Pure and Applied Functional Analysis
Volume 4, Number 2, 2019, 407–426

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL
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A STRONG CONVERGENCE THEOREM BY HALPERN TYPE ITERATION FOR A FINITE FAMILY OF GENERALIZED DEMIMETRIC MAPPINGS IN A HILBERT SPACE

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Dedicated to Professor Boris Mordukhovich on the occasion of his 70th birthday

Abstract. In this paper, using Halpern type iteration, we prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of generalized demimetric mappings and the set of common solutions of generalized variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using this result, we obtain well-known and new strong convergence theorems in a Hilbert space.

1. Introduction

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let η be a real number with $\eta \in (-\infty, 1)$. A mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called η-demimetric [28] if

$$2\langle x - q, J(x - Ux)\rangle \geq (1 - \eta)\|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where $F(U)$ is the set of fixed points of U and J is the dualty mapping on E. Then we have from [28] that the set $F(U)$ of fixed points of U is closed and convex. Using this property, we proved weak and strong convergence theorems in Hilbert spaces and Banach spaces; see [15, 27, 28, 29, 31]. Very recently, Kawasaki and Takahashi [11] generalized the concept of demimetric mappings as follows: Let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [11] if

$$\theta\langle x - q, J(x - Ux)\rangle \geq \|x - Ux\|^2 \tag{1.1}$$

for all $x \in C$ and $q \in F(U)$. This mapping U is called θ-generalized demimetric. We can also prove that the set $F(U)$ of fixed points of such a mapping U is closed and convex; see [11].

On the other hand, in 1967, Halpern [8] gave an iteration process as follows: Take $x_0, x_1 \in C$ arbitrarily and define $\{x_n\}$ recursively by

$$x_{n+1} = \alpha_n x_0 + (1 - \alpha_n)Tx_n, \quad \forall n \in \mathbb{N},$$

2010 *Mathematics Subject Classification.* 47H05, 47H09.

Key words and phrases. Common fixed point, demimetric mapping, variational inequality problem, metric projection, Halpern iteration.

where $\{\alpha_n\}$ is a sequence in $[0,1]$. There are many investigations of Halpern iterative process for finding fixed points of nonexpansive mappings. Takahashi [29] proved a strong convergence theorem of Halpern type iteration for demimetric mappings in a Hilbert space.

In this paper, using Halpern type iteration, we prove a strong convergence theorem for finding a common element of the set of common fixed points for a finite family of generalized demimetric mappings and the set of common solutions of generalized variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Using the result, we obtain well-known and new strong convergence theorems in a Hilbert space. In particular, we extend the result of Takahashi [29] to that of generalized demimetric mappings in a Hilbert space.

2. Preliminaries

Throughout this paper, let $\mathbb{N}$ be the set of positive integers and let $\mathbb{R}$ be the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for every $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [23] and [24].

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. When $\{x_n\}$ is a sequence in H, we denote the strong convergence of $\{x_n\}$ to $x \in H$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. We have from [25] that for any $x, y \in H$ and $\lambda \in \mathbb{R}$,

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y \rangle, \tag{2.2}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x - y\|^2. \tag{2.3}$$

Furthermore, we have that for $x, y, u, v \in H$,

$$2\langle x - y, u - v \rangle = \|x - v\|^2 + \|y - u\|^2 - \|x - u\|^2 - \|y - v\|^2. \tag{2.4}$$

Let C be a nonempty, closed and convex subset of a Hilbert space H. A mapping $T : C \to H$ is called nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A mapping $T : C \to H$ with $F(T) \neq \emptyset$ is called quasi-nonexpansive if $\|Tx - y\| \leq \|x - y\|$ for all $x \in C$ and $y \in F(T)$. If $T : C \to H$ is quasi-nonexpansive, then $F(T)$ is closed and convex; see [10]. For a nonempty, closed and convex subset D of H, the nearest point projection of H onto D is denoted by P_D, that is, $\|x - P_D x\| \leq \|x - y\|$ for all $x \in H$ and $y \in D$. Such a mapping P_D is called the metric projection of H onto D. We know that the metric projection P_D is firmly nonexpansive, i.e.,

$$\|P_D x - P_D y\|^2 \leq \langle P_D x - P_D y, x - y\rangle$$

for all $x, y \in H$. Furthermore, $\langle x - P_D x, y - P_D x\rangle \leq 0$ holds for all $x \in H$ and $y \in D$; see [23, 25]. Using this inequality and (2.4), we have that

$$\|P_D x - y\|^2 + \|P_D x - x\|^2 \leq \|x - y\|^2, \quad \forall x \in H,\ y \in D. \tag{2.5}$$

More information on the metric projection and on firmly nonexpansive mappings can be found in the book by Goebel and Reich [7]. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $A : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ax - Ay\rangle \geq \alpha \|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping A is called α-inverse strongly monotone. If $A : C \to H$ is α-inverse strongly monotone and $0 < \lambda \leq 2\alpha$, then $I - \lambda A : C \to H$ is nonexpansive. In fact, we have that for all $x, y \in C$,

$$\begin{aligned} \|(I - \lambda A)x - (I - \lambda A)y\|^2 &= \|x - y - \lambda(Ax - Ay)\|^2 \\ &= \|x - y\|^2 - 2\lambda\langle x - y, Ax - Ay\rangle + \lambda^2\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2 - 2\lambda\alpha\|Ax - Ay\|^2 + \lambda^2\|Ax - Ay\|^2 \\ &= \|x - y\|^2 + \lambda(\lambda - 2\alpha)\|Ax - Ay\|^2 \\ &\leq \|x - y\|^2. \end{aligned} \tag{2.6}$$

Thus, $I - \lambda A : C \to H$ is nonexpansive; see [1, 19, 25] for more results of inverse strongly monotone mappings. The variational inequalty problem for $A : C \to H$ is to find a point $u \in C$ such that

$$\langle Au, x - u\rangle \geq 0, \quad \forall x \in C. \tag{2.7}$$

The set of solutions of (2.7) is denoted by $VI(C, A)$. We also have that, for $\lambda > 0$, $u = P_C(I - \lambda A)u$ if and only if $u \in VI(C, A)$. In fact, let $\lambda > 0$. Then, for $u \in C$,

$$\begin{aligned} u = P_C(I - \lambda A)u &\Longleftrightarrow \langle (I - \lambda A)u - u, u - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle -\lambda Au, u - y\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Au, u - y\rangle \leq 0, \quad \forall y \in C \\ &\Longleftrightarrow \langle Au, y - u\rangle \geq 0, \quad \forall y \in C \\ &\Longleftrightarrow u \in VI(C, A). \end{aligned}$$

Let G be a mapping of H into 2^H. The effective domain of G is denoted by $D(G)$, that is, $D(G) = \{x \in H : Gx \neq \emptyset\}$. A multi-valued mapping G is said to be a monotone operator on H if $\langle x - y, u - v\rangle \geq 0$ for all $x, y \in D(G)$, $u \in Gx$, and $v \in Gy$. A monotone operator G on H is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on H. For a maximal monotone operator G on H and $r > 0$, we may define a single-valued operator $J_r = (I + rG)^{-1} \colon H \to D(G)$, which is called the resolvent of G for r. We denote by $A_r = \frac{1}{r}(I - J_r)$ the Yosida approximation of G for $r > 0$. We know from [24] that

$$A_r x \in G J_r x, \quad \forall x \in H, \ r > 0. \tag{2.8}$$

Let G be a maximal monotone operator on H and let

$$G^{-1}0 = \{x \in H : 0 \in Gx\}.$$

Then $G^{-1}0 = F(J_r)$ for all $r > 0$ and the resolvent J_r is firmly nonexpansive, i.e.,

$$\|J_r x - J_r y\|^2 \le \langle x - y, J_r x - J_r y\rangle, \quad \forall x, y \in H. \tag{2.9}$$

We also know the following lemma from [22].

Lemma 2.1 ([22]). *Let H be a Hilbert space and let G be a maximal monotone operator on H. For $r > 0$ and $x \in H$, define the resolvent $J_r x$. Then the following holds:*

$$\frac{s-t}{s}\langle J_s x - J_t x, J_s x - x\rangle \ge \|J_s x - J_t x\|^2$$

for all $s, t > 0$ and $x \in H$.

From Lemma 2.1, we have that

$$\|J_\lambda x - J_\mu x\| \le (|\lambda - \mu| / \lambda) \|x - J_\lambda x\| \tag{2.10}$$

for all $\lambda, \mu > 0$ and $x \in H$; see also [6, 23].

Using the ideas of [20, 33], Alsulami and Takahashi [2] proved the following lemma.

Lemma 2.2 ([2]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $G : H \to 2^H$ be a maximal monotone mapping and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $\kappa > 0$ and let $U : C \to H$ be a κ-inverse strongly monotone mapping. Suppose that $G^{-1}0 \cap U^{-1}0 \neq \emptyset$. Let $\lambda, r > 0$ and $z \in C$. Then the following are equivalent:*

(i) $z = J_\lambda(I - rU)z$;
(ii) $0 \in Uz + Gz$;
(iii) $z \in G^{-1}0 \cap U^{-1}0$.

Let E be a smooth Banach space, let C be a nonempty, closed and convex subset of E and let θ be a real number with $\theta \neq 0$. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called generalized demimetric [11] if it satisfies (1.1), i.e.,

$$\theta\langle x - q, J(x - Ux)\rangle \ge \|x - Ux\|^2$$

for all $x \in C$ and $q \in F(U)$, where J is the duality mapping on E.

Examples We know examples of generalized demimetric mappings.

(1) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let k be a real number with $0 \le k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [5] if

$$\|Ux - Uy\|^2 \le \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $\frac{2}{1-k}$-generalized demimetric; see [11].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [12] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1-\alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1-\beta)\|x - y\|^2 \tag{2.11}$$

for all $x, y \in C$. Such a mapping U is called $(\alpha,\ \beta)$-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is 2-generalized demimetric; see [11]. In fact, setting $x = u \in F(U)$ and $y = x \in C$ in (2.11), we have that

$$\alpha\|u - Ux\|^2 + (1-\alpha)\|u - Ux\|^2 \leq \beta\|u - x\|^2 + (1-\beta)\|u - x\|^2$$

and hence

$$\|Ux - u\|^2 \leq \|x - u\|^2.$$

From $\|Ux - u\|^2 = \|Ux - x\|^2 + \|x - u\|^2 + 2\langle Ux - x, x - u\rangle$, we have that

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2$$

for all $x \in C$ and $u \in F(U)$. This means that U is 2-generalized demimetric.

Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [13, 14] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [26] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [9].

(3) Let E be a mooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let P_C be the metric projection of E onto C. Then P_C is 1-generalized demimetric; see [11].

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Then the metric resolvent J_λ for $\lambda > 0$ is 1-generalized demimetric; see [11].

(5) Let H be a Hilbert space, let C be a nonempty subset of H and let T be a mapping from C into H. Suppose that T is Lipschitzian, that is, there exists $L > 0$ such that

$$\|Tx - Ty\| \leq L\|x - y\|$$

for all $x, y \in C$. Let $S = (L+1)I - T$. Then S is $(-2L)$-generalized demimetric; see [11, 30].

(6) Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $\alpha > 0$. If B be an α-inverse strongly monotone mapping from C into H with $B^{-1}0 \neq \emptyset$, then $T = I + B$ is $\left(-\frac{1}{\alpha}\right)$-generalized demimetric; see [11, 30].

The following lemmas are important and crucial in the proof of our main result.

Lemma 2.3 ([11]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. If a mapping $U : C \to E$ is θ-generalized demimetric and $\theta > 0$, then U is $\left(1 - \frac{2}{\theta}\right)$-demimetric.*

Lemma 2.4 ([11]). *Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping of C into E. Then $F(T)$ is closed and convex.*

Lemma 2.5 ([11]). *Let E be a smooth Banach space, let C be a nonempty subset of E and let θ be a real number with $\theta \neq 0$. Let T be a θ-generalized demimetric mapping from C into E and let $k \in \mathbb{R}$ with $k \neq 0$. Then $(1-k)I + kT$ is θk-generalized demimetric from C into E.*

We also know the following lemma from [31]:

Lemma 2.6 ([31]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $k \in (-\infty, 1)$ and let T be a k-demimetric mapping of C into H such that $F(T)$ is nonempty. Let λ be a real number with $0 < \lambda \leq 1-k$ and define $S = (1-\lambda)I + \lambda T$. Then S is a quasi-nonexpansive mapping of C into H.*

We also know the following lemmas from Aoyama, Kimura, Takahashi and Toyoda [3], Xu [35] and Maingé [16].

Lemma 2.7 ([3], [35]). *Let $\{s_n\}$ be a sequence of nonnegative real numbers, let $\{\alpha_n\}$ be a sequence in $[0,1]$ with $\sum_{n=1}^{\infty} \alpha_n = \infty$, let $\{\beta_n\}$ be a sequence of nonnegative real numbers with $\sum_{n=1}^{\infty} \beta_n < \infty$, and let $\{\gamma_n\}$ be a sequence of real numbers with $\limsup_{n\to\infty} \gamma_n \leq 0$. Suppose that*

$$s_{n+1} \leq (1-\alpha_n)s_n + \alpha_n \gamma_n + \beta_n$$

for all $n = 1, 2, \dots$. Then $\lim_{n\to\infty} s_n = 0$.

Lemma 2.8 ([16]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that does not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_i}\}$ of $\{\Gamma_n\}$ which satisfies $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. Define the sequence $\{\tau(n)\}_{n \geq n_0}$ of integers as follows:*

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\},$$

where $n_0 \in \mathbb{N}$ satisfies $\{k \leq n_0 : \Gamma_k < \Gamma_{k+1}\} \neq \emptyset$. Then, the following hold:

(i) $\tau(n_0) \leq \tau(n_0+1) \leq \cdots$ *and* $\tau(n) \to \infty$*;*

(ii) $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$ *and* $\Gamma_n \leq \Gamma_{\tau(n)+1}$, $\forall n \geq n_0$.

3. Strong Convergence Theorem

In this section, we prove a strong convergence theorem of Halpern type iteration for finding a common element of the set of common fixed points for a finite family of generalized demimetric mappings and the set of common solutions of generalized variational inequality problems for a finite family of inverse strongly monotone mappings in a Hilbert space. Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called demiclosed if, for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup w$ and $x_n - Ux_n \to 0$, $w = Uw$ holds. For example, if C is a nonempty, closed and convex subset of H and T is a nonexpansive mapping of C of H, then T is demiclosed; see [4] and [25, p. 114].

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{\theta_1, \dots, \theta_M\} \subset \mathbb{R}$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of θ_j-generalized demimetric and demiclosed mappings of C into H and let $\{k_j\}_{j=1}^M$ be a finite family of real numbers with $\theta_j k_j > 0$. Let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Let G be a maximal monotone operator on H and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1} 0) \neq \emptyset.$$

Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j ((1-\lambda_n)I + \lambda_n T_j) x_n, \\ w_n = \sum_{i=1}^N \sigma_i J_{\eta_n}(I - \eta_n B_i) x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset \mathbb{R}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions:

(1) *for any $n \in \mathbb{N}$ and $j \in \{1, \dots, M\}$,*

$$0 < a \leq \frac{\lambda_n}{k_j} \leq 2 \min\Big\{\frac{1}{\theta_1 k_1}, \dots, \frac{1}{\theta_M k_M}\Big\}, \ 0 < b \leq \eta_n \leq 2 \min\{\mu_1, \dots, \mu_N\};$$

(2) *$\sum_{j=1}^M \xi_j = 1$ and $\sum_{i=1}^N \sigma_i = 1$;*
(3) *$0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ and $\alpha_n + \beta_n + \gamma_n = 1$;*
(4) *$\lim_{n\to\infty} \delta_n = 0$ and $\sum_{i=1}^\infty \delta_n = \infty$.*

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1} 0)$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i+G)^{-1}0)} u$.

Proof. Since B_i is μ_i-inverse strongly monotone and $0 < b \leq \eta_n \leq 2\mu_i$ for all $i \in \{1, \dots, N\}$, we have that $J_{\eta_n}(I - \eta_n B_i)$ is nonexpansive from (2.6) and (2.9) and hence $F(J_{\eta_n}(I - \eta_n B_i))$ is closed and convex. Since

$$F(J_{\eta_n}(I - \eta_n B_i)) = (B_i + G)^{-1} 0$$

from Lemma 2.2, we have that $(B_i + G)^{-1}0$ is closed and convex. Furthermore, we know from Lemma 2.4 that $F(T_j)$ is closed and convex. Therefore, we have that $\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1}0)$ is nonempty, closed and convex. Thus, we obtain that $P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i+G)^{-1}0)}$ is well defined.

We know from Lemma 2.5 that $(1 - k_j)I + k_j T_j$ is $\theta_j k_j$-generalized demimetric. From Lemma 2.3 and $\theta_j k_j > 0$, we have that $(1 - k_j)I + k_j T_j$ is $\Big(1 - \frac{2}{\theta_j k_j}\Big)$-demimetric in the sense of Takahashi [28]. Since

$$0 < \frac{\lambda_n}{k_j} \leq \frac{2}{\theta_j k_j} = 1 - \Big(1 - \frac{2}{\theta_j k_j}\Big)$$

and

$$(1 - \lambda_n)I + \lambda_n T_j = \bigg(1 - \frac{\lambda_n}{k_j}\bigg) I + \frac{\lambda_n}{k_j}((1 - k_j)I + k_j T_j),$$

we have from Lemma 2.6 that $(1-\lambda_n)I + \lambda_n T_j$ is quasi-nonexpansive. Thus, we have that for $z \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N}(B_i+G)^{-1}0)$,

$$\begin{aligned} \|z_n - z\| &= \|\sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n - z\| \\ &\leq \sum_{j=1}^{M} \xi_j \|((1-\lambda_n)I + \lambda_n T_j)x_n - z\| \\ &\leq \sum_{j=1}^{M} \xi_j \|x_n - z\| = \|x_n - z\|. \end{aligned} \tag{3.1}$$

Furthermore, since $J_{\eta_n}(I - \eta_n B_i)$ is nonexpansive, we have that

$$\begin{aligned} \|w_n - z\| &= \|\sum_{i=1}^{N} \sigma_i J_{\eta_n}(I - \eta_n B_i)x_n - z\| \\ &\leq \sum_{i=1}^{N} \sigma_i \|J_{\eta_n}(I - \eta_n B_i)x_n - z\| \\ &\leq \sum_{i=1}^{N} \sigma_i \|x_n - z\| = \|x_n - z\|. \end{aligned} \tag{3.2}$$

Put $y_n = P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)$. Then we have that

$$\begin{aligned} \|y_n - z\| &\leq \|\alpha_n x_n + \beta_n z_n + \gamma_n w_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|z_n - z\| + \gamma_n \|w_n - z\| \\ &\leq \alpha_n \|x_n - z\| + \beta_n \|x_n - z\| + \gamma_n \|x_n - z\| \\ &= \|x_n - z\|. \end{aligned} \tag{3.3}$$

Using this, we get that

$$\begin{aligned} \|x_{n+1} - z\| &= \|\delta_n(u_n - z) + (1-\delta_n)(y_n - z)\| \\ &\leq \delta_n \|u_n - z\| + (1-\delta_n)\|y_n - z\| \\ &\leq \delta_n \|u_n - z\| + (1-\delta_n)\|x_n - z\|. \end{aligned}$$

Since $\{u_n\}$ is bounded, there exists $M > 0$ such that $\sup_{n\in\mathbb{N}} \|u_n - z\| \leq M$. Putting $K = \max\{\|x_1 - z\|, M\}$, we have that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. In fact, it is obvious that $\|x_1 - z\| \leq K$. Suppose that $\|x_k - z\| \leq K$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned} \|x_{k+1} - z\| &\leq \delta_k \|u_k - z\| + (1-\delta_k)\|x_k - z\| \\ &\leq \delta_k K + (1-\delta_k)K = K. \end{aligned}$$

By induction, we obtain that $\|x_n - z\| \leq K$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ is bounded. Take $z_0 = P_{\cap_{j=1}^M F(T_j)\cap(\cap_{i=1}^N (B_i+G)^{-1}0)}u$. Using [18], we have that

$$\begin{aligned}
\|y_n - z_0\|^2 &\leq \|\alpha_n x_n + \beta_n z_n + \gamma_n w_n - z_0\|^2 \\
&= \alpha_n\|x_n - z_0\|^2 + \beta_n\|z_n - z_0\|^2 + \gamma_n\|w_n - z_0\|^2 \\
&\quad - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|w_n - x_n\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2 \\
&\leq \alpha_n\|x_n - z_0\|^2 + \beta_n\|x_n - z_0\|^2 + \gamma_n\|x_n - z_0\|^2 \\
&\quad - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|w_n - x_n\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2 \\
&= \|x_n - z_0\| - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|w_n - x_n\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2
\end{aligned}$$

and hence

$$\begin{aligned}
\|x_{n+1} - z_0\|^2 &= \|\delta_n(u_n - z_0) + (1-\delta_n)(y_n - z_0)\|^2 \\
&\leq \delta_n\|u_n - z_0\|^2 + (1-\delta_n)\|y_n - z_0\|^2 \\
&\leq \delta_n\|u_n - z_0\|^2 + \|y_n - z_0\|^2 \\
&\leq \delta_n\|u_n - z_0\|^2 + \|x_n - z_0\|^2 \\
&\qquad - \alpha_n\beta_n\|z_n - x_n\|^2 - \alpha_n\gamma_n\|w_n - x_n\|^2 - \gamma_n\beta_n\|z_n - w_n\|^2.
\end{aligned}$$

Using $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$, we have that

$$\begin{aligned}
&c^2\|x_n - z_n\|^2 + c^2\|w_n - x_n\|^2 + c^2\|z_n - w_n\|^2 \\
&\quad\leq \alpha_n\beta_n\|z_n - x_n\|^2 + \alpha_n\gamma_n\|w_n - x_n\|^2 + \gamma_n\beta_n\|z_n - w_n\|^2 \\
&\quad\leq \delta_n\|u_n - z_0\|^2 + \|x_n - z_0\|^2 - \|x_{n+1} - z_0\|^2.
\end{aligned} \tag{3.4}$$

We also have that

$$\begin{aligned}
\|x_{n+1} - x_n\| &= \|\delta_n u_n + (1-\delta_n)y_n - x_n\| \\
&\leq \delta_n\|u_n - x_n\| + (1-\delta_n)\|y_n - x_n\| \\
&\leq \delta_n\|u_n - x_n\| + \|y_n - x_n\| \\
&\leq \delta_n\|u_n - x_n\| + \|\alpha_n x_n + \beta_n z_n + \gamma_n w_n - x_n\| \\
&\leq \delta_n\|u_n - x_n\| + \|\beta_n(z_n - x_n)\| + \|\gamma_n(w_n - x_n)\| \\
&\leq \delta_n\|u_n - x_n\| + \|z_n - x_n\| + \|w_n - x_n\|.
\end{aligned} \tag{3.5}$$

We will divide the proof into two cases.

Case 1: Put $\Gamma_n = \|x_n - z_0\|^2$ for all $n \in \mathbb{N}$. Suppose that there exists a natural number N such that $\Gamma_{n+1} \leq \Gamma_n$ for all $n \geq N$. In this case, $\lim_{n\to\infty}\Gamma_n$ exists and then $\lim_{n\to\infty}(\Gamma_{n+1} - \Gamma_n) = 0$. Using $\delta_n \to 0$, we have from (3.4) that

$$\lim_{n\to\infty}\|z_n - x_n\| = 0,\ \lim_{n\to\infty}\|w_n - x_n\| = 0,\ \lim_{n\to\infty}\|z_n - w_n\| = 0. \tag{3.6}$$

From (3.5), we also have that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0. \tag{3.7}$$

For $z_0 = P_{\cap_{j=1}^M F(T_j)\cap(\cap_{i=1}^N (B_i+G)^{-1}0)}u$, we show that

$$\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle \le 0.$$

Put $s = \limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle$. Without loss of generality, there exists a subsequence $\{x_l\}$ of $\{x_n\}$ such that

$$s = \lim_{l\to\infty}\langle u - z_0, x_l - z_0\rangle$$

and $\{x_l\}$ converges weakly to some point w. On the other hand, since T_j is θ_j-generalized demimetric and hence $(1-k_j)I + k_jT_j$ is θ_jk_j-generalized demimetric for all $j \in \{1,\dots,M\}$, we have from $\theta_jk_j > 0$ that, for $z \in \cap_{j=1}^M F(T_j)$,

$$\begin{aligned}
\langle x_n - z, x_n - z_n\rangle &= \Big\langle x_n - z, x_n - \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n\Big\rangle \\
&= \sum_{j=1}^M \xi_j\langle x_n - z, x_n - ((1-\lambda_n)I + \lambda_n T_j)x_n\rangle \\
&= \sum_{j=1}^M \xi_j\Big\langle x_n - z, x_n - \Big(\Big(1-\frac{\lambda_n}{k_j}\Big)I + \frac{\lambda_n}{k_j}\Big((1-k_j)I + k_jT_j\Big)\Big)x_n\Big\rangle \\
&= \sum_{j=1}^M \xi_j\Big\langle x_n - z, \frac{\lambda_n}{k_j}x_n - \frac{\lambda_n}{k_j}\Big((1-k_j)I + k_jT_j\Big)x_n\Big\rangle \\
&= \sum_{j=1}^M \xi_j\frac{\lambda_n}{k_j}\langle x_n - z, x_n - ((1-k_j)I + k_jT_j)x_n\rangle \\
&\ge \sum_{j=1}^M \xi_j\frac{\lambda_n}{k_j}\frac{1}{\theta_jk_j}\|x_n - ((1-k_j)I + k_jT_j))x_n\|^2 \\
&= \sum_{j=1}^M \xi_j\frac{\lambda_n}{k_j}\frac{1}{\theta_jk_j}k_j^2\|x_n - T_jx_n\|^2 \\
&= \sum_{j=1}^M \xi_j\frac{\lambda_n}{\theta_j}\|x_n - T_jx_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and $\frac{\xi_j\lambda_n}{\theta_j} > 0$ that

$$\lim_{n\to\infty}\|x_n - T_jx_n\| = 0, \quad \forall j \in \{1,\dots,M\}.$$

Since T_j is demiclosed for all $j \in \{1,\dots,M\}$, we have $w \in \cap_{j=1}^M F(T_j)$.

Let us show that $w \in \cap_{i=1}^N (B_i + G)^{-1}0$. Since $J_{\eta_n}(I - \eta_n B_i)$ is nonexpansive for all $i \in \{1, \ldots, N\}$, we have from (2.4) that, for $z \in \cap_{1=1}^N (B_i + G)^{-1}0$,

$$\begin{aligned}
2\langle x_n - z, x_n - w_n\rangle &= 2\Big\langle x_n - z, x_n - \sum_{i=1}^N \sigma_i J_{\eta_n}(I - \eta_n B_i)x_n\Big\rangle \\
&= \sum_{i=1}^N 2\sigma_i \langle x_n - z, x_n - J_{\eta_n}(I - \eta_n B_i)x_n\rangle \\
&= \sum_{i=1}^N \sigma_i (\|x_n - J_{\eta_n}(I - \eta_n B_i)x_n\|^2 \\
&\qquad + \|x_n - z\|^2 - \|J_{\eta_n}(I - \eta_n B_i)x_n - z\|^2) \\
&\geq \sum_{i=1}^N \sigma_i \|x_n - J_{\eta_n}(I - \eta_n B_i)x_n\|^2.
\end{aligned}$$

We have from $\lim_{n\to\infty} \|w_n - x_n\| = 0$ that

$$\lim_{n\to\infty} \|x_n - J_{\eta_n}(I - \eta_n B_i)x_n\| = 0, \quad \forall i \in \{1, \ldots, N\}.$$

Consider a subsequence $\{\eta_l\}$ of $\{\eta_n\}$ corresponding to the sequence $\{x_l\}$. Since the subsequence $\{\eta_l\}$ of $\{\eta_n\}$ is bounded, there exists a subsequence $\{\eta_h\}$ of $\{\eta_l\}$ such that $\lim_{h\to\infty} \eta_h = \eta$ and $0 < b \leq \eta \leq 2\min\{\mu_1, \ldots, \mu_N\}$. For such η, we have from (2.10) that for any $i \in \{1, \ldots, N\}$,

$$\begin{aligned}
\|x_h - J_\eta(I - \eta B_i)x_h\| &\leq \|x_h - J_{\eta_h}(I - \eta_h B_i)x_h\| \\
&\quad + \|J_{\eta_h}(I - \eta_h B_i)x_h - J_{\eta_h}(I - \eta B_i)x_h\| \\
&\quad + \|J_{\eta_h}(I - \eta B_i)x_h - J_\eta(I - \eta B_i)x_h\| \\
&\leq \|x_h - J_{\eta_h}(I - \eta_h B_i)x_h\| \\
&\quad + \|(I - \eta_h B_i)x_h - (I - \eta B_i)x_h\| \\
&\quad + \|J_{\eta_h}(I - \eta B_i)x_h - J_\eta(I - \eta B_i)x_h\| \\
&\leq \|x_h - J_{\eta_h}(I - \eta_h B_i)x_h\| + |\eta_h - \eta|\|B_i x_h\| \\
&\quad + \frac{|\eta_h - \eta|}{\eta}\|J_\eta(I - \eta B_i)x_h - (I - \eta B_i)x_h\|.
\end{aligned}$$

On the other hand, we have that for $y \in C$ and $i \in \{1, \ldots, N\}$,

$$\begin{aligned}
b\|B_i x_n\| &\leq \eta_n \|B_i x_n\| = \|\eta_n B_i x_n\| \\
&= \|x_n - (y - \eta_n B_i y) + y - \eta_n B_i y - (x_n - \eta_n B_i x_n)\| \\
&\leq \|x_n - y\| + \eta_n \|B_i y\| + \|(I - \eta_n B_i)y - (I - \eta_n B_i)x_n\| \\
&\leq \|x_n - y\| + \max\{\mu_1, \ldots, \mu_N\}\|B_i y\| + \|y - x_n\|.
\end{aligned}$$

Since $\{x_n\}$ is bounded, we have that $\{B_i x_n\}$ is bounded for all $i \in \{1, \ldots, N\}$. Thus we have that

$$\lim_{h\to\infty} \|x_h - J_\eta(I - \eta B_i)x_h\| = 0, \quad \forall i \in \{1, \ldots, N\}.$$

Since $\{x_h\}$ converges weakly to w and $J_\eta(I-\eta B_i)$ is demiclosed for all $i \in \{1,\dots,N\}$, we have $w \in F(J_\eta(I-\eta B_i))$. From Lemma 2.2, we have $w \in \cap_{i=1}^N (B_i+G)^{-1}0$. Therefore, we have

$$w \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i+G)^{-1}0).$$

Since $\{x_l\}$ converges weakly to $w \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i+G)^{-1}0)$, we have that

$$s = \lim_{l\to\infty} \langle u - z_0, x_l - z_0\rangle = \langle u - z_0, w - z_0\rangle \leq 0.$$

Since $x_{n+1} - z_0 = \delta_n(u_n - z_0) + (1-\delta_n)(y_n - z_0)$, we have from (2.2) that

$$\begin{aligned} \|x_{n+1}-z_0\|^2 &\leq (1-\delta_n)^2\|y_n - z_0\|^2 + 2\delta_n\langle u_n - z_0, x_{n+1} - z_0\rangle \\ &\leq (1-\delta_n)\|x_n - z_0\|^2 + 2\delta_n\langle u_n - u, x_{n+1} - z_0\rangle \\ &\quad + 2\delta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &= (1-\delta_n)\|x_n - z_0\|^2 + 2\delta_n\langle u_n - u, x_{n+1} - z_0\rangle \\ &\quad + 2\delta_n\langle u - z_0, x_{n+1} - x_n\rangle + 2\delta_n\langle u - z_0, x_n - z_0\rangle. \end{aligned}$$

Since $\sum_{n=1}^\infty \delta_n = \infty$, we obtain from Lemma 2.7 that $x_n \to z_0$.

Case 2: Suppose that there exists a subsequence $\{\Gamma_{n_i}\}$ of the sequence $\{\Gamma_n\}$ such that $\Gamma_{n_i} < \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we define $\tau : \mathbb{N} \to \mathbb{N}$ by

$$\tau(n) = \max\{k \leq n : \Gamma_k < \Gamma_{k+1}\}.$$

Then we have from Lemma 2.8 that $\Gamma_{\tau(n)} \leq \Gamma_{\tau(n)+1}$. Thus we have from (3.4) that for all $n \in \mathbb{N}$,

$$\begin{aligned} &c^2\|x_{\tau(n)}-z_{\tau(n)}\|^2 + c^2\|w_{\tau(n)} - x_{\tau(n)}\|^2 + c^2\|z_{\tau(n)} - w_{\tau(n)}\|^2 \\ &\quad\leq \delta_{\tau(n)}\|u_{\tau(n)} - z_0\|^2 + \left\|x_{\tau(n)} - z_0\right\|^2 - \|x_{\tau(n)+1} - z_0\|^2 \\ &\quad\leq \delta_{\tau(n)}\|u_{\tau(n)} - z_0\|^2. \end{aligned} \tag{3.8}$$

Using $\alpha_{\tau(n)} \to 0$, we have from (3.8) that

$$\lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0,\ \lim_{n\to\infty}\|w_{\tau(n)} - x_{\tau(n)}\| = 0,\ \lim_{n\to\infty}\|z_{\tau(n)} - w_{\tau(n)}\| = 0.$$

As in the proof of Case 1, we have from $\lim_{n\to\infty}\|z_{\tau(n)} - x_{\tau(n)}\| = 0$ that

$$\lim_{n\to\infty}\|x_{\tau(n)} - T_j x_{\tau(n)}\| = 0, \quad \forall j \in \{1,\dots,M\}. \tag{3.9}$$

As in the proof of Case 1, we also have that

$$\lim_{n\to\infty}\|x_{\tau(n)+1} - x_{\tau(n)}\| = 0. \tag{3.10}$$

For $z_0 = P_{\cap_{j=1}^M F(T_j)\cap(\cap_{i=1}^N (B_i+G)^{-1}0)}u$, let us show that

$$\limsup_{n\to\infty}\langle z_0 - u, x_{\tau(n)} - z_0\rangle \geq 0.$$

Put $s = \limsup_{n\to\infty}\langle z_0 - u, x_{\tau(n)} - z_0\rangle$. Without loss of generality, there exists a subsequence $\{x_{\tau(l)}\}$ of $\{x_{\tau(n)}\}$ such that $s = \lim_{l\to\infty}\langle z_0 - u, x_{\tau(l)} - z_0\rangle$ and $\{x_{\tau(l)}\}$ converges weakly to some point $w \in C$. Since T_j is demiclosed for all $j \in \{1,\dots,M\}$, we have from (3.9) that $w \in \cap_{j=1}^{M} F(T_j)$. Let us show that $w \in \cap_{i=1}^{N}(B_i + G)^{-1}0$. As in the proof of Case 1, we have from $\lim_{n\to\infty} \|w_{\tau(n)} - x_{\tau(n)}\| = 0$ that

$$\lim_{n\to\infty} \|x_{\tau(n)} - J_{\eta_{\tau(n)}}(I - \eta_{\tau(n)} B_i)x_{\tau(n)}\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Consider a subsequence $\{\eta_{\tau(l)}\}$ of $\{\eta_{\tau(n)}\}$ corresponding to the sequence $\{x_{\tau(l)}\}$. Since the subsequence $\{\eta_{\tau(l)}\}$ of $\{\eta_{\tau(n)}\}$ is bounded, we have that there exists a subsequence $\{\eta_{\tau(h)}\}$ of $\{\eta_{\tau(l)}\}$ such that $\lim_{h\to\infty} \eta_{\tau(h)} = \eta$ and $0 < b \le \eta \le 2\min\{\mu_1, \dots, \mu_N\}$. As in the proof of Case 1, we have that for any $i \in \{1, \dots, N\}$,

$$\begin{aligned}
&\|x_{\tau(h)} - J_\eta(I - \eta B_i)x_{\tau(h)}\| \\
&\qquad \le \|x_{\tau(h)} - J_{\eta_{\tau(h)}}(I - \eta_{\tau(h)} B_i)x_{\tau(h)}\| \\
&\qquad\quad + \|J_{\eta_{\tau(h)}}(I - \eta_{\tau(h)} B_i)x_{\tau(h)} - J_{\eta_{\tau(h)}}(I - \eta B_i)x_{\tau(h)}\| \\
&\qquad\quad + \|J_{\eta_{\tau(h)}}(I - \eta B_i)x_{\tau(h)} - J_\eta(I - \eta B_i)x_{\tau(h)}\| \\
&\qquad \le \|x_{\tau(h)} - J_{\eta_{\tau(h)}}(I - \eta_{\tau(h)} B_i)x_{\tau(h)}\| \\
&\qquad\quad + \|(I - \eta_{\tau(h)} B_i)x_{\tau(h)} - (I - \eta B_i)x_{\tau(h)}\| \\
&\qquad\quad + \|J_{\eta_{\tau(h)}}(I - \eta B_i)x_{\tau(h)} - J_\eta(I - \eta B_i)x_{\tau(h)}\| \\
&\qquad \le \|x_{\tau(h)} - J_{\eta_{\tau(h)}}(I - \eta_{\tau(h)} B_i)x_{\tau(h)}\| + |\eta_{\tau(h)} - \eta|\|B_i x_{\tau(h)}\| \\
&\qquad\quad + \frac{|\eta_{\tau(h)} - \eta|}{\eta}\|J_\eta(I - \eta B_i)x_{\tau(h)} - (I - \eta B_i)x_{\tau(h)}\|.
\end{aligned}$$

Thus we have that

$$\lim_{h\to\infty} \|x_{\tau(h)} - J_\eta(I - \eta B_i)x_{\tau(h)}\| = 0, \quad \forall i \in \{1, \dots, N\}.$$

Since $\{x_{\tau(h)}\}$ converges weakly to w and $J_\eta(I - \eta B_i)$ are demiclosed, we have $w \in \cap_{i=1}^{N}(B_i + G)^{-1}0$. Therefore, we have

$$w \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N}(B_i + G)^{-1}0).$$

Then we have

$$s = \lim_{l\to\infty} \langle z_0 - u, x_{\tau(l)} - z_0\rangle = \langle z_0 - u, w - z_0\rangle \ge 0.$$

As in the proof of Case 1, we also have that

$$\begin{aligned}
\|x_{\tau(n)+1} - z_0\|^2 &\le (1 - \delta_{\tau(n)})\|x_{\tau(n)} - z_0\|^2 + 2\delta_{\tau(n)}\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\
&\quad + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)} - z_0\rangle.
\end{aligned}$$

From $\Gamma_{\tau(n)} \le \Gamma_{\tau(n)+1}$, we have that

$$\begin{aligned}
\delta_{\tau(n)}\|x_{\tau(n)} - z_0\|^2 &\le 2\delta_{\tau(n)}\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\
&\quad + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\delta_{\tau(n)}\langle u - z_0, x_{\tau(n)} - z_0\rangle.
\end{aligned}$$

Since $\delta_{\tau(n)} > 0$, we have that

$$\begin{aligned}
\|x_{\tau(n)} - z_0\|^2 &\le 2\langle u_{\tau(n)} - u, x_{\tau(n)+1} - z_0\rangle \\
&\quad + 2\langle u - z_0, x_{\tau(n)+1} - x_{\tau(n)}\rangle + 2\langle u - z_0, x_{\tau(n)} - z_0\rangle.
\end{aligned}$$

Thus we have that

$$\limsup_{n\to\infty} \left\|x_{\tau(n)} - z_0\right\|^2 \le 0$$

and hence $\|x_{\tau(n)} - z_0\| \to 0$. From (3.10), we have also that $x_{\tau(n)} - x_{\tau(n)+1} \to 0$. Thus $\|x_{\tau(n)+1} - z_0\| \to 0$ as $n \to \infty$. Using Lemma 2.8 again, we obtain that

$$\|x_n - z_0\| \leq \|x_{\tau(n)+1} - z_0\| \to 0$$

as $n \to \infty$. This completes the proof. □

4. Applications

In this section, we apply Theorem 3.1 to obtain well-known and new strong convergence theorems in Hilbert spaces. Let H be a Hilbert space and let f be a proper, lower semicontinuous and convex function of H into $(-\infty, \infty]$. The subdifferential ∂f of f is defined as follows:

$$\partial f(x) = \{z \in H : f(x) + \langle z, y - x\rangle \leq f(y),\ \forall y \in H\}$$

for all $x \in H$. From Rockafellar [21], we know that ∂f is a maximal monotone operator. Let C be a nonempty, closed and convex subset of H and let i_C be the indicator function of C, i.e.,

$$i_C(x) = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases}$$

Then $i_C : H \to (-\infty, \infty]$ is a proper, lower semicontinuous and convex function on H and then the subdifferential ∂i_C of i_C is a maximal monotone operator. Thus we can define the resolvent J_λ of ∂i_C for $\lambda > 0$, i.e.,

$$J_\lambda x = (I + \lambda \partial i_C)^{-1} x$$

for all $x \in H$. We have that, for any $x \in H$ and $u \in C$,

$$\begin{aligned} u = J_\lambda x &\iff x \in u + \lambda \partial i_C u \iff x \in u + \lambda N_C u \\ &\iff x - u \in \lambda N_C u \\ &\iff \frac{1}{\lambda}\langle x - u, v - u\rangle \leq 0,\ \forall v \in C \\ &\iff \langle x - u, v - u\rangle \leq 0,\ \forall v \in C \\ &\iff u = P_C x, \end{aligned} \tag{4.1}$$

where $N_C u$ is the normal cone to C at u, i.e.,

$$N_C u = \{z \in H : \langle z, v - u\rangle \leq 0,\ \forall v \in C\}.$$

We know the following lemmas obtained by Marino and Xu [17] and Kocourek, Takahashi and Yao [12]; see also [32, 34].

Lemma 4.1 ([17, 32]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Lemma 4.2 ([12, 34]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We first prove a strong convergence theorem for a finite family of strict pseudo-contractions and a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.3. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{s_1, \dots, s_M\} \subset [0, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of s_j-strict pseudo-contractions of C into H. Let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Let G be a maximal monotone operator on H and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1}0) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i J_{\eta_n}(I - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset \mathbb{R}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy the following conditions:

(1) *for any $n \in \mathbb{N}$,*

$$0 < a \leq \lambda_n \leq \min\{1 - s_1, \dots, 1 - s_M\},\ 0 < b \leq \eta_n \leq 2\min\{\mu_1, \dots, \mu_N\};$$

(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;
(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^\infty \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1}0)$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1}0)} u$.

Proof. Since T_j is a s_j-strict pseudo-contraction of C into H such that $F(T_j) \neq \emptyset$, from (1) in Examples, T_j is $\frac{2}{1-s_j}$-generalized demimetric. Take $k_j = 1$ in Theorem 3.1. Then we get that $\frac{2}{\theta_j k_j} = 1 - s_j$ in Theorem 3.1. Furthermore, from Lemma 4.1, T_j is demiclosed. Thus, we have the desired result from Theorem 3.1. □

We prove a strong convergence theorem for a finite family of generalized hybrid mappings and a finite family of inverse strongly monotone mappings in a Hilbert space.

Theorem 4.4. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of generalized hybrid mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Let G be a maximal monotone operator on H and let $J_\lambda = (I + \lambda G)^{-1}$ be the resolvent of G for $\lambda > 0$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that*

$$\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N (B_i + G)^{-1}0) \neq \emptyset.$$

For any $x_1 = x \in C$, define $\{x_n\}$ as follows:

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\} \subset \mathbb{R}, \{\eta_n\} \subset (0,\infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ satisfy the following conditions:

(1) *for any $n \in \mathbb{N}$,*

$$0 < a \le \lambda_n \le 1, \; 0 < b \le \eta_n \le 2\min\{\mu_1, \dots, \mu_N\};$$

(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} (B_i + G)^{-1}0)$, where $z_0 = P_{\cap_{j=1}^{M} F(T_j) \cap (\cap_{i=1}^{N} (B_i+G)^{-1}0)} u$.

Proof. Since T_j is a generalized hybrid mapping of C into H such that $F(T_j) \neq \emptyset$, from (2) in Examples, T_j is 2-generalized demimetric. Take $k_j = 1$ in Theorem 3.1. Then we get that $\frac{2}{2} = 1$ in Theorem 3.1. Furthermore, from Lemma 4.2, T_j is demiclosed. Therefore, we have the desired result from Theorem 3.1. □

We prove a strong convergence theorem for a finite family of Lipschitzian mappings and a finite family of nonexpansive mappings in a Hilbert space.

Theorem 4.5. *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{L_1, \dots, L_M\} \subset (0,\infty)$ and let $\{S_j\}_{j=1}^{M}$ be a finite family of L_j-Lipschitzian mappings of C into H and let $\{U_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into H. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. Assume that $\cap_{j=1}^{M} F(\frac{S_j}{L_j}) \cap (\cap_{i=1}^{N} F(U_i)) \neq \emptyset$. For any $x_1 = x \in C$, define $\{x_n\}$ as follows:*

$$\begin{cases} z_n = \sum_{j=1}^{M} \xi_j((1+\lambda_n L_j)I - \lambda_n S_j)x_n, \\ w_n = \sum_{i=1}^{N} \sigma_i((1-\eta_n)I + \eta_n U_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \end{cases}$$

where $\{\lambda_n\}, \{\eta_n\} \subset \mathbb{R}$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0,1)$, $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0,1)$ and $a, b, c \in \mathbb{R}$ satisfy the following conditions:

(1) $0 < a \le \frac{\lambda_n}{-1} \le \min\left\{\frac{1}{L_1}, \dots, \frac{1}{L_M}\right\}$, $0 < b \le \eta_n \le 1$ *for all* $n \in \mathbb{N}$;
(2) $\sum_{j=1}^{M} \xi_j = 1$ *and* $\sum_{i=1}^{N} \sigma_i = 1$;
(3) $0 < c \le \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;
(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^{\infty} \delta_n = \infty$.

Then the sequence $\{x_n\}$ converges strongly to $z_0 \in \cap_{j=1}^{M} F(\frac{S_j}{L_j}) \cap (\cap_{i=1}^{N} F(U_i))$, where $z_0 = P_{\cap_{j=1}^{M} F(\frac{S_j}{L_j}) \cap (\cap_{i=1}^{N} F(U_i))} u$.

Proof. Since S_j is L_j-Lipschitzian and $F(\frac{S_j}{L_j}) \neq \emptyset$, $T_j = (L_j + 1)I - S_j$ is $-2L_j$-generalized demimetric. Take $k_j = -1$ in Theorem 3.1. Then we have that $\theta_j k_j = 2L_j$ and

$$(1-\lambda_n)I + \lambda_n T_j = (1-\lambda_n + \lambda_n L_j + \lambda_n)I - \lambda_n S_j = (1+\lambda_n L_j)I - \lambda_n S_j.$$

Furthermore, from Lemma 4.1, T_j is demiclosed. In fact, if $x_n \rightharpoonup z$ and $x_n - T_j x_n \to 0$, then

$$\frac{1}{L_j}(x_n - T_j x_n) = \frac{1}{L_j}(S_j x_n - L_j x_n) = \frac{S_j}{L_j}x_n - x_n \to 0.$$

Since $\frac{S_j}{L_j}$ is nonexpansive and hence demiclosed, we have that $z \in F(\frac{S_j}{L_j}) = F(T_j)$. Since U_i is nonexpansive, $B_i = I - U_i$ is a $\frac{1}{2}$-inverse strongly monotone mapping. Putting $G = 0$ in Theorem 3.1, we have that $J_{\eta_n} = I$ and

$$\cap_{i=1}^N (B_i + G)^{-1}0 = \cap_{i=1}^N (B_i)^{-1}0 = \cap_{i=1}^N F(U_i).$$

Furthermore, we have that

$$I - \eta_n B_i = I - \eta_n(I - U_i) = (1-\eta_n)I + \eta_n U_i.$$

Therefore, we have the desired result from Theorem 3.1. □

Finally, using Theorem 3.1, we obtain the following theorem by Takahashi [29].

Theorem 4.6 ([29]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let $\{t_1, \dots, t_M\} \subset (-\infty, 1)$ and $\{\mu_1, \dots, \mu_N\} \subset (0, \infty)$. Let $\{T_j\}_{j=1}^M$ be a finite family of t_j-demimetric and demiclosed mappings of C into H and let $\{B_i\}_{i=1}^N$ be a finite family of μ_i-inverse strongly monotone mappings of C into H. Assume that $\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i)) \neq \emptyset$. Let $\{u_n\}$ be a sequence in C such that $u_n \to u$. For $x_1 = x \in C$, let $\{x_n\} \subset C$ be a sequence generated by*

$$\begin{cases} z_n = \sum_{j=1}^M \xi_j((1-\lambda_n)I + \lambda_n T_j)x_n, \\ w_n = \sum_{i=1}^N \sigma_i P_C(I - \eta_n B_i)x_n, \\ x_{n+1} = \delta_n u_n + (1-\delta_n)\big(P_C(\alpha_n x_n + \beta_n z_n + \gamma_n w_n)\big), \quad \forall n \in \mathbb{N}, \end{cases}$$

where $a, b, c \in \mathbb{R}$, $\{\lambda_n\}, \{\eta_n\} \subset (0, \infty)$, $\{\xi_1, \dots, \xi_M\}, \{\sigma_1, \dots, \sigma_N\} \subset (0, 1)$ and $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\delta_n\} \subset (0, 1)$ satisfy the following conditions:

(1) *for any $n \in \mathbb{N}$,*

$$0 < a \leq \lambda_n \leq \min\{1-t_1, \dots, 1-t_M\}, \ 0 < b \leq \eta_n \leq 2\min\{\mu_1, \dots, \mu_N\};$$

(2) $\sum_{j=1}^M \xi_j = 1$ *and* $\sum_{i=1}^N \sigma_i = 1$;

(3) $0 < c \leq \alpha_n, \beta_n, \gamma_n < 1$ *and* $\alpha_n + \beta_n + \gamma_n = 1$ *for all* $n \in \mathbb{N}$;

(4) $\lim_{n\to\infty} \delta_n = 0$ *and* $\sum_{i=1}^\infty \delta_n = \infty$.

Then $\{x_n\}$ converges strongly to a point $z_0 \in \cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C, B_i))$, where $z_0 = P_{\cap_{j=1}^M F(T_j) \cap (\cap_{i=1}^N VI(C,B_i))} u$.

Proof. Since T_j is a t_j-demimetric mapping of C into H such that $F(T_j) \neq \emptyset$, T_j is $\frac{2}{1-t_j}$-generalized demimetric. Take $k_j = 1$ in Theorem 3.1. Then we get that $\frac{2}{\theta k_j} = 1 - t_j$ in Theorem 3.1. Put $G = \partial i_C$ in Theorem 3.1. Then we have from (4.1) that for $\eta_n > 0$, $J_{\eta_n} = P_C$. Furthermore, we have $(\partial i_C)^{-1}0 = C$ and $(B_i + \partial i_C)^{-1}0 = VI(C, B_i)$. In fact, we have that, for any $z \in C$,

$$\begin{aligned} z \in (B_i + \partial i_C)^{-1}0 &\iff 0 \in B_i z + \partial i_C z \\ &\iff 0 \in B_i z + N_C z \\ &\iff -B_i z \in N_C z \\ &\iff \langle -B_i z, v - z\rangle \le 0, \ \forall v \in C \\ &\iff \langle B_i z, v - z\rangle \ge 0, \ \forall v \in C \\ &\iff z \in VI(C, B_i). \end{aligned}$$

Therefore, we have the desired result from Theorem 3.1. □

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Manuscript received January 14 2018
revised May 19 2018

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