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MINIMIZERS OF STRICTLY CONVEX FUNCTIONS

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Abstract. A proper use of strictly convex lower semi-continuous functions is shown to prove classical results from various fields in analysis: Haar measure on compact groups, complex analysis, ergodic theory, Lipschitz retractions. All proofs are elementary.

1. Introduction

Strictly convex functions and their extrema are basic tools of functional analysis: if for instance C is a closed bounded convex subset of $\mathbb{R}^n$ equipped with its Euclidean norm, the infimum of the norm provides a unique nearest point to 0 in C, while its maximum provides an extreme point of C. In the infinite dimensional setting, it frequently happens that natural convex functions f on convex compact sets K are merely lower semi-continuous, and then only the infimum is (usually) reached by f, with uniqueness in the strictly convex case. This may provide a distinguished point of K when f is properly chosen.

The purpose of this work is to show that such lower semi-continuous strictly convex functions (which we will call energy functionals) provide unified and very simple proofs of a number of fundamental results in analysis, from various fields: Haar measure on compact groups, Schwarz lemma and related results from complex analysis, ergodic theorems, Lipschitz retractions. Such techniques are used for a long time: we refer for instance to [10] for an early application of strictly convex functions to the Choquet-Bishop-de Leeuw integral representation theorem. Among many more recent references, we single out for instance Section III.7 in [16], and [2], [7], [8]. However, connections with complex analysis (see Section 3) or renorming arguments (see Corollary 4.4) could be less classical.

It is our hope that this note could be used in any graduate course in functional analysis (or possibly for the corresponding final exam). We therefore include basic examples which give concrete applications, and sometimes also show that our statements are optimal. It might be interesting to notice that apparently unrelated statements such as Schwarz's lemma and the existence of the Haar measure both follow from an abstract observation from Choquet theory. However, such simple-minded arguments have limitations: our approach seems to be powerless on topics such as Schauder-type results for analytic maps on $\mathbb{C}^n$, extensions of Schwarz's

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lemma to boundary points (see [4] and subsequent articles) or fixed points results for 1-Lipschitz maps on weakly compact convex sets.

Let us describe the content of this note. Section 2 contains the basic lemmas from convex analysis, with a first application to the existence of the Haar measure on metrizable compact groups. Lemmas 2.2, 2.3 and 2.4 are applied throughout our work. Section 3 gathers applications to complex analysis. Section 4 gives applications to ergodic theorems and the action of uniformly bounded linear groups on reflexive spaces. Section 5 provides applications to the construction of 1-Lipschitz retractions in strictly convex reflexive spaces. All our proofs are elementary.

The notation we use is classical. We recall that the weak operator topology on the space $L(X)$ of continuous linear operators on a locally convex space X is the topology generated by the semi-norms

$$N_{x,x^*}(T) = | < x^*, T(x) > |$$

with $x \in X$ and $x^* \in X^*$. This topology is denoted by w_o. We denote by $GL(X)$ the group of all invertible elements of $L(X)$. A real-valued function ϕ from a convex set C is strictly convex if $\phi(\lambda x + (1-\lambda)y) < \lambda\phi(x) + (1-\lambda)\phi(y)$ for all pairs of distinct points $(x, y) \in C^2$ and all $\lambda \in (0, 1)$.

2. Basic lemmas

We first recall a well-known fact.

Lemma 2.1. *Let K be a convex compact subset of a locally convex Hausdorff vector space V. Then K is metrizable if and only if there exists a strictly convex continuous function $\varphi : K \to \mathbb{R}$.*

Proof. Since V is locally convex Hausdorff, the space $A(K)$ of affine continuous real-valued functions on K separates the points of K. If K is metrizable, the space $C(K)$ is norm-separable and so is its subspace $A(K)$. Therefore there exists a sequence (f_n) of affine continuous functions which separate K. We may and do assume that $\sup_K |f_n| \leq 1$ for all n. Then the map $F : K \to l_2(\mathbb{N})$ defined by $F(x) = (n^{-1}f_n(x))_n$ is an affine continuous embedding when $l_2(\mathbb{N})$ is equipped with the norm topology. Then the function $\varphi = \|.\|_2^2 \circ F$ works.

Conversely, assume that such a function φ exists. Let Δ be the diagonal subset of $K \times K = K^2$. For $n \geq 1$, we set

$$U_n = \{(x, y) \in K^2; [\varphi(x) + \varphi(y)]/2 - \varphi((x + y)/2) < 1/n\}$$

.

The sequence (U_n) if a countable basis of neighbourhoods of Δ in K^2, and this countability implies that K is metrizable (see IX.15, Thm.1 in [3])

□

Remark: We will use below strictly convex lower semi-continuous functions. It has been shown in [6] that if K is a convex compact subset of a locally convex Hausdorff vector space V, there exists a strictly convex lower semi-continuous function defined on K if and only if K embeds linearly into a strictly convex dual space equipped with its weak* topology. Note however that there exists convex compact subsets of L_p ($0 \leq p < 1$) without extreme points on which exist strictly convex lower semi-continuous functions (as follows from [11] and [17]). In what follows, we will focus on metrizable convex compact sets in locally convex spaces.

Our next lemma on "increasing" strictly convex functions is trivial but useful. We will call below such a strictly convex function Φ an *energy functional*.

Lemma 2.2. *Let K be a metrizable convex compact subset of a locally convex Hausdorff vector space V. Let $\Phi : K \to \mathbb{R}$ be a strictly convex lower semi-continuous function. Assume that $\prec$ is a relation on K such that $x \prec y$ implies $\Phi(x) \leq \Phi(y)$. Then there exists $m \in K$ which is $\prec$-minimal. That is, if $x \prec m$ then $x = m$.*

Proof. Since K is compact and Φ is lower semi-continuous, Φ attains its minimum at some $m \in K$. if $x \prec m$, we have $\Phi(x) = \Phi(m)$. But then $x = m$ since Φ is strictly convex.

□

The relations $\prec$ we will use in this work are obtained as follows. Let $\star : K^2 \to K$ be a map. We will call it a *product*. We may define $\prec$ by: $x \prec y$ if there exists z such that $x = z \star y$, or alternatively by $x \prec y$ if there exists z such that $x = y \star z$. Both definitions will prove to be useful. Note that the relations $\prec$ are transitive when the product $\star$ is associative. In practice we will consider only associative products.

Our next lemma is a simple tool for showing the existence of a "left zero" in a convex semigroup, under quite general assumptions. We recall that an element $x \in K$ is called *right regular* for the product $\star$ if $y \star x = z \star x$ implies $y = z$. With this notation the following holds.

Lemma 2.3. *Let K be metrizable convex compact subset of a locally convex space. Let $\star$ be an associative product defined on K. We assume that $\star$ is separately continuous, and left-linear, that is: $(x+y)/2 \star z = (x \star z + y \star z)/2$ for all x, y and z in K. Then:*
(1) if every $z \in K$ is right-regular, or
(2) if $\star$ is bilinear and every $e \in Ext(K)$ is right-regular
there exists $m \in K$ such that $m \star z = m$ for every $z \in K$.

Proof. Let φ be a strictly convex continuous function on K. We consider the energy functional defined by

$$\Phi(x) = \sup\{\varphi(x \star z);\ z \in K\}.$$

Since $\star$ is left-continuous, the function Φ is lower semi-continuous. Since $\star$ is associative, the function Φ is increasing for the relation $\prec$ defined by: $x \prec y$ if there exists $z \in K$ such that $x = y \star z$. We claim that Φ is strictly convex. Indeed, assume (1) and pick x and y in K with $x \neq y$. Since $\star$ is right-continuous and φ is continuous, there exists $z \in K$ such that

$$\Phi((x+y)/2) = \varphi((x+y)/2 \star z) = \varphi((x \star z + y \star z)/2).$$

Since z is right-regular we have $x \star z \neq y \star z$. But then

$$\Phi((x+y)/2) = \varphi((x \star z + y \star z)/2) < [\varphi(x \star z) + \varphi(y \star z)]/2 \leq [\Phi(x) + \Phi(y)]/2.$$

The conclusion follows from the definition of $\prec$ and Lemma 2.2. If we assume (2), the proof follows the same lines once we notice that if $\star$ is bilinear, the function ψ defined by $\psi(z) = \varphi((x+y)/2 \star z)$ is convex and continuous and thus it attains its supremum at an extreme point.

□

Example: Let us show by an example that assuming right-regularity is necessary in Lemma 2.3. We recall that a square matrix $M \in M_n(\mathbb{R})$ is *stochastic* if $m_{ij} \geq 0$ for all (i,j) and $\sum_{j=1}^{n} m_{ij} = 1$ for all i. The set St_n of stochastic $n \times n$-matrices is compact convex and stable under product. However, if $A = (a_{ij})$ is such that $a_{i1} = 1$ for all i and $a_{ij} = 0$ for all i and all $j \geq 2$, while $B = (b_{ij})$ is such that $b_{i2} = 1$ for all i and $b_{ij} = 0$ for all i and all $j \neq 2$, then $TA = A$ and $TB = B$ for all $T \in St_n$. Hence St_n fails to satisfy the conclusion of Lemma 2.3.

Let us single out a special case of Lemma 2.3.

Lemma 2.4. *Let K be metrizable convex compact subset of a locally convex space. Let $\star$ be an associative, separately continuous and bilinear product defined on K. We assume that every $e \in Ext(K)$ is both left- and right-regular for $\star$. Then there exists $m \in K$ such that $x \star m = m \star x = m$ for every $x \in K$.*

Proof. We may apply (2) from Lemma 2.3 to get $m \in K$ such that $m \star x = m$ for every $x \in K$. But since every extreme point is also left-regular, we may use the same argument with left and right swapped, to get $m' \in K$ such that $x \star m' = m'$ for all $x \in K$. But then $m \star m' = m = m'$ and this concludes the proof.

□

Examples: Lemma 2.4 immediately shows the existence of the Haar measure for any metrizable compact group G. Our proof follows [8]. Let K be the set $\mathcal{P}(G)$ of probability measures on G equipped with the weak* topology. The convolution product $\star$ satisfies the assumptions of Lemma 2.4. Indeed $Ext(K)$ consists of the Dirac measures (δ_x) which are $\star$-regular since the convolution with δ_x is a translation operator.

The Markov-Kakutani fixed point theorem for compact groups follows from the existence of the Haar measure, but it can be shown directly with an energy functional: let K be a compact convex metrizable set, and ϕ a strictly convex continuous function on K. Let G be a pointwise compact group of affine continuous maps from K to K. Then the map Φ defined on K by $\Phi(x) = \sup\{\phi(T(x));\ T \in G\}$ is l.s.c. and strictly convex, and its unique minimizer is a common fixed point to all $T \in G$.

3. Applications to complex analysis

In Lemma 2.3 we assumed linearity of $\star$ on the left only. This will allow us to apply this lemma to a very important product: the composition of functions $\circ$. We prove now a quite general result on convex $\circ$-stable sets of holomorphic maps.

Let U be an open subset of $\mathbb{C}^n$. We denote by $\mathcal{H}(U;U)$ the set of analytic maps from U to U. We denote by τ_K the topology of uniform convergence on the compact subsets of U. We denote by $Id \in \mathcal{H}(U;U)$ the identity function such that $Id(z) = z$ for all z. With this notation, the following holds.

Proposition 3.1. *Let U be a bounded connected open subset of $\mathbb{C}$. Let $C \subset \mathcal{H}(U;U)$ be a convex $\circ$-stable set. Then either $C = \{Id\}$ or the τ_K- closure $\overline{C}$ of C contains a constant function.*

Proof. Assume that $\overline{C}$ contains no constant function. Then every $f \in \overline{C}$ is a non-constant holomorphic function, and thus $f(U)$ is an open subset of $\mathbb{C}$. But since $f(U)$ is contained in $\overline{U}$ and is open, we have in fact $f(U) \subset U$. Hence $\overline{C} \subset \mathcal{H}(U;U)$. It is now easy to check that the convex set $\overline{C}$ is $\circ$-stable, and that $\circ$ is τ_K-separately continuous. Moreover since U is bounded, the set $\overline{C}$ is τ_K-compact. Every $f \in \overline{C}$ is right-regular for $\circ$. Indeed if $g \circ f = h \circ f$, the holomorphic functions g and h coincide on the open set $f(U)$ and thus everywhere on the connected set U.

We may now apply Lemma 2.3 which provides $m \in \overline{C}$ such that $m \circ f = m$ for every $f \in \overline{C}$. In particular $m \circ m = m$, and thus $m = Id$ on $m(U)$. It follows that $m = Id$ or m is a constant function. The second option was ruled out, hence $m = Id$ and then $m \circ f = m = f$ for all $f \in \overline{C}$. Thus $C = \{Id\}$.

□

A first corollary of this result is Schwarz's uniqueness lemma in convex domains.

Corollary 3.2. *Let U be a bounded convex open subset of $\mathbb{C}$, and let $f \in \mathcal{H}(U;U)$. Assume that there exists $z_0 \in U$ such that $f(z_0) = z_0$ and $f'(z_0) = 1$. Then $f(z) = z$ for all $z \in U$.*

Proof. We let $C = \{f \in \mathcal{H}(U;U); f(z_0) = z_0 \ , f'(z_0) = 1\}$. Since U is convex, the set C is convex as well. The chain rule shows that it is stable under $\circ$. Hence if the conclusion fails, $\overline{C}$ contains a constant function by Prop. 3.1, but this cannot be since the map $D(f) = f'(z_0)$ is τ_K- continuous on $\mathcal{H}(U;\mathbb{C})$.

□

Our next corollary concerns stable subsets under a holomorphic action on a convex domain. Note that Schwarz's uniqueness lemma can be understood as what we obtain when two fixed points of a holomorphic function merge into one.

Corollary 3.3. *Let U be a convex bounded open subset of $\mathbb{C}$, and let $f \in \mathcal{H}(U;U)$. Assume that there exists two disjoint convex compact subsets L and M of U such that $f(L) \subset L$ and $f(M) \subset M$. Then $f(z) = z$ for all $z \in U$.*

Proof. Let $C = \{f \in \mathcal{H}(U;U); f(L) \subset L,\ f(M) \subset M\}$. It is clear that C is convex and $\circ$-stable. Hence if the conclusion fails, by Proposition 3.1 $\overline{C}$ contains a constant function equal to μ. But clearly $\mu \in L \cap M$ and this cannot be.

□

It is classical that a holomorphic function on a simply connected domain with two fixed points is Id. This is an immediate consequence of Corollary 3.3 if U is convex, and the general case follows if we use Riemann's conformal mapping theorem. We recall that the"two fixed points" result fails in general for bounded open sets, as shown by the example of $f(z) = 1/z$ on the set $U = \{z \in \mathbb{C}; 1/2 < |z| < 2\}$. On the other hand, Corollary 3.3 follows from the "two fixed points" result for L and M *homeomorphic* to compact convex sets if we use Schauder's fixed point theorem.

Note that Corollary 3.3 and Schauder's theorem show that if there exists a compact convex set $C \subset U$ such that $f(C) \subset C$ and $f \neq Id$, then the intersection of all such sets C is the unique fixed point of f.

We now consider analytic functions of several complex variables. It is well-known that such functions are harder to handle. For instance the dichotomy open range vs. constant function completely fails, and Proposition 3.1 fails as well. A simple example is provided by the set

$$C = \{f_\mu \in \mathcal{H}(D^2; D^2); |\mu| < 1\}$$

where $f_\mu(z_1, z_2) = (\mu z_1, z_2)$, which satisfies the assumptions of Proposition 3.1 but not its conclusion. However, we observe that the arguments which proved Proposition 3.1 show in dimensions greater than 1 the following weaker statement: instead of finding an element of $\overline{C}$ which is everywhere constant, we can find such an element which is constant on a non-compact set. Indeed:

Proposition 3.4. *Let U be a bounded connected open subset of $\mathbb{C}^n$. Let $C \subset \mathcal{H}(U; U)$ be a convex $\circ$-stable set. Then either $C = \{Id\}$ or the τ_K- closure $\overline{C}$ of C contains f such that $f^{-1}(z)$ is not compact for some $z \in f(U)$.*

Proof. If the conclusion fails, then by Theorem 15.1.6 in [15], every $f \in \overline{C}$ is an open map. The rest of the proof follows the same lines, once we observe that if $m \circ m = m$ and m is an open analytic function then $m = Id$.

□

Schwarz's uniqueness lemma has an extension to analytic functions of several complex variables due to H. Cartan ([5], see [15],p. 23). In fact, Cartan's uniqueness theorem holds in all Banach spaces (see Theorem 12.1 in [9]). Proposition 3.4 appears to fall short from implying Cartan's theorem. Note however that this theorem easily follows in the case of convex domains from Schauder's fixed point theorem, under an extra assumption on f which prevents trouble at the boundary.

Proposition 3.5. *Let U be a bounded convex open subset of $\mathbb{C}^n$, and let $f \in \mathcal{H}(U; U)$. Assume that there exists $z_0 \in U$ such that $f(z_0) = z_0$ and $f'(z_0) = Id$, and that f extends to an analytic function on a neighborhood of $\overline{U}$. Then $f(z) = z$ for all $z \in U$.*

Proof. We denote by $\mathcal{H}(U; \overline{U})$ the set of analytic functions on U whose range is contained in $\overline{U}$. We define $D = \{g \in \mathcal{H}(U; \overline{U}); g(z_0) = z_0,\ g'(z_0) = Id\}$. The set D is convex and τ_K-compact. Moreover the map $\psi : D \to D$ defined by $\psi(g) = f \circ g$ is continuous from (D, τ_K) to itself, hence by Schauder's fixed point theorem there is $g \in D$ such that $f \circ g = g$. But since $g'(z_0) = Id$, the range of g has non-empty interior, and thus $f \circ g = g$ implies that $f = Id$.

□

Let us mention in passing an easy case of Cartan's result, where we do not need any regularity assumption at the boundary: if U is the unit ball of a norm N on $\mathbb{C}^n$ and $f \in \mathcal{H}(U; U)$ is such that $f(0) = 0$ and $f'(0) = Id$, then $f = Id$. Indeed one can apply Corollary 3.2 to functions g defined by $g(\lambda) =< z^*, f(\lambda z) >$, where we have $N(z) = N^*(z^*) =< z^*, z >= 1$ and $\lambda \in \mathbb{C}$ with $|\lambda| < 1$. Cartan's theorem follows for transitive domains in $\mathbb{C}^n$ but this is a quite restrictive condition (see e.g. [12]).

4. Applications to ergodic theory

In this section, the product we consider will simply be the composition of linear maps, hence a bilinear product. Our first proposition is a special case of Lemma 2.4.

Corollary 4.1. *Let E be a locally convex Hausdorff space, and $K \subset L(E)$ a convex w_o-metrizable compact subset of $L(E)$, stable under products of operators. We assume that $Ext(K)$ is contained in the set $GL(E)$ of invertible elements of $L(E)$. Then there exists $P \in K$ such that $PT = TP = P$ for all $T \in K$.*

Indeed it suffices to apply Lemma 2.4 to product of operators - which is w_0 separately continuous - since regularity of the extreme points follows immediately from their invertibility. This corollary extends for instance Theorem III.7.9 in [16]. Note that P is a projection on the closed space $\{x \in E; T(x) = x \ for \ all \ T \in K\}$ and that $T(ker(P)) \subset ker(P)$ for all $T \in K$.

We now focus on reflexive spaces equipped with special norms. Our next result provides a projection of norm 1 on the space of common fixed points of a convex bounded semigroup.

Proposition 4.2. *Let X be a separable reflexive strictly convex Banach space. Let C be a convex $\circ$-stable subset of the unit ball of $L(X)$. Then there exists P in the w_o-closure K of C such that $TP = P$ for all $T \in K$.*

Proof. We equip $L(X)$ with the weak operator topology w_o. The product is w_o-separately continuous, and the w_o-closure K of C is w_o-compact and $\circ$-stable. Let $(x_n)_{n\geq 1}$ be a dense sequence in B_X. We consider the following energy functional:

$$\Phi(T) = \sum_{n\geq 1} 2^{-n} \|T(x_n)\|_X^2.$$

Since X is strictly convex, it is easily seen that Φ is strictly convex on $L(X)$. Moreover, it is w_0-l.s.c. as supremum of such functions. Finally, it is increasing for the relation on K defined by: $S \prec T$ if there is $L \in K$ such that $S = LT$. Indeed since K is contained in the unit ball of $L(X)$, we have $\|S(x_n)\| = \|LT(x_n)\| \leq \|T(x_n)\|$ for all n if $S \prec T$. Now Lemma 2.2 concludes the proof.

□

Example: Let E be a linear subspace of the separable reflexive strictly convex space X. Let us consider the set

$$C_E = \{T \in L(X);\ \|T\| \leq 1,\ T = Id \ on \ E\}.$$

Proposition 4.2 applied to C_E shows the existence of a smallest contractively complemented subspace $\tilde{E} \subset X$ containing E. The space $\tilde{E}$ is such that any $T \in L(X)$ of norm 1 which is identity on E is also identity on $\tilde{E}$.

This proposition can be dualized, as follows.

Proposition 4.3. *Let X be a separable reflexive Gâteaux smooth Banach space. Let C be a convex $\circ$-stable subset of the unit ball of $L(X)$. Then there exists P in the w_o-closure K of C such that $PT = P$ for all $T \in K$.*

Proof. Since X is reflexive and Gâteaux smooth, its dual X^* is strictly convex. We may therefore apply Proposition 4.2 to the convex set $K^* = \{T^*; T \in K\}$, to obtain P^* such that $T^*P^* = P^*$ for all $T \in K$. The conclusion follows since $T^*P^* = (PT)^*$. □

The proof of Proposition 4.3 consists in fact into applying Lemma 2.2 through the energy functional $\Phi^*(T) = \sum_{n\geq 1} 2^{-n}\|T^*(x_n^*)\|^2_{X^*}$, where (x_n^*) is a dense sequence in B_{X^*}.

Examples: Propositions 4.2 and 4.3 fail even in finite dimensional spaces when no specific assumption is made on the norm. We may consider for instance the convex semigroup St_n of stochastic matrices in the unit ball of $L(l_\infty^n)$, where l_∞^n denotes $\mathbb{R}^n$ equipped by the supremum norm, and the conclusion of Proposition 4.3 fails for this set. Dually, the set St_n^* provides a example in the unit ball of $L(l_1^n)$ where the conclusion of Proposition 4.2 fails. We can also observe that if P_1 and P_2 are two distinct norm 1-projections with the same kernel, then $C = conv(\{P_1, P_2\})$ satisfies the assumptions of Proposition 4.2 but not its conclusion. Such pair of projections exist on every non strictly convex 2-dimensional space.

Of course, gathering Propositions 4.2 and 4.3 and the usual uniqueness trick proves that if X reflexive and separable is Gâteaux smooth and strictly convex, then in the above notation there exists a contractive projection $P \in K$ such that $PT = TP = P$ for all $T \in K$. This applies for instance to the classical spaces $L_p(1 < p < \infty)$. In the special case where K is the smallest convex w_o-compact $\circ$-stable subset containing a single contractive operator T, our arguments work under a smoothness *or* a convexity assumption since then K is $\circ$- commutative.

We are now ready to apply a renorming argument, in order to show an isomorphic result.

Corollary 4.4. *Let X be a separable reflexive space, and $G \subset GL(X)$ be a uniformly bounded group. Then there exists a projection P in the w_o-closed convex hull of G such that $PT = TP = P$ for all $T \in G$.*

Proof. First, we equip the space X with the equivalent norm defined by $N(x) = \sup\{\|T(x)\|; T \in G\}$. The group G is contained into the group of invertible isometries of the norm N. Now we may apply the renorming construction from [14] which provides an equivalent locally uniformly rotund norm L on X such that every invertible N-isometry is an L-isometry. Since L is in particular strictly convex, we can apply Proposition 4.2 which provides a projection P in the w_o-closed convex hull of G such that $TP = P$ for all $T \in G$. But applying [14] to the dual space X^* gives a Fréchet smooth equivalent norm M such that every invertible N-isometry is an M-isometry. Since M is in particular Gâteaux smooth, Proposition 4.3 gives a projection P' in the w_o-closed convex hull of G such that $P'T = P'$ for all $T \in G$. As usual $P'P = P' = P$ and this concludes the proof. □

Note that the renorming procedure of [14] is canonical enough to provide a useful property which, in the finite-dimensional case, follows from the uniqueness of John's ellipsoid: every invertible isometry for the original norm remains an isometry for the new norm.

It is appropriate to compare Corollaries 4.1 and 4.4: when G is assumed to be w_o-compact, Corollary 4.4 is a special case of Corollary 4.1. In general however, it is not easy to control the regularity of the extreme points of the w_o-closed convex hull of G. Moreover Corollary 4.4 typically applies to groups G which are not locally compact, such as the group of unitary operators on L_2.

Linear algebra shows that Corollary 4.4 actually states the following: if X is separable and reflexive, and G is a uniformly bounded subgroup of $GL(X)$, then $X = F \oplus H$, where $F = \{x \in X; T(x) = x \; for \; all \; T \in G\}$ and H is a closed vector space such that $T(H) \subset H$ for all $T \in G$, and for all $x \in H \backslash \{0\}$, there is $T \in G$ with $T(x) \neq x$.

Examples: We consider some finite-dimensional examples, with the above notation. Note that if G is contained in the orthogonal group $O_n(\mathbb{R})$ on $\mathbb{R}^n$, then P is an orthogonal projection since it has norm 1 on l_2^n. If G is the group of all linear rotations of $\mathbb{R}^3$ with a given axis A, then $F = A$ and P is the orthogonal projection on A. If $G = O_n(\mathbb{R})$, then $conv(G) = B_{L(l_2^n)}$ and $P = 0$. If $G \subset O_n(\mathbb{R})$ is the subgroup of permutation matrices, then by a result of Birkhoff $conv(G)$ is the set of bistochastic matrices. If we set $e = (1, 1, 1, ...1) \in \mathbb{R}^n$, the projection P is the orthogonal projection with range $F = span(e)$. Note that the example of the convex set St_n of stochastic matrices, which is $\circ$-stable but is not a group and for which the conclusion of Corollary 4.4 fails, shows that our algebraic assumption on G is necessary.

If X is any separable reflexive space and G is the group of all invertible isometries of X, then $P = 0$. Actually, when G is such that there is $T \in G$ with $(-T) \in G$ then $P = 0$, since by Corollary 4.4 $-(TP) = -P = (-T)P = P$. Note now that for any normed space X, Id and $-Id$ are invertible isometries.

5. Applications to Lipschitz maps

In this last section, we return to non-linear maps, more precisely to 1-Lipschitz maps. We investigate the non expansive hull $N(E)$ of a subset E of an appropriate Banach space, as defined in [13] (see page 166). For convenience, we will use below the notation $\tilde{E}$. We refer to [13] and to Chapter 1, Section 13 in [9] for more results on 1-Lipschitz retractions. What follows is a non-linear version of Proposition 4.2.

Proposition 5.1. *Let X be a separable reflexive strictly convex space. Let E be a non-empty subset of X. Then there exists a subset $\tilde{E}$ of X containing E such that if $F : X \to X$ is 1-Lipschitz and such that $F(x) = x$ for all $x \in E$, then $F(x) = x$ for all $x \in \tilde{E}$, and moreover $\tilde{E}$ is the range of a 1-Lipchitz retraction defined on X.*

Proof. We may and do assume that $0 \in E$. We let $E_1 = E \cap B_X$. Let

$$K_1 = \{F : B_X \to B_X;\; F \; 1-Lipschitz, F = Id \; on \; E_1\}$$

The set K_1 is convex, $\circ$-stable and compact for the pointwise topology with values in (B_X, w). We denote this topology by τ. Let (x_n) be a dense sequence in B_X with $x_0 = 0$. We consider again the energy functional $\Phi : K_1 \to \mathbb{R}$ defined by

$$\Phi(F) = \sum_{n\geq 0} 2^{-n} \|F(x_n)\|^2.$$

It is easily seen that Φ is τ-s.c.i. and strictly convex on K_1. Moreover, for all f and $g \in K_1$, one has $\Phi(g \circ f) \leq \Phi(f)$. Indeed, since g is 1-Lipschitz and $g(0) = 0$, it follows that $\|g(y)\| \leq \|y\|$ for all $y \in B_X$. We may now apply Lemma 2.2 for the relation: $h \prec f$ if there is g such that $h = g \circ f$. Lemma 2.2 provides $m_1 \in K_1$ such that $g \circ m_1 = m_1$ for all $g \in K_1$. Let $m_1(B_X) = \tilde{E}_1$. It follows from the fact that $g \circ m_1 = m_1$ for all $g \in K_1$ that if $x \in B_X$, we have $x \in \tilde{E}_1$ if and only if $g(x) = x$ for all $g \in K_1$. Note that the strict convexity of X implies that $\tilde{E}_1$ is a convex set.

For any $n \geq 2$, we now reproduce the above argument on nB_X, with $E_n = E \cap nB_X$ and $K_n = \{F : nB_X \to nB_X;\ F\ 1-Lipschitz, F = Id\ on\ E_n\}$. This provides a sequence (m_n) of 1-Lipschitz retractions from nB_X on an increasing sequence $\tilde{E}_n$ of convex subsets of X. Since $\|m_n(x)\| \leq \|x\|$ for all x and $n \geq \|x\|$, there exists a subsequence (m_{n_k}) such that $\lim_k m_{n_k}(x) = m(x)$ exists in (X, w) for all $x \in X$.

The map m is 1-Lipschitz since the norm of X is weakly l.s.c. Since the sequence $(\tilde{E}_n)$ is increasing, $m(x) = x$ for every $x \in \bigcup_{n\geq 1} \tilde{E}_n$. We now denote

$$\tilde{E} = \overline{\bigcup_{n\geq 1} \tilde{E}_n}.$$

Note that this closure is the same for the weak and norm topologies since the relevant set is convex. Hence $m(X) \subset \tilde{E}$ since $m_n(X) \subset \tilde{E}$ for all n, and $m(x) = x$ for every $x \in \tilde{E}$ since m is 1-Lipschitz. Therefore m is a 1-Lipschitz retraction from X onto $\tilde{E}$. Finally, if $F : X \to X$ is 1-Lipschitz and $F = Id$ on E, then the restriction of F to nB_X belongs to K_n for every n and thus $F = Id$ on $\tilde{E}_n$ for every n, hence on $\tilde{E}$. This concludes the proof.

$\square$

Note that $\tilde{E}$ clearly contains $\overline{conv}(E)$. Actually, $\tilde{E} = \overline{conv}(E)$ for all bounded two-dimensional subsets E of X if and only if X is isometric to the Hilbert space (see (15.1) in [1]), in which case $\tilde{E} = \overline{conv}(E)$ for every non-empty subset of X.

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