle$ where their topological dual spaces X^*, Y_i^* are endowed respectively with the weak-star topology $w(X^*, X)$, $w(Y_i^*, Y_i)$, $i = 0, ..., p$. Further, we assume that Y_i is partially ordered by the nonempty convex cone $K_i \subseteq Y_i$, for $i = 0, ..., p$. On $X \times Y_0 \times Y_1 \times ... \times Y_p$ we use the norm

$$\|(x, y_0, y_1, ..., y_p)\|_{X \times Y_0 \times Y_1 \times ... \times Y_p} = \sqrt{\|x\|_X^2 + \|y_0\|_{Y_0}^2 + \|y_1\|_{Y_1}^2 + ... + \|y_p\|_{Y_p}^2}.$$

We recall that the topological dual space $(X \times Y_0 \times Y_1 \times ... \times Y_p)^*$ is identical to the product Banach space $X^* \times Y_0^* \times Y_1^* \times ... \times Y_p^*$ endowed with the product topology denoted by τ.

The aim of this section is to give a sequential formula for the subdifferential of the multi-composed function $f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p$ where

- $f : X \to \overline{\mathbb{R}}$ is proper, convex and lower semicontinuous,
- $\varphi : Y_0 \to \overline{\mathbb{R}}$ is proper, convex, K_0-nondecreasing on $\mathrm{dom}\varphi$ and lower semicontinuous,
- $\psi : X \to Y_0 \cup \{+\infty_{Y_0}\}$ is proper, K_0-convex, K_0-epi closed and $\psi(\mathrm{dom}\psi) \subseteq \mathrm{dom}\varphi$,
- $g : Y_1 \to \overline{\mathbb{R}}$ is proper, convex, K_1-nondecreasing on $\mathrm{dom}g$ and lower semicontinuous,
- $h_1 : Y_2 \to Y_1 \cup \{+\infty_{Y_1}\}$ is proper, K_1-convex, (K_2, K_1)-nondecreasing on $\mathrm{dom}h_1$, K_1-epi closed and $h_1(\mathrm{dom}h_1) \subseteq \mathrm{dom}g$,
- $h_i : Y_{i+1} \to Y_i \cup \{+\infty_{Y_i}\}$ is proper, K_i-convex, (K_{i+1}, K_i)-nondecreasing on $\mathrm{dom}h_i$, K_i-epi closed and $h_i(\mathrm{dom}h_i) \subseteq \mathrm{dom}h_{i-1}$, $i = 2, ..., p-1$,
- $h_p : X \to Y_p \cup \{+\infty_{Y_p}\}$ is proper, K_p-convex, K_p-epi closed and $h_p(\mathrm{dom}h_p) \subseteq \mathrm{dom}h_{p-1}$,
- $\mathrm{dom}f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom}g) \cap \mathrm{dom}h_p \neq \emptyset$,
- $\varphi(+\infty_{Y_0}) = +\infty$, $g(+\infty_{Y_1}) = +\infty$, and $h_i(+\infty_{Y_{i+1}}) = +\infty_{Y_i}, i = 1, ..., p-1$.

Let us consider the following auxiliary functions

$$\begin{array}{rccl} F: & X \times \prod_{k=0}^p Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x, y_0, y_1, ..., y_p) & \longmapsto & F(x, y_0, y_1, ...y_p) := f(x), \end{array}$$

$$\begin{array}{rccl} \Phi: & X \times \prod_{k=0}^p Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x, y_0, y_1, ..., y_p) & \longmapsto & \Phi(x, y_0, y_1, ...y_p) := \varphi(y_0), \end{array}$$

$$\begin{array}{rccl} \Psi: & X \times \prod_{k=0}^p Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x, y_0, y_1, ..., y_p) & \longmapsto & \Psi(x, y_0, y_1, ...y_p) := \delta_{\mathrm{epi}\psi}(x, y_0), \end{array}$$

$$\begin{array}{rccl} G: & X \times \prod_{k=0}^p Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x, y_0, y_1, ..., y_p) & \longmapsto & G(x, y_0, y_1, ...y_p) := g(y_1), \end{array}$$

for $i = 1, ..., p-1$

$$\begin{array}{rccl} H_i: & X \times \prod_{k=0}^p Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x, y_0, y_1, ..., y_p) & \longmapsto & H_i(x, y_0, y_1, ...y_p) := \delta_{\mathrm{epi}h_i}(y_{i+1}, y_i), \end{array}$$

and

$$\begin{array}{llll} H_p: & X\times\prod_{k=0}^{p}Y_k & \longrightarrow & \overline{\mathbb{R}} \\ & (x,y_0,y_1,...,y_p) & \longmapsto & H_p(x,y_0,y_1,...y_p):=\delta_{\mathrm{epi}h_p}(x,y_p). \end{array}$$

Remark 4.1. Let us note that
• $\mathrm{dom}F=\mathrm{dom}f\times\prod_{k=0}^{p}Y_k$,
• $\mathrm{dom}\Phi=X\times\mathrm{dom}\varphi\times\prod_{k=1}^{p}Y_k$,
• $\mathrm{dom}\Psi=\{(x,y_0,y_1,...y_p)\in X\times\prod_{k=0}^{p}Y_k:(x,y_0)\in\mathrm{epi}\psi\}$,
• $\mathrm{dom}G=X\times Y_0\times\mathrm{dom}g\times\prod_{k=2}^{p}Y_k$,
• $\mathrm{dom}H_i=\{(x,y_0,y_1,...y_p)\in X\times\prod_{k=0}^{p}Y_k:(y_{i+1},y_i)\in\mathrm{epi}h_i\}$ $(i=1,...,p-1)$,
• $\mathrm{dom}H_p=\{(x,y_0,y_1,...y_p)\in X\times\prod_{k=0}^{p}Y_k:(x,y_p)\in\mathrm{epi}h_p\}$,
• F,Φ,Ψ,G and $H_i, i=1,...,p$, are proper, convex and lower semicontinuous functions.

Before stating the main result of this section, we need the following results.

Lemma 4.2. *For any $x\in X$, one has*

$$\begin{aligned} & f(x)+(\varphi\circ\psi)(x)+(g\circ h_1\circ h_2\circ...\circ h_p)(x) \\ & =\inf_{(y_0,y_1,...,y_p)\in\prod_{i=0}^{p}Y_i}\{F(x,y_0,y_1,...,y_p)+\Phi(x,y_0,y_1,...,y_p)+\Psi(x,y_0,y_1,...,y_p) \\ & \qquad +G(x,y_0,y_1,...,y_p)+\sum_{i=1}^{p}H_i(x,y_0,y_1,...,y_p)\}. \end{aligned}$$

Proof. If $x\notin\mathrm{dom}\psi\cap\mathrm{dom}h_p$, the equality is obvious, since

$$f(x)+(\varphi\circ\psi)(x)+(g\circ h_1\circ h_2\circ...\circ h_p)(x)=+\infty$$

and

$$\begin{aligned} & F(x,y_0,y_1,...,y_p)+\Phi(x,y_0,y_1,...,y_p)+\Psi(x,y_0,y_1,...,y_p) \\ & \qquad +G(x,y_0,y_1,...,y_p)+\sum_{i=1}^{p}H_i(x,y_0,y_1,...,y_p)=+\infty, \end{aligned}$$

for any $(y_0,y_1,...,y_p)\in\prod_{i=0}^{p}Y_i$. Suppose that $x\in\mathrm{dom}\psi\cap\mathrm{dom}h_p$. By setting

$$\begin{aligned} \mathcal{A}_x:=\{(y_0,y_1,...,y_p)\in\prod_{i=0}^{p}Y_i:(x,y_0)\in\mathrm{epi}\psi,(x,y_p)\in\mathrm{epi}h_p, \\ \text{and } (y_{i+1},y_i)\in\mathrm{epi}h_i, i=1,...,p-1\}, \end{aligned}$$

we have for any $(y_0,y_1,...,y_p)\notin\mathcal{A}_x$

$$\begin{aligned} & F(x,y_0,y_1,...,y_p)+\Phi(x,y_0,y_1,...,y_p)+\Psi(x,y_0,y_1,...,y_p) \\ & \qquad +G(x,y_0,y_1,...,y_p)+\sum_{i=1}^{p}H_i(x,y_0,y_1,...,y_p)=+\infty, \end{aligned}$$

and hence we get

$$\text{(4.1)}\qquad \inf_{(y_0,y_1,...,y_p)\in\prod_{i=0}^{p}Y_i}\{F(x,y_0,y_1,...,y_p)+\Phi(x,y_0,y_1,...,y_p)+\Psi(x,y_0,y_1,...,y_p)$$

$$+ G(x,y_0,y_1,...,y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p)\}$$
$$= \inf_{(y_0,y_1,...,y_p)\in\mathcal{A}_x} \{F(x,y_0,y_1,...,y_p) + \Phi(x,y_0,y_1,...,y_p) + \Psi(x,y_0,y_1,...,y_p)$$
$$+ G(x,y_0,y_1,...,y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p)\}.$$

Let $(y_0,y_1,...,y_p) \in \mathcal{A}_x$, we have $h_p(x) \leqq_{K_p} y_p$, $h_i(y_{i+1}) \leqq_{K_i} y_i$, $i = 1,...,p-1$, $\psi(x) \leqq_{K_0} y_0$ and by monotonicity of φ, g and h_i, $i = 1,...,p-1$, it follows that

$$(\varphi \circ \psi)(x) \leq \varphi(y_0) \text{ and } (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x) \leq g(y_1),$$

which yields

$$f(x) + (\varphi\circ\psi)(x) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x)$$
$$\leq f(x) + \varphi(y_0) + g(y_1)$$
$$= F(x,y_0,y_1,...,y_p) + \Phi(x,y_0,y_1,...,y_p) + \Psi(x,y_0,y_1,...,y_p)$$
$$+ G(x,y_0,y_1,...,y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p),$$

and hence

$$\text{(4.2)}\quad f(x) + (\varphi\circ\psi)(x) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x)$$
$$\leq \inf_{(y_0,y_1,...,y_p)\in\mathcal{A}_x} \{F(x,y_0,y_1,...,y_p) + \Phi(x,y_0,y_1,...,y_p) + \Psi(x,y_0,y_1,...,y_p)$$
$$+ G(x,y_0,y_1,...,y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p)\}.$$

By setting $z_p := h_p(x)$, $z_{p-1} := h_{p-1}(z_p)$, ..., $z_1 := h_1(z_2)$ and $z_0 := \psi(x)$, we have $(x,z_0,z_1,...,z_p) \in \mathcal{A}_x$, and thus we obtain that

$$\text{(4.3)}\quad f(x) + (\varphi\circ\psi)(x) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x)$$
$$\geq \inf_{(y_0,y_1,...,y_p)\in\mathcal{A}_x} \{F(x,y_0,y_1,...,y_p) + \Phi(x,y_0,y_1,...,y_p) + \Psi(x,y_0,y_1,...,y_p)$$
$$+ G(x,y_0,y_1,...,y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p)\},$$

and according to (4.1), (4.2) and (4.3) we get

$$f(x) + (\varphi\circ\psi)(x) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x)$$
$$= \inf_{(y_0,y_1,...,y_p)\in\prod_{i=0}^{p} Y_i} \{F(x,y_0,y_1,...,y_p) + \Phi(x,y_0,y_1,...,y_p) + \Psi(x,y_0,y_1,...y_p)$$
$$+ G(x,y_0,y_1,...y_p) + \sum_{i=1}^{p} H_i(x,y_0,y_1,...,y_p)\}.$$

□

Remark 4.3. Let us mention that the above lemma extends some earlier results due to Wanka and Wilfer. Indeed, if C is a nonempty subset of X, $f := \delta_C$ and $\varphi := \delta_{-K_0}$, then we can apply Lemma 4.2 to obtain Theorem 2 in [12].

Lemma 4.4. *Assume that* $\overline{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_p$, $\overline{y}_p := h_p(\overline{x})$, $\overline{y}_{p-1} := h_{p-1}(\overline{y}_p)$, ..., $\overline{y}_1 := h_1(\overline{y}_2)$ *and* $\overline{y}_0 := \psi(\overline{x})$. *Then*

$$x^* \in \partial(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p)(\overline{x})$$
$$\Longleftrightarrow$$
$$(x^*, 0, 0, ..., 0) \in \partial(F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p).$$

Proof. ($\Rightarrow$) Let $x^* \in \partial(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p)(\overline{x})$. Then, for any $x \in X$ we have

$$\begin{aligned} &f(x) + (\varphi \circ \psi)(x) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x) \\ &\quad \geq f(\overline{x}) + (\varphi \circ \psi)(\overline{x}) + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(\overline{x}) + \langle x^*, x - \overline{x} \rangle \\ &\quad = (F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) + \langle x^*, x - \overline{x} \rangle. \end{aligned}$$

According to Lemma 4.2, we have for any $(x, y_0, y_1, ..., y_p) \in X \times \prod_{i=0}^{p} Y_i$

$$\begin{aligned} (F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(x, y_0, y_1, ..., y_p) &\geq f(x) + (\varphi \circ \psi)(x) \\ &\quad + (g \circ h_1 \circ h_2 \circ ... \circ h_p)(x) \end{aligned}$$

and hence we get

$$\begin{aligned} &(F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(x, y_0, y_1, ..., y_p) \\ &\qquad \geq (F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) + \langle x^*, x - \overline{x} \rangle \end{aligned}$$

i.e.

$$(x^*, 0, 0, ..., 0) \in \partial(F + \Phi + \Psi + G + \sum_{i=1}^{p} H_i)(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p).$$

($\Leftarrow$) It is immediate by using Lemma 4.2. □

Lemma 4.5. (1) *Let* $i \in \{1, ..., p-1\}$ *and* $(x, y_0, y_1, ..., y_p) \in \mathrm{dom} H_i$. *Then*

$$(x^*, y_0^*, y_1^*, ..., y_p^*) \in \partial H_i(x, y_0, y_1, ..., y_p)$$

if and only if

(a) $x^* = 0$ *and* $y_k^* = 0$ *for* $k \in \{0, ..., p\} \backslash \{i, i+1\}$,

(b) $-y_i^* \in K_i^*$ *and* $\langle -y_i^*, y_i - h_i(y_{i+1}) \rangle = 0$,

(c) $y_{i+1}^* \in \partial(-y_i^* \circ h_i)(y_{i+1})$.

(2) *Let* $(x, y_0, y_1, ..., y_p) \in \text{dom}H_p$. *Then*

$$(x^*, y_0^*, y_1^*, ..., y_p^*) \in \partial H_p(x, y_0, y_1, ..., y_p)$$

if and only if

$$\begin{aligned} &\text{(a)} \quad y_k^* = 0 \textit{ for } k \in \{0, ..., p-1\}, \\ &\text{(b)} \quad -y_p^* \in K_p^* \textit{ and } \langle -y_p^*, y_p - h_p(x)\rangle = 0, \\ &\text{(c)} \quad x^* \in \partial(-y_p^* \circ h_p)(x). \end{aligned}$$

(3) *Let* $(x, y_0, y_1, ..., y_p) \in \text{dom}F$. *Then*

$$\partial F(x, y_0, y_1, ..., y_p) = \partial f(x) \times \{0\} \times \{0\} \times ... \times \{0\}.$$

(4) *Let* $(x, y_0, y_1, ..., y_p) \in \text{dom}G$. *Then*

$$\partial G(x, y_0, y_1, ..., y_p) = \{0\} \times \{0\} \times \partial g(y_1) \times ... \times \{0\}.$$

(5) *Let* $(x, y_0, y_1, ..., y_p) \in \text{dom}\Psi$. *Then*

$$(x^*, y_0^*, y_1^*, ..., y_p^*) \in \partial \Psi(x, y_0, y_1, ..., y_p)$$

if and only if

$$\begin{aligned} &\text{(a)} \quad y_k^* = 0 \textit{ for } k \in \{1, ..., p\}, \\ &\text{(b)} \quad -y_0^* \in K_0^* \textit{ and } \langle -y_0^*, y_0 - \psi(x)\rangle = 0, \\ &\text{(c)} \quad x^* \in \partial(-y_0^* \circ \psi)(x). \end{aligned}$$

(6) *Let* $(x, y_0, y_1, ..., y_p) \in \text{dom}\Phi$. *Then*

$$\partial \Phi(x, y_0, y_1, ..., y_p) = \{0\} \times \partial\varphi(y_0) \times \{0\} \times ... \times \{0\}.$$

Proof. (1) Let $i \in \{1, ..., p-1\}$ and $(x, y_0, y_1, ..., y_p) \in \text{dom}H_i$. It is easy to see that for any $(x^*, y_0^*, y_1^*, ..., y_p^*) \in X^* \times \prod_{k=0}^{p} Y_k^*$, we have

$$H_i^*(x^*, y_0^*, y_1^*, ..., y_p^*) = \delta_{\{0\}}(x^*) + \sum_{\substack{k=0 \\ k \notin \{i, i+1\}}}^{p} \delta_{\{0\}}(y_k^*) + \delta_{K_i}^*(y_i^*) + (-y_i^* \circ h_i)^*(y_{i+1}^*).$$

Now,

$$(x^*, y_0^*, y_1^*, ..., y_p^*) \in \partial H_i(x, y_0, y_1, ..., y_p)$$

if and only if

$$\begin{aligned} (4.4) \quad & H_i^*(x^*, y_0^*, y_1^*, ..., y_p^*) + H_i(x, y_0, y_1, ..., y_p) \\ & \qquad - \langle x^*, x\rangle - \sum_{\substack{k=0 \\ k \notin \{i, i+1\}}}^{p} \langle y_k^*, y_k\rangle - \langle y_i^*, y_i\rangle - \langle y_{i+1}^*, y_{i+1}\rangle = 0 \end{aligned}$$

which implies

$$x^* = 0 \text{ and } y_k^* = 0 \text{ for } k \in \{0, ..., p\}\backslash\{i, i+1\}.$$

As $(y_{i+1}, y_i) \in \text{epi}h_i$, the equality (4.4) becomes equivalent to

$$\begin{aligned} &[\delta_{K_i}^*(y_i^*) + \delta_{K_i}(y_i - h_i(y_{i+1})) - \langle y_i^*, y_i - h_i(y_{i+1})\rangle] \\ &\qquad + [(-y_i^* \circ h_i)^*(y_{i+1}^*) + (-y_i^* \circ h_i)(y_{i+1}) - \langle y_{i+1}^*, y_{i+1}\rangle] = 0. \end{aligned}$$

According to the Fenchel-Young inequality and the fact that K_i is a convex cone, (4.4) is equivalent to

$$\begin{aligned} &\text{(a)} \quad x^* = 0 \text{ and } y_k^* = 0 \text{ for } k \in \{0, ..., p\}\backslash\{i, i+1\}, \\ &\text{(b)} \quad -y_i^* \in K_i^* \text{ and } \langle -y_i^*, y_i - h_i(y_{i+1})\rangle = 0, \\ &\text{(c)} \quad y_{i+1}^* \in \partial(-y_i^* \circ h_i)(y_{i+1}). \end{aligned}$$

The proof of (2) – (6) is similar to (1). □

Now we state our main result.

Theorem 4.6. *Suppose that* $\overline{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom} g) \cap \mathrm{dom} h_p$, $\overline{y}_p := h_p(\overline{x})$, $\overline{y}_{p-1} := h_{p-1}(\overline{y}_p)$, ..., $\overline{y}_1 := h_1(\overline{y}_2)$ *and* $\overline{y}_0 := \psi(\overline{x})$. *Then* $x^* \in \partial(f + \varphi \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p)(\overline{x})$ *if and only if there exist nets* $\{x_j\}_{j\in J} \subseteq \mathrm{dom} f$, $\{y_{0,j}\}_{j\in J} \subseteq \mathrm{dom}\varphi$, $\{(z_j, z_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi$, $\{u_{1,j}\}_{j\in J} \subseteq \mathrm{dom} g$, $\{(v_{i+1,j}^i, v_{i,j}^i)\}_{j\in J} \subseteq \mathrm{epi} h_i$, $i = 1, ..., p-1$, $\{(v_j^p, v_{p,j}^p)\}_{j\in J} \subseteq \mathrm{epi} h_p$, $\{x_j^*\}_{j\in J} \subseteq X^*$, $\{y_{0,j}^*\}_{j\in J} \subseteq Y_0^*$, $\{(z_j^*, z_{0,j}^*)\}_{j\in J} \subseteq X^* \times Y_0^*$, $\{u_{1,j}^*\}_{j\in J} \subseteq Y_1^*$, $\{(v_{i,j}^{i*}, v_{i+1,j}^{i*})\}_{j\in J} \subseteq Y_i^* \times Y_{i+1}^*$, $i = 1, ..., p-1$, *and* $\{(v_j^{p*}, v_{p,j}^{p*})\}_{j\in J} \subseteq X^* \times Y_p^*$ *satisfying*

$$\begin{cases} x_j^* \in \partial f(x_j),\ y_{0,j}^* \in \partial\varphi(y_{0,j}),\ u_{1,j}^* \in \partial g(u_{1,j}), \\ -z_{0,j}^* \in K_0^*,\ \langle -z_{0,j}^*,\ z_{0,j} - \psi(z_j)\rangle = 0 \text{ and } z_j^* \in \partial(-z_{0,j}^* \circ \psi)(z_j), \\ -v_{i,j}^{i*} \in K_i^* \text{ and } \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0\ (i = 1, ..., p-1), \\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i)\ (i = 1, ..., p-1), \\ -v_{p,j}^{p*} \in K_p^*,\ \langle -v_{p,j}^{p*}, v_{p,j}^p - h_p(v_j^p)\rangle = 0 \text{ and } v_j^{p*} \in \partial(-v_{p,j}^{p*} \circ h_p)(v_j^p), \end{cases}$$

$$\begin{cases} x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ z_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_j^p \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x}, \\ y_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ z_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ u_{1,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_1}} \overline{y}_1, \\ v_{p,j}^p \xrightarrow[j\in J]{\|\cdot\|_{Y_p}} \overline{y}_p,\ v_{i,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_i}} \overline{y}_i,\ v_{i+1,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_{i+1}}} \overline{y}_{i+1}\ (i = 1, ..., p-1), \end{cases}$$

$$\begin{cases} f(x_j) - f(\overline{x}) - \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0, \\ \varphi(y_{0,j}) - \varphi(\overline{y}_0) - \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ g(u_{1,j}) - g(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0, \\ \langle -z_j^*, z_j - \overline{x}\rangle + \langle -z_{0,j}^*, z_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0\ (i = 1, ..., p-1), \\ \langle -v_j^{p*}, v_j^p - \overline{x}\rangle + \langle -v_{p,j}^{p*}, v_{p,j}^p - \overline{y}_p\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} x_j^* + z_j^* + v_j^{p*} \xrightarrow[j\in J]{w(X^*,X)} x^*,\ y_{0,j}^* + z_{0,j}^* \xrightarrow[j\in J]{w(Y_0{}^*,Y_0)} 0, \\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w(Y_1{}^*,Y_1)} 0,\ v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w(Y_i{}^*,Y_i)} 0\ (i=2,...,p). \end{cases}$$

Proof. By using Lemma 4.4, $x^* \in \partial(f+\varphi\circ\psi+g\circ h_1\circ h_2\circ...\circ h_p)(\overline{x})$ is equivalent to $(x^*,0,0,...,0) \in \partial(F+\Phi+\Psi+G+\sum_{i=1}^p H_i)(\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p)$. So by virtue of Theorem 3.2 and by taking into account Remark 4.1, there exist nets $\{(x_j,x_{0,j},x_{1,j},...,x_{p,j})\}_{j\in J} \subseteq \mathrm{dom}F = \mathrm{dom}f \times \prod_{k=0}^p Y_k$, $\{(x_j^*,x_{0,j}^*,x_{1,j}^*,...,x_{p,j}^*)\}_{j\in J} \subseteq X^* \times \prod_{k=0}^p Y_k^*$, $\{(y_j,y_{0,j},y_{1,j},...,y_{p,j})\}_{j\in J} \subseteq \mathrm{dom}\Phi = X\times\mathrm{dom}\varphi\times\prod_{k=1}^p Y_k$, $\{(y_j^*,y_{0,j}^*,y_{1,j}^*,...,y_{p,j}^*)\}_{j\in J} \subseteq X^*\times\prod_{k=0}^p Y_k^*$, $\{(z_j,z_{0,j},z_{1,j},...,z_{p,j})\}_{j\in J} \subseteq \mathrm{dom}\Psi$ (i.e. $\{z_j\}_{j\in J} \subseteq X$ and $\{z_{k,j}\}_{j\in J} \subseteq Y_k$, $k=0,...,p$, with $\{(z_j,z_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi$), $\{(z_j^*,z_{0,j}^*,z_{1,j}^*,...,z_{p,j}^*)\}_{j\in J} \subseteq X^*\times\prod_{k=0}^p Y_k^*$, $\{(u_j,u_{0,j},u_{1,j},...,u_{p,j})\}_{j\in J} \subseteq \mathrm{dom}G = X\times Y_0\times\mathrm{dom}g\times\prod_{k=2}^p Y_k$, $\{(u_j^*,u_{0,j}^*,u_{1,j}^*,...,u_{p,j}^*)\}_{j\in J} \subseteq X^*\times\prod_{k=0}^p Y_k^*$, $\{(v_j^i,v_{0,j}^i,v_{1,j}^i,...,v_{p,j}^i)\}_{j\in J} \subseteq \mathrm{dom}H_i$, $i=1,...,p-1$ (i.e. for $i=1,...,p-1$, $\{v_j^i\}_{j\in J} \subseteq X$ and $\{v_{k,j}^i\}_{j\in J} \subseteq Y_k$, $k=0,...,p$, with $\{(v_{i+1,j}^i,v_{i,j}^i)\}_{j\in J} \subseteq \mathrm{epi}h_i$), $\{(v_j^{i*},v_{0,j}^{i*},v_{1,j}^{i*},...,v_{p,j}^{i*})\}_{j\in J} \subseteq X^*\times\prod_{k=0}^p Y_k^*$, $i=1,...,p-1$, $\{(v_j^p,v_{0,j}^p,v_{1,j}^p,...,v_{p,j}^p)\}_{j\in J} \subseteq \mathrm{dom}H_p$ (i.e. $\{v_j^p\}_{j\in J} \subseteq X$ and $\{v_{k,j}^p\}_{j\in J} \subseteq Y_k$, $k=0,...,p$, with $\{(v_j^p,v_{p,j}^p)\}_{j\in J} \subseteq \mathrm{epi}h_p$), $\{(v_j^{p*},v_{0,j}^{p*},v_{1,j}^{p*},...,v_{p,j}^{p*})\}_{j\in J} \subseteq X^*\times\prod_{k=0}^p Y_k^*$ such that

$$(x_j^*,x_{0,j}^*,x_{1,j}^*,...,x_{p,j}^*) \in \partial F(x_j,x_{0,j},x_{1,j},...,x_{p,j}), \tag{4.5a}$$

$$(y_j^*,y_{0,j}^*,y_{1,j}^*,...,y_{p,j}^*) \in \partial \Phi(y_j,y_{0,j},y_{1,j},...,y_{p,j}), \tag{4.5b}$$

$$(z_j^*,z_{0,j}^*,z_{1,j}^*,...,z_{p,j}^*) \in \partial \Psi(z_j,z_{0,j},z_{1,j},...,z_{p,j}), \tag{4.5c}$$

$$(u_j^*,u_{0,j}^*,u_{1,j}^*,...,u_{p,j}^*) \in \partial G(u_j,u_{0,j},u_{1,j},...,u_{p,j}), \tag{4.5d}$$

$$(v_j^{i*},v_{0,j}^{i*},v_{1,j}^{i*},...,v_{p,j}^{i*}) \in \partial H_i(v_j^i,v_{0,j}^i,v_{1,j}^i,...,v_{p,j}^i)\ (i=1,...,p-1), \tag{4.5e}$$

$$(v_j^{p*},v_{0,j}^{p*},v_{1,j}^{p*},...,v_{p,j}^{p*}) \in \partial H_p(v_j^p,v_{0,j}^p,v_{1,j}^p,...,v_{p,j}^p), \tag{4.5f}$$

$$(x_j,x_{0,j},x_{1,j},...,x_{p,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times...\times Y_p}} (\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p), \tag{4.6a}$$

$$(y_j,y_{0,j},y_{1,j},...,y_{p,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times...\times Y_p}} (\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p), \tag{4.6b}$$

$$(z_j,z_{0,j},z_{1,j},...,z_{p,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times...\times Y_p}} (\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p), \tag{4.6c}$$

$$(u_j,u_{0,j},u_{1,j},...,u_{p,j}) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times...\times Y_p}} (\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p), \tag{4.6d}$$

$$(v_j^i,v_{0,j}^i,v_{1,j}^i,...,v_{p,j}^i) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times...\times Y_p}} (\overline{x},\overline{y}_0,\overline{y}_1,...,\overline{y}_p)\ (i=1,...,p-1), \tag{4.6e}$$

$$(v_j^p, v_{0,j}^p, v_{1,j}^p, ..., v_{p,j}^p) \xrightarrow[j\in J]{\|\cdot\|_{X\times Y_0\times Y_1\times ...\times Y_p}} (\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p), \tag{4.6f}$$

$$F(x_j, x_{0,j}, x_{1,j}, ..., x_{p,j}) - F(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle x_j^*, x_j - \overline{x}\rangle - \sum_{k=0}^{p} \langle x_{k,j}^*, x_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \tag{4.7a}$$

$$\Phi(y_j, y_{0,j}, y_{1,j}, ..., y_{p,j}) - \Phi(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle y_j^*, y_j - \overline{x}\rangle - \sum_{k=0}^{p} \langle y_{k,j}^*, y_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \tag{4.7b}$$

$$\Psi(z_j, z_{0,j}, z_{1,j}, ..., z_{p,j}) - \Psi(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle z_j^*, z_j - \overline{x}\rangle - \sum_{k=0}^{p} \langle z_{k,j}^*, z_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \tag{4.7c}$$

$$G(u_j, u_{0,j}, u_{1,j}, ..., u_{p,j}) - G(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle u_j^*, u_j - \overline{x}\rangle - \sum_{k=0}^{p} \langle u_{k,j}^*, u_{k,j} - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \tag{4.7d}$$

$$H_i(v_j^i, v_{0,j}^i, v_{1,j}^i, ..., v_{p,j}^i) - H_i(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle v_j^{i*}, v_j^i - \overline{x}\rangle - \sum_{k=0}^{p} \langle v_{k,j}^{i*}, v_{k,j}^i - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0 \ (i = 1, ..., p-1), \tag{4.7e}$$

$$H_p(v_j^p, v_{0,j}^p, v_{1,j}^p, ..., v_{p,j}^p) - H_p(\overline{x}, \overline{y}_0, \overline{y}_1, ..., \overline{y}_p) - \langle v_j^{p*}, v_j^p - \overline{x}\rangle - \sum_{k=0}^{p} \langle v_{k,j}^{p*}, v_{k,j}^p - \overline{y}_k\rangle \xrightarrow[j\in J]{} 0, \tag{4.7f}$$

and

$$\Big(x_j^* + y_j^* + z_j^* + u_j^* + \sum_{i=1}^{p} v_j^{i*}, x_{0,j}^* + y_{0,j}^* + z_{0,j}^* + u_{0,j}^* + \sum_{i=1}^{p} v_{0,j}^{i*}, x_{1,j}^* + y_{1,j}^* + z_{1,j}^* + u_{1,j}^* + \sum_{i=1}^{p} v_{1,j}^{i*}, ..., x_{p,j}^* + y_{p,j}^* + z_{p,j}^* + u_{p,j}^* + \sum_{i=1}^{p} v_{p,j}^{i*}\Big) \xrightarrow[j\in J]{\tau} (x^*, 0, 0, ..., 0). \tag{4.8}$$

By Lemma 4.5, the relations in (4.5a)-(4.5f) become

$$\begin{cases} x_j^* \in \partial f(x_j),\ y_{0j}^* \in \partial\varphi(y_{0,j}),\ u_{1,j}^* \in \partial g(u_{1,j}), \\ -z_{0,j}^* \in K_0^*,\ \langle -z_{0,j}^*,\ z_{0,j} - \psi(z_j)\rangle = 0 \text{ and } z_j^* \in \partial(-z_{0,j}^* \circ \psi)(z_j), \\ -v_{i,j}^{i*} \in K_i^* \text{ and } \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0\ (i = 1, ..., p-1), \\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i)\ (i = 1, ..., p-1), \\ -v_{p,j}^{p*} \in K_p^*,\ \langle -v_{p,j}^{p*}, v_{p,j}^p - h_p(v_j^p)\rangle = 0 \text{ and } v_j^{p*} \in \partial(-v_{p,j}^{p*} \circ h_p)(v_j^p), \end{cases}$$

with

$$\begin{cases} x_{0,j}^* = 0,\ x_{1,j}^* = 0,\ \ldots,\ x_{p,j}^* = 0, \\ y_j^* = 0 \text{ and } y_{1,j}^* = 0,\ y_{2,j}^* = 0,\ \ldots,\ y_{p,j}^* = 0, \\ z_{1,j}^* = 0,\ z_{2,j}^* = 0,\ \ldots,\ z_{p,j}^* = 0, \\ u_j^* = 0,\ u_{0,j}^* = 0 \text{ and } u_{2,j}^* = 0,\ u_{3,j}^* = 0,\ \ldots,\ u_{p,j}^* = 0, \\ v_j^{i*} = 0 \text{ and } v_{k,j}^{i*} = 0,\ k \in \{0, ..., p\}\backslash\{i, i+1\}\ (i = 1, ..., p-1), \\ v_{0,j}^{p*} = 0,\ v_{1,j}^{p*} = 0,\ \ldots,\ v_{p-1,j}^{p*} = 0. \end{cases} \tag{4.9}$$

We see from (4.6a)-(4.6f) that

$$\begin{cases} x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ y_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ x_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ y_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k\ (k = 0, ..., p), \\ z_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ u_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ z_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k,\ u_{k,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k\ (k = 0, ..., p), \\ v_j^i \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_{k,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_k}} \overline{y}_k\ (i = 1, ..., p)(k = 0, ..., p), \end{cases}$$

and by (4.9), the conditions (4.7a)-(4.7f) can be expressed as

$$\begin{cases} f(x_j) - f(\overline{x}) - \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0, \\ \varphi(y_{0,j}) - \varphi(\overline{y}_0) - \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ g(u_{1,j}) - g(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0, \\ \langle -z_j^*, z_j - \overline{x}\rangle + \langle -z_{0,j}^*, z_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0\ (i = 1, ..., p-1), \\ \langle -v_j^{p*}, v_j^p - \overline{x}\rangle + \langle -v_{p,j}^{p*}, v_{p,j}^p - \overline{y}_p\rangle \xrightarrow[j\in J]{} 0. \end{cases}$$

On the other hand, it is clear that (4.8) is nothing else than

$$\begin{cases} x_j^* + y_j^* + z_j^* + u_j^* + \sum_{i=1}^{p} v_j^{i*} \xrightarrow[j\in J]{w(X^*,X)} x^*, \\ x_{0,j}^* + y_{0,j}^* + z_{0,j}^* + u_{0,j}^* + \sum_{i=1}^{p} v_{0,j}^{i*} \xrightarrow[j\in J]{w({Y_0}^*,Y_0)} 0, \\ x_{1,j}^* + y_{1,j}^* + z_{1,j}^* + u_{1,j}^* + \sum_{i=1}^{p} v_{1,j}^{i*} \xrightarrow[j\in J]{w({Y_1}^*,Y_1)} 0, \\ \vdots \\ x_{p,j}^* + y_{p,j}^* + z_{p,j}^* + u_{p,j}^* + \sum_{i=1}^{p} v_{p,j}^{i*} \xrightarrow[j\in J]{w({Y_p}^*,Y_p)} 0, \end{cases}$$

and hence, by taking into account (4.9), we get

$$\begin{cases} x_j^* + z_j^* + v_j^{p*} \xrightarrow[j\in J]{w(X^*,X)} x^*,\ y_{0,j}^* + z_{0,j}^* \xrightarrow[j\in J]{w({Y_0}^*,Y_0)} 0, \\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w({Y_1}^*,Y_1)} 0 \text{ and } v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w({Y_i}^*,Y_i)} 0\ (i = 2, ..., p). \end{cases}$$

□

Remark 4.7. Let us observe in the above proof that for $k \in \{1, ..., p\}$, $k' \in \{2, ..., p\}$, $i \in \{1, ..., p-1\}$ and $q \in \{0, ..., p\}\backslash\{i, i+1\}$ the nets $\{x_{0,j}\}_j$, $\{x_{k,j}\}_j$, $\{y_j\}_j$, $\{y_{k,j}\}_j$, $\{z_{k,j}\}_j$, $\{u_j\}_j$, $\{u_{0,j}\}_j$, $\{u_{k',j}\}_j$, $\{v_{k-1,j}^p\}_j$, $\{v_j^i\}_j$ and $\{v_{q,j}^i\}_j$ are superfluous.

Remark 4.8. If we assume that X, Y_0, ..., Y_p are reflexive Banach spaces, then we can establish Theorem 4.6 in terms of sequences and strong limits.

By taking $g \equiv 0$, $h_i \equiv 0$ and $K_i = Y_i$, $i = 1, ..., p$, we get the following corollary.

Corollary 4.9. *Suppose that* $\overline{x} \in \mathrm{dom} f \cap \psi^{-1}(\mathrm{dom}\varphi) \cap \mathrm{dom}\psi$ *and* $\overline{y}_0 = \psi(\overline{x})$*. Then,* $x^* \in \partial(f + \varphi \circ \psi)(\overline{x})$ *if and only if there exist nets* $\{x_j\}_{j\in J} \subseteq \mathrm{dom} f$, $\{y_{0,j}\}_{j\in J} \subseteq \mathrm{dom}\varphi$, $\{(z_j, z_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi$, $\{x_j^*\}_{j\in J} \subseteq X^*$, $\{y_{0,j}^*\}_{j\in J} \subseteq Y_0^*$ *and* $\{(z_j^*, z_{0,j}^*)\}_{j\in J} \subseteq X^* \times Y_0^*$ *satisfying*

$$\begin{cases} x_j^* \in \partial f(x_j),\ y_{0,j}^* \in \partial\varphi(y_{0,j}), \\ -z_{0,j}^* \in K_0^*,\ \langle -z_{0,j}^*,\ z_{0,j} - \psi(z_j)\rangle = 0,\ z_j^* \in \partial(-z_{0,j}^* \circ \psi)(z_j), \end{cases}$$

$$\begin{cases} x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ z_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x}, \\ y_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ z_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0, \end{cases}$$

$$\begin{cases} f(x_j) - f(\overline{x}) - \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0, \\ \varphi(y_{0,j}) - \varphi(\overline{y}_0) - \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ \langle -z_j^*, z_j - \overline{x}\rangle + \langle -z_{0,j}^*, z_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} x_j^* + z_j^* \xrightarrow[j\in J]{w(X^*,X)} x^*, \\ y_{0,j}^* + z_{0,j}^* \xrightarrow[j\in J]{w(Y_0{}^*,Y_0)} 0. \end{cases}$$

Remark 4.10. Some results given in [1] and [10] can also be derived from Corollary 4.9 when we suppose that X and Y_0 are reflexive Banach spaces.

5. Applications

In this section, we apply the main result obtained in the previous section to two broad classes of optimization problems. The first is a general multi-composed problem with geometric and cone constraints and the second is a constrained location problem without set-up costs.

5.1. **Sequential optimality conditions for a general multi-composed optimization problem.** Let us consider the following multi-composed problem with geometric and cone constraints

$$(P_{\mathcal{M}}) \qquad \inf_{\substack{x\in C \\ \psi(x)\in -K_0}} (g\circ h_1\circ h_2\circ ...\circ h_p)(x)$$

where
- C is nonempty closed convex subset of X and K_0 is supposed to be closed,
- $\psi : X \to Y_0 \cup \{+\infty_{Y_0}\}$ is proper, K_0-convex, K_0-epi closed and $\psi(\mathrm{dom}\psi) \subseteq -K_0$,
- $g : Y_1 \to \overline{\mathbb{R}}$ is proper, convex, K_1-nondecreasing on dom$g$ and lower semicontinuous,
- $h_1 : Y_2 \to Y_1 \cup \{+\infty_{Y_1}\}$ is proper, K_1-convex, (K_2, K_1)-nondecreasing on domh_1, K_1-epi closed and $h_1(\mathrm{dom}h_1) \subseteq \mathrm{dom}g$,
- $h_i : Y_{i+1} \to Y_i\cup\{+\infty_{Y_i}\}$ is proper, K_i-convex, (K_{i+1}, K_i)-nondecreasing on domh_i, K_i-epi closed and $h_i(\mathrm{dom}h_i) \subseteq \mathrm{dom}h_{i-1}$, $i = 2, ..., p-1$,
- $h_p : X \to Y_p \cup \{+\infty_{Y_p}\}$ is proper, K_p-convex, K_p-epi closed and $h_p(\mathrm{dom}h_p) \subseteq \mathrm{dom}h_{p-1}$,
- $C \cap \psi^{-1}(-K_0) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom}g) \cap \mathrm{dom}h_p \neq \emptyset$,
- $g(+\infty_{Y_1}) = +\infty$, and $h_i(+\infty_{Y_{i+1}}) = +\infty_{Y_i}$ for $i = 1, ..., p-1$.

We mention that the problem $(P_{\mathcal{M}})$ has been investigated by Wanka et al. ([4, 12, 14]) by using the Lagrange duality approach.

In order to give sequential optimality conditions for the problem $(P_{\mathcal{M}})$, let us note that $(P_{\mathcal{M}})$ can be written equivalently as

$$(P_{\mathcal{M}}) \quad \inf_{x\in X}\{\delta_C(x) + (\delta_{-K_0}\circ\psi)(x) + (g\circ h_1\circ h_2\circ ...\circ h_p)(x)\}.$$

Theorem 5.1. *Suppose that* $\overline{x} \in C \cap \psi^{-1}(-K_0) \cap \mathrm{dom}\psi \cap (h_p^{-1} \circ h_{p-1}^{-1} \circ ... \circ h_1^{-1})(\mathrm{dom}g) \cap \mathrm{dom}h_p$, $\overline{y}_p := h_p(\overline{x})$, $\overline{y}_{p-1} := h_{p-1}(\overline{y}_p), ..., \overline{y}_1 := h_1(\overline{y}_2)$ *and* $\overline{y}_0 := \psi(\overline{x})$. *Then,* $\overline{x}$ *is an optimal solution of the problem* $(P_{\mathcal{M}})$ *if and only if there exist nets* $\{x_j\}_{j\in J} \subseteq C$, $\{y_{0,j}\}_{j\in J} \subseteq -K_0$, $\{(z_j, z_{0,j})\}_{j\in J} \subseteq \mathrm{epi}\psi$, $\{u_{1,j}\}_{j\in J} \subseteq \mathrm{dom}g$, $\{(v_{i+1,j}^i, v_{i,j}^i)\}_{j\in J} \subseteq \mathrm{epi}h_i, i = 1, ..., p-1$, $\{(v_j^p, v_{p,j}^p)\}_{j\in J} \subseteq \mathrm{epi}h_p$, $\{x_j^*\}_{j\in J} \subseteq X^*$,

$\{y_{0,j}^*\}_{j\in J} \subseteq Y_0^*$, $\{(z_j^*, z_{0,j}^*)\}_{j\in J} \subseteq X^* \times Y_0^*$, $\{u_{1,j}^*\}_{j\in J} \subseteq Y_1^*$, $\{(v_{i,j}^{i*}, v_{i+1,j}^{i*})\}_{j\in J} \subseteq Y_i^* \times Y_{i+1}^*, i = 1, ..., p-1$, *and* $\{(v_j^{p*}, v_{p,j}^{p*})\}_{j\in J} \subseteq X^* \times Y_p^*$ *satisfying*

$$\begin{cases} x_j^* \in N_C(x_j),\ y_{0,j}^* \in K_0^*,\ \langle y_{0,j}^*, y_{0,j}\rangle = 0 \text{ and } u_{1,j}^* \in \partial g(u_{1,j}),\\ -z_{0,j}^* \in K_0^*,\ \langle -z_{0,j}^*,\ z_{0,j} - \psi(z_j)\rangle = 0 \text{ and } z_j^* \in \partial(-z_{0,j}^* \circ \psi)(z_j),\\ -v_{i,j}^{i*} \in K_i^* \text{ and } \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0\ (i = 1, ..., p-1),\\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i)\ (i = 1, ..., p-1),\\ -v_{p,j}^{p*} \in K_p^*,\ \langle -v_{p,j}^{p*}, v_{p,j}^p - h_p(v_j^p)\rangle = 0 \text{ and } v_j^{p*} \in \partial(-v_{p,j}^{p*} \circ h_p)(v_j^p), \end{cases}$$

$$\begin{cases} x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ z_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_j^p \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\\ y_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ z_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ u_{1,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_1}} \overline{y}_1,\\ v_{p,j}^p \xrightarrow[j\in J]{\|\cdot\|_{Y_p}} \overline{y}_p,\ v_{i,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_i}} \overline{y}_i,\ v_{i+1,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_{i+1}}} \overline{y}_{i+1}\ (i = 1, ..., p-1), \end{cases}$$

$$\begin{cases} \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0,\\ \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0,\\ g(u_{1,j}) - g(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0,\\ \langle -z_j^*, z_j - \overline{x}\rangle + \langle -z_{0,j}^*, z_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0,\\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0\ (i = 1, ..., p-1),\\ \langle -v_j^{p*}, v_j^p - \overline{x}\rangle + \langle -v_{p,j}^{p*}, v_{p,j}^p - \overline{y}_p\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} x_j^* + z_j^* + v_j^{p*} \xrightarrow[j\in J]{w(X^*,X)} 0,\ y_{0,j}^* + z_{0,j}^* \xrightarrow[j\in J]{w(Y_0{}^*,Y_0)} 0,\\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w(Y_1{}^*,Y_1)} 0,\ v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w(Y_i{}^*,Y_i)} 0\ (i = 2, ..., p). \end{cases}$$

Proof. It is clear that $\overline{x}$ is an optimal solution of $(P_{\mathcal{M}})$ if and only if $0 \in \partial(\delta_C + \delta_{-K_0} \circ \psi + g \circ h_1 \circ h_2 \circ ... \circ h_p)(\overline{x})$. Since C and $-K_0$ are nonempty, closed and convex, it follows that δ_C and δ_{-K_0} are proper, convex and lower semicontinuous functions, with δ_{-K_0} is K_0-nondecreasing (for more details see [2]). So, as $f := \delta_C$, $\varphi := \delta_{-K_0}$, ψ, g and h_i, $i = 1, ..., p$, satisfy all the assumptions of Theorem 4.6, it follows that there exist nets $\{x_j\}_{j\in J} \subseteq C$, $\{y_{0,j}\}_{j\in J} \subseteq -K_0$, $\{(z_j, z_{0,j})\}_{j\in J} \subseteq \text{epi}\psi$, $\{u_{1,j}\}_{j\in J} \subseteq \text{dom}g$, $\{(v_{i+1,j}^i, v_{i,j}^i)\}_{j\in J} \subseteq \text{epi}h_i, i = 1, ..., p-1$, $\{(v_j^p, v_{p,j}^p)\}_{j\in J} \subseteq \text{epi}h_p$, $\{x_j^*\}_{j\in J} \subseteq X^*$, $\{y_{0,j}^*\}_{j\in J} \subseteq Y_0^*$, $\{(z_j^*, z_{0,j}^*)\}_{j\in J} \subseteq X^* \times Y_0^*$, $\{u_{1,j}^*\}_{j\in J} \subseteq Y_1^*$, $\{(v_{i,j}^{i*}, v_{i+1,j}^{i*})\}_{j\in J} \subseteq Y_i^* \times Y_{i+1}^*, i = 1, ..., p-1$, and $\{(v_j^{p*}, v_{p,j}^{p*})\}_{j\in J} \subseteq X^* \times Y_p^*$, such that

$$\begin{cases} x_j^* \in N_C(x_j),\ y_{0,j}^* \in N_{-K_0}(y_{0,j}) \text{ and } u_{1,j}^* \in \partial g(u_{1,j}), \\ -z_{0,j}^* \in K_0^*,\ \langle -z_{0,j}^*,\ z_{0,j} - \psi(z_j)\rangle = 0 \text{ and } z_j^* \in \partial(-z_{0,j}^* \circ \psi)(z_j), \\ -v_{i,j}^{i*} \in K_i^* \text{ and } \langle -v_{i,j}^{i*}, v_{i,j}^i - h_i(v_{i+1,j}^i)\rangle = 0\ (i = 1, ..., p-1), \\ v_{i+1,j}^{i*} \in \partial(-v_{i,j}^{i*} \circ h_i)(v_{i+1,j}^i)\ (i = 1, ..., p-1), \\ -v_{p,j}^{p*} \in K_p^*,\ \langle -v_{p,j}^{p*}, v_{p,j}^p - h_p(v_j^p)\rangle = 0 \text{ and } v_j^{p*} \in \partial(-v_{p,j}^{p*} \circ h_p)(v_j^p), \end{cases}$$

$$\begin{cases} x_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ z_j \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x},\ v_j^p \xrightarrow[j\in J]{\|\cdot\|_X} \overline{x}, \\ y_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ z_{0,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_0}} \overline{y}_0,\ u_{1,j} \xrightarrow[j\in J]{\|\cdot\|_{Y_1}} \overline{y}_1, \\ v_{p,j}^p \xrightarrow[j\in J]{\|\cdot\|_{Y_p}} \overline{y}_p,\ v_{i,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_i}} \overline{y}_i,\ v_{i+1,j}^i \xrightarrow[j\in J]{\|\cdot\|_{Y_{i+1}}} \overline{y}_{i+1}\ (i = 1, ..., p-1), \end{cases}$$

$$\begin{cases} \langle x_j^*, x_j - \overline{x}\rangle \xrightarrow[j\in J]{} 0, \\ \langle y_{0,j}^*, y_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ g(u_{1,j}) - g(\overline{y}_1) - \langle u_{1,j}^*, u_{1,j} - \overline{y}_1\rangle \xrightarrow[j\in J]{} 0, \\ \langle -z_j^*, z_j - \overline{x}\rangle + \langle -z_{0,j}^*, z_{0,j} - \overline{y}_0\rangle \xrightarrow[j\in J]{} 0, \\ \langle -v_{i,j}^{i*}, v_{i,j}^i - \overline{y}_i\rangle + \langle -v_{i+1,j}^{i*}, v_{i+1,j}^i - \overline{y}_{i+1}\rangle \xrightarrow[j\in J]{} 0\ (i = 1, ..., p-1), \\ \langle -v_j^{p*}, v_j^p - \overline{x}\rangle + \langle -v_{p,j}^{p*}, v_{p,j}^p - \overline{y}_p\rangle \xrightarrow[j\in J]{} 0, \end{cases}$$

and

$$\begin{cases} x_j^* + z_j^* + v_j^{p*} \xrightarrow[j\in J]{w(X^*,X)} 0,\ y_{0,j}^* + z_{0,j}^* \xrightarrow[j\in J]{w(Y_0{}^*,Y_0)} 0, \\ u_{1,j}^* + v_{1,j}^{1*} \xrightarrow[j\in J]{w(Y_1{}^*,Y_1)} 0,\ v_{i,j}^{(i-1)*} + v_{i,j}^{i*} \xrightarrow[j\in J]{w(Y_i{}^*,Y_i)} 0\ (i = 2, ..., p). \end{cases}$$

We end up the proof by observing that

$$y_{0,j}^* \in N_{-K_0}(y_{0,j}) \iff y_{0,j}^* \in K_0^*, \text{ and } \langle y_{0,j}^*, y_{0,j}\rangle = 0.$$

□

5.2. **Sequential optimality conditions to constrained location problems without set-up costs.** In this subsection we consider the following single minimax location problem without set-up costs, treated by Wanka and Wilfer [13],

$$(LP) \quad \inf_{x\in C} \max_{1\le i\le q} \{\gamma_{C_i}(x - e_i)\},$$

where

• C and $C_1, ..., C_q$ are nonempty, closed and convex subsets of a reflexive Banach space X with $0 \in int\, C_i$, $i = 1, ..., q$,

• $e_1, ..., e_q \in X$ are distinct points,
• $\gamma_{C_i} : X \to \overline{\mathbb{R}}$ is the Minkowski gauge of the subset C_i, defined by

$$\gamma_{C_i}(x) := \begin{cases} \inf\{\lambda > 0 : x \in \lambda C_i\}, & \text{if } \{\lambda > 0 : x \in \lambda C_i\} \neq \emptyset, \\ +\infty, & \text{otherwise}, \end{cases} \qquad (i = 1, ..., q).$$

Let us note that the defined gauges $\gamma_{C_1}, ..., \gamma_{C_q}$ are convex and continuous functions (see Theorem 1 and Remark 2 in [13]), which implies that the problem (LP) is a convex optimization problem.

For obtaining sequential optimality conditions for the problem (LP), we set $Y_1 = Y_2 := \mathbb{R}^q$ and $K_1 = K_2 := \mathbb{R}^q_+$. To write (LP) as a convex multi-composed optimization problem, we introduce the following functions
• $l^+_\infty : \mathbb{R}^q \to \mathbb{R}$ defined by

$$l^+_\infty(x_1, ..., x_q) := l_\infty(x_1^+, ..., x_q^+), \ (x_1, ..., x_q) \in \mathbb{R}^q$$

where

$$l_\infty(x_1, ..., x_q) := \max_{1 \le i \le q} \{|x_i|\}, \ x_i^+ := \max\{0, x_i\}, i = 1, .., q,$$

and

$$l^+_\infty(+\infty_{\mathbb{R}^q}) = +\infty,$$

• $h_1 : \mathbb{R}^q \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ defined by

$$h_1(x_1, ..., x_q) := \begin{cases} (x_1, ..., x_q), \ \text{if } (x_1, ..., x_q) \in \mathbb{R}^q_+, \\ +\infty_{\mathbb{R}^q}, \ \text{otherwise}, \end{cases}$$

• $h_2 : X \to \mathbb{R}^q$ defined by

$$h_2(x) := (\gamma_{C_1}(x - e_1), ..., \gamma_{C_q}(x - e_q)), \ x \in X.$$

These definitions yield the following equivalent representation for the considered problem

$$(LP) \quad \inf_{x \in C} (l^+_\infty \circ h_1 \circ h_2)(x)$$

Remark 5.2. Let us note that
• $l^+_\infty : \mathbb{R}^q \to \mathbb{R}$ is proper, convex, lower semicontinuous and $\mathbb{R}^q_+$-nondecreasing on $\mathrm{dom} l^+_\infty = \mathbb{R}^q$ (see [11]),
• $h_1 : \mathbb{R}^q \to \mathbb{R}^q \cup \{+\infty_{\mathbb{R}^q}\}$ is proper, $\mathbb{R}^q_+$-convex, $(\mathbb{R}^q_+, \mathbb{R}^q_+)$-nondecreasing on $\mathrm{dom} h_1 = \mathbb{R}^q_+$, $\mathbb{R}^q_+$-epi closed and $h_1(\mathrm{dom} h_1) \subseteq \mathbb{R}^q_+$,
• $h_2 : X \to \mathbb{R}^q$ is proper, $\mathbb{R}^q_+$-convex, $\mathbb{R}^q_+$-epi closed, $\mathrm{dom} h_2 = X$ and $h_2(\mathrm{dom} h_2) \subseteq \mathbb{R}^q_+$.

Lemma 5.3. *Let $(x_1, ..., x_q) \in \mathbb{R}^q$, then*

$$\partial l^+_\infty(x_1, ..., x_q) = \Big\{(x_1^*, ..., x_q^*) \in \mathbb{R}^q_+ : \sum_{i=1}^q x_i^* \le 1 \ \text{ and } \max_{1 \le i \le q} \{x_i^+\} = \sum_{i=1}^q x_i^* x_i\Big\}.$$

Proof. From Proposition 4.2 in [11], it follows that for all $(x_1^*, ..., x_q^*) \in \mathbb{R}^q$

$$\left(l_\infty^+\right)^*(x_1^*, ..., x_q^*) = \begin{cases} 0, \text{ if } (x_1^*, ..., x_q^*) \in \mathbb{R}_+^q \text{ and } \displaystyle\sum_{i=1}^q |x_i^*| \leq 1, \\ +\infty, \text{ otherwise.} \end{cases}$$

Therefore, the proof is straightforward since

$$\partial l_\infty^+(x_1, ..., x_q) = \Big\{(x_1^*, ..., x_q^*) \in \mathbb{R}^q : \left(l_\infty^+\right)^*(x_1^*, ..., x_q^*) + l_\infty^+(x_1, ..., x_q) = \sum_{i=1}^q x_i^* x_i\Big\}.$$

□

Lemma 5.4. *Let* $(x_1, ..., x_q) \in \mathbb{R}_+^q$ *and* $(y_1, ..., y_q) \in \mathbb{R}_+^q$*, then*

$$(x_1^*, ..., x_q^*) \in \partial\Big((y_1, ..., y_q) \circ h_1\Big)(x_1, ..., x_q) \iff x_i^* \in \Omega(y_i, x_i),\ \forall i \in \{1, ..., q\}$$

where

$$\Omega(y_k, x_k) := \Big\{x \in]-\infty, y_k] :\ (x - y_k)x_k = 0\Big\}\ (k = 1, ..., q).$$

Proof. Let $(x_1, ..., x_q) \in \mathbb{R}_+^q$ and $(y_1, ..., y_q) \in \mathbb{R}_+^q$, then

$$(x_1^*, ..., x_q^*) \in \partial\Big((y_1, ..., y_q) \circ h_1\Big)(x_1, ..., x_q)$$

$\iff$

$$[y_1 z_1 + ... + y_q z_q] \geq [y_1 x_1 + ... + y_q x_q] + [x_1^*(z_1 - x_1) + ... + x_q^*(z_q - x_q)], \quad \forall (z_1, ..., z_q) \in \mathbb{R}_+^q$$

$\iff$

$$y_i z_i \geq y_i x_i + x_i^*(z_i - x_i), \quad \forall z_i \in \mathbb{R}_+, \forall i \in \{1, ..., q\}$$

$\iff$

$$x_i^* - y_i \in N_{\mathbb{R}_+}(x_i), \quad \forall i \in \{1, ..., q\}$$

$\iff$

$$x_i^* \leq y_i \text{ and } (x_i^* - y_i)x_i = 0, \quad \forall i \in \{1, ..., q\}$$

i.e.

$$x_i^* \in \Omega(y_i, x_i), \quad \forall i \in \{1, ..., q\}.$$

Therefore, the proof is complete. □

Lemma 5.5. *Let* $x \in X$*,* $(y_1, ..., y_q) \in \mathbb{R}_+^q$ *and* $I := \Big\{i \in \{1, ..., q\} :\ y_i > 0\Big\}$*. Then*

$$\partial\Big((y_1, ..., y_q) \circ h_2\Big)(x) = \Gamma\Big((y_1, ..., y_q), x\Big)$$

where

$$\Gamma\Big((y_1, ..., y_q), x\Big) := \Big\{x^* \in X^* :\ \exists\, x_1^*, ..., x_q^* \in X^*, x_1^* + ... + x_q^* = x^*,\ \text{with } x_i^* \in \partial(y_i \gamma_{C_i})(x - e_i) \text{ for } i \in I \text{ and } x_i^* = 0 \text{ for } i \notin I\Big\}.$$

Proof. Let $x \in X$, $(y_1, ..., y_q) \in \mathbb{R}^q_+$ and $I := \{i \in \{1, ..., q\} : \ y_i > 0\}$. Since the functions $\gamma_{C_1}, ..., \gamma_{C_q} : X \to \mathbb{R}$ are convex and continuous, it follows that

$$\partial\Big((y_1, ..., y_q) \circ h_2\Big)(x) = \partial\Big(y_1\gamma_{C_1}(. - e_1) + ... + y_q\gamma_{C_q}(. - e_q)\Big)(x)$$
$$= \partial\Big(y_1\gamma_{C_1}(. - e_1)\Big)(x) + ... + \partial\Big(y_q\gamma_{C_q}(. - e_q)\Big)(x).$$

Further, it is easy to check that

$$\partial\Big(y_i\gamma_{C_i}(. - e_i)\Big)(x) = \begin{cases} \partial(y_i\gamma_{C_i})(x - e_i), \text{ if } i \in I, \\ \{0\}, \text{ otherwise,} \end{cases} \qquad (i = 1, ..., q).$$

Therefore, the proof is complete. □

Let us introduce the following sets

$$\mathbb{E}_1 := \Big\{(x_1, ..., x_q, r_1, ..., r_q) \in \mathbb{R}^q_+ \times \mathbb{R}^q_+ : \ x_i \le r_i, \quad \forall i \in \{1, ..., q\}\Big\}$$

and

$$\mathbb{E}_2 := \Big\{(x, r_1, ..., r_q) \in X \times \mathbb{R}^q_+ : \ \gamma_{C_i}(x - e_i) \le r_i, \quad \forall i \in \{1, ..., q\}\Big\}.$$

It is clear that $\mathbb{E}_1$ and $\mathbb{E}_2$ represent the epigraphs of h_1 and h_2, respectively.

Now, we give sequential optimality conditions for the problem (LP).

Theorem 5.6. *Let $\overline{x} \in C$. Then, $\overline{x}$ is an optimal solution of the problem (LP) if and only if there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq C$, $\{(y_{1,n}, ..., y_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q$, $\{(z_{1,n}, ..., z_{q,n}, \alpha_{1,n}, ..., \alpha_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_1$, $\{(z_n, \beta_{1,n}, ..., \beta_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_2$, $\{x^*_n\}_{n\in\mathbb{N}} \subseteq X^*$, $\{(y^*_{1,n}, ..., y^*_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q_+$, with $\sum_{i=1}^q y^*_{i,n} \le 1$, $\{(z^*_{1,n}, ..., z^*_{q,n}, \alpha^*_{1,n}, ..., \alpha^*_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q \times \mathbb{R}^q_+$ and $\{(z^*_n, \beta^*_{1,n}, ..., \beta^*_{q,n})\}_{n\in\mathbb{N}} \subseteq X^* \times \mathbb{R}^q_+$ satisfying*

$$\begin{cases} x^*_n \in N_C(x_n), \ \max\limits_{1\le i\le q}\{y^+_{i,n}\} = \sum\limits_{i=1}^q y^*_{i,n} y_{i,n}, \\ \alpha^*_{i,n}(\alpha_{i,n} - z_{i,n}) = 0, \ z^*_{i,n} \in \Omega(\alpha^*_{i,n}, z_{i,n}) \ (i = 1, .., q), \\ \beta^*_{i,n}(\beta_{i,n} - \gamma_{C_i}(z_n - e_i)) = 0, \ z^*_n \in \Gamma\Big((\beta^*_{1,n}, ..., \beta^*_{q,n}), z_n\Big) \ (i = 1, .., q), \end{cases}$$

$$\begin{cases} x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x}, \ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x}, \\ (y_{1,n}, ..., y_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x} - e_1), ..., \gamma_{C_q}(\overline{x} - e_q)), \\ (z_{1,n}, ..., z_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x} - e_1), ..., \gamma_{C_q}(\overline{x} - e_q)), \\ (\alpha_{1,n}, ..., \alpha_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x} - e_1), ..., \gamma_{C_q}(\overline{x} - e_q)), \\ (\beta_{1,n}, ..., \beta_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x} - e_1), ..., \gamma_{C_q}(\overline{x} - e_q)), \end{cases}$$

$$\begin{cases} \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \sum_{i=1}^{q} y_{i,n}^*\gamma_{C_i}(\overline{x}-e_i) \xrightarrow[n\to+\infty]{} \max\limits_{1\le i\le q}\{\gamma_{C_i}(\overline{x}-e_i)\}, \\ \sum_{i=1}^{q} \alpha_{i,n}^*(\alpha_{i,n}-\gamma_{C_i}(\overline{x}-e_i)) - \sum_{i=1}^{q} z_{i,n}^*(z_{i,n}-\gamma_{C_i}(\overline{x}-e_i)) \xrightarrow[n\to+\infty]{} 0, \\ \sum_{i=1}^{q} \beta_{i,n}^*(\beta_{i,n}-\gamma_{C_i}(\overline{x}-e_i)) - \langle z_n^*, z_n-\overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} x_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0,\ (y_{1,n}^*-\alpha_{1,n}^*, ..., y_{q,n}^*-\alpha_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0), \\ (z_{1,n}^*-\beta_{1,n}^*, ..., z_{q,n}^*-\beta_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,...,0). \end{cases}$$

Proof. By taking into consideration Remark 5.2, it follows that the functions $f := \delta_C$, $\varphi \equiv 0$, $\psi \equiv 0$, $g := l_\infty^+$, h_1 and h_2 satisfy all the assumptions of Theorem 5.1. Therefore, by applying Theorem 5.1 and also Lemma 5.3, Lemma 5.4 and Lemma5.5, it follows that $\overline{x}$ is an optimal solution of (LP) if and only if there exist sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq C$, $\{(y_{1,n}, ..., y_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q$, $\{(z_{1,n}, ..., z_{q,n}, \alpha_{1,n}, ..., \alpha_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_1$, $\{(z_n, \beta_{1,n}, ..., \beta_{q,n})\}_{n\in\mathbb{N}} \subseteq \mathbb{E}_2$, $\{x_n^*\}_{n\in\mathbb{N}} \subseteq X^*$, $\{(y_{1,n}^*, ..., y_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}_+^q$, with $\sum_{i=1}^q y_{i,n}^* \le 1$,
$\{(z_{1,n}^*, ..., z_{q,n}^*, \alpha_{1,n}^*, ..., \alpha_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq \mathbb{R}^q \times \mathbb{R}_+^q$, $\{(z_n^*, \beta_{1,n}^*, ..., \beta_{q,n}^*)\}_{n\in\mathbb{N}} \subseteq X^* \times \mathbb{R}_+^q$ such that

$$\begin{cases} x_n^* \in N_C(x_n),\ \max\limits_{1\le i\le q}\{y_{i,n}^+\} = \sum_{i=1}^{q} y_{i,n}^* y_{i,n}, \\ \sum_{k=1}^{q} \alpha_{k,n}^*(\alpha_{k,n}-z_{k,n}) = 0,\ z_{i,n}^* \in \Omega(\alpha_{i,n}^*, z_{i,n})\ (i=1,..,q), \\ \sum_{k=1}^{q} \beta_{k,n}^*(\beta_{k,n}-\gamma_{C_k}(z_n-e_k)) = 0,\ z_n^* \in \Gamma\Big((\beta_{1,n}^*, ..., \beta_{q,n}^*), z_n\Big), \end{cases}$$

$$\begin{cases} x_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x},\ z_n \xrightarrow[n\to+\infty]{\|\cdot\|_X} \overline{x}, \\ (y_{1,n}, ..., y_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x}-e_1), ..., \gamma_{C_q}(\overline{x}-e_q)), \\ (z_{1,n}, ..., z_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x}-e_1), ..., \gamma_{C_q}(\overline{x}-e_q)), \\ (\alpha_{1,n}, ..., \alpha_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x}-e_1), ..., \gamma_{C_q}(\overline{x}-e_q)), \\ (\beta_{1,n}, ..., \beta_{q,n}) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (\gamma_{C_1}(\overline{x}-e_1), ..., \gamma_{C_q}(\overline{x}-e_q)), \end{cases}$$

$$\begin{cases} \langle x_n^*, x_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \\ \max\limits_{1\le i\le q}\{y_{i,n}^+\} - \max\limits_{1\le i\le q}\{\gamma_{C_i}(\overline{x}-e_i)\} - \sum\limits_{i=1}^{q} y_{i,n}^*(y_{i,n} - \gamma_{C_i}(\overline{x}-e_i)) \xrightarrow[n\to+\infty]{} 0, \\ \sum\limits_{i=1}^{q} \alpha_{i,n}^*(\alpha_{i,n} - \gamma_{C_i}(\overline{x}-e_i)) - \sum\limits_{i=1}^{q} z_{i,n}^*(z_{i,n} - \gamma_{C_i}(\overline{x}-e_i)) \xrightarrow[n\to+\infty]{} 0, \\ \sum\limits_{i=1}^{q} \beta_{i,n}^*(\beta_{i,n} - \gamma_{C_i}(\overline{x}-e_i)) - \langle z_n^*, z_n - \overline{x}\rangle \xrightarrow[n\to+\infty]{} 0, \end{cases}$$

and

$$\begin{cases} x_n^* + z_n^* \xrightarrow[n\to+\infty]{\|\cdot\|_{X^*}} 0, \ (y_{1,n}^* - \alpha_{1,n}^*, \dots, y_{q,n}^* - \alpha_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,\dots,0), \\ (z_{1,n}^* - \beta_{1,n}^*, \dots, z_{q,n}^* - \beta_{q,n}^*) \xrightarrow[n\to+\infty]{\|\cdot\|_{\mathbb{R}^q}} (0,\dots,0). \end{cases}$$

Since $\max\limits_{1\le i\le q}\{y_{i,n}^+\} = \sum_{i=1}^q y_{i,n}^* y_{i,n}$, we have

$$\max_{1\le i\le q}\{y_{i,n}^+\} - \max_{1\le i\le q}\{\gamma_{C_i}(\overline{x}-e_i)\} - \sum_{i=1}^{q} y_{i,n}^*(y_{i,n} - \gamma_{C_i}(\overline{x}-e_i)) \xrightarrow[n\to+\infty]{} 0$$

is equivalent to

$$\sum_{i=1}^{q} y_{i,n}^*\gamma_{C_i}(\overline{x}-e_i) \xrightarrow[n\to+\infty]{} \max_{1\le i\le q}\{\gamma_{C_i}(\overline{x}-e_i)\}.$$

As $\alpha_{i,n}^* \ge 0$, $\alpha_{i,n} - z_{i,n} \ge 0$, $\beta_{i,n}^* \ge 0$ and $\beta_{i,n} - \gamma_{C_i}(z_n - e_i) \ge 0$, $i = 1, \dots, q$, we conclude that

$$\begin{cases} \sum\limits_{k=1}^{q} \alpha_{k,n}^*(\alpha_{k,n} - z_{k,n}) = 0 \iff \alpha_{i,n}^*(\alpha_{i,n} - z_{i,n}) = 0, i = 1, \dots, q, \\ \sum\limits_{k=1}^{q} \beta_{k,n}^*(\beta_{k,n} - \gamma_{C_k}(z_n - e_k)) = 0 \iff \begin{cases} \beta_{i,n}^*(\beta_{i,n} - \gamma_{C_i}(z_n - e_i)) = 0, \\ i = 1, \dots, q, \end{cases} \end{cases}$$

and hence the proof is complete. □

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