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FURTHER THEORY FOR A HIGHER-ORDER WATER WAVE MODEL

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Abstract. This essay is about a class of higher-order models for the unidirectional propagation of small amplitude long waves on the surface of an ideal fluid. These models are particularly interesting because certain specializations possess a helpful Hamiltonian structure. The present contribution brings forth new theory for the initial-value problem for these evolution equations.

1. Introduction

The original theory for the unidirectional propagation of long-crested water waves of small amplitude and long wavelength was the work of 19^{th}-century giants such as Airy, Stokes, Barré de Saint-Venant, Boussinesq, Korteweg and de Vries. One of the earlier models of these sort of waves over a flat, featureless bottom is the classical Korteweg-de Vries (KdV) equation

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x + \frac{1}{6}\beta\eta_{xxx} = 0,$$

first derived by Boussinesq [5]. The independent variables x and t are proportional to distance in the direction of propagation, taken to be toward increasing values of x, and elapsed time, respectively, while the depth of the water column at the point x at time t is $h(x,t) = h_0 + \eta(x,t)$ with h_0 being the undisturbed depth. Thus $\eta(x,t)$ is the deviation of the free surface from its rest position at the point corresponding to x at time t. As written, the variables are non-dimensionalized by the undisturbed depth h_0 in the vertical coordinate while the wavelength ℓ of a typical wave is used to non-dimensionalize the horizontal coordinate. Thus h is scaled by h_0, the horizontal variable x is scaled by ℓ and the unit of time is taken to be $\sqrt{h_0/g}$, with g being the gravity constant. This non-dimensionalization throws up the two parameters, $\alpha = a/h_0$ and $\beta = (h_0/\ell)^2$ appearing in the equation. Here, a is a typical wave amplitude of the flow being modeled. This model and its BBM alternative

$$\eta_t + \eta_x + \frac{3}{2}\alpha\eta\eta_x - \frac{1}{6}\beta\eta_{xxt} = 0,$$

(see [1]) are known to be good approximations of solutions of the full, inviscid water-wave problem on the so-called Boussinesq time scale. That is, the approximation is good provided the waves are long-crested, have small amplitude and long

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wavelength, and the initial data $\eta(x,0) = \eta_0(x)$ is of sufficient regularity while both α and β are small compared to one and the Stokes' number $S = \alpha/\beta$ is of order one (see [8], [11], [13] for example). In principle, any positive value of h_0 can be countenanced, but in practice if the depth is too small, damping and even surface tension effects come strongly into play and the models need additional terms to be good approximations. For example, the experiments reported in [4] run in about 3 cm of water required damping to be taken into account. They showed very good agreement for values of β of about .01 and a range of amplitudes spanning Stokes numbers from about 1/30 to 25. In the larger scale experiments in [14], damping was not so important but the agreement was only qualitative.

By the Boussinesq time scale, we mean the time interval $[0,T]$ where $T = O(1/\alpha) = O(1/\beta)$. In a laboratory or field setting, this often translates into the equation approximating well small amplitude, long wavelength, long-crested waves for perhaps a dozen wavelengths (see [4]). If an approximation needs to be accurate on longer spatial or temporal scales, as for example in coastal engineering applications [2], higher-order models naturally present themselves. Formally, such higher-order models could follow the wave motion accurately on much longer spatial and temporal intervals. (However, it is worth stressing that while there is rigorous theory for the KdV and BBM approximations, only the linear versions of the higher-order models are known to work on the longer time scale $O(1/\beta^2)$.)

There are several versions of higher-order accurate, unidirectional models in the literature; see, for example, [9], [10], [12]. The present paper is devoted to theory for the models having the form

$$\begin{aligned}\eta_t + \eta_x - \gamma_1\beta\eta_{xxt} + \gamma_2\beta\eta_{xxx} + \delta_1\beta^2\eta_{xxxxt} + \delta_2\beta^2\eta_{xxxxx}\\ +\frac{3}{4}\alpha(\eta^2)_x + \alpha\beta\Big(\gamma(\eta^2)_{xx} - \frac{7}{48}\eta_x^2\Big)_x - \frac{1}{8}\alpha^2(\eta^3)_x = 0\end{aligned} \tag{1.1}$$

derived in [3]. The variables in (1.1) are the same as those appearing in the KdV and BBM models. The five parameters $\gamma_1, \gamma_2, \delta_1, \delta_2$ and γ are not arbitrary. Indeed, they are determined by the choice of five more fundamental parameters $\theta, \lambda, \mu, \lambda_1$ and μ_1. The constant θ has physical significance. It is related to the height at which the horizontal velocity is specified, a dependent variable which does not appear explicitly in these unidirectional models. The parameter θ lies in $[0,1]$ because the vertical coordinate has been scaled by the undisturbed depth h_0. The constants λ, μ, λ_1 and μ_1 are modeling parameters and in principle can take any real value. However, the five resulting parameters $\gamma_1, \gamma_2, \delta_1, \delta_2$ and γ appearing in (1.1) are not independent. For example, it must be the case that $\gamma_1 + \gamma_2 = \frac{1}{6}$. Others of these restrictions will appear presently. This is all spelled out in detail in [3].

The model's validity subsists formally on the assumptions that α and β are comparably-sized small quantities while η and its first few partial derivatives start and remain of order one during the evolution.

The pure initial-value problem for (1.1) in which the initial data

$$\eta(x,0) = \eta_0(x) \tag{1.2}$$

is viewed as known and the subsequent evolution of η determined by this specification and the equation (1.1) was studied in [3] and later in [6]. Initial-boundary-value problems were also considered in [7], but these will not be taken up here. In [3], it was shown that if the parameters γ_1 and δ_1 appearing in front of the η_{xxt}- and η_{xxxxt}-terms are positive, and if the initial data lies in the L_2-based Sobolev space $H^s(\mathbb{R})$ for $s \geq 1$, then the initial-value problem (1.1)-(1.2) is locally well posed in $H^s(\mathbb{R})$. That is, there exists a positive time T and a unique solution $\eta \in C(0,T;H^s(\mathbb{R}))$ of (1.1) satisfying (1.2). Note that if δ_1 is negative or even if $\delta_1 > 0$ but γ_1 is sufficiently negative, the initial-value problem is linearly ill posed. This corresponds to when the quadratic form $Q(p,q) = p^2 + \gamma_1 pq + \delta_1 q^2$ is not positive definite (see (2.4)).

A very attractive aspect of the models displayed in (1.1) is that if $\gamma = \frac{7}{48}$, the equation has a Hamiltonian structure. In that case, the initial-value problem was shown in [3] to be globally well posed in $H^s(\mathbb{R})$ if $s \geq \frac{3}{2}$. Moreover, for $s \geq 2$, the solution is globally bounded in $H^2(\mathbb{R})$, so providing one piece of the information that would be needed to confirm it as an accurate approximation of a solution to the full water wave problem on the longer time scale $O(1/\beta^2)$. It was also shown that within the restrictions on the parameters implying that solutions of (1.1) formally track solutions of the water-wave problem to the second order, there were choices of $\theta, \lambda, \mu, \lambda_1$ and μ_1 for which γ_1 and δ_1 are positive and $\gamma = \frac{7}{48}$.

More recently, in [6], Carvajal and Panthee were able to refine the Fourier splitting technique that led to global well-posedness in $H^s(\mathbb{R})$ for $3/2 \leq s < 2$ and show that the initial-value problem is globally well posed in $H^s(\mathbb{R})$ for $s \geq 1$. However, as is usual when using Fourier splitting, no bound on the temporal growth of solutions was obtained in these larger spaces. They also indicated that the value $s = 1$ is sharp by showing that if $s < 1$, then the map that takes initial data to the associated solution cannot be C^2. While this is not exactly ill posedness, it is a strong indicator that the initial-value problem (1.1)-(1.2) is not well posed in $H^s(\mathbb{R})$ for any $s < 1$.

The goal in the present essay is to improve further the theory for (1.1)-(1.2). Our results are summarized in the following theorems. Definitions of the function classes referred to below are standard, but are briefly reviewed in the introduction to Section 3.

Theorem 1.1 (local well-posedness; continuous function spaces)**.** *The initial-value problem* (1.1)-(1.2) *is well posed locally in time if η_0 lies in the class $C_b^1(\mathbb{R})$ of bounded continuous functions whose derivative is likewise bounded and continuous. That is to say, there is a positive number $T = T(\|\eta_0\|_{C_b^1(\mathbb{R})})$ and a unique solution $\eta \in C(0,T;C_b^1(\mathbb{R}))$ of* (1.1) *with initial condition η_0. The mapping from initial data to the solution is Lipschitz continuous from the relevant spaces. Moreover, if $\eta_0 \in C_b^k(\mathbb{R})$ for $k > 1$, then the solution $\eta \in C(0,T;C_b^k(\mathbb{R}))$ for the same value of T.*

Remark 1.2. These results are all new. The previous theory only countenanced Sobolev-space regularity. The fact that if the data is smoother, this smoothness propagates as long as the solution remains in $C_b^1(\mathbb{R})$ is an example of propagation of regularity. As will be seen in the proof, this property also holds for Hölder regularity.

Theorem 1.3. *(local well-posedness; Sobolev spaces) Problem* (1.1) *with initial condition* (1.2) *is well posed locally in time in the Sobolev space $H^1(\mathbb{R})$. That is to say,* *there is a positive number $T = T(\|\eta_0\|_{H^1(\mathbb{R})})$ such that the unique solution η of* (1.1)-(1.2) *lies in $C(0,T;H^1(\mathbb{R}))$. The solution map that associates to initial data the corresponding solution is Lipschitz continuous. Moreover, if $\eta_0 \in H^s(\mathbb{R})$ for $s > 1$, then the solution $\eta \in C(0,T;H^s(\mathbb{R}))$.*

Remark 1.4. As already mentioned, the first part of this theorem is known. However, the second part, the fact that the time interval of existence in $H^s(\mathbb{R})$ for $s > 1$ only depends upon the $H^1(\mathbb{R})$-norm of the initial data, is new. This is another example of propagation of regularity. The proof offered here has the advantage of providing the Lipschitz bounds on the solution map. As mentioned above, the recent work [6] indicates that the value $s = 1$ is in fact sharp for local well-posedness.

Theorem 1.5. *In Theorem 1.3 above, if $\gamma = \frac{7}{48}$ and $s \geq 2$, then* (1.1) *is globally well posed, the solution η lies in the space $C([0,\infty);H^s(\mathbb{R}))$ and its $H^s(\mathbb{R})$-norm has the temporal growth bound*

$$\|\eta(\cdot,t)\|_s \leq c(1+t)^{\frac{s-2+\sigma}{2}}$$

for all $t \geq 0$, where c is a constant depending only on the initial data, $\sigma = s - \lfloor s \rfloor$ and $\lfloor s \rfloor = \max\{n \in \mathbb{N} : n \leq s\}$.

Remark 1.6. Again, the first part of this result is known. The time-dependent bounds on the $H^s(\mathbb{R})$-norms for $s > 2$ are new.

Theorem 1.7. *In* (1.1), *assume that $\gamma = \frac{7}{48}$, $\delta_2 > \delta_1 > 0$ and $\gamma_1, \gamma_2 > 0$. For any $s \in [1,2)$, if $\eta_0 \in H^s(\mathbb{R})$ and its $H^1(\mathbb{R})$-norm $\|\eta_0\|_{H^1(\mathbb{R})}$ is of order 1 while $\alpha \sim \beta << 1$, then* (1.1) *is globally well posed in $H^s(\mathbb{R})$ and*

$$\|\eta(\cdot,t)\|_s \leq c(1+t)^{s-1}$$

for all $t \geq 0$, where c again depends only on the initial data.

Remark 1.8. Notice in particular that the $H^1(\mathbb{R})$-norm is bounded, uniformly in time. The global well-posedness was already established in [6]. The temporal bounds are new. The hypothesis $\delta_2 > \delta_1 > 0$ is always satisfied for physically relevant choices of the parameters in the Hamiltonian case $\gamma = \frac{7}{48}$ (see Section 4 of [3]).

Our analysis proceeds as follows. In the next section, an integral equation equivalent to (1.1)-(1.2) is derived. This equation is different from the one used in [3] and [6]. It is this difference that allows the further progress reported here. In addition, this integral equation provides a vehicle for constructing simpler proofs of some of the known results. Section 3 consists of preliminaries and a local well-posedness theory from which the time-dependent bounds in Theorems 1.5 and 1.7 are derived. Section 4 concentrates on the global result in Theorem 1.7.

2. Derivation of an integral equation

An integral equation that is equivalent to the initial-value problem (1.1)-(1.2) is derived here.

2.1. **Change of Variables.** In (1.1), make the change of variables $\tilde{x} = \beta^{-\frac{1}{2}}(x - \frac{\delta_2}{\delta_1}t)$, $\tilde{t} = \beta^{-\frac{1}{2}}t$ and let $u(\tilde{x}, \tilde{t}) = \alpha\eta(x, t)$. In these variables, and with the tildes suppressed for ease of reading, the initial-value problem becomes

$$\begin{cases} u_t + \Big(1 - \frac{\delta_2}{\delta_1}\Big)u_x - \gamma_1 u_{xxt} + \Big(\gamma_2 + \gamma_1\frac{\delta_2}{\delta_1}\Big)u_{xxx} + \delta_1 u_{xxxxt} \\ \qquad + \frac{3}{4}\big(u^2\big)_x + \gamma\big(u^2\big)_{xxx} - \frac{7}{48}\big(u_x^2\big)_x - \frac{1}{8}\big(u^3\big)_x = 0, \\ u(x,0) = u_0(x) = \alpha\eta_0\big(\beta^{\frac{1}{2}}x\big). \end{cases} \tag{2.1}$$

Remark 2.1. As η_0 is independent of the parameters α and β, the initial condition $u_0 = \alpha\eta_0(\beta^{\frac{1}{2}}x)$ is a waveform of small amplitude and long wavelength. For future reference, note that for non-negative integers k,

$$\|\partial_{\tilde{x}}^k u\|_{L_2(\mathbb{R})} = \alpha\beta^{\frac{k}{2}-\frac{1}{4}}\|\partial_x^k \eta\|_{L_2(\mathbb{R})}.$$

Thus, temporal bounds on the spatial norms of u and η coexist. The advantage of the problem (2.1) that emerges after this change of variables is that the η_{xxxxx}-term has been eliminated.

2.2. **Integral equation for** (2.1)**.** Write the initial value-problem (2.1) in the form

$$\begin{cases} \big(I - \gamma_1\partial_{xx} + \delta_1\partial_{xxxx}\big)u_t \\ \qquad = -\partial_x\left[\left(1 - \dfrac{\delta_2}{\delta_1}\right)u + \dfrac{3}{4}u^2 - \dfrac{7}{48}u_x^2 - \dfrac{1}{8}u^3\right] \\ \qquad - \partial_x^3\left[\left(\gamma_2 + \gamma_1\dfrac{\delta_2}{\delta_1}\right)u + \gamma u^2\right], \\ u(x,0) = u_0(x). \end{cases} \tag{2.2}$$

Since $\gamma_1, \delta_1 > 0$, the operator $I - \gamma_1\partial_{xx} + \delta_1\partial_{xxxx}$ is positive, hence invertible on all of the Sobolev spaces H^s, for $s \in \mathbb{R}$. Define f_1 and f_2 by

$$\begin{aligned} f_1(x,t) &= \left(1 - \frac{\delta_2}{\delta_1}\right)u + \frac{3}{4}u^2 - \frac{7}{48}u_x^2 - \frac{1}{8}u^3 \\ \text{and}\ \ f_2(x,t) &= \left(\gamma_2 + \gamma_1\frac{\delta_2}{\delta_1}\right)u + \gamma u^2. \end{aligned} \tag{2.3}$$

Take the Fourier transform,

$$\mathcal{F}u(\xi) = \widehat{u}(\xi) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x)e^{-i\xi x}\,dx,$$

in the spatial variable of both sides of (2.2) to obtain

$$\begin{cases} \big(1 + \gamma_1\xi^2 + \delta_1\xi^4\big)\,\widehat{u}_t(\xi,t) = -i\xi\widehat{f_1} + i\xi^3\widehat{f_2}, \\ \widehat{u}(\xi,0) = \widehat{u}_0(\xi). \end{cases} \tag{2.4}$$

Solve (2.4) for $\widehat{u}_t(\xi,t)$ and take the inverse transform of the result to reach the formula

$$\begin{cases} u_t(x,t) = \displaystyle\int_{-\infty}^{\infty} K(x-y)f_1(y,t)\,dt + \int_{-\infty}^{\infty} L(x-y)f_2(y,t)\,dt, \\ u(x,0) = u_0(x), \end{cases} \tag{2.5}$$

where

$$K(x) = \mathcal{F}^{-1}\Big\{\frac{-i\xi}{1+\gamma_1\xi^2+\delta_1\xi^4}\Big\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{-i\xi e^{ix\xi}}{1+\gamma_1\xi^2+\delta_1\xi^4}d\xi$$

and

$$L(x) = \mathcal{F}^{-1}\Big\{\frac{i\xi^3}{1+\gamma_1\xi^2+\delta_1\xi^4}\Big\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{i\xi^3 e^{ix\xi}}{1+\gamma_1\xi^2+\delta_1\xi^4}d\xi.$$

The Residue Theorem reveals that

$$K(x) = \begin{cases} \dfrac{\operatorname{sgn}(x)}{2\sqrt{\gamma_1^2-4\delta_1}}\Big\{e^{-\rho_1|x|}-e^{-\rho_2|x|}\Big\} & \text{for } \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{1}{4\delta_1^{\frac{3}{4}}}xe^{-\rho_0\,|x|} & \text{for } \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)}{\sqrt{4\delta_1-\gamma_1^2}}e^{-\rho|x|\cos\omega}\sin\big(\rho|x|\sin\omega\big) & \text{for } 0<\gamma_1 < 2\sqrt{\delta_1}, \end{cases} \tag{2.6}$$

while

$$L(x) = \begin{cases} \dfrac{\operatorname{sgn}(x)}{2\sqrt{\gamma_1^2-4\delta_1}}\Big\{\rho_1^2e^{-\rho_1|x|}-\rho_2^2e^{-\rho_2|x|}\Big\} & \text{for } \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{1}{4\delta_1^{\frac{3}{4}}}(-2\rho_0\operatorname{sgn}(x)+\rho_0^2x)e^{-\rho_0\,|x|} & \text{for } \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)\rho^2}{\sqrt{4\delta_1-\gamma_1^2}}e^{-\rho|x|\cos\omega}\sin\big(2\omega-\rho|x|\sin\omega\big) & \text{for } \gamma_1 < 2\sqrt{\delta_1}. \end{cases} \tag{2.7}$$

Here, ρ_1, ρ_2 ρ_0, ρ and ω depend on γ_1 and δ_1 as follows:

Case I. When $\gamma_1 > 2\sqrt{\delta_1}$, then

$$\rho_1 = \Big\{\frac{\gamma_1-\sqrt{\gamma_1^2-4\delta_1}}{2\delta_1}\Big\}^{\frac{1}{2}} \quad \text{and} \quad \rho_2 = \Big\{\frac{\gamma_1+\sqrt{\gamma_1^2-4\delta_1}}{2\delta_1}\Big\}^{\frac{1}{2}}.$$

Here and below, the positive branch of square roots of positive numbers is intended.

Case II. When $\gamma_1 = 2\sqrt{\delta_1}$, then

$$\rho_0 = \delta_1^{-\frac{1}{4}} = \frac{\sqrt{2}}{\sqrt{\gamma_1}}.$$

Case III. If $0 < \gamma_1 < 2\sqrt{\delta_1}$,

$$\rho = \delta_1^{-\frac{1}{4}} \quad \text{and} \quad \omega = \frac{1}{2}\arcsin\left(\frac{\sqrt{4\delta_1-\gamma_1^2}}{2\sqrt{\delta_1}}\right).$$

Remark 2.2. The integral kernel K may also be found using a direct Green's function argument as in [7] rather than by way of the Residue Theorem applied to its Fourier transform representation.

In (2.5), integrate with respect to t and formally use the Fundamental Theorem of Calculus to obtain the integral equation

$$\begin{aligned} u &= u_0 + \int_0^t K * \left[\left(1 - \frac{\delta_2}{\delta_1}\right) u + \frac{3}{4}u^2 - \frac{7}{48}u_x^2 - \frac{1}{8}u^3\right](x,s)\,ds \\ &\quad + \int_0^t L * \left[\left(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1\right) u + \gamma u^2\right](x,s)\,ds \\ &= u_0 + \int_0^t \left[\left(\left(1 - \frac{\delta_2}{\delta_1}\right) K + \left(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1\right) L\right) * u \right. \\ &\quad \left. + \left(\frac{3}{4}K + \gamma L\right) * u^2 - \frac{7}{48} K * u_x^2 - \frac{1}{8} K * u^3\right](x,s)\,ds \\ &=: \mathcal{A}u, \end{aligned} \tag{2.8}$$

where $*$ connotes convolution as usual. Solving (2.1) has thus been transformed into obtaining a fixed point of the operator $\mathcal{A}$.

3. Function classes and local well-posedness

The following function classes will intervene in our analysis.

(1) The Banach space $C_b^m(\mathbb{R})$ consists of all continuous and bounded functions defined on $\mathbb{R}$ whose derivatives up to order m are also continuous and bounded. The norm on this space is

$$\|u\|_{C_b^m(\mathbb{R})} = \sup_{x\in\mathbb{R}} \left\{|u(x)|, \cdots, |u^{(m)}(x)|\right\}.$$

If $m = 0$, the superscript will be omitted.

(2) The Banach space $L_p(\mathbb{R})$, for $1 \le p < \infty$, is the standard collection of p^{th}-power Lebesgue-integrable functions with the usual modification for $p = \infty$. The norm of a function $f \in L_p(\mathbb{R})$ with $1 \le p \le \infty$ is written $|f|_p$. For finite values of $p, 1 \le p < \infty$, the norm is normalized by the factor $1/\sqrt{2\pi}$. The $L_2(\mathbb{R})$–based Sobolev space $H^s(\mathbb{R})$ of order $s \ge 0$ is

$$H^s(\mathbb{R}) = \left\{v \in L_2(\mathbb{R}) : \int_{-\infty}^{\infty} (1+\xi^2)^s |\widehat{f}(\xi)|^2\,d\xi < \infty\right\}$$

equipped with the norm

$$\|f\|_s = \|f\|_{H^s(\mathbb{R})} = \left\{\int_{-\infty}^{\infty} (1+\xi^2)^s |\widehat{f}(\xi)|^2\,d\xi\right\}^{\frac{1}{2}}.$$

When $s = m$ is a positive integer, the norm may also be written as

$$\|f\|_s = \left\{\sum_{j=0}^{m} \binom{m}{j} \int_{-\infty}^{\infty} \left|f^{(j)}(x)\right|^2 dx\right\}^{\frac{1}{2}}.$$

When $s = 0$, the norm $\|v\|_0 = |v|_2$ and both these are written unadorned as $\|v\|$.

(3) If X is a Banach space, the space $C(0,T;X)$ of continuous functions from $[0,T]$ to X will be useful. The norm on this space is the usual one, *viz.*

$$\|f\|_{C(0,T;X)} = \max_{0\le t\le T} \|f(\cdot,t)\|_X.$$

3.1. **Preliminaries.** Our attention now turns to the integral equation (2.8). Some properties of the integral kernel K in (2.6) are first up on the agenda.

Proposition 3.1. *The function K defined in* (2.6) *is continuous on $\mathbb{R}$. It is infinitely smooth for $x \neq 0$. Moreover, its odd order derivatives $K^{(2k+1)}(x)$, for $x \neq 0$, $k = 1, 2, \cdots$, are continuous functions on $\mathbb{R}$ in the sense that the discontinuity at $x = 0$ is removable. The even order derivatives $K^{(2k)}(x)$, for $x \neq 0, k = 1, 2, \cdots$ have jump discontinuities at $x = 0$ of the form*

$$K''(0-) - K''(0+) = {\delta_1}^{-1} \tag{3.1}$$

for $k = 1$ and, inductively, for $k = 2, 3, \cdots$,

$$\begin{aligned} K^{(2k+2)}(0-) - K^{(2k+2)}(0+) = \delta_1^{-1}\Big\{&\gamma_1\big(K^{(2k)}(0-) - K^{(2k)}(0+)\big) \\ &- \big(K^{(2k-2)}(0-) - K^{2k-2)}(0+)\big)\Big\}. \end{aligned} \tag{3.2}$$

Remark 3.2. Notice the function $L(x)$ in (2.7) is, except at $x = 0$, exactly $K''(x)$, as is clear from its definition in Section 2.2 in terms of the Fourier transform. Henceforth, we write K'' in place of L.

Proof. From (2.6), it is a straightforward to ascertain that the odd order derivatives $K^{(2k+1)}(x)$ of $K(x)$ are equal to

$$K^{(2k+1)}(x) = \begin{cases} \dfrac{-\rho_1^{2k+1}e^{-\rho_1|x|} + \rho_2^{2k+1}e^{-\rho_2|x|}}{2\sqrt{\gamma_1^2 - 4\delta_1}} & \text{for} \quad \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{1}{4\delta_1^{\frac{3}{4}}}\big\{-|x|\rho_0^{2k+1} + (2k+1)\rho_0^{2k}\big\}e^{-\rho_0\,|x|} & \text{for } \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\rho^{2k+1}}{\sqrt{4\delta_1 - \gamma_1^2}}e^{-\rho|x|\cos\omega}\sin\big((2k+1)\omega - \rho|x|\sin\omega\big) & \\ & \text{for}\, 0 < \gamma_1 < 2\sqrt{\delta_1}, \end{cases}$$

for $x \neq 0$ and $k = 0, 1, 2, \cdots$. A straightforward examination reveals these are all continuous at $x = 0$ and hence on all of $\mathbb{R}$.

The even order derivatives $K^{(2k+2)}$ of K are

$$K^{(2k+2)}(x) = \begin{cases} \dfrac{\operatorname{sgn}(x)\Big(\rho_1^{2k+2}e^{-\rho_1|x|} - \rho_2^{2k+2}e^{-\rho_2|x|}\Big)}{2\sqrt{\gamma_1^2 - 4\delta_1}} & \text{for } \gamma_1 > 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)\Big(|x|\rho_0^{2k+2} - (2k+2)\rho_0^{2k+1}\Big)e^{-\rho_0\,|x|}}{4\delta_1^{\frac{3}{4}}} & \text{for } \gamma_1 = 2\sqrt{\delta_1}, \\ \dfrac{\operatorname{sgn}(x)\rho^{2k+2}e^{-\rho|x|\cos\omega}\sin\big((2k+2)\omega - \rho|x|\sin\omega\big)}{\sqrt{4\delta_1 - \gamma_1^2}} & \\ & \text{for } 0 < \gamma_1 < 2\sqrt{\delta_1}, \end{cases}$$

for $x \neq 0$ and $k = 0, 1, 2, \cdots$. They are continuous on $\mathbb{R}$ except at $x = 0$ where they have a simple jump discontinuity. Inspection shows that

$$K''(0-) - K''(0+) = \delta_1^{-1}.$$

Further calculation yields

$$K^{(2k+2)}(x) = \delta_1^{-1}\Big(\gamma_1 K^{(2k)}(x) - K^{(2k-2)}(x)\Big),$$

where $K^{(0)} = K$. Evaluating the last formula for x near to, but both below and above 0, taking the relevant one-sided limits and applying induction starting at $k = 1$ leads to the formula (3.2) and concludes the proof. □

The following elementary lemma will be needed.

Lemma 3.3. *Let $M \in L_1(\mathbb{R})$. If $v \in C_b(\mathbb{R})$, then so is the convolution $M * v$ and, for all x,*

$$|M * v(x)| \le \|M * v\|_{C_b(\mathbb{R})} \le |M|_1 \|v\|_{C_b(\mathbb{R})}.$$

Thus the mapping $v \mapsto M * v$ is a bounded linear operator of $C_b(\mathbb{R})$ to itself.

If $M \in C_b(\mathbb{R}) \cap L_1(\mathbb{R})$, $M' \in L_1(\mathbb{R})$ and $v \in C_b(\mathbb{R})$, then $M * v \in C_b^1(\mathbb{R})$ and

$$\frac{d}{dx}\big(M * v(x)\big) \;=\; M' * v(x).$$

Notice that the kernel K introduced above and all its derivatives taken at $x \neq 0$ have exponentially decreasing tails and are bounded and either continuous or have a single jump discontinuity at $x = 0$. As a consequence, they all lie in $L_1(\mathbb{R})$ and so convolution with $K^{(j)}$ is a bounded linear operator on $C_b(\mathbb{R})$ for all $j \ge 0$.

The notation $\mathcal{K}v = K * v$ will be convenient, as will the convolution $\mathcal{L}v = K'' * v$.

Proposition 3.4. *The operator $\mathcal{K}$ is bounded and linear from $C_b^k(\mathbb{R})$ to $C_b^{k+3}(\mathbb{R})$ for $k = 0, 1, 2, \cdots$. Moreover, if $v \in C_b(\mathbb{R})$ and $\lim_{|x|\to\infty} v(x) = 0$, then*

$$\lim_{|x|\to\infty} \mathcal{K}v(x) = \lim_{|x|\to\infty} \frac{d}{dx}\mathcal{K}v(x) = \lim_{|x|\to\infty} \frac{d^2}{d^2x}\mathcal{K}v(x) = \lim_{|x|\to\infty} \frac{d^3}{dx^3}\mathcal{K}v(x) = 0.$$

Proof. Since K and K' are continuous and K, K' and K'' lie in $L_1(\mathbb{R})$, Lemma 3.3 assures that $\mathcal{K}v$ lies in $C_b^2(\mathbb{R})$ and that

$$\frac{d^j}{dx^j}\mathcal{K}v = \; K^{(j)} * v$$

for $j = 0, 1, 2$. Thus, $\mathcal{K}$ is a bounded linear operator from $C_b(\mathbb{R})$ to $C_b^2(\mathbb{R})$.

Write $(\mathcal{K}v)''$ as

$$(\mathcal{K}v)''(x) \;=\; K'' * v(x) \;=\; \int_{-\infty}^{x} K''(x-y)v(y)dy + \int_{x}^{\infty} K''(x-y)v(y)dy$$

and use Leibniz integral rule to conclude that both terms on the right-hand side are differentiable. Performing these differentiations and using (3.1) leads to

$$\begin{aligned} \frac{d^3}{dx^3}\mathcal{K}v(x) =& \big(K''(0-) - K''(0+)\big)v(x) \\ &+ \int_{-\infty}^{x} K'''(x-y)v(y)\,dy + \int_{x}^{\infty} K'''(x-y)v(y)\,dy \\ =& -\frac{1}{\delta_1}v(x) + \int_{-\infty}^{\infty} K'''(x-y)v(y)\,dy. \end{aligned} \tag{3.3}$$

As K''' lies in $L_1(\mathbb{R})$, the right-hand side lies in $C_b(\mathbb{R})$. Moreover, it is clear that

$$\left\| \frac{d^3}{dx^3}\mathcal{K}v \right\|_{C_b(\mathbb{R})} \le \left(\frac{1}{\delta_1} + |K'''|_1 \right) \|v\|_{C_b(\mathbb{R})},$$

and thus $\mathcal{K}$ is seen to be a bounded linear map of $C_b(\mathbb{R})$ to $C_b^3(\mathbb{R})$, as advertised.

This line of argument can be continued. For if $k > 0$ and $v \in C_b^k(\mathbb{R})$, then the third derivative of $\mathcal{K}v$ is composed of the two terms on the right-hand side of (3.3). The first of these lies in $C_b^k(\mathbb{R})$ by assumption. Since K''' lies in $L_1(\mathbb{R})$, it follows that $K''' * v$ also lies in $C_b^k(\mathbb{R})$ because

$$\frac{d^j}{dx^j}\int_{-\infty}^{\infty} K'''(x-y)v(y)dy \;=\; \int_{-\infty}^{\infty} K'''(x-y)\frac{d^j}{dy^j}v(y)dy$$

for $j = 1, \cdots, k$. This shows at once that $\mathcal{K}v$ lies in $C_b^{k+3}(\mathbb{R})$ and that the mapping

$$v \;\longmapsto\; \mathcal{K}v$$

is a bounded operator from $C_b^k(\mathbb{R})$ to $C_b^{k+3}(\mathbb{R})$ for all $k \ge 0$.

Now suppose $v \in C_b(\mathbb{R})$ and $\lim_{|x|\to\infty} v(x) = 0$. Given $\epsilon > 0$, there exists $M = M_\epsilon > 0$ such that $|v(x)| \le \epsilon$ for $|x| > M$. Estimate $\mathcal{K}v(x)$ thusly:

$$\begin{aligned} \Big| \int_{-\infty}^{\infty} K(x-y)v(y)\,dy \Big| &\le \int_{-M}^{M} |K(x-y)v(y)|\,dy \\ &+ \int_{|y|>M} |K(x-y)v(y)|\,dy \le \int_{-M}^{M} |K(x-y)|dy\, \|v\|_{C_b(\mathbb{R})} + \epsilon |K|_1. \end{aligned}$$

Since $\lim_{|x|\to\infty} K(x-y) = 0$ uniformly for $y \in [-M, M]$, it transpires that

$$\limsup_{|x|\to\infty} \Big| \int_{-\infty}^{\infty} K(x-y)v(y)\,dy \Big| \le |K|_1 \epsilon,$$

and since $\epsilon > 0$ was arbitrary, $\mathcal{K}v(x)$ goes to zero at $\pm\infty$. The same argument may be applied to conclude

$$\lim_{|x|\to\infty} \frac{d^j}{dx^j}\mathcal{K}v(x) = \lim_{|x|\to\infty} \int_{-\infty}^{\infty} \frac{d^j}{dx^j} K(x-y)v(y)\,dy = 0,$$

for $j = 1, 2$. Finally, formula (3.3) implies that

$$\lim_{|x|\to\infty} \frac{d^3}{dx^3}\mathcal{K}v(x) = -\frac{1}{\delta_1}\lim_{|x|\to\infty} v(x) + \lim_{|x|\to\infty}\int_{-\infty}^{\infty} K'''(x-y)v(y)\,dy = 0,$$

as required. □

Corollary 3.5. *The operator $\mathcal{L}$ is a bounded linear mapping from $C_b^k(\mathbb{R})$ to $C_b^{k+1}(\mathbb{R})$ for $k = 0, 1, 2, \cdots$. Moreover, if $v \in C_b(\mathbb{R})$ and $\lim_{|x|\to\infty} v(x) = 0$, then*

$$\lim_{|x|\to\infty} \mathcal{L}v(x) = \lim_{|x|\to\infty} \frac{d}{dx}\mathcal{L}v(x) = 0.$$

Proposition 3.6. *The operator $\mathcal{K}$ maps $H^s(\mathbb{R})$ to $H^{s+3}(\mathbb{R})$ continuously for any $s \in \mathbb{R}$. It also maps $L_1(\mathbb{R})$ to $H^{\frac{5}{2}-}(\mathbb{R})$ continuously.*

Proof. The first statement is obvious since the symbol of the Fourier multiplier operator $\mathcal{K}$ decays like $|\xi|^{-3}$ as $\xi \to \pm\infty$.

For the second statement, suppose $\sigma \in \mathbb{R}$. Calculate the $H^\sigma(\mathbb{R})$-norm of $\mathcal{K}v$, *viz.*

$$\int_{-\infty}^{\infty}(1+\xi^2)^\sigma|\widehat{\mathcal{K}v}(\xi)|^2\,d\xi = \int_{-\infty}^{\infty}\frac{\xi^2(1+\xi^2)^\sigma}{(1+\gamma_1\xi^2+\delta_1\xi^2)^2}|\widehat{v}(\xi)|^2\,d\xi \le c_*|v|_1^2,$$

where

$$c_* = \int_{-\infty}^{\infty}\frac{(1+\xi^2)^\sigma\xi^2}{(1+\gamma_1\xi^2+\delta_1\xi^4)^2}\,d\xi.$$

The quantity c_* is finite if $\sigma < 5/2$. □

Corollary 3.7. *The operator $\mathcal{L}$ maps $H^s(\mathbb{R})$ continuously onto $H^{s+1}(\mathbb{R})$. It maps $L_1(\mathbb{R})$ continuously to $H^{\frac{1}{2}-}(\mathbb{R})$.*

3.2. **Local well-posedness in continuous function spaces.**

Lemma 3.8. *The integral equation (2.8) is locally well posed in the space $C_b^1(\mathbb{R})$. Precisely, for any value $r > 0$, there is $T = T_r$ such that for all initial data $u_0 \in C_b^1(\mathbb{R})$ with $\|u_0\|_{C_b^1(\mathbb{R})} \le r$, there is a unique solution $u \in C(0, T_r; C_b^1(\mathbb{R}))$ of (2.8). Moreover, the correspondence between the initial data u_0 and the associated solution u in $C(0, T_r; C_b^1(\mathbb{R}))$ of (2.8) is a uniformly Lipschitz continuous mapping of the ball of radius r about 0 in $C_b^1(\mathbb{R})$.*

Proof. The operator $\mathcal{A}$ defined in (2.8) may be written

$$\mathcal{A}u = u_0 + \int_0^t (\mathcal{K}f_1 + \mathcal{L}f_2)(x, s)\,ds \tag{3.4}$$

as follows from Remark 3.2 and the definitions of f_1 and f_2 in (2.3). The operator $\mathcal{K}$ maps $C_b^k(\mathbb{R})$ to $C_b^{k+3}(\mathbb{R})$ whilst $\mathcal{L}$ maps $C_b^k(\mathbb{R})$ to $C_b^{k+1}(\mathbb{R})$ for $k = 0, 1, \cdots$. Hence, if u belongs to $C(0, T; C_b^1(\mathbb{R}))$ for some $T > 0$, then the first and the second integrands belong to $C(0, T; C_b^4(\mathbb{R}))$ and $C(0, T; C_b^2(\mathbb{R}))$, respectively. Consequently, $\mathcal{A}u$ lies in $C(0,T;C_b^1(\mathbb{R}))$, which is to say, the operator maps $C(0,T;C_b^1(\mathbb{R}))$ to itself. To estimate the norm of $\mathcal{A}u$ in $C(0,T;C_b^1(\mathbb{R}))$, rewrite $\mathcal{A}u$ in its equivalent form,

$$\begin{aligned}\mathcal{A}u = u_0 + \int_0^t \Bigg[&\bigg(\Big(1-\frac{\delta_2}{\delta_1}\Big)K + \Big(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1\Big)K''\bigg) * u(x,s)\\ &+ \Big(\frac{3}{4}K + \gamma K''\Big) * u^2(x,s)\\ &- \frac{7}{48}K * u_x^2(x,s) - \frac{1}{8}K * u^3(x,s)\Bigg]\, ds.\end{aligned}$$

The analysis in Proposition 3.4 allows the conclusion that

$$\begin{aligned}\partial_x \mathcal{A}u(x,t) = u_0'(x) + \int_0^t\int_{-\infty}^{\infty} K'(x-y)\Bigg[&\Big(1-\frac{\delta_2}{\delta_1}\Big)u(y,s)\\ &+\frac{3}{4}u^2(y,s) - \frac{7}{48}u_y^2(y,s) - \frac{1}{8}u^3(y,s)\Bigg]\,dy\,ds\\ + \int_0^t\int_{-\infty}^{\infty} K''(x-y)\Bigg[\Big(\gamma_2 + \frac{\delta_2}{\delta_1}\gamma_1\Big)&u_y(y,s) + 2\gamma u(y,s)u_y(y,s)\Bigg]\,dy\,ds.\end{aligned}$$

It follows readily that

$$\begin{aligned}\|\mathcal{A}u\|_{C(0,T;C_b^1(\mathbb{R}))} \le &\|u_0\|_{C_b^1(\mathbb{R})} + cT\Big\{\|u\|_{C(0,T;C_b^1(\mathbb{R}))}\\ &+ \|u\|^2_{C(0,T;C_b^1(\mathbb{R}))} + \|u\|^3_{C(0,T;C_b^1(\mathbb{R}))}\Big\},\end{aligned}\tag{3.5}$$

where the constant c depends only on $|K|_1, |K'|_1$, $|K''|_1$ and the various fixed parameters γ, γ_1 etc. Similar considerations show that if $u_0 \in C_b^1(\mathbb{R})$ and u and v lie in $C(0,T;C_b^1(\mathbb{R}))$, then

$$\begin{aligned}\|\mathcal{A}u - \mathcal{A}v\|_{C(0,T;C_b^1(\mathbb{R}))} \le cT\|u-v\|_{C(0,T;C_b^1(\mathbb{R}))}&\Big\{1 + \|u\|_{C(0,T;C_b^1(\mathbb{R}))}\\ + \|v\|_{C(0,T;C_b^1(\mathbb{R}))} + \|u\|^2_{C(0,T;C_b^1(\mathbb{R}))} &+ \|v\|^2_{C(0,T;C_b^1(\mathbb{R}))}\Big\},\end{aligned}\tag{3.6}$$

for the same constant c.

If r is chosen to be $2\|u_0\|_{C_b^1(\mathbb{R})}$ and T_r taken to be

$$T_r = \frac{1}{2c\Big(1+2r+2r^2\Big)},\tag{3.7}$$

then (3.5) and (3.6) imply that $\mathcal{A}$ is a contraction mapping on the ball

$$B_r(0) = \{u \in C(0,T_r;C_b^1(\mathbb{R})) : \|u\|_{C(0,T_0;C_b^1(\mathbb{R}))} \le r\}.$$

It follows that there is a unique element $u \in C(0,T_r;C_b^1(\mathbb{R}))$ such that

$$\mathcal{A}u = u.$$

To see that the correspondence between initial data and solutions is Lipschitz, let $u_0, v_0 \in C_b^1(\mathbb{R})$ with

$$\|u_0\|_{C_b^1(\mathbb{R})},\ \|v_0\|_{C_b^1(\mathbb{R})} \ \le\ M,$$

say. Let $r = 2M$ and choose T_r as in (3.7). Denote by u and v the solutions of (2.8) corresponding to the auxiliary data u_0 and v_0, respectively. Write $\mathcal{A}_{u_0}$ for the operator $\mathcal{A}$ corresponding to initial data u_0 and similarly $\mathcal{A}_{v_0}$ is the operator corresponding to v_0. Because of the way things are set up, it follows that

$$u = \mathcal{A}_{u_0} u \quad \text{and} \quad v = \mathcal{A}_{v_0} v.$$

Calculate as follows:

$$\begin{aligned}
\|u - v\|_{C(0,T;C_b^1(\mathbb{R}))} &= \|\mathcal{A}_{u_0} u - \mathcal{A}_{v_0} v\|_{C(0,T;C_b^1(\mathbb{R}))} \\
&\le \|\mathcal{A}_{u_0} u - \mathcal{A}_{u_0} v\|_{C(0,T;C_b^1(\mathbb{R}))} + \|\mathcal{A}_{u_0} v - \mathcal{A}_{v_0} v\|_{C(0,T;C_b^1(\mathbb{R}))} \\
&\le \theta \|u - v\|_{C(0,T;C_b^1(\mathbb{R}))} + \|u_0 - v_0\|_{C_b^1(\mathbb{R})},
\end{aligned}$$

at least if $T \le T_r$, where $\theta < 1$ is the contractive constant for the operator $\mathcal{A}_{u_0}$. In consequence, it transpires that

$$\|u - v\|_{C(0,T_r;C_b^1(\mathbb{R}))} \le \frac{1}{1-\theta} \|u_0 - v_0\|_{C_b^1(\mathbb{R})},$$

thereby establishing local Lipschitz continuity.

A straightforward analysis using the integral equation shows that

$$\lim_{t \to 0} \|u(\cdot, t) - u_0\|_{C_b^1(\mathbb{R})} = 0.$$

The lemma is proved. □

Remark 3.9. If $u_0(x), u_0'(x) \to 0$ as $|x| \to \infty$, then $u(x,t), u_x(x,t) \to 0$ as $|x| \to \infty$. This follows since the functions tending to zero at $\pm\infty$ form a closed subspace of $C(0,T;C_b^1(\mathbb{R}))$ and convolution with K or K'' maps such functions into the same class.

Corollary 3.10. *Let initial data $u_0 \in C_b^1(\mathbb{R})$ be given and suppose u solves* (2.8) *in $C(0,T;C_b^1(\mathbb{R}))$. If u_0 in fact lies in $C_b^k(\mathbb{R})$ for some $k \ge 2$, respectively lies in the Hölder space $C_b^{k,\sigma}(\mathbb{R})$ for some $k \ge 1$ and $0 < \sigma \le 1$, then it follows that the solution $u \in C(0,T;C_b^k(\mathbb{R}))$, respectively $u \in C(0,T;C_b^{k,\sigma}(\mathbb{R}))$.*

Remark 3.11. The preceding contraction mapping argument may be used to show local well-posedness in all of these spaces. However, the interval of existence depends upon the norm of the initial data in the relevant space. As the space gets smaller, the norm of the data gets larger and the time interval over which the contraction argument applies gets smaller. Corollary 3.10 states that if the data starts life in a space smoother than $C_b^1(\mathbb{R})$, then this extra spatial smoothness continues as long as the solution remains in $C_b^1(\mathbb{R})$; hence the moniker "propagation of regularity".

Proof. This follows from the smoothing properties of $\mathcal{K}$ discussed earlier. To wit, if $u \in C(0,T;C_b^1(\mathbb{R}))$, then

$$u - u_0 = \int_0^t (\mathcal{K} f_1 + \mathcal{L} f_2)\, ds \tag{3.8}$$

lies in $C(0,T;C_b^2(\mathbb{R}))$. Thus, if u_0 lies in $C_b^2(\mathbb{R})$ or the Hölder space $C_b^{1,\sigma}(\mathbb{R})$, then clearly u lies in

$$C_b^2(\mathbb{R}) + C(0,T;C_b^2(\mathbb{R})), \text{ respectively}$$
$$C_b^{1,\sigma}(\mathbb{R}) + C(0,T;C_b^2(\mathbb{R})) \subset C(0,T;C_b^{1,\sigma}(\mathbb{R})).$$

Iterating this argument establishes the result. □

The formula (3.8) reveals that the solution cannot be more spatially regular than the initial data since the right-hand side is more regular than u. However, the solution is infinitely differentiable with respect to time.

Corollary 3.12. *The solution u obtained above is infinitely smooth in t. More exactly, if $u_0 \in C_b^1(\mathbb{R})$, then for all $j \geq 1$, $\partial_t^j u \in C(0,T;C_b^2(\mathbb{R}))$.*

Proof. It follows from (3.4) that

$$\partial_t u = \partial_t \mathcal{A} u = \mathcal{K} f_1 + \mathcal{L} f_2. \tag{3.9}$$

The mapping properties of $\mathcal{K}$ and $\mathcal{L}$ then insure that $\partial_t u$ belongs to the space $C(0,T;C_b^2(\mathbb{R}))$. Differentiating (3.9) with respect to t and using the now known smoothness of $\partial_t u$ yields the same conclusion about $\partial_t^2 u$. A simple induction then leads to the advertised result. □

Remark 3.13. In fact, one can show that the solution u is an analytic function of t. We do not stop off to prove this, but an argument for this conclusion follows as in [1] or [15], though the contexts are slightly different.

The same conclusion holds if u_0 lies in a smaller space. Thus, if $u_0 \in C_b^k(\mathbb{R})$ for some integer $k \geq 2$, then the solution emanating from u_0 is infinitely differentiable in time and $\partial_t^j u \in C(0,T;C_b^{k+1}(\mathbb{R}))$ for all $j = 1, 2, \cdots$. The proof is the same as that above for $k = 1$. The same conclusion also obtains for higher order Hölder spaces.

The formula (3.9) shows that a solution $u \in C(0,T;C_b^1(\mathbb{R}))$ of the integral equation (3.4) is a solution of the initial-value problem, at least in the sense of, say, tempered distributions. This follows by applying the operator $I - \gamma_1 \partial_x^2 + \delta_1 \partial_x^4$ to both side of (3.9), taking the Fourier transform and using the definition of $\mathcal{K}$ as a Fourier multiplier operator.

The latter remark and the preceding regularity results allow the conclusion that if $u_0 \in C_b^3(\mathbb{R})$, then the associated solution of the initial-value problem is a classical solution of the partial differential equation. That is, all the derivatives appearing in the equation are continuous functions and the equation is satisfied identically in space and time. Notice that since a solution cannot acquire more spatial regularity than that of the initial data, it follows that if $u_0 \notin C_b^3(\mathbb{R})$, then the resulting solution will not be classical.

3.3. **Local well-posedness in Sobolev spaces.** Whether or not there are global solutions to the initial-value problem (1.1)-(1.2) when initial data in $C_b^k(\mathbb{R})$ is specified is an open question. However, if the initial data is localized, there are some results of global well-posedness, as mentioned earlier.

In this section, local well-posedness for data in the L_2-based Sobolev classes $H^s(\mathbb{R})$ is considered. As much of the theory echoes that appearing in the last section, the arguments will only be sketched.

Theorem 3.14. *If the initial data $u_0 \in H^1(\mathbb{R})$, then the integral equation* (2.8) *is locally well posed in time. That is, given $M > 0$, there is a time $T = T_M$ such that if $\|u_0\|_1 \le M$, then* (2.8) *has a unique solution $u \in C(0, T_M; H^1(\mathbb{R}))$ for which $\lim_{t\to 0} \|u(\cdot, t) - u_0\|_1 = 0$. Moreover, the correspondence between the initial data u_0 and the associated solution u of* (2.8) *is a uniformly Lipschitz continuous mapping from $\{v \in H^1(\mathbb{R}) : \|v\|_1 \le M\}$ into $C(0, T_M; H^1(\mathbb{R}))$.*

Proof. The strategy is the same as in Lemma 3.8: show that the operator $\mathcal{A}$ defined in (2.8) has a fixed point in $C(0, T; H^1(\mathbb{R}))$ for suitably chosen values of T.

From Proposition 3.6, we know that convolution with K is a bounded linear operator from $H^s(\mathbb{R})$ into $H^{s+3}(\mathbb{R})$ while convolution with K'' is bounded and linear into $H^{s+1}(\mathbb{R})$. Also, convolution with K maps $L_1(\mathbb{R})$ boundedly and linearly into $H^{\frac{5}{2}-}(\mathbb{R})$. Thus, if $u \in C(0, T; H^1(\mathbb{R}))$, then $K'' * f_2 \in C(0, T; H^2(\mathbb{R}))$ whereas $K * f_1 \in C(0, T; H^4(\mathbb{R}))$ except for the term $K * u_x^2$. For the latter term, note that $u_x^2 \in C(0, T; L_1(\mathbb{R}))$ so that $K * u_x^2$ certainly lies in $C(0, T; H^2(\mathbb{R}))$. In sum, if $u \in C(0, T; H^1(\mathbb{R}))$, then $\mathcal{A}u - u_0$ lies in $C(0, T; H^2(\mathbb{R}))$. Thus $\mathcal{A}$ maps $C(0, T; H^1(\mathbb{R}))$ into itself.

Lemma 3.15 (bilinear estimates)**.** *For $f \in H^1(\mathbb{R})$, the following inequalities hold:*

$$\|K * f^2\|_1 \lesssim \|f\|_1^2, \qquad \|K * f^3\|_1 \lesssim \|f\|_1^3,$$
$$\|K'' * f^2\|_1 \lesssim \|f\|_1^2, \qquad \|K * f_x^2\|_1 \lesssim \|f\|_1^2.$$

Proof. These are elementary and the inequalities are far from sharp. They suffice for the analysis to follow. □

The results in Lemma 3.15 may be used systematically to deduce that if $u, v \in C(0, T; H^1(\mathbb{R}))$ and they both belong to the ball $B_r(0)$ of radius r about zero in this space, then

$$\begin{aligned}\|\mathcal{A}u - \mathcal{A}v\|_{C(0,T;H^1(\mathbb{R}))} &\le cT(1 + r + r^2)\|u - v\|_{C(0,T;H^1(\mathbb{R}))} \\ &\le 2cT(1 + r^2)\|u - v\|_{C(0,T_1;H^1(\mathbb{R}))},\end{aligned}$$

where the constant c is independent of u, v and T.

Proceeding as before, fix $u_0 \in H^1(\mathbb{R})$ and let $r = 2\|u_0\|_1$. Choose $T = \big(4c(1 + r^2)\big)^{-1}$ so that if u, v are in the ball $B_r(0)$ just defined, then

$$\|\mathcal{A}u - \mathcal{A}v\|_{C(0,T;H^1(\mathbb{R})} \le \frac{1}{2}\|u - v\|_{C(0,T;H^1(\mathbb{R})}.$$

On the other hand, if $u \in B_r(0)$, then

$$\|\mathcal{A}u\|_{C(0,T;H^1(\mathbb{R}))} \le \|\mathcal{A}u - \mathcal{A}0\|_{C(0,T;H^1(\mathbb{R}))} + \|\mathcal{A}0\|_{C(0,T;H^1(\mathbb{R}))},$$

where 0 is the zero function in $C(0, T; H^1(\mathbb{R}))$. Clearly, for all $t \ge 0$, $\mathcal{A}0(x, t) = u_0(x)$. Hence, the last inequality can be extended to

$$\|\mathcal{A}u\|_{C(0,T;H^1(\mathbb{R}))} \le \frac{1}{2}\|u - 0\|_{C(0,T;H^1(\mathbb{R}))} + \|u_0\|_1 \le r.$$

Thus, $\mathcal{A}$ is a contraction that maps $B_r(0)$ into itself. Consequently, there is a unique element u in $B_r(0)$ such that $\mathcal{A}u = u$.

The argument showing that the mapping from initial data u_0 to the associated solution u of (2.8) is a uniformly Lipschitz continuous mapping from $B_r(0)$ into $C(0,T;H^1(\mathbb{R}))$ is identical to that offered in the proof of Lemma 3.8. That the initial value u_0 is taken on in the $H^1(\mathbb{R})$-norm follows directly from the integral equation. □

Theorem 3.16. *If the initial data $u_0 \in H^s(\mathbb{R})$ for some $s > 1$, then the solution u lies in the space $C^m(0,T;H^s(\mathbb{R}))$ for some $T > 0$ and $m = 0, 1, \cdots$. If $m \geq 1$, then $\partial_t^m u \in C(0,T;H^{s+1}(\mathbb{R}))$. If the solution is obtained via the contraction mapping principle as in the last theorem, then there is a time interval $[0,T]$ whose length only depends on the $H^1(\mathbb{R})$-norm $\|u_0\|_1$ of the initial data such that the above conclusions holds.*

Proof. Since it is already established that problem (2.8) is locally well posed in $H^1(\mathbb{R})$ in the last theorem, there are values $T > 0$ for which there is a solution $u \in C(0,T;H^1(\mathbb{R}))$. If such a u is obtained from the last theorem, then T only depends upon $\|u_0\|_1$. But, in any case, take any value of T for which an $H^1(\mathbb{R})$-solution exists. Reorganize (2.8) in the form

$$u(x,t) - u_0(x) = \int_0^t \big(\mathcal{K}f_1(x,s) + \mathcal{L}f_2(x,s)\big)ds, \tag{3.10}$$

where f_1 and f_2 are as in (2.3). Because of the mapping properties of $\mathcal{K}$ and $\mathcal{L}$, the right-hand side of the last equation lies in $C(0,T;H^2(\mathbb{R}))$. Thus, if $1 \leq s \leq 2$, then $u \in C(0,T;H^s(\mathbb{R}))$. If $s > 2$, let $k = \lfloor s \rfloor$. A bootstrap argument shows the $u - u_0 \in C(0,T;H^{k+1}(\mathbb{R}))$. It then follows that $u \in C(0,T;H^s(\mathbb{R}))$.

For $m > 1$, the same bootstrap argument featured in the proof of Corollary 3.12 gives the advertised smoothness in time. □

3.4. **Invariants.** Throughout this subsection, it is assumed that $\gamma = 7/48$. That this is consistent with modeling long-crested, surface gravity waves is shown in Section 4 of [3]. For this special value of γ, if $u \in C(0,T;H^2(\mathbb{R}))$ is a solution of (2.1), then the quantity

$$\int_{-\infty}^{\infty} \Big(u^2 + \gamma_1 u_x^2 + \delta_1 u_{xx}^2\Big)dx \tag{3.11}$$

is independent of time and so determined by its initial value u_0. Similarly, if $u \in C(0,T;H^1(\mathbb{R}))$ is a solution of (2.1), then

$$\int_{-\infty}^{\infty} \left[\Big(\frac{\delta_2}{\delta_1} - 1 - \frac{1}{2}u\Big)u^2 + \Big(\gamma_2 + \gamma_1\frac{\delta_2}{\delta_1} + \frac{7}{24}u\Big)u_x^2 + \frac{1}{16}u^4\right]dx \tag{3.12}$$

is likewise time-independent. These two results are easily derived for smooth solutions by differentiating the quantities displayed above with respect to t and using the equation (2.1). For solutions in the larger spaces indicated, one approximates the initial data by smooth functions and uses the continuous dependence result to take an appropriate limit. This standard ploy will be used throughout this section without further comment.

3.5. **Global well-posedness and temporal growth.** As mentioned, when $\gamma = \frac{7}{48}$, the previous works [3] and [6] on this initial-value problem established global well-posedness in $H^s(\mathbb{R})$ for $s \geq 1$ and if $s \geq 2$, the solution was shown to be bounded in $H^2(\mathbb{R})$, independently of t. The latter fact follows directly from the invariant (3.11).

The principal concern in this subsection is with the temporal growth of the $H^s(\mathbb{R})$-norms of solutions for $s \neq 2$. There is implicitly a temporal growth estimate contained in the theory developed in [3] for the $H^3(\mathbb{R})$-norm. However, for other values of s, there are no results. We proceed by first deriving temporal bounds for the $H^k(\mathbb{R})$-norms for $k = 3, 4, \cdots$. An interpolation argument then provides growth rates for non-integer values of s. We are even able to provide bounds for $1 \leq s < 2$, including a time-independent bound on the $H^1(\mathbb{R})$-norm for small data.

Theorem 3.17. *The initial-value problem* (2.2) *is globally well posed in $H^k(\mathbb{R})$ for any integer $k \geq 2$. If the initial data u_0 lies in $H^m(\mathbb{R})$ for a particular integer $m \geq 2$ and the solution u emanating from u_0 is bounded in $H^2(\mathbb{R})$, independently of t, then the temporal growth bounds*

$$\|u(\cdot,t)\|_k \leq c(1+t)^{\frac{k-2}{2}}, \quad \text{for } t \geq 0 \text{ and } k = 2, 3, \cdots m \tag{3.13}$$

hold. Here, the constant c depends only on $\|u_0\|_m$ and the assumed bound on the $H^2(\mathbb{R})$-norm of u. In particular, if $\gamma = \frac{7}{48}$, then the inequalities (3.13) *are valid and the constant c depends only on $\|u_0\|_m$.*

Proof. Note first that because the functional defined in (3.11) is time independent, the $H^2(\mathbb{R})$-norm of a solution u is bounded, independently of t, when $\gamma = \frac{7}{48}$. So the overlying hypothesis that the solution u is bounded in $H^2(\mathbb{R})$ holds in this case. The boundedness of the $H^2(\mathbb{R})$-norm is exactly the statement (3.13) for $k = 2$. If $\gamma \neq \frac{7}{48}$, we simply assume that the $H^2(\mathbb{R})$-norm is bounded, independently of t. In any case, the argument proceeds by induction on k.

Turning to $k = 3$, multiply (2.2) by $-u_{xx}$ and integrate the result over $\mathbb{R}$ with respect to x. After integrating by parts several times, there appears the differential-integral equation

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left[\big(\partial_x u(x,t)\big)^2 + \gamma_1\big(\partial_x^2 u(x,t)\big)^2 + \delta_1\big(\partial_x^3 u(x,t)\big)^2\right]dx \\
&= \int_{-\infty}^{\infty}\left[\frac{3}{4}u^2 + \gamma(u^2)_{xx} - \frac{7}{48}u_x^2 - \frac{1}{8}u^3\right]_x u_{xx}\,dx \\
&= \int_{-\infty}^{\infty}\left[\frac{3}{4}u^2 + 2\gamma u u_{xx} + 2\gamma u_x^2 - \frac{7}{48}u_x^2 - \frac{1}{8}u^3\right]_x u_{xx}\,dx \\
&= \int_{-\infty}^{\infty}\left[\frac{3}{2}u u_x u_{xx} + \left(5\gamma - \frac{7}{24}\right)u_x u_{xx}^2 - \frac{3}{8}u^2 u_x u_{xx}\right]dx \\
&\leq \frac{3}{2}|u|_\infty\|u_x\|\|u_{xx}\| + \left|5\gamma - \frac{7}{24}\right||u_x|_\infty\|u_{xx}\|^2 + \frac{3}{8}|u|_\infty^2\|u_x\|\|u_{xx}\| \leq c,
\end{aligned}$$

where c is a constant only dependent on the time-independent bound on the $H^2(\mathbb{R})$-norm assumed for u. (In case $\gamma = \frac{7}{48}$, this means only on $\|u_0\|_2$.) It follows that

$$\int_{-\infty}^{\infty} \Big[\big(\partial_x u\big)^2 + \gamma_1 \big(\partial_x^2 u\big)^2 + \delta_1 \big(\partial_x^3 u\big)^2 \Big]\, dx \\ \leq \int_{-\infty}^{\infty} \Big[u_0'(x)^2 + \gamma_1 u_0''(x)^2 + \delta_1 u_0'''(x)^2 \Big] dx + ct.$$

Further bounds on the temporal behavior of the $H^s(\mathbb{R})$-norms, which is our next task, were not provided in [3] or [6].

Assume that the estimate (3.13) is true for some integer $k \geq 3$. To establish the result for $k+1$, we need an appropriate growth bound on $\|\partial_x^{k+1} u\|$. To this end, multiply (2.2) by $(-1)^{k-1}\partial_x^{2k-2} u$ and integrate again with respect to x. (For definiteness, the constant γ has been set to $\gamma = 7/48$, but the reader will see that the calculations to follow do not depend upon this choice.) Noting that

$$\int_{-\infty}^{\infty} (u + \gamma_2 u_{xx})_x \partial_x^{2k-2} u \, dx = 0,$$

it follows, after integrating by parts that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt} \int_{-\infty}^{\infty} \Big[\big(\partial_x^{k-1} u\big)^2 + \gamma_1 \big(\partial_x^k u\big)^2 + \delta_1 \big(\partial_x^{k+1} u\big)^2 \Big]\, dx \\ &\quad = (-1)^k \int_{-\infty}^{\infty} \left[\frac{3}{4} u^2 - \frac{7}{48}(u^2)_{xx} + \frac{7}{48} u_x^2 - \frac{1}{8} u^3 \right]_x \partial_x^{2k-2} u \, dx \\ &\quad = \int_{-\infty}^{\infty} \bigg[\frac{3}{4}\partial_x^{k-1}(u^2)\partial_x^k u + \frac{7}{48}\partial_x^k(u^2)\partial_x^{k+1} u \\ &\qquad\qquad + \frac{7}{48}\partial_x^{k-1}(u_x^2)\partial_x^k u - \frac{1}{8}\partial_x^{k-1}(u^3)\partial_x^k u \bigg] dx. \end{aligned} \tag{3.14}$$

Note that for any $k \geq 1$, if $f, g \in H^k(\mathbb{R})$, then there is a number c such that

$$\|\partial_x^k (fg)\| \leq c\big(\|f\|_1 \|\partial_x^k g\| + \|g\|_1 \|\partial_x^k f\|\big).$$

Hence,

$$\|\partial_x^k (u^2)\| \leq c \|u\|_1 \|\partial_x^k u\|. \tag{3.15}$$

And similarly,

$$\|\partial_x^k (u^3)\| \leq c(\|u^2\|_1 \|\partial_x^k u\| + \|u\|_1 \|\partial_x^k (u^2)\|) \leq c \|u\|_1^2 \|\partial_x^k u\|. \tag{3.16}$$

The time-dependent quantity A composed of the first and fourth terms on the right-hand side of (3.14) is

$$A = \int_{-\infty}^{\infty} \Big(\frac{3}{4}\partial_x^{k-1}(u^2)\partial_x^k u - \frac{1}{8}\partial_x^{k-1}(u^3)\partial_x^k u \Big)\, dx.$$

The time-independent $H^2(\mathbb{R})$-bound of the solution u and the last inequalities together with the induction hypothesis provide the estimate

$$\begin{aligned}|A| &\le c\Big[\|\partial_x^{k-1}(u^2)\|\|\partial_x^k u\| + \|\partial_x^{k-1}(u^3)\|\|\partial_x^k u\|\Big]\\ &\le c\Big[\|u\|_1\|\partial_x^{k-1}u\|\|\partial_x^k u\| + \|u\|_1^2\|\partial_x^{k-1}u\|\|\partial_x^k u\|\Big]\\ &\le c(1+t)^{\frac{2k-5}{2}}.\end{aligned}$$

With suitable integrations by parts and applications of Leibniz formula, the middle two terms together become

$$\begin{aligned}B &= \int_{\infty}^{\infty}\Big[-\partial_x^{k+1}(u^2)\,\partial_x^k u + \partial_x^{k-1}(u_x^2)\,\partial_x^k u\Big]dx\\ &= \int_{\infty}^{\infty}\Big[-2u\partial_x^{k+1}u\,\partial_x^k u - \partial_x^k u\sum_{j=1}^{k}\binom{k+1}{j}\partial_x^j u\,\partial_x^{k+1-j}u\\ &\qquad + \partial_x^k u\sum_{j=0}^{k-1}\binom{k-1}{j}\partial_x^{j+1}u\,\partial_x^{k-j}u\Big]dx.\end{aligned}$$

Integrate the first term above by parts and pull out all the terms of the form $u_x\,(\partial_x^k u)^2$ from the two sums to reach the formula

$$\begin{aligned}B = \int_{-\infty}^{\infty}\Big[&u_x(\partial_x^k u)^2 - 2(k+1)u_x(\partial_x^k u)^2\\ &- \partial_x^k u\sum_{j=2}^{k-1}\binom{k+1}{j}\partial_x^j u\,\partial_x^{k+1-j}u\\ &+ 2u_x(\partial_x^k u)^2 + \partial_x^k u\sum_{j=1}^{k-2}\binom{k-1}{j}\partial_x^{j+1}u\,\partial_x^{k-j}\Big]dx.\end{aligned}\tag{3.17}$$

where again, a non-essential constant has been dropped. The same elementary inequalities used in the analysis of the term A together with the fact that $|u_x|_\infty$ is bounded, independently of t, and that for $2 \le j \le k-1$, $|\partial_x^j u|_\infty \le \|\partial_x^j u\|^{\frac{1}{2}}\|\partial_x^{j+1}u\|^{\frac{1}{2}}\| \le c(1+t)^{\frac{2j-3}{4}}$, come to our rescue and it is deduced that

$$|B| \le c(1+t)^{k-2}.$$

Putting together the foregoing bounds on A and B, it is determined that (3.13) holds for $k+1$, thus completing the proof of the theorem. □

Now consider values of s that are non-integer, say $s = k+\sigma$ where $0 < \sigma < 1$ and $k \ge 2$. The following growth bounds are obtained in this case.

Theorem 3.18. *If $u_0 \in H^s(\mathbb{R})$ where $s = k+\sigma$ is as above and u is the associated solution of* (3.4)*, then*

$$\|u(\cdot,t)\|_s \le c(1+t)^{\frac{s-2+\sigma}{2}}\tag{3.18}$$

for $t \geq 0$*.*

Remark 3.19. We conjecture that the σ appearing in (3.18) does not express the real state of affairs, but is simply a consequence of our proof.

Proof. This result follows from the integer result in Theorem 3.17 together with the formula (3.10) and the smoothing properties of the operators $\mathcal{K}$ and $\mathcal{L}$. Formula (3.10), repeated here for the reader's convenience, states that

$$u(x,t) - u_0(x) = \int_0^t (\mathcal{K} f_1(x,s) + \mathcal{L} f_2(x,s)) ds,$$

where f_1 and f_2 are the polynomials in u and u_x displayed in (2.3).

Suppose that $s = k+\sigma$ with $0 < \sigma < 1$ and $k \geq 2$. Since $\mathcal{K}$ and $\mathcal{L}$ smooth by three orders and one order, respectively, in the $H^s(\mathbb{R})$ spaces, $u - u_0 \in C(0,T; H^{k+1}(\mathbb{R}))$ as remarked before. Differentiating $k+1$ times with respect to x gives

$$\partial_x^{k+1}(u - u_0) = \int_0^t \left(\partial_x^{k+1} \mathcal{K} f_1(x,s) + \partial_x^{k+3} \mathcal{K} f_2(x,s) \right) ds.$$

Taking the $L_2(\mathbb{R})$-norm of both sides of this equation leads to the inequality

$$\|\partial_x^{k+1}(u - u_0)\| \leq c \int_0^t \left(\|\partial_x^{k-2} f_1(x,s)\| + \|\partial_x^k f_2(x,s)\| \right) ds, \tag{3.19}$$

where c is a constant depending only on the symbols of $\mathcal{K}$ and $\mathcal{L}$. Using the properties (3.15) and (3.16), it follows that

$$\begin{aligned} \|\partial_x^{k-2} f_1(x,s)\| &\leq c(\|\partial_x^{k-2} u\| + \|\partial_x^{k-2}(u^2)\| + \|\partial_x^{k-2}(u_x)^2\| + \|\partial_x^{k-2}(u^3)\|) \\ &\leq c(\|\partial_x^{k-2} u\| + \|u\|_1 \|\partial_x^{k-2} u\| + \|u_x\|_1 \|\partial_x^{k-1} u\| + \|u\|_1^2 \|\partial_x^{k-2} u\|) \\ &\leq c(1+s)^{\frac{k-3}{2}}, \end{aligned}$$

and similarly,

$$\begin{aligned} \|\partial_x^k f_2(x,s)\| &\leq c(\|\partial_x^k u\| + \|\partial_x^k (u^2)\|) \\ &\leq c(\|\partial_x^k u\| + \|u\|_1 \|\partial_x^k u\|) \\ &\leq c(1+s)^{\frac{k-2}{2}}. \end{aligned}$$

Once the last two inequalities are in hand, one simply integrates this bound on the integrand with respect to s in (3.19) to obtain

$$\|\partial_x^{k+1}(u - u_0)\| \; \leq \; c(1+t)^{\frac{k}{2}}. \tag{3.20}$$

for $t \geq 0$.

Return now to $s = k + \sigma$ and use interpolation to write

$$\begin{aligned} \|u(\cdot,t)\|_s \; &\leq \; \|u(\cdot,t) - u_0\|_s + \|u_0\|_s \\ &\leq \|u(\cdot,t) - u_0\|_k^{1-\sigma} \|u(\cdot,t) - u_0\|_{k+1}^{\sigma} + \|u_0\|_s \end{aligned} \tag{3.21}$$

Apply (3.20) to the $H^{k+1}(\mathbb{R})$-norm and (3.13) to the $H^k(\mathbb{R})$-norm appearing on the right-hand side of (3.21) to come to

$$\|u(\cdot,t)\|_s \; \leq \; c(1+t)^{\frac{k-2}{2}(1-\sigma)}(1+t)^{\frac{k}{2}\sigma} \; = \; c(1+t)^{\frac{s-2+\sigma}{2}}$$

as stated. ∎

Our final results are concerned with the time-independent $H^1(\mathbb{R})$-bound on solutions emanating from sufficiently small initial data.

Theorem 3.20. *Assume that*

$$\gamma = \frac{7}{48}, \quad \delta_2 > \delta_1 > 0 \text{ and } \gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} > 0.$$

Under these hypotheses, there are positive numbers τ and τ_1, depending only on $\gamma_1, \gamma_2, \delta_1$ and δ_2, such that if $u_0 \in H^1(\mathbb{R})$ and $\|u_0\|_1 \le \tau$, then

$$|u(\cdot,t)|_\infty \le \|u(\cdot,t)\|_1 \le \tau_1$$

for all $t \ge 0$.

Proof. Define χ_1 and χ_2 to be

$$\chi_1 = \frac{\delta_2}{\delta_1} - 1, \quad \chi_2 = \gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} \text{ and } \chi = \min\{\chi_1, \chi_2\}.$$

According to the hypotheses, χ_1 and χ_2 are both positive.

Suppose that $\|u_0\|_1 \le \tau$ where τ will be specified presently. Consider the invariant functional $I(u)$ defined in (3.12) by the integral

$$\int_{-\infty}^{\infty} \left[\left(\frac{\delta_2}{\delta_1} - 1 - \frac{1}{2}u \right) u^2 + \left(\gamma_2 + \gamma_1 \frac{\delta_2}{\delta_1} + \frac{7}{24}u \right) u_x^2 + \frac{1}{16}u^4 \right] dx.$$

Estimate this functional from above and below as follows:

$$\begin{aligned} I(u) &\ge \chi \|u\|_1^2 - \frac{1}{2}|u|_\infty \|u\|^2 - \frac{7}{24}|u|_\infty \|u_x\|^2 + \frac{1}{16}|u|_4^4 \\ &\ge \chi \|u\|_1^2 - \frac{1}{2}\|u\|_1^3 \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} I(u) = I(u_0) &\le \left(\chi_1 + \frac{1}{2}|u_0|_\infty \right) \|u_0\|^2 \\ &\quad + \left(\chi_2 + \frac{7}{24}|u_0|_\infty \right) \|u_0'\|^2 + \frac{1}{16}|u_0|_\infty^2 \|u_0\|^2 \\ &\le \max\{\chi_1, \chi_2\} \|u_0\|_1^2 + \frac{1}{2}\|u_0\|_1^3 + \frac{1}{16}\|u_0\|_1^4 \\ &\le \max\{\chi_1, \chi_2\} \tau^2 + \frac{1}{2}\tau^3 + \frac{1}{16}\tau^4 < \frac{1}{2}\chi^3. \end{aligned} \tag{3.23}$$

Of course the last step in the above inequalities needs τ to be taken small enough.

The inequalities (3.23) and (3.22) together imply

$$\frac{1}{2}\chi^3 > I(u_0) = I(u) \ge \chi \|u\|_1^2 - \frac{1}{2}\|u\|_1^3,$$

or, what is the same,

$$\begin{aligned} 0 &< \chi^3 - 2\chi \|u\|_1^2 + \|u\|_1^3 \\ &= \left(\|u\|_1 - \chi \right) \left(\|u\|_1 - \frac{\sqrt{5}+1}{2}\chi \right) \left(\|u\|_1 + \frac{\sqrt{5}-1}{2}\chi \right). \end{aligned} \tag{3.24}$$

This analysis holds for any $t \ge 0$. As the last factor on the right-hand side of (3.24) is always positive, so for any $t > 0$, either

$$\|u(\cdot,t)\|_1 < \chi \quad \text{or} \quad \|u(\cdot,t)\|_1 > \frac{\sqrt{5}+1}{2}\chi.$$

Because of (3.22), if $\|u(\cdot,t)\|_1 \le \chi$, then

$$I(u_0) = I(u) \ge \frac{\chi}{2}\|u(\cdot,t)\|_1^2. \tag{3.25}$$

Let $\bar\tau$ be chosen so that

$$I(u_0) \le \frac{\chi^3}{8}. \tag{3.26}$$

The inequality (3.23) shows this to be possible. Take $\tau = \min\{\frac{\chi}{2}, \bar\tau\}$. By continuity, there are time intervals $[0,T]$ such that $\|u(\cdot,t)\|_1 \le \chi$ for $0 \le t \le T$. Let $[0,T^*]$ be a maximal such interval. Suppose T^* to be finite. Then, $\|u(\cdot,T^*)\|_1 = \chi$ and for t near to, but greater than T^*, $\|u(\cdot,t)\|_1 > \chi$. However, combining (3.25) and (3.26) leads to the conclusion that $\|u(\cdot,T^*)\|_1 \le \frac{\chi}{2}$, and so $\|u(\cdot,t)\|_1 \le \chi$ for t in a neighborhood of T^*, which is a contradiction. Thus, $\|u(\cdot,t)\|_1 \le \chi$ for all $t \ge 0$. $\square$

Remark 3.21. Recall that in the current variables, the initial data u_0 has the form $u_0 = \alpha\eta_0\big(\beta^{\frac{1}{2}}x\big)$. In consequence,

$$\|u_0\|_1^2 = \frac{\alpha^2}{\beta^{\frac{1}{2}}}\|\eta_0\|^2 + \alpha^2\beta^{\frac{1}{2}}\|\eta_0'\|^2.$$

As η_0 is of order one in the original variables and $\alpha \sim \beta$ are both small compared to one, it follows that $\|u_0\|_1 = O(\alpha^{\frac{3}{4}})$, so the assumption of small $H^1(\mathbb{R})$-norm is consistent with the original physical situation that gave rise to the model.

Remark 3.22. Referring to Section 4 of [3] where the restrictions on the parameter values for the equation to be physically relevant and at the same time have $\gamma = \frac{7}{48}$ are laid out, we see that,

$$\gamma_1 = \gamma_2 = \frac{1}{12} \qquad \delta_2 - \delta_1 = \frac{7}{180} \quad \text{or} \quad \frac{\delta_2}{\delta_1} - 1 = \frac{7}{180\delta_1} > 0$$

and

$$\gamma_1 + \gamma_2\frac{\delta_2}{\delta_1} = \frac{1}{12}\left(1 + \frac{\delta_2}{\delta_1}\right) = \frac{1}{12}\left(2 + \frac{7}{180\delta_1}\right) > \frac{1}{6}.$$

Thus, for physically relevant choices of the parameters, it is always the case that χ_1 and χ_2 and hence χ, are positive.

Corollary 3.23. *If $\gamma = \frac{7}{48}$ and the initial data $u_0 \in H^1(\mathbb{R})$ with sufficiently small $H^1(\mathbb{R})$-norm, the solution lies in $C_b(0,\infty; C_b^\sigma(\mathbb{R}))$ for any $0 < \sigma < \frac{1}{2}$. Moreover, if $u_0 \in H^1(\mathbb{R}) \cap C_b^\sigma(\mathbb{R})$ for some σ in the range $[\frac{1}{2}, 1]$, then the solution $u \in C_b(0,\infty; C_b^\sigma(\mathbb{R}))$.*

By using the same interpolation argument that prevailed in the proof of Theorem 3.18, the foregoing $H^1(\mathbb{R})$-bound provides a temporal growth estimate in the fractional-order spaces $H^s(\mathbb{R})$ for $1 \le s < 2$.

Corollary 3.24. *If $\gamma = \frac{7}{48}$, the initial-value problem* (2.1) *is globally well-posed in $H^s(\mathbb{R})$ for any s with $1 \le s < 2$ provided the $\|u_0\|_1$ is sufficiently small. Furthermore, the solution $u \in C(0,\infty; H^s(\mathbb{R}))$ has the temporal growth bound*

$$\|u(\cdot,t)\|_s \le c(1+t)^{s-1}$$

where $c = O(1)$ is a constant only dependent on $\|u_0\|_s$.

3.6. **Results in order-one variables.** The authors find it useful to translate the preceding theory, set in the dimensionless, but unscaled variables $u(\tilde{x},\tilde{t})$, back into the original scaled variables $\eta(x,t)$. In the latter variables, solutions and their first few partial derivatives are all formally of order one and it is perhaps easier to judge how well the model is doing in various respects. Because of Remark 2.1, this is an easy task. Our last result simply translates some aspects of the foregoing theory back into the variables as they appear in the original problem (1.1).

Theorem 3.25. *Suppose the initial-value problem* (1.1)-(1.2) *has γ_1 and δ_1 positive. Then this problem is locally well posed in $H^s(\mathbb{R})$ for any $s \ge 1$, in $C_b^k(\mathbb{R})$ for any $k = 1, 2, \cdots$ and in $C_b^{k,\sigma}(\mathbb{R})$ for any $\sigma \in (0,1]$ and $k = 1, 2, \cdots$. If in addition $\gamma = \frac{7}{48}$, then it is globally well posed in $H^s(\mathbb{R})$ for $s \ge 1$. If $s \ge 2$ then the solution is globally bounded in $H^2(\mathbb{R})$ and satisfies the growth bounds in Theorem 3.18 for $t \ge 0$. Also in the case $\gamma = \frac{7}{48}$, for α and β sufficiently small and $s \ge 1$, the solution is globally bounded in $H^1(\mathbb{R})$.*

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