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CONSTRAINED DIFFERENTIAL INCLUSIONS WITH NONLOCAL INITIAL CONDITIONS

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ABSTRACT. We show existence for the perturbed sweeping process with nonlocal initial conditions under very general hypotheses. Periodic, anti-periodic, mean value and multipoints conditions are included in this study. We give abstract results for differential inclusions with nonlocal initial conditions through bounding functions and tangential conditions. Some applications to differential complementarity systems and to vector hysteresis are given.

1. INTRODUCTION

The sweeping process is a first-order differential inclusion, involving the normal cone to a moving set depending on time. Roughly speaking, a point is swept by a moving closed set. This differential inclusion was introduced and deeply studied by Moreau in a series of papers (see [32–35]) to model an elastoplastic mechanical system. Since then, many other applications of the sweeping process have been given, e.g., in electrical circuits [1], crowd motion [29], hysteresis in elastoplastic models [26], among others. While the study of the sweeping process with Cauchy initial conditions is well known (see, for instance, [9, 22, 24, 27, 32, 33, 35, 38, 39]), the sweeping process with nonlocal initial condition has received relatively little attention. In the context of periodic sweeping processes, we can mention the works of Castaing and Monteiro-Marques [11, 12] for convex sets in Hilbert spaces and Gavioli [17] for wedges sets in finite dimensions.

The first part of the paper is devoted to establishing some sufficient conditions for the existence of perturbed sweeping processes with nonregular sets and nonlocal initial conditions, that is, we consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{1.1}$$

where H is a separable Hilbert space, $C\colon [T_0, T] \rightrightarrows H$ is a set-valued map with nonempty and closed values, $N\left(S; \cdot\right)$ denotes the Clarke normal cone to S and $F\colon [T_0, T] \times H \rightrightarrows H$ is a given set-valued map with nonempty, closed and convex

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values. Here $M\colon C([T_0,T];H)\to H$ is an operator (possibly nonlinear) satisfying

$$\|Mx-My\|\leq m\|x-y\|_\infty \quad \text{for all } x,y\in C([T_0,T];H), \tag{1.2}$$

with $m\in[0,1]$. The class of operators M satisfying the condition (1.2) is sufficiently large and includes the following well-known nonlocal initial conditions:

(i) $Mx=x_0$ (general Cauchy initial condition $x(T_0)=x_0$);
(ii) $Mx=\pm x(T)$ (periodic and anti-periodic initial conditions);
(iii) $Mx=\frac{1}{T-T_0}\int_{T_0}^{T}x(s)ds$ (mean value initial condition);
(iv) $Mx=\sum_{i=1}^{k_0}\alpha_i x(t_i)$ with $\alpha_i\in\mathbb{R}$ and $\sum_{i=1}^{k_0}|\alpha_i|\leq 1$, where $T_0<t_1<\cdots<t_{k_0}\leq T$ (multi-point initial condition).

Our study is achieved through the Galerkin-Like method, introduced by Jourani and Vilches in [23]. This method to solve differential inclusions, consists in approaching the original problem by projecting the state into a n-dimensional Hilbert space but not the velocity. The approached problems always have a solution and, under some compactness conditions, they converge strongly pointwisely (up to a subsequence) to a solution of the original differential inclusion. We combine the Galerkin-Like method with the reduction technique for the sweeping process (see, e.g., [18,38]). The reduction technique associates to every sweeping process an unconstrained differential inclusion, whose solutions are also solutions of the sweeping process. In order to apply this method, the moving sets must to be positively α-far (see definition below). The class of positively α-far sets is very general and includes convex sets, uniformly prox-regular sets, subsmooth sets, compact wedged sets, etc.

In Sections 4 and 5, we present the main results of the first part of the paper, namely, the existence for perturbed sweeping process with nonlocal initial conditions. As a consequence, we obtain the existence of periodic solutions for the perturbed sweeping process, which extends the results from [11,12,17]. We believe that these results can be used for further developments in the theory of periodic perturbations and stability for the sweeping process (see [25]).

The second part of the paper is concerned with existence of abstract differential inclusions with nonlocal initial conditions. To deal with it, we use the concept of bounding functions and some tangential conditions. We say that V is a (weak/strong) bounding function for a differential inclusion (see Definition 6.1), when the existence of this function implies the existence of an a priori bound for the solutions of the differential inclusion. Typically, the bounding function has to satisfy some conditions involving the derivatives of V (in some sense) and the right-hand side of the differential inclusion. The idea of bounding functions was introduced by Mawhin [30] to deal with second order boundary value problems. Since then, it was systematically used for the study of various boundary problems (see [7,36] and the references therein). In [30], Mawhin imposes a specific condition relative to the second order derivatives of V, which implies the boundedness of the solution for the second order boundary value problem. For the case of first order differential inclusions, the concept of bounding function involves conditions on the first order derivatives of V and the right-hand side of the differential inclusion, in some ring or localized in the boundary of some bounded set. Thus, the concept of bounding function is vague and highly depending on the method to deal with the differential inclusion. Our definition of weak bounding function (see Definition 6.1) is taken from [7]. The use of bounding functions, generally, is related to the Leray-Schauder continuation principle and the topological degree theory (see [7] for more details). We emphasize that our approach make no appeal to these tools from nonlinear analysis but merely basic elements of set-valued and variational analysis.

In Section 6 we use bounding functions to study a first order differential inclusion with nonlocal initial conditions when H is compactly embedded in a separable Banach space E.

In Section 7, we use some tangential conditions to get the existence of abstract nonlocal differential inclusion in finite dimensions. These tangential conditions, related with the weak invariance of differential inclusions, typically, involves the intersection between the Bouligand tangent cone and the right-hand side of the differential inclusion. Since we apply a fixed point theorem to the solution map of the differential inclusion, a strong property is needed, namely, the intersection between the Clarke tangent cone and the right-hand side of the differential inclusion is nonempty (see Remark 7.2).

Finally, in Sections 8 and 9, we give, respectively, some applications to nonlocal differential complementarity systems and to vector hysteresis.

2. Preliminaries

From now on H stands for a separable Hilbert space whose norm is denoted by $\|\cdot\|$. The closed unit ball is denoted by $\mathbb{B}$. The notation H_w stands for H equipped with the weak topology and $x_n \rightharpoonup x$ denotes the weak convergence of a sequence $(x_n)_n$ to x. The support function of a set $S \subseteq H$ is defined, for any $v \in H$, by $\sigma(v, S) := \sup_{s\in S} \langle v, s\rangle$.

Recall that a vector $h \in H$ belongs to the *Clarke tangent cone* $T^C(S; x)$ (see [13]); when for every sequence $(x_n)_n$ in S converging to x and every sequence of positive numbers $(t_n)_n$ converging to 0, there exists some sequence $(h_n)_n$ in H converging to h such that $x_n + t_n h_n \in S$ for all $n \in \mathbb{N}$. This cone is closed and convex, and its negative polar $N(S; x)$ is the *Clarke normal cone* to S at $x \in S$, that is,

$$N(S; x) := \left\{v \in H\colon\ \langle v, h\rangle \le 0 \quad \forall h \in T^C(S; x)\right\}.$$

As usual, $N(S; x) := \emptyset$ if $x \notin S$. Through that normal cone, the *Clarke subdifferential* of a function $f\colon H \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\partial f(x) := \{v \in H\colon (v, -1) \in N(\operatorname{epi} f, (x, f(x)))\},$$

where $\operatorname{epi} f := \{(y, r) \in H \times \mathbb{R}\colon f(y) \le r\}$ is the epigraph of f. When the function f is finite and locally Lipschitzian around x, the Clarke subdifferential is characterized (see [14]) in the following simple and amenable way

$$\partial f(x) = \{v \in H\colon \langle v, h\rangle \le f^\circ(x; h) \text{ for all } h \in H\},$$

where

$$f^\circ(x; h) := \limsup_{(t,y)\to(0^+,x)} t^{-1}[f(y + th) - f(y)],$$

is the *generalized directional derivative* of the locally Lipschitzian function f at x in the direction $h \in H$. The function $f^\circ(x; \cdot)$ is in fact the support function of $\partial f(x)$. That characterization easily yields that the Clarke subdifferential of any locally Lipschitzian function has the important property of upper semicontinuity from H into H_w (see definition below).

Given a lower semicontinuous function $f\colon H \to \mathbb{R}$, we define the *Dini directional derivative* of f at x in the direction v, denoted $Df(x;v)$, as

$$Df(x;v) := \liminf_{w\to v, t\downarrow 0} \frac{f(x+tw)-f(x)}{t}.$$

Moreover, if f is locally Lipschitz, then

$$Df(x;v) = \liminf_{t\downarrow 0} t^{-1}\left[f(x+tv)-f(x)\right].$$

Given $x \in S$, we say that $v \in H$ belong to the *Bouligand tangent cone* $T^B(S;x)$ (see [13]), when there exist $v_n \to v$ and $t_n \to 0^+$ such that $x + t_n v_n \in S$ for all $n \in \mathbb{N}$. By the very definition of $T^B(S;x)$, it is clear that

$$T^C(S;x) \subseteq T^B(S;x) \quad \text{for all } x \in S.$$

Moreover, equality holds when S is convex.

Let $([T_0,T],\mathcal{L},\lambda)$ be the Lebesgue measure space over the interval $[T_0,T]$ and $\Phi\colon [T_0,T] \rightrightarrows H$ be a set-valued map. We say that Φ is *measurable* if its support function $t \mapsto \sigma(v,\Phi(t))$ is $\mathcal{L}$-measurable for all $v \in H$. Furthermore, let $\Psi\colon H \rightrightarrows H$ be a set-valued map with nonempty, closed, convex and bounded values. We say that Ψ is *upper semicontinuous* from H into H_w if its support function $x \mapsto \sigma(v,\Psi(x))$ is upper semicontinuous for all $v \in H$.

The distance function to a set $S \subseteq H$ at the point $x \in H$ is defined by $d_S(x) := \inf_{y\in S}\|x-y\|$. We denote $\mathrm{Proj}_S(x)$ the set (possibly empty)

$$\mathrm{Proj}_S(x) := \{y \in S\colon d_S(x) = \|x-y\|\}.$$

The equality (see [14])

$$N(S;x) = \mathrm{cl}^*\left(\mathbb{R}_+ \partial d_S(x)\right) \quad \text{for } x \in S, \tag{2.1}$$

gives an expression of the Clarke normal cone in terms of the distance function. As usual, it will be convenient to write $\partial d(x,S)$ in place of $\partial d(\cdot,S)(x)$.

Given $\alpha \in (0,1]$, we say that $S \subseteq H$ is *positively α-far* if there exists $\rho \in (0,+\infty)$ such that

$$\alpha \le \inf_{x\in U_\rho(S)} d\left(0,\partial d_S(x)\right),$$

where $U_\rho(S) := \{x \in H\colon 0 < d_S(x) < \rho\}$ is the open ρ-tube around S. This class, introduced in [18], is broad enough to include convex sets, uniformly prox-regular sets and uniformly subsmooth sets, among others. See [22] for more details.

We recall the Kakutani-Fan-Glicksberg fixed point (see [4]), which will be used in the sequel.

Theorem 2.1 (Kakutani-Fan-Glicksberg). *Let X be a nonempty compact convex subset of a locally convex Hausdorff space, and let $\mathcal{F}\colon K \rightrightarrows K$ be a set-valued map with closed graph and nonempty convex values. Then the set of fixed point of $\mathcal{F}$ is compact and nonempty.*

A metric space X is called contractible if there exist a point $x_0 \in X$ and a continuous map (homotopy) $h\colon X \times [0,1] \to X$ such that $h(x,0) = x$ and $h(x,1) = x_0$ for all $x \in X$. It is clear that convex sets are contractible. Moreover, a compact metric space A is called an R_δ-set if there exists a decreasing sequence $\{A_n\}_n$ of compact contractible sets whose intersection is exactly A. Furthermore, we say that $\Phi\colon X \rightrightarrows Y$ is an R_δ-map if it is upper semicontinuous and takes R_δ-values. The following result is a generalization of the Bohnenblust-Karlin fixed point theorem (see [36, Proposition 1.23]).

Proposition 2.2. *Let X be a nonempty, compact and contractible topological space, $\Phi\colon X \rightrightarrows Y$ an R_δ-map and $f\colon Y \to X$ a continuous function. If $\mathcal{P}\colon X \rightrightarrows X$ is the composition map $x \rightrightarrows f(\Phi(x))$, then $\mathcal{P}$ admits a fixed point.*

We denote by $L^1([T_0,T];H)$ the space of H-valued Lebesgue integrable functions defined over $[T_0,T]$. We write $L^1_w([T_0,T];H)$ to mean the space $L^1([T_0,T];H)$ endowed with the weak topology. A set $K \subseteq L^1([T_0,T];H)$ is *uniformly integrable* if

$$\lim_{\lambda\to+\infty}\left[\sup_{f\in K}\int_{\{\|f\|\geq\lambda\}}\|f(s)\|ds\right] = 0.$$

Moreover, if there exists $\psi \in L^1(T_0,T)$ such that for all $f \in K$

$$\|f(t)\| \leq \psi(t) \quad \text{a.e. } t \in [T_0,T],$$

then K is uniformly integrable. The following result is a well-known characterization of bounded relatively weakly compact sets in $L^1([T_0,T];H)$ (see [16, Theorem 2.3.24]).

Theorem 2.3 (Dunford-Pettis)**.** *Let $K \subseteq L^1([T_0,T];H)$ be a bounded set. Then, K is relatively weakly compact in $L^1([T_0,T];H)$ if and only if K is uniformly integrable.*

We recall the following characterization of weak convergence in the space $C([T_0,T];H)$ (see [8, Theorem 4.2]).

Lemma 2.4. *$(x_n)_n \subseteq C([T_0,T];H)$ weakly converges in $C([T_0,T];H)$ to x if and only if $(x_n)_n$ is uniformly bounded in $C([T_0,T];H)$ and $x_n(t) \rightharpoonup x(t)$ for all $t \in [T_0,T]$.*

A function u belong to $W^{1,1}([T_0,T];H)$ if there exists $f \in L^1([T_0,T];H)$ and a fixed element $u_0 \in H$ such that $u(t) = u_0 + \int_{T_0}^t f(s)ds$ for all $t \in [T_0,T]$.

A separable Hilbert space $(H, \|\cdot\|)_H$ is *compactly embedded* in a separable Banach space $(E, \|\cdot\|_E)$, we write $H \hookrightarrow E$, if there exists $C > 0$ such that $\|x\|_E \leq C\|x\|_H$ for all $x \in H$ and every bounded set in H is relatively compact in E.

Let $(e_n)_{n\in\mathbb{N}}$ be an orthonormal basis of H. For every $n \in \mathbb{N}$ we consider the linear operator P_n from H into $\operatorname{span}\{e_1,\ldots,e_n\}$ defined by

$$P_n\left(\sum_{k=1}^{\infty}\langle x, e_k\rangle e_k\right) = \sum_{k=1}^{n}\langle x, e_k\rangle e_k.$$

These operators are the key elements of the Galerkin-like method, introduced in [23]. The following result summarize the main properties of the linear operator P_n .

Lemma 2.5.

(i) $\|P_n(x)\| \leq \|x\|$ *for all* $x \in H$;
(ii) $\langle P_n(x), x - P_n(x)\rangle = 0$ *for all* $x \in H$;
(iii) $P_n(x) \to x$ *as* $n \to +\infty$ *for all* $x \in H$;
(iv) *if* $(x_n)_n$ *is a bounded sequence with* $x_n \rightharpoonup x$ *as* $n \to +\infty$ *then* $P_n(x_n) \rightharpoonup x$ *as* $n \to +\infty$;
(v *if* $B \subseteq H$ *is relatively compact then* $\sup_{x \in B} \|x - P_n(x)\| \to 0$ *as* $n \to +\infty$.

Lemma 2.6 ([24])**.** *If* $S \subseteq H$ *is a ball compact set, then*

$$\partial d_S(x) = \frac{x - \overline{\mathrm{co}}\,\mathrm{Proj}_S(x)}{d_S(x)} \quad x \notin S.$$

The following result may be proved in the same way as [24, Lemma 4.4] (see also [22, Lemma 5.7]).

Lemma 2.7. *Let* $x\colon [T_0, T] \to H$ *be an absolutely continuous function,* $P\colon H \to H$ *be a linear operator and* $C\colon [T_0, T] \rightrightarrows H$ *be a set-valued map with nonempty and closed values satisfying*

$$|d_{C(t)}(x) - d_{C(s)}(x)| \leq |\zeta(t) - \zeta(s)| \quad \text{for all } x \in H \text{ and } s, t \in [T_0, T],$$

for some $\zeta \in W^{1,1}(T_0, T)$*. Then, the following assertions hold true:*

(i) *The function* $t \to d(P(x(t)); C(t))$ *is absolutely continuous over* $[T_0, T]$.
(ii) *For all* $t \in]T_0, T[$*, where* $\dot{\zeta}(t)$ *and* $\dot{x}(t)$ *exist,*

$$\begin{aligned}&\limsup_{s \downarrow 0} \frac{1}{s}\left[d_{C(t+s)}(P(x(t+s))) - d_{C(t)}(P(x(t)))\right]\\ &\leq |\dot{\zeta}(t)| + \limsup_{s \downarrow 0} \frac{1}{s}\left[d_{C(t)}(P(x(t+s))) - d_{C(t)}(P(x(t)))\right].\end{aligned}$$

(iii) *For all* $t \in]T_0, T[$*, where* $\dot{x}(t)$ *exists,*

$$\limsup_{s \downarrow 0} \frac{1}{s}\left[d_{C(t)}(P(x(t+s))) - d_{C(t)}(P(x(t)))\right] \leq \max_{y^* \in \partial d_{C(t)}(P(x(t)))} \langle y^*, \dot{x}(t)\rangle.$$

(iv) *For all* $t \in \{s \in [T_0, T]\colon P(x(s)) \notin C(s)\}$*, where* $\dot{x}(t)$ *exists,*

$$\lim_{s \downarrow 0} \frac{1}{s}\left[d_{C(t)}(P(x(t+s))) - d_{C(t)}(P(x(t)))\right] = \min_{y^* \in \partial d(P(x(t)), C(t))} \langle y^*, P(\dot{x}(t))\rangle.$$

(v) *For every* $x \in H$*, the set-valued map* $t \rightrightarrows \partial d(P(x), C(t))$ *is measurable.*

3. Technical assumptions

For the sake of readability, in this section we collect the hypotheses used along the paper.

Hypotheses on the set-valued map $C\colon [T_0, T] \rightrightarrows H$*:* C is a set-valued map with nonempty and closed values. Moreover, we will consider the following conditions:

($\mathcal{H}_1$) There exists $\zeta \in W^{1,1}(T_0, T)$ such that for $s, t \in [T_0, T]$ and all $x \in H$

$$|d(x, C(t)) - d(x, C(s))| \leq |\zeta(t) - \zeta(s)|.$$

($\mathcal{H}_2$) There exist two constants $\alpha_0 \in (0,1]$ and $\rho \in (0,+\infty)$ such that

$$0 < \alpha_0 \leq \inf_{x \in U_\rho(C(t))} d\left(0, \partial d(x, C(t))\right) \quad \text{a.e. } t \in [T_0, T],$$

where $U_\rho(C(t)) := \{y \in H : 0 < d(y, C(t)) < \rho\}$ for all $t \in [T_0, T]$.

($\mathcal{H}_3$) For all $t \in [T_0, T]$, the set $C(t)$ is ball compact, that is, for every $r > 0$ the set $C(t) \cap r\mathbb{B}$ is compact in H.

Remark 3.1. The condition ($\mathcal{H}_2$) holds true for a big family of sets, e.g., convex sets, r-uniformly prox-regular sets, equi-uniformly subsmooth sets, etc (see [22]).

Hypotheses on the set-valued map $F\colon [T_0, T] \times H \rightrightarrows H$*:* F is a set-valued map with nonempty, closed and convex values. Moreover, we will consider the following conditions:

($\mathcal{H}_1^F$) For all $x \in H$, $F(\cdot, x)$ is measurable.

($\mathcal{H}_2^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is upper semicontinuous from H into H_w.

($\mathcal{H}_3^F$) There exists $\beta \in L^1(T_0, T)$ such that

$$d\left(0, F(t,x)\right) := \inf\{\|w\| \colon w \in F(t,x)\} \leq \beta(t),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$.

($\mathcal{H}_4^F$) For all $r > 0$ there exists $v_r \in L^1(T_0, T)$ such that for a.e. $t \in [T_0, T]$ and all $x \in H$ with $\|x\| \leq r$

$$d\left(0, F(t,x)\right) := \inf\{\|w\| \colon w \in F(t,x)\} \leq v_r(t).$$

Moreover, in the case where $(H, \|\cdot\|_H)$ is compactly embedded in a separable Banach space $(E, \|\cdot\|_E)$, we will consider the following hypothesis on F:

($\mathcal{H}_5^F$) For a.e. $t \in [T_0, T]$, $F(t, \cdot)$ is closed from E into E_w, that is, $\operatorname{graph} F(t, \cdot)$ is closed in $E \times E_w$.

Hypotheses on the map $M\colon C\left([T_0, T]; H\right) \to H$*:*

($\mathcal{H}_1^M$) There exists $m \in [0,1)$ such that

$$\|Mx - My\| \leq m\|x - y\|_\infty \quad \text{for all } x, y \in C\left([T_0, T]; H\right).$$

($\mathcal{H}_2^M$) For all $x, y \in C\left([T_0, T]; H\right)$

$$\|Mx - My\| \leq \|x - y\|_\infty.$$

($\mathcal{H}_3^M$) M is sequentially weakly upper semicontinuous, that is, if $x_n \rightharpoonup x$ in $C\left([T_0, T]; H\right)$ (see Lemma 2.4), then $Mx_k \rightharpoonup Mx$ in H, for some subsequence $(x_k)_k$ of $(x_n)_n$.

Remark 3.2. a) If $M\colon C\left([T_0, T]; H\right) \to H$ is linear and continuous, then ($\mathcal{H}_3^M$) holds.

b) The conditions ($\mathcal{H}_2^M$) and ($\mathcal{H}_3^M$) hold for the following operators:

i) $Mx = x_0$;

ii) $Mx = \pm x(T)$;

iii) $Mx = \frac{1}{T - T_0} \int_{T_0}^T x(s)ds$;

iv) $Mx = \sum_{i=1}^{k_0} \alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}$ and $\sum_{i=1}^{k_0} |\alpha_i| \leq 1$, where $T_0 < t_1 < \cdots < t_{k_0} \leq T$.

4. Perturbed sweeping process with nonlocal initial conditions

In this section, we prove existence results for (1.1) in infinite dimensional Hilbert spaces. We distinguish between the contractive case $(\mathcal{H}_1^M)$ (see Theorem 4.2) and the nonexpansive case $(\mathcal{H}_2^M)$ (see Theorem 4.4). Our results are associated with the existence of a closed convex set D so that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where

$$\mathcal{C} := \{x \in W^{1,1}([T_0,T]) : x(t) \in C(t) \text{ for all } t \in [T_0,T]\}. \tag{4.1}$$

This condition seems very natural because the constrained nature of the sweeping process. Moreover, unlike the contractive case, we have to impose a boundedness condition on the set D to assure the existence of solutions of (1.1).

Before presenting the main results of this section, we want to emphasize the role of condition $(\mathcal{H}_3)$. Indeed, the compactness hypothesis $(\mathcal{H}_3)$ seems to be a strong assumption, but it is not. We refer to [23] for an example of a perturbed sweeping process with Cauchy initial condition, governed by a ball, without solutions.

The following lemma will be used in the construction of the fixed point operator used in the proof of Theorem 4.2.

Lemma 4.1. *Assume that $(\mathcal{H}_1^M)$ holds. If $f \in L^1([T_0,T];H)$, then there exists a unique solution of the following differential equation:*

$$\begin{cases} \dot{x}(t) = f(t) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = Mx. \end{cases}$$

Moreover, $\|x(t)\| \leq \dfrac{1}{1-m}\left(\|M0\| + \int_{T_0}^T \|f(s)\|ds\right)$ *for all* $t \in [T_0,T]$.

Proof. Fix $x_0 \in H$. For each $n \in \mathbb{N}$ we define

$$x_{n+1}(t) = Mx_n + \int_{T_0}^t f(s)ds \quad \text{for all } t \in [T_0,T]. \tag{4.2}$$

Then, for all $n \geq 1$

$$\|x_{n+1}(t) - x_n(t)\| = \|Mx_n - Mx_{n-1}\| \leq m\|x_n - x_{n-1}\|_\infty.$$

Therefore, $\|x_{n+1} - x_n\|_\infty \leq m\|x_n - x_{n-1}\|_\infty$ for all $n \geq 1$, which proves, since $m \in [0,1)$, that $(x_n)_n$ is a Cauchy sequence in $C([T_0,T];H)$. Therefore, by passing to the limit in (4.2), we obtain the result. □

The following result asserts the existence of solutions for (1.1), when the operator M is a contraction.

Theorem 4.2. *Let $F\colon [T_0,T] \times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume, in addition to $(\mathcal{H}_1^M)$, $(\mathcal{H}_3^M)$, that there exists a convex set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by (4.1) and*

$$\left(1 + \frac{1}{\alpha_0^2}\right)\int_{T_0}^T \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho. \tag{4.3}$$

Then, there exists at least one solution of (1.1)*. Moreover,*

$$\|\dot{x}(t)\| \leq \frac{1}{\alpha_0^2}|\dot{\zeta}(t)| + \left(1+\frac{1}{\alpha_0^2}\right)\beta(t) \quad a.e.\ t \in [T_0,T].$$

Proof. Let us define the set-valued map $G\colon [T_0,T]\times H \rightrightarrows H$ by:

$$G(t,x) := -\frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)|+\beta(t)\right)\partial d_{C(t)}(x) + F(t,x)\cap \beta(t)\mathbb{B}.$$

It is clear that G satisfy $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$.

The idea of the proof is to use the reduction technique for the sweeping process together with the Galerkin-like method. The reduction technique consists in showing the existence of solutions of the following unconstrained differential inclusion:

$$\text{(4.4)} \qquad \begin{cases} \dot{x}(t) \in G(t,x(t)) & \text{a.e. } t\in[T_0,T], \\ x(T_0) = Mx, \end{cases}$$

Thus, by formula (2.1), every solution of (4.4) together with the condition $x(t)\in C(t)$ for all $t\in[T_0,T]$, is a solution of (1.1). Since it is not possible to prove directly the existence of (4.4), we use the Galerkin like-method, that is, we approach (4.4) by projecting the state into a n-dimensional Hilbert space.

The proof will be divided into several steps.

Step 1: For each $n\in\mathbb{N}$ there exists at least one solution x_n of

$$\text{(4.5)} \qquad \begin{cases} \dot{x}(t) \in G(t,P_n(x(t))) & \text{a.e. } t\in[T_0,T], \\ x(T_0) = \text{proj}_D\left(P_n\left(Mx\right)\right), \end{cases}$$

where $P_n\colon H\to \text{span}\{e_1,\ldots,e_n\}$ is the linear operator defined in Lemma 2.5.

Proof of Step 1: Let $K\subseteq L^1([T_0,T];H)$ be the set defined by

$$K := \left\{f\in L^1([T_0,T];H) : \|f(t)\|\leq \psi(t) \text{ a.e. } t\in[T_0,T]\right\},$$

where

$$\text{(4.6)} \qquad \psi(t) := \frac{1}{\alpha_0^2}|\dot{\zeta}(t)| + \left(1+\frac{1}{\alpha_0^2}\right)\beta(t) \quad \text{for all } t\in[T_0,T].$$

It is clear that K is nonempty, closed and convex. In addition, since $\psi\in L^1(T_0,T)$, K is bounded and uniformly integrable, hence, it is compact in $L^1_w([T_0,T];H)$ (see Theorem 2.3). We observe that K, endowed with the relative $L^1_w([T_0,T];H)$ topology is a metric space (see [15, Theorem V.6.3]). Define the map $\mathcal{F}_n\colon K\rightrightarrows L^1([T_0,T];H)$ as

$$\mathcal{F}_n(f) := \{v\in L^1([T_0,T];H) : v(t)\in G(t,P_n(x_f(t))) \text{ a.e. } t\in[T_0,T]\},$$

where for each $f\in K$, x_f is the unique solution (see Lemma 4.1) of

$$\begin{cases} \dot{x}(t) = f(t) & \text{a.e. } t\in[T_0,T], \\ x(T_0) = \text{proj}_D\left(P_n(Mx)\right). \end{cases}$$

By $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and [3, Lemma 6], we conclude that $\mathcal{F}_n(f)$ has nonempty, closed and convex values. Moreover, $\mathcal{F}_n(K)\subseteq K$. Indeed, if $f\in K$ and $v\in\mathcal{F}_n(f)$ then,

$$\|v(t)\| \leq \sup\{\|w\|\colon w\in G(t,P_n(x_f(t)))\} \leq \psi(t) \quad \text{for a.e. } t\in[T_0,T].$$

We denote by K_w the set K seen as a compact and convex subset of $L^1_w([T_0,T];H)$.
Claim 1.: $\mathcal{F}_n$ is upper semicontinuous from K_w into K_w.
Proof of Claim 1.: By virtue of [21, Proposition 1.2.23], it is sufficient to prove that the graph graph$(\mathcal{F}_n)$ of $\mathcal{F}_n$ is sequentially closed in $K_w \times K_w$.

Let $((f_j,v_j))_j \subseteq \text{graph}(\mathcal{F}_n)$ with $f_j \to f$ and $v_j \to v$ in $L^1_w([T_0,T];H)$ as $j \to +\infty$. We have to show that $(f,v) \in \text{graph}(\mathcal{F}_n)$. We first note that,

$$v_j(t) \in G(t, P_n\left(x_{f_j}(t)\right)) \quad \text{for a.e. } t \in [T_0,T]. \tag{4.7}$$

Moreover, since $f_j \in K$ and Lemma 4.1, we have that

$$\|\dot{x}_{f_j}(t)\| = \|f_j(t)\| \le \psi(t) \quad \text{a.e. } t \in [T_0,T], \tag{4.8}$$

and

$$\|x_{f_j}(t)\| \le \frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^T \psi(s)ds\right) \quad \forall t \in [T_0,T]. \tag{4.9}$$

On the one hand, let us consider $P := \{\dot{x}_{f_j} \colon j \in \mathbb{N}\} \subseteq L^1([T_0,T];H)$. According to (4.8), the set P is bounded and uniformly integrable. Thus, as a result of the Dunford-Pettis theorem (see Theorem 2.3), P is relatively compact in $L^1_w([T_0,T];H)$. Therefore, there exists a subsequence of $(\dot{x}_{f_j})$ (without relabeling) converging to some $v \in L^1_w([T_0,T];H)$. Now, let $S := \{x_{f_j} \colon j \in \mathbb{N}\} \subseteq L^1([T_0,T];H)$. Then, due to (4.9) and the Dunford-Pettis theorem (see Theorem 2.3), S is relatively compact in $L^1_w([T_0,T];H)$. Consequently, there exists a subsequence of $(x_{f_j})_j$ (without relabeling) converging to some $x \in L^1_w([T_0,T];H)$.

On the other hand, due to (4.8) and (4.9), the sequence $(x_{f_j})_j$ is uniformly bounded in $W^{1,1}([T_0,T];H)$ and in $L^\infty([T_0,T];H)$. Therefore, by result of [31, Theorem 0.2.2.1], there exists a subsequence of $(x_{f_j})_j$ (without relabeling) and a function $\tilde{x}$ such that

$$x_{f_j}(t) \to \tilde{x}(t) \text{ weakly as } j \to +\infty \text{ for all } t \in [T_0,T]. \tag{4.10}$$

Moreover, by virtue of [16, Proposition 2.3.31], $x \equiv \tilde{x}$. Now, we prove that $v = \dot{x}$. Indeed, let $w \in H$ and $t \in [T_0,T]$ be fixed. Then,

$$\left\langle x_{f_j}(t) - x_{f_j}(T_0), w\right\rangle = \int_{T_0}^t \left\langle \dot{x}_{f_j}(s), w\right\rangle = \int_{T_0}^T \left\langle \dot{x}_{f_j}(s), w \cdot 1_{[T_0,t]}(s)\right\rangle ds, \tag{4.11}$$

where

$$1_{[T_0,t]}(s) := \begin{cases} 1, & \text{if } s \in [T_0,t], \\ 0, & \text{if } s \in]t,T], \end{cases}$$

belongs to $L^\infty([T_0,T];H)$. Moreover, due to $(\mathcal{H}_3^M)$ and Lemma 2.4, $Mx_{f_j} \rightharpoonup Mx$ in H (without relabeling), which implies, by the definition of P_n, that $P_n\left(Mx_{f_j}\right) \to P_n(Mx)$. Thus, $x_{f_j}(T_0) \to \operatorname{proj}_D(P_n(Mx))$. Therefore, using (4.10), the weak convergence of $\dot{x}_{f_j}$ to v in $L^1([T_0,T];H)$ and passing to the limit in (4.11), we obtain

$$\langle x(t) - \operatorname{proj}_D(P_n(Mx)), w\rangle = \int_{T_0}^t \langle v(s), w\rangle\, ds \quad \text{for all } w \in H.$$

Thus

$$x(t) = \text{proj}_D\left(P_n(Mx)\right) + \int_{T_0}^{t} v(s)ds \quad \text{for all } t \in [T_0, T].$$

Therefore, we have proved the existence of a subsequence of $(x_{f_j})_j$ (without relabeling) and an absolutely continuous function $x\colon [T_0, T] \to H$ such that

$$\begin{aligned}
&x_{f_j}(t) \to x(t) \text{ weakly for all } t \in [T_0, T],\\
&x_{f_j} \to x \text{ in } L^1_w\left([T_0, T]; H\right),\\
&\dot{x}_{f_j} \to \dot{x} \text{ in } L^1_w\left([T_0, T]; H\right),\\
&x(t) = \text{proj}_D\left(P_n(Mx)\right) + \int_{T_0}^{t} f(s)ds \qquad \text{for all } t \in [T_0, T].
\end{aligned}$$

Moreover, by the definition of P_n, $P_n(x_{f_j}(t)) \to P_n(x(t))$ for every $t \in [T_0, T]$. Consequently, by virtue of (4.7), the upper semicontinuity of $G(t, \cdot)$ from H into H_w and [16, Proposition 2.3.1], we obtain, for a.e. $t \in [T_0, T]$

$$v(t) \in \overline{\text{conv}}\, w\text{-}\limsup_{m\to+\infty}\{v_m(t)\} \subseteq \overline{\text{conv}}\, G(t, P_n(x(t))) = G(t, P_n(x(t))),$$

which shows that $(f, v) \in \text{graph}\,(\mathcal{F}_n)$, as claimed. □

Now, we apply the Kakutani-Fan-Glicksberg fixed point theorem (see Theorem 2.1) to the set-valued map $\mathcal{F}_n\colon K_w \rightrightarrows K_w$, to deduce the existence of $\hat{f}_n \in K$ such that $\hat{f}_n \in \mathcal{F}_n\left(\hat{f}_n\right)$. Thus, the function $x_n := x_{\hat{f}_n} \in W^{1,1}\left([T_0, T]; H\right)$ is a solution of (4.5), which proves Step 1. □

<u>Step 2.</u>: There exists $x \in W^{1,1}\left([T_0, T]; H\right)$ solution of

$$\text{(4.12)} \qquad \begin{cases} \dot{x}(t) \in G(t, x(t)) & \text{a.e. } t \in [T_0, T],\\ x(t) \in C(t) & \text{for all } t \in [T_0, T],\\ x(T_0) = \text{proj}_D\left(Mx\right). \end{cases}$$

Proof of Step 2.: For each $n \in \mathbb{N}$, let x_n be a solution of (4.5) and for all $t \in [T_0, T]$ define

$$\varphi_n(t) := d_{C(t)}(P_n(x_n(t))) \quad \text{ and } \quad \Gamma_n(t) := \partial d_{C(t)}(P_n(x_n(t))).$$

Then, according to Step 1, there exist $f_n(t) \in F(t, P_n(x_n(t))) \cap \beta(t)\mathbb{B}$ and $d_n(t) \in \Gamma_n(t)$ such that

$$\text{(4.13)} \qquad \begin{cases} \dot{x}_n(t) = -\dfrac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right) d_n(t) + f_n(t) & \text{a.e. } t \in [T_0, T],\\ x_n(T_0) = \text{proj}_D(P_n(Mx_n)). \end{cases}$$

Moreover, according to (4.9), for all $t \in [T_0, T]$

$$\text{(4.14)} \qquad \|x_n(t)\| \leq \frac{1}{1-m}\left(\|\,\text{proj}_D\left(P_n(M0)\right)\| + \int_{T_0}^{T} \varphi(s)ds\right),$$

where ψ is defined by (4.6). Therefore, $(x_n)_n$ and $(P_n(x_n))_n$ are uniformly bounded in $C\left([T_0, T]; H\right)$.

Claim 2. $\lim_{n\to+\infty} \varphi_n(T_0) = 0$.

Proof of Claim 2.: Indeed, since $(x_n(T_0))_n$ is bounded (see (4.14)), there exists a positive number $\tilde{R}$ such that $(x_n(T_0))_n \subseteq D \cap \tilde{R}\mathbb{B} \subseteq C(T_0) \cap \tilde{R}\mathbb{B}$. Hence, due to the ball compactness of $C(T_0)$ and Lemma 2.5,

$$\begin{aligned}\limsup_{n\to+\infty} \varphi_n(T_0) &= \limsup_{n\to+\infty} \left[d_{C(T_0)}(P_n x_n(T_0)) - d_{C(T_0)}(x_n(T_0))\right] \\ &\leq \limsup_{n\to+\infty} \|x_n(T_0) - P_n(x_n(T_0))\| \\ &\leq \limsup_{n\to+\infty} \sup_{x\in D\cap\tilde{R}\mathbb{B}} \|x - P_n(x)\| \\ &= 0,\end{aligned}$$

which proves the claim.

From now on, without loss of generality, due to Claim 2 and condition (4.3), we may assume that for all $n \in \mathbb{N}$

$$\varphi_n(T_0) + \left(1 + \frac{1}{\alpha_0^2}\right) \int_{T_0}^{T} \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho. \tag{4.15}$$

Claim 3.: For all $t \in [T_0, T]$

$$\varphi_n^3(t) \leq \varphi_n(T_0)^3 + \frac{3}{\alpha_0^2} \int_{T_0}^{t} \left(|\dot{\zeta}(s)| + \beta(s)\right) \sup_{x\in A(s)} \|x - P_n(x)\|^2 ds, \tag{4.16}$$

where

$$R := \rho + \frac{1}{1-m} \left(\sup_{n\in\mathbb{N}} \|\operatorname{proj}_D(P_n(M0))\| + \int_{T_0}^{T} \psi(s) ds\right)$$

and, due to $(\mathcal{H}_3)$, $A(t) := \overline{\text{co}}\,(C(t) \cap R\mathbb{B})$ is relatively compact for every $t \in [T_0, T]$.
Proof of Claim 3.: The idea of the proof is to use $(\mathcal{H}_2)$. To do that, we proceed to show first that $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$. Indeed, let $t \in [T_0, T]$ where $\dot{x}_n(t)$ exists. Then, due to (ii) and (iii) of Lemma 2.7 and (4.13),

$$\begin{aligned}\dot{\varphi}_n(t) &\leq |\dot{\zeta}(t)| + \max_{y^*\in\Gamma_n(t)} \langle y^*, P_n(\dot{x}_n(t))\rangle \\ &\leq |\dot{\zeta}(t)| + \|\dot{x}_n(t)\| \\ &\leq \left(1 + \frac{1}{\alpha_0^2}\right)\left(|\dot{\zeta}(t)| + \beta(t)\right).\end{aligned}$$

Therefore, according to (4.15), for all $t \in [T_0, T]$

$$\varphi_n(t) \leq \varphi_n(T_0) + \left(1 + \frac{1}{\alpha_0^2}\right) \int_{T_0}^{t} \left(|\dot{\zeta}(s)| + \beta(s)\right) ds < \rho,$$

as claimed. □

Now, let $t \in \Omega_n := \{t \in [T_0, T] : P_n(x_n(t)) \notin C(t)\}$ where $\dot{x}_n(t)$ exists. Then, due to Lemma 2.7,

$$\begin{aligned}\dot{\varphi}_n(t) &\le |\dot{\zeta}(t)| + \min_{y^* \in \Gamma_n(t)} \langle y^*, P_n(\dot{x}_n(t))\rangle \\ &\le |\dot{\zeta}(t)| - \frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right)\langle d_n(t), P_n(d_n(t))\rangle \\ &+ \langle d_n(t), P_n(f_n(t))\rangle \\ &\le \left(|\dot{\zeta}(t)| + \beta(t)\right)\left(1 - \frac{1}{\alpha_0^2}\langle d_n(t), P_n(d_n(t))\rangle\right).\end{aligned}$$

Moreover, due to $(\mathcal{H}_2)$,

$$\begin{aligned}-\langle d_n(t), P_n(d_n(t))\rangle &= \langle d_n(t), d_n(t) - P_n(d_n(t))\rangle + \langle d_n(t), -d_n(t)\rangle \\ &\le \langle d_n(t), d_n(t) - P_n(d_n(t))\rangle - \alpha_0^2 \\ &= \|d_n(t) - P_n(d_n(t))\|^2 - \alpha_0^2.\end{aligned}$$

Hence, for a.e. $t \in \Omega_n$,

$$\dot{\varphi}_n(t) \le \frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)| + \beta(t)\right)\|d_n(t) - P_n(d_n(t))\|^2.$$

Furthermore, for $t \in \Omega_n$, since $d_n(t) \in \Gamma_n(t)$, Lemma 2.6 ensures the existence of

$$g_n(t) \in \overline{\text{co}}\ \text{Proj}_{C(t)}(P_n(x_n(t)))$$

such that

$$d_n(t) = \frac{1}{\varphi_n(t)}\left(P_n(x_n(t)) - g_n(t)\right).$$

Thus, by virtue of (4.14) and the inequality $\varphi_n(t) < \rho$ for all $t \in [T_0, T]$,

$$\begin{aligned}\|g_n(t)\| &\le \varphi_n(t) + \|P_n(x_n(t))\| \\ &\le \rho + \frac{1}{1-m}\left(\|\,\text{proj}_D(P_n(M0))\| + \int_{T_0}^t \psi(s)ds\right) \\ &\le R,\end{aligned}$$

which entails that $g_n(t) \in A(t)$ for all $t \in \Omega_n$. Thus, for every $t \in \Omega_n$

$$\varphi_n(t)^2\|d_n(t) - P_n(d_n(t))\|^2 = \|g_n(t) - P_n(g_n(t))\|^2 \le \sup_{x \in A(t)}\|x - P_n(x)\|^2.$$

Therefore, for $t \notin \Omega_n$, we obtain that for $t \in [T_0, T]$

$$\begin{aligned}\varphi_n^3(t) &= \varphi_n^3(T_0) + 3\int_{T_0}^t \varphi_n^2(s)\dot{\varphi}_n(s)ds \\ &\le \varphi_n^3(T_0) + \frac{3}{\alpha_0^2}\int_{T_0}^t \left(|\dot{\zeta}(s)| + \beta(s)\right)\sup_{x \in A(s)}\|x - P_n(x)\|^2 ds,\end{aligned}$$

as claimed. □

Claim 4.: $\lim_{n\to+\infty} \varphi_n(t) = 0$ for all $t \in [T_0, T]$.

Proof of Claim 4.: Fix $t \in [T_0, T]$. Since $A(t)$ is relatively compact, Lemma 2.5 2.5 asserts that

$$\lim_{n\to+\infty}\sup_{x \in A(t)}\|x - P_n(x)\| = 0.$$

Hence, by Fatou's lemma and (4.16),

$$\begin{aligned}\limsup_{n\to+\infty}\varphi_n^3(t) &\le \frac{3}{\alpha_0^2}\limsup_{n\to+\infty}\int_{T_0}^t \left(|\dot{\zeta}(s)|+\beta(s)\right)\sup_{x\in A(s)}\|x-P_n(x)\|^2 ds\\ &\le \frac{3}{\alpha_0^2}\int_{T_0}^t \left(|\dot{\zeta}(s)|+\beta(s)\right)\limsup_{n\to+\infty}\sup_{x\in A(s)}\|x-P_n(x)\|^2 ds\\ &= 0,\end{aligned}$$

as required. □

Claim 5.: $(P_n(x_n(t)))_n$ is relatively compact for all $t\in[T_0,T]$.

Proof of Claim 5.: Fix $t\in[T_0,T]$ and let $s_n(t)\in \operatorname{Proj}_{C(t)}(P_n(x_n(t)))$ (this projection is well defined because $(P_n(x_n))_n$ is uniformly bounded in $C([T_0,T];H)$). Then, as a result of (4.14),

$$\begin{aligned}\|s_n(t)\| &\le \varphi(t)+\|P_n(x_n(t))\|\\ &\le \rho+\frac{1}{1-m}\left(\|\operatorname{proj}_D(P_n(M0))\|+\int_{T_0}^T\psi(s)ds\right)\\ &\le R,\end{aligned}$$

where we have used (4.14) and the definition of R. Hence, $s_n(t)\in C(t)\cap R\mathbb{B}$. Thus, due to the ball compactness of $C(t)$, there exists a subsequence of $(s_n(t))_n$ (without relabeling) such that $s_n(t)\to s(t)$ as $n\to+\infty$. Therefore, by virtue of Claim 4,

$$\begin{aligned}\limsup_{n\to+\infty}\|P_n(x_n(t))-s(t)\| &\le \limsup_{n\to+\infty}\left[\|P_n(x_n(t))-s_n(t)\|+\|s_n(t)-s(t)\|\right]\\ &\le \limsup_{n\to+\infty}\left[\varphi_n(t)+\|s_n(t)-s(t)\|\right]\\ &= 0,\end{aligned}$$

which proves the claim. □

Claim 6.: There exists a subsequence $(x_k)_k$ of $(x_n)_n$ and an absolutely continuous function x such that

(i) $x_k(t)\rightharpoonup x(t)$ in H as $k\to+\infty$ for all $t\in[T_0,T]$,
(ii) $x_k\rightharpoonup x$ in $L^1([T_0,T];H)$ as $k\to+\infty$,
(iii) $\dot{x}_k\rightharpoonup \dot{x}$ in $L^1([T_0,T];H)$ as $k\to+\infty$,
(iv) $\|\dot{x}(t)\|\le\psi(t)$ a.e. $t\in[T_0,T]$, where ψ is the function defined in (4.6).

Proof of Claim 6.: It follows from similar arguments given in Claim 1. □

Claim 7.: $P_k(x_k(t))\to x(t)$ as $k\to+\infty$ for all $t\in[T_0,T]$.

Proof of Claim 7.: Fix $t\in[T_0,T]$. Since $x_k(t)\rightharpoonup x(t)$ as $k\to+\infty$, from 2.5 of Lemma 2.5, it follows that $P_k(x_k(t))\rightharpoonup x(t)$. Hence, due to the relative compactness of $(P_k(x_k(t)))_k$ (see Claim 5), the claim is proved. □

Claim 8.: For all $t\in[T_0,T]$, $x(t)\in C(t)$.

Proof of Claim 8.: Fix $t \in [T_0, T]$. Then, due to Claim 4 and Claim 7,

$$\begin{aligned} d_{C(t)}(x(t)) &= \limsup_{k\to+\infty} \left(d_{C(t)}(x(t)) - d_{C(t)}(P_k(x_k(t))) + d_{C(t)}(P_k(x_k(t)))\right) \\ &\leq \limsup_{k\to+\infty} \left(\|x(t) - P_k\left(x_k(t)\right)\| + \varphi_k(t)\right) \\ &= 0, \end{aligned}$$

which proves the claim. □

Summarizing, we have

(i) For each $x \in H$, $G(\cdot, x)$ is measurable,
(ii) for a.e. $t \in [T_0, T]$, $G(t, \cdot)$ is upper semicontinuous from H into H_w,
(iii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0, T]; H)$,
(iv) $P_k(x_k(t)) \to x(t)$ as $k \to +\infty$ for a.e. $t \in [T_0, T]$,
(v) for all $k \in \mathbb{N}$, $\dot{x}_k(t) \in G(t, P_k(x_k(t)))$ for a.e. $t \in [T_0, T]$.

These conditions and the convergence theorem (see [3, Proposition 5] for more details) imply that x is a solution of (4.12), which finishes the proof of Step 2. □

Step 3: The theorem holds.
Proof of Step 3: Since $x(t) \in C(t)$ for all $t \in [T_0, T]$, $x \in \mathcal{C}$ (see (4.1)). Thus, $Mx \in D$ and, hence, $x(T_0) = \text{proj}_D(Mx) = Mx$, which proves the theorem. □

Remark 4.3.

(1) The hypothesis $(\mathcal{H}_3^F)$ in Theorem 4.2 can be replaced by the following more general condition: There exist $\alpha, \beta \in L^1(T_0, T)$ with $\int_{T_0}^T \alpha(s)ds < 1 - m$ such that

$$d(0, F(t, x(t))) := \inf\{\|w\| \colon w \in F(t, x)\} \leq \alpha(t)\|x\| + \beta(t),$$

for all $x \in H$ and a.e. $t \in [T_0, T]$. Indeed, by virtue of Gronwall's inequality, it is possible to prove that every solution of (4.4) satisfies

$$\|x\|_\infty \leq R := \frac{1}{1 - m - \int_{T_0}^T \alpha(s)ds} \left(\|M0\| + \int_{T_0}^T \psi(s)ds\right),$$

where ψ is given by (4.6). Define

$$p_R(x) = \begin{cases} x & \text{if } \|x\| \leq R, \\ R\frac{x}{\|x\|} & \text{if } \|x\| > R. \end{cases}$$

Then, by using the set-valued map $\tilde{G}(t, x) := G(t, p_R(x))$ instead of G in (4.4), we have that $\tilde{G}$ satisfies $(\mathcal{H}_3^F)$ and the same proof applies.

(2) When H is a finite dimensional Hilbert space, the condition $(\mathcal{H}_3^F)$ in Theorem 4.2 can be removed. Indeed, it suffices to use Arzela-Ascoli's theorem in Step 1 and Step 6 instead of [31, Theorem 0.2.2.1].

The following result deals with the nonexpansive case. We emphasize that contrary to the contractive case (Theorem 4.2), it is not posible to assure the boundedness of solutions of (1.1) without any additional condition. Therefore, to overcome this difficulty, we assume the boundedness of the convex set D.

Theorem 4.4. *$F\colon [T_0,T]\times H \rightrightarrows H$ satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume, in addition to $(\mathcal{H}_2^M)$, $(\mathcal{H}_3^M)$, that there exists a convex and bounded set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by* (4.1) *and*

$$\left(1+\frac{1}{\alpha_0^2}\right)\int_{T_0}^{T}\left(|\dot{\zeta}(s)|+\beta(s)\right)ds<\rho. \tag{4.17}$$

Then, there exists at least one solution of (1.1). *Moreover,*

$$\|\dot{x}(t)\| \le \frac{1}{\alpha_0^2}|\dot{\zeta}(t)| + \left(1+\frac{1}{\alpha_0^2}\right)\beta(t) \quad a.e.\ t\in[T_0,T].$$

Proof. For each $k\in\mathbb{N}$, let x_k be a solution (whose existence is guaranteed by Step 2. of the proof of Theorem 4.2) of the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in -\frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)|+\beta(t)\right)\partial d_{C(t)}(x(t)) & \\ \qquad + F(t,x(t))\cap\beta(t)\mathbb{B} & \text{a.e. } t\in[T_0,T], \\ x(t)\in C(t) & \text{for all } t\in[T_0,T], \\ x(T_0) = \operatorname{proj}_D\left(\frac{k}{k+1}Mx\right). & \end{cases}$$

Then, $(x_k(T_0))_k \subseteq D \subseteq C(T_0)$. Thus, since D is bounded, there exists $R>0$ such that $(x_k(T_0))_k \subseteq D \subseteq R\mathbb{B}$. Hence, for all $k\in\mathbb{N}$ and all $t\in[T_0,T]$

$$\|x_k(t)\| \le \|x_k(T_0)\| + \int_{T_0}^{t}\|\dot{x}_k(s)\|ds \le R + \int_{T_0}^{t}\left(\frac{|\dot{\zeta}(s)|}{\alpha_0^2} + \left(1+\frac{1}{\alpha_0^2}\right)\beta(s)\right)ds.$$

This inequality shows that $(x_k)_k$ is bounded in $C([T_0,T];H)$ and, due to $(\mathcal{H}_3)$, this gives that the sequence $(x_{k(t)})_k$ is relatively compact for all $t\in[T_0,T]$. Therefore, by using Arzela-Ascoli and Dunford-Pettis theorems, we obtain the existence of a subsequence of $(x_k)_k$ (without relabeling) and an absolutely continuous function $x\colon[T_0,T]\to H$ such that

(i) $(x_k)_k$ converges uniformly to x on $[T_0,T]$,
(ii) $\dot{x}_k \rightharpoonup \dot{x}$ in $L^1([T_0,T];H)$.

These conditions and the convergence theorem (see [3, Proposition 5] for more details) imply that x satisfies

$$\begin{cases} \dot{x}(t) \in -\frac{1}{\alpha_0^2}\left(|\dot{\zeta}(t)|+\beta(t)\right)\partial d_{C(t)}(x(t)) & \\ \qquad + F(t,x(t))\cap\beta(t)\mathbb{B} & \text{a.e. } t\in[T_0,T], \\ x(t)\in C(t) & \text{for all } t\in[T_0,T], \\ x(T_0) = \operatorname{proj}_D(Mx). & \end{cases}$$

Moreover, since $x(t)\in C(t)$ for all $t\in[T_0,T]$, $x\in\mathcal{C}$ (see (4.1)). Thus, $Mx\in D$. Therefore, $x(T_0)=\operatorname{proj}_D(Mx)=Mx$, which proves the theorem. □

Remark 4.5. When H is a finite dimensional Hilbert space, the condition $(\mathcal{H}_3^M)$ in Theorem 4.4 can be removed (see Remark 4.3).

When M is a positively homogeneous, conditions (4.3) and (4.17) in Theorems 4.2 and 4.4, respectively, can be removed.

Theorem 4.6. *Let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume that M is positively homogeneous, satisfies $(\mathcal{H}_3^M)$ and there exists a convex set D such that $M\mathcal{C} \subseteq D \subseteq C(T_0)$, where $\mathcal{C}$ is given by* (4.1). *Assume that one of the following two conditions is satisfied:*

i) *$(\mathcal{H}_1^M)$ holds.*
ii) *D is bounded and $(\mathcal{H}_2^M)$ holds.*

Then, there exists at least one solution of (1.1).

Proof. Let us consider the set-valued map $C_\lambda(t) := \frac{1}{\lambda}C(t)$ and $\tilde{F}(t,x) := \frac{1}{\lambda}F(t,\lambda x)$, where $\lambda > 0$ is such that

$$\left(1+\frac{1}{\alpha_0^2}\right)\int_{T_0}^{T}\left(|\dot{\zeta}(s)|+\beta(s)\right)ds < \lambda\rho.$$

Then, for all $s,t \in [T_0,T]$ and $x \in H$

$$\left|d_{C_\lambda(t)}(x) - d_{C_\lambda(s)}(x)\right| = \frac{1}{\lambda}\left|d_{C(t)}(\lambda x) - d_{C(s)}(\lambda x)\right| \leq \frac{1}{\lambda}|\zeta(t)-\zeta(s)|.$$

Therefore, according to Theorem 4.2, in the first case, and Theorem 4.4, in the second case, there exists a solution x_λ of

$$\begin{cases} \dot{x}_\lambda(t) \in -N\left(C_\lambda(t); x_\lambda(t)\right) + \tilde{F}_\lambda(t,x(t)) \cap \dfrac{\beta(t)}{\lambda}\mathbb{B} & \text{a.e. } t \in [T_0,T], \\ x_\lambda(T_0) = Mx_\lambda. \end{cases}$$

Define $x(t) := \lambda x_\lambda(t)$. Then, since M is positively homogeneous, it is not difficult to verify that x is a solution of (1.1). □

Remark 4.7. The argument given in the proof of Theorem 4.6 shows that there are infinitely many solutions of the nonlocal problem (1.1).

As a consequence of Theorem 4.6, we obtain the existence of periodic solutions of the perturbed sweeping process. The following corollary extends the results given in [17] and [11,12], where the authors showed the existence, respectively, for wedged and convex sets compact sets.

Corollary 4.8. *Let $F\colon [T_0,T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C\colon [T_0,T] \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1)$, $(\mathcal{H}_2)$ and $(\mathcal{H}_3)$. Assume that there exists a convex and bounded set D such that $C(T) \subseteq D \subseteq C(T_0)$. Then, there exists at least one solution of*

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t,x(t)) & a.e.\ t \in [T_0,T], \\ x(T_0) = x(T). \end{cases}$$

Proof. Let $Mx = x(T)$. Then, M satisfies $(\mathcal{H}_2^M)$, $(\mathcal{H}_3^M)$ and $M\mathcal{C} \subseteq C(T) \subseteq D$. Therefore, the result follows from Theorem 4.6. □

The following result, which is a direct consequence of Theorem 4.6, deals with several common nonlocal initial conditions for the sweeping process governed by a fixed set C.

Corollary 4.9. *Let $F\colon [T_0,T]\times H \rightrightarrows H$ satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ and $C \subseteq H$ be a fixed compact and convex set. Assume that the operator $M\colon C([T_0,T];H)\to H$ is one of the following operators:*

(i) *$Mx = x(T)$ (periodic initial condition);*
(ii) *$Mx = \frac{1}{T-T_0}\int_{T_0}^{T} x(s)ds$ (mean value initial condition);*
(iii) *$Mx = \sum_{i=1}^{k_0}\alpha_i x(t_i)$ with $\alpha_i \in \mathbb{R}^+$ and $\sum_{i=1}^{k_0}\alpha_i = 1$, where $T_0 < t_1 < \cdots < t_{k_0} \le T$ (multi-point initial condition).*

Then, there exists at least one solution of

$$\begin{cases} \dot{x}(t) \in -N(C;x(t)) + F(t,x(t)) & a.e.\ t\in[T_0,T],\\ x(T_0) = Mx. \end{cases}$$

Moreover, $\|\dot{x}(t)\| \le 2\beta(t)$ for a.e. $t\in[T_0,T]$.

5. Sweeping process governed by convex sets without compactness

In Theorems 4.2 and 4.4, we gave existence results for the sweeping process with nonlocal initial conditions. In these two theorems, due to the presence of the strong-weak upper semicontinuity and set-valuedness of the perturbation, it is assumed that the moving sets are ball compact (see $(\mathcal{H}_3)$. In fact, the ball compactness hypothesis $(\mathcal{H}_3)$ seems to be a strong assumption, but it is not. We refer to [23] for an example of perturbed sweeping processes with Cauchy initial condition, governed by a ball, without existence of solutions. Nevertheless, in the case where the moving sets $C(t)$ are convex for all $t\in[T_0,T]$ and the perturbation is single-valued and one-sided Lipschitz (see Assumption 3. in Theorem 5.1), it is possible to obtain similar results to Theorems 4.2 and 4.4, without the ball compactness of the moving sets.

The following result improves [11, Theorem 4.7], where the existence of periodic solutions of the sweeping processes is addressed for compact convex moving sets in finite dimensions.

Theorem 5.1. *Let $F\colon [T_0,T]\times H\to H$ be a function satisfying*

(1) *For every $x\in H$, $F(\cdot,x)$ is measurable.*
(2) *For every $t\in[T_0,T]$, $F(t,\cdot)$ is strongly-weakly continuous.*
(3) *For a.e. $t\in[T_0,T]$ and all $x,y\in H$*

$$\langle F(t,x)-F(t,y), x-y\rangle \le \omega(t)\|x-y\|^2,$$

where $\omega\in L^2(T_0,T)$.

(4) *There exists $d\ge 0$ such that, for all $t\in[T_0,T]$ and all $x,y\in H$*

$$\|F(t,x)\| \le d\left(1+\|x\|\right).$$

Assume, in addition to $(\mathcal{H}_1)$, that the sets $C(t)$ are convex for all $t\in[T_0,T]$ and there exists a convex a bounded set D such that $M\mathcal{C}\subseteq D\subseteq C(T_0)$, where $\mathcal{C}$ is given by (4.1). Assume that one of the following conditions is satisfied:

(i) $(\mathcal{H}_1^M)$ *holds and* $\alpha := m \times \max_{t\in[T_0,T]} \exp\left(\int_{T_0}^t \omega(s)ds\right) < 1$.

(ii) D *is bounded,* $(\mathcal{H}_2^M)$ *holds and* ω *is nonpositive.*

Then, there exists at least one solution of (1.1).

Proof. Let us consider the perturbed sweeping process with Cauchy initial condition:

$$\begin{cases} \dot{x}(t) \in -N\left(C(t); x(t)\right) + F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = a \in C(T_0). \end{cases}$$

Due to [40, Theorem 3.1] (see also [41, Theorem 5.7]), this differential inclusion has a unique solution, denoted by x_a. Moreover, if $a_1, a_2 \in C(T_0)$, then for $i = 1, 2$

$$\|x_{a_1}(t) - x_{a_2}(t)\| \leq \exp\left(\int_{T_0}^t \omega(s)ds\right) \|a_1 - a_2\| \quad \text{a.e. } t \in [T_0, T]. \tag{5.1}$$

Indeed, if $a_1, a_2 \in C(T_0)$, then for $i = 1, 2$

$$\dot{x}_{a_i} \in -N\left(C(t); x_{a_i}(t)\right) + F(t, x_{a_i}(t)) \quad \text{a.e. } t \in [T_0, T],$$

that is, for $i = 1, 2$ and for a.e. $t \in [T_0, T]$

$$\langle -\dot{x}_{a_i}(t) + F(t, x_{a_i}(t)), y - x_{a_i}(t)\rangle \leq 0 \quad \text{for all } y \in C(t).$$

Since for $i = 1, 2$, $x_{a_i} \in C(t)$ for all $t \in [T_0, T]$, for a.e. $t \in [T_0, T]$

$$\langle -\dot{x}_{a_1}(t) + F(t, x_{a_1}(t)), x_{a_2}(t) - x_{a_1}(t)\rangle \leq 0,$$
$$\langle -\dot{x}_{a_2}(t) + F(t, x_{a_2}(t)), x_{a_1}(t) - x_{a_2}(t)\rangle \leq 0.$$

By adding these two inequalities, we obtain for a.e. $t \in [T_0, T]$

$$\frac{1}{2}\frac{d}{dt}\|x_{a_1}(t) - x_{a_2}(t)\|^2 \leq \langle F(t, x_{a_1}(t)) - F(t, x_{a_2}(t)), x_{a_1}(t) - x_{a_2}(t)\rangle.$$

Hence, due to Assumption 3., for a.e. $t \in [T_0, T]$

$$\frac{1}{2}\frac{d}{dt}\|x_{a_1}(t) - x_{a_2}(t)\|^2 \leq \omega(t)\|x_{a_1}(t) - x_{a_2}(t)\|^2,$$

which, by elementary calculations, proves (5.1).
Let $\mathcal{F}\colon D \to H$ de the operator defined by $\mathcal{F}(a) = Mx_a$. Then, since $M\mathcal{C} \subseteq D \subseteq C(T_0)$, $\mathcal{F}(D) \subseteq D$. We now distinguish two cases:

(a) Assume that (i) holds: The hypothesis $(\mathcal{H}_1^M)$ and (5.1) implies that for any $a, b \in C(T_0)$

$$\begin{aligned} \|\mathcal{F}(a) - \mathcal{F}(b)\| &= \|Mx_a - Mx_b\| \\ &\leq m \times \sup_{t\in[T_0,T]} \exp\left(\int_{T_0}^t \omega(s)ds\right) \|a - b\| \\ &\leq \alpha\|a - b\|, \end{aligned}$$

where, by assumption, $\alpha < 1$. Therefore, $\mathcal{F}$ is a contraction, thus, it admits a fixed point.

(b) Assume that (ii) holds: The hypothesis $(\mathcal{H}_2^M)$ and (5.1) implies that for any $a, b \in C(T_0)$

$$\begin{aligned}\|\mathcal{F}(a)-\mathcal{F}(b)\| &= \|Mx_a - Mx_b\| \\ &\leq \sup_{t\in[T_0,T]} \exp\left(\int_{T_0}^t \omega(s)ds\right)\|a-b\| \\ &\leq \|a-b\|,\end{aligned}$$

where we have used that ω is nonpositive. Therefore, $\mathcal{F}\colon D \to D$ is a nonexpansive operator between bounded sets in Hilbert spaces, thus, due to [10, Theorem 1], it admits a fixed point.

Therefore, in any case, the operator $\mathcal{F}$ admits a fixed point. Finally, it is clear that any fixed point of $\mathcal{F}$ is a solution of (1.1). □

6. The case of H is compactly embedded in a Banach space E

In this section we assume that $(H, \|\cdot\|_H)$ is compactly embedded in a separable Banach space $(E, \|\cdot\|_E)$ (for example, $H = H^1(\Omega)$ and $E = L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^n$ is an open domain with Lipschitz boundary).

Let $F\colon [T_0, T] \times H \rightrightarrows H$ be a set-valued map satisfying hypotheses $(\mathcal{H}_1^F)$ and $(\mathcal{H}_2^F)$. In this section we study existence of solutions for the following differential inclusion:

$$\left\{\begin{array}{l}\dot{x}(t) \in F(t, x(t)) \quad \text{a.e. } t \in [T_0, T],\\ x(T_0) = Mx,\end{array}\right. \tag{6.1}$$

where $M\colon C([T_0, T]; H) \to H$ is a (possibly nonlinear) operator and F satisfies the additional hypothesis $(\mathcal{H}_5^F)$ (see Section 3). We emphasize that several control problem for first-order partial integro-differential equations (e.g., with $H = H^1(\Omega)$ and $E = L^2(\Omega)$) can be formulated as (6.1) (see, e.g., [6, 7]).

Now we introduce the concept of bounding function. We distinguish between weak and strong bounding function according to whether the infimum or the supremum over F is considered. We point out that our definition of weak bounding function coincides with the given in [7] under the name of merely "bounding function".

Definition 6.1. Let $V\colon H \to \mathbb{R}$ be a locally Lipschitz function such that $V(x) = 0$ for $\|x\|_H = R_0$ and $V(x) < 0$ for $r_0 < \|x\|_H < R_0$.

(a) We say that V is a *weak bounding function* if V is C^1 in the ring $\{x \in H\colon r_0 < \|x\|_H < R_0\}$ and there exists a sequence $(n_m)_m \subseteq \mathbb{N}$ converging to $+\infty$ such that for a.e. $t \in [T_0, T]$

$$\inf_{d\in F(t,P_{n_m}(x))} \langle \nabla V(P_{n_m}(x)), P_{n_m}(d)\rangle \leq 0 \quad \text{for all } r_0 < \|P_{n_m}(x)\|_H < R_0. \tag{6.2}$$

(b) We say that V is a *strong bounding function* if there exists a sequence $(n_m)_m \subseteq \mathbb{N}$ converging to $+\infty$ such that for a.e. $t \in [T_0, T]$ and all $r_0 < \|P_{n_m}(x)\|_H < R_0$

$$\sup_{d\in F(t,P_{n_m}(x))} \min\{DV(P_{n_m}(x); P_{n_m}(d)), D(-V)(P_{n_m}(x); -P_{n_m}(d))\} \leq 0. \tag{6.3}$$

Remark 6.2. (i) If V is differentiable at x, then

$$\min\{DV(x;d), D(-V)(x;-d)\} = \langle \nabla V(x), d\rangle .$$

Thus, if V is differentiable in the ring $\{x \in H\colon r_0 < \|x\|_H < R_0\}$, then every strong bounding function is indeed a weak bounding function.

(ii) The bounding function is unaffected by changing F outside the ball $R_0\mathbb{B}$.

(iii) When $V(x) := \frac{1}{2}\left(\|x\|_H^2 - R_0^2\right)$, the notion of weak bounding function is equivalent to the well known "Hartman's type condition" (see Example 7.5): For a.e. $t \in [T_0, T]$

$$\inf_{d\in F(t,x)} \langle \nabla V(x), d\rangle \le 0 \quad \text{for all } r_0 < \|x\|_H < R_0.$$

By using the notion of bounding function, we can prove an existence result for (6.1). The statement (ii) of the following theorem extends the results of [6] by allowing to M be a nonlinear map. Moreover, statement (iii) of the following theorem extends [28, Theorem 7] to infinite dimensions and extends the main result of [7], by allowing to M to be a nonlinear map and F to be multivalued and merely upper semicontinuous from E into E_w.

Theorem 6.3. *Assume that H is compactly embedded in E. Let $F\colon [T_0, T]\times H \rightrightarrows H$ be a set-valued map satisfying $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_5^F)$. Assume that one of the following conditions is verified:*

(i) *$(\mathcal{H}_3^F)$, $(\mathcal{H}_1^M)$ and $(\mathcal{H}_3^M)$ hold.*

(ii) *$(\mathcal{H}_4^F)$, $(\mathcal{H}_2^M)$ and $(\mathcal{H}_3^M)$ hold, $M\left(C\left([T_0, T]; R_0\mathbb{B}_H\right)\right) \subseteq R_0\mathbb{B}_H$ and there exists a weak bounding function V for F.*

(iii) *$(\mathcal{H}_4^F)$, $(\mathcal{H}_2^M)$ and $(\mathcal{H}_3^M)$ hold, $M\left(C\left([T_0, T]; R_0\mathbb{B}_H\right)\right) \subseteq R_0\mathbb{B}_H$ and there exists a strong bounding function V for F.*

Then, there exists at least one solution of (6.1).

Proof. (i) According to Step 1 from the proof of Theorem 4.2, for each $n \in \mathbb{N}$, there exists x_n solution of

$$\begin{cases} \dot{x}_n(t) \in F(t, P_n(x_n(t))) \cap \beta(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T],\\ x_n(T_0) = P_n(Mx_n). \end{cases}$$

Define

$$L := \frac{1}{1-m}\left(\|M0\|_H + \int_{T_0}^{T} \beta(s)ds\right).$$

Then, $\|\dot{x}_n(t)\|_H \le \beta(t)$ for a.e. $t \in [T_0, T]$ and $\|x_n(t)\|_H \le L$ for all $t \in [T_0, T]$. Indeed, for all $t \in [T_0, T]$

$$\begin{aligned} \|x_n(t)\|_H &\le \|x_n(T_0)\|_H + \int_{T_0}^{t} \beta(s)ds \\ &\le \|Mx_n\|_H + \int_{T_0}^{t} \beta(s)ds \\ &\le m \sup_{t\in[T_0,T]} \|x_n(t)\|_H + \|M0\|_H + \int_{T_0}^{t} \beta(s)ds. \end{aligned}$$

Therefore, as in the proof of Claim 6 from Theorem 4.2, there exists a subsequence of $(x_n)_n$ (without relabeling) and a absolutely continuous function $x\colon [T_0,T]\to H$ such that

$$\begin{aligned}&x_n(t)\to x(t) \text{ weakly in } H \text{ for all } t\in[T_0,T],\\ &x_n\to x \text{ in } L^1_w\left([T_0,T];H\right),\\ &\dot{x}_n\to \dot{x} \text{ in } L^1_w\left([T_0,T];H\right).\end{aligned}$$

Moreover, due to the compactness of the embedding $H\hookrightarrow E$,

$$x_n(t)\to x(t) \text{ in } E \text{ for every } t\in[T_0,T].$$

These conditions, $(\mathcal{H}_5^F)$ and the convergence theorem (see [3, Proposition 5] for more details) imply that x satisfies $\dot{x}(t)\in F(t,x(t))$ for a.e. $t\in[T_0,T]$. Finally, due to $(\mathcal{H}_3^M)$, $P_nMx_n\to Mx$ weakly in H (up to a subsequence), which finishes the proof.

Define

$$\begin{aligned}\tilde{F}(t,x)&:=\{d\in F(t,x)\colon \alpha(x)\left\langle \nabla V(x),d\right\rangle_H\leq 0\},\\ G(t,x)&:=\tilde{F}\left(t,\operatorname{proj}_{R_0\mathbb{B}_H}(x)\right)\cap v_{R_0}(t)\mathbb{B}_H,\end{aligned}$$

where

$$\alpha(x)=\begin{cases}1 & \text{if } r_0<\|x\|_H<R_0,\\ 0 & \text{otherwise.}\end{cases}$$

By similar arguments as in [6], the set-valued map G satisfies $(\mathcal{H}_1^F)$,$(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$.

Fix $r\in(r_0,R_0)$. For each $n\in\mathbb{N}$ let x_n be a solution (whose existence is guaranteed by Step 1 from the proof of Theorem 4.2) of

$$\begin{cases}\dot{x}_n(t)\in G(t,P_n(x_n(t))) & \text{a.e. } t\in[T_0,T],\\ x_n(T_0)=\operatorname{proj}_{r\mathbb{B}_H}\left(\dfrac{r}{R_0}P_n(M(P_nx_n))\right). & \end{cases}$$

Therefore, for all $t\in[T_0,T]$,

$$\|x_n(t)\|_H\leq r+\int_{T_0}^t v_{R_0}(s)ds.$$

After taking a subsequence (without relabeling), we can assume that (6.2) holds. Now we proceed to prove that $P_n(x_n(t))\in R_0\mathbb{B}_H$. Indeed, otherwise, since

$$\|P_n(x_n(T_0))\|_H\leq r,$$

we can find $t_0\in(T_0,T]$ and $\varepsilon>0$ such that $\|P_n(x_n(t_0))\|_H=R_0$ and $r_0<\|P_n(x_n(t))\|_H<R_0$ for $t\in(t_0-\varepsilon,t_0)$. We observe that for all $t\in(t_0-\epsilon,t_0)$

$$\begin{aligned}G(t,P_n(x(t)))&=\tilde{F}(t,\operatorname{proj}_{R_0\mathbb{B}}(P_n(x_n(t))))\cap v_{R_0}(t)\mathbb{B}_H\\ &=\tilde{F}(t,P_n(x_n(t)))\cap v_{R_0}(t)\mathbb{B}_H\\ &\subseteq F(t,P_n(x_n(t)))\cap v_{R_0}(t)\mathbb{B}_H.\end{aligned}\tag{6.4}$$

Define $g_n(t):=V\left(P_n(x_n(t))\right)$ in $(t_0-\delta,t_0)$, where $\delta\in(0,\varepsilon)$ is such that g_n is absolutely continuous in $(t_0-\delta,t_0)$. Then, $\dot{g}_n(t)$ exists for a.e. $t\in(t_0-\delta,t_0)$. On the one hand,

$$\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds = V(P_n(x_n(t_0))) - V(P_n(x_n(t_0-\delta))) \\ = -V(P_n(x_n(t_0-\delta))) > 0. \tag{6.5}$$

On the other hand, for a.e. $t \in (t_0-\delta, t_0)$,

$$\dot{g}_n(t) = \langle \nabla V\left(P_n(x_n(t))\right), P_n(\dot{x}_n(t))\rangle_H \quad \leq 0,$$

where we have used the definition of G, (6.4) and the definition of weak bounding function. Thus, $\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds \leq 0$, which gives a contradiction with (6.5). Hence, $P_n(x_n(t)) \in R_0\mathbb{B}_H$ for all $t \in [T_0, T]$.

So, by the assumptions of (ii), $M(P_nx_n) \in R_0\mathbb{B}_H$, which implies that $\frac{r}{R_0}P_nM(P_nx_n) \in r\mathbb{B}_H$. Thus,

$$x_n(T_0) = \frac{r}{R_0}P_nM(P_nx_n).$$

Therefore, for each $n \in \mathbb{N}$, there exists x_n solution of

$$\begin{cases} \dot{x}_n(t) \in F(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \dfrac{r}{R_0}P_n(M(P_nx_n)). \end{cases}$$

Then, by passing to the limit (up to a subsequence), as in (i), we obtain the existence of a solution $x\colon [T_0, T] \to R_0\mathbb{B}_H$ of

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x(T_0) = \dfrac{r}{R_0}Mx. \end{cases}$$

Let $(r_k)_k$ be a sequence converging to R_0 with $r_k \in (r_0, R_0)$. Then, for each $k \in \mathbb{N}$, there exists x_k solution of

$$\begin{cases} \dot{x}_k(t) \in F(t, x_k(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_k(T_0) = \dfrac{r_k}{R_0}Mx_k, \end{cases}$$

with $x_k(t) \in R_0\mathbb{B}_H$ for all $t \in [T_0, T]$. Therefore, by passing to the limit (up to a subsequence), as in (i), we obtain the existence of a solution x of

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases}$$

which finishes the proof.

(ii) Fix $r \in (r_0, R_0)$. For each $n \in \mathbb{N}$ let x_n be a solution (whose existence is guaranteed by Step 1 from the proof of Theorem 4.2) of

$$\begin{cases} \dot{x}_n(t) \in F(t, \operatorname{proj}_{R_0\mathbb{B}_H}(P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H & \text{a.e. } t \in [T_0, T], \\ x_n(T_0) = \operatorname{proj}_{r\mathbb{B}_H}\left(\dfrac{r}{R_0}P_n(M(P_nx_n))\right). \end{cases}$$

Therefore, for all $t \in [T_0, T]$,

$$\|x_n(t)\|_H \leq r + \int_{T_0}^{t} v_{R_0}(s)ds.$$

After taking a subsequence (without relabeling), we can assume that (6.3) holds. We proceed to prove that $P_n(x_n(t)) \in R_0\mathbb{B}_H$. Indeed, otherwise, since

$$\|P_n(x_n(T_0))\|_H \leq r,$$

we can find $t_0 \in (T_0, T]$ and $\varepsilon > 0$ such that $\|P_n(x(t_0))\|_H = R_0$ and $r_0 < \|P_n(x_n(t))\|_H < R_0$ for $t \in (t_0 - \varepsilon, t_0)$. We observe that for all $t \in (t_0 - \varepsilon, t_0)$

$$\begin{aligned} F(t, \operatorname{proj}_{R_0\mathbb{B}}(P_n(x_n(t)))) \cap v_{R_0}(t)\mathbb{B}_H &= F(t, P_n(x_n(t))) \cap v_{R_0}(t)\mathbb{B}_H \\ &\subseteq F(t, P_n(x_n(t))). \end{aligned} \tag{6.6}$$

Define $g_n(t) := V(P_n(x_n(t)))$ in $(t_0 - \delta, t_0)$, where $\delta \in (0, \varepsilon)$ is such that g_n is absolutely continuous in $(t_0 - \delta, t_0)$. Then, $\dot{g}_n(t)$ exists for a.e. $t \in (t_0 - \delta, t_0)$. On the one hand,

$$\begin{aligned} \int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds &= V(P_n(x_n(t_0))) - V(P_n(x_n(t_0 - \delta))) \\ &= -V(P_n(x_n(t_0 - \delta))) > 0, \end{aligned} \tag{6.7}$$

because $\|P_n(x(t_0))\|_H = R_0$. On the other hand, for a.e. $t \in (t_0 - \delta, t_0)$,

$$\begin{aligned} \dot{g}_n(t) &= \lim_{h \to 0} \frac{V(P_n(x_n(t+h))) - V(P_n(x_n(t)))}{h} \\ &= \lim_{h \to 0} \frac{V(P_n(x_n(t)) + hP_n\dot{x}_n(t)) - V(P_n(x_n(t)))}{h} \\ &= DV\left(P_n(x_n(t)); P_n\dot{x}_n(t)\right) \\ &= D(-V)\left(P_n(x_n(t)); -P_n\dot{x}_n(t)\right) \\ &\leq 0, \end{aligned}$$

where we have used (6.6) and the definition of the strong bounding function for F. Thus, $\int_{t_0-\delta}^{t_0} \dot{g}_n(s)ds \leq 0$, which gives a contradiction with (6.7). The rest of the proof, follows as in (ii). □

Remark 6.4. If $M\colon C([T_0, T]; H) \to H$ satisfies $M(0) = 0$, which is true if, e.g., M is linear, then with the notation of Theorem 6.3,

$$M\left(C\left([T_0, T]; R_0\mathbb{B}\right)\right) \subseteq R_0\mathbb{B}.$$

Indeed, if $x \in C([T_0, T]; R_0\mathbb{B})$, then $\|Mx\| \leq \|x\| \leq R_0$.

7. Tangential conditions

In this section, we give an abstract result for the abstract problem (6.1) in finite dimensions.

Let us consider the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in F(t, x(t)) & \text{a.e. } t \in [T_0, T], \\ x(t) \in D & \text{for all } t \in [T_0, T], \\ x(T_0) = x_0 \in D, \end{cases} \tag{7.1}$$

and the set-valued map $\mathcal{S}\colon D \rightrightarrows D$ defined for each $x_0 \in D$ as, $\mathcal{S}(x_0)$, the set of solutions of (7.1). A classical approach to find solutions of (6.1) is to apply some fixed point theorem to the set-valued map $M \circ \mathcal{S}$. Of course, some conditions are needed to obtain the nonemptiness of the values of $\mathcal{S}$. These conditions, generally, are of tangential type for some appropriate tangent cone. In fact, it is well known that existence solutions of (7.1) can be obtained when the set-valued map F has a nonempty intersection with the Bouligand tangent cone (see [5, Theorem 2]). However, in order to get some topological properties of the values of $\mathcal{S}$, we will ask for a strong property, namely, the intersection of the set-valued map F with the Clarke tangent cone is nonempty, i.e.,

$$F(t, x) \cap T^C(D; x) \neq \emptyset \quad \text{for all } (t, x) \in [T_0, T] \times D. \tag{7.2}$$

The following proposition is a direct consequence of [5, Theorem 16].

Proposition 7.1. *Let H be a finite-dimensional Hilbert space. Let $D \subseteq H$ be a closed and bounded set. Assume that D is positively α-far and* (7.2) *holds. Then, for any $x_0 \in D$, the set $\mathcal{S}(x_0)$ of solutions of* (7.1) *is nonempty, compact and an R_δ-set. Moreover, the set-valued map $\mathcal{S}\colon D \rightrightarrows C([T_0, T]; D)$ is upper semicontinuous.*

Remark 7.2.

i) If F is single-valued with $F(t, \cdot)$ continuous for all $t \in [T_0, T]$, then (7.2) is equivalent to

$$F(t, x) \cap T^B(D; x) \neq \emptyset \quad \text{for all } (t, x) \in [T_0, T] \times D.$$

Indeed, let $(x_n)_n \subseteq D$ converging to $x \in D$. Then, for all $t \in [T_0, T]$, $F(t, x_n) \in T^B(D; x_n)$ and,

$$F(t, x) = \lim_{n \to +\infty} F(t, x) \in \liminf_{y \to x, y \in D} T^B(D; y) = T^C(D; x).$$

ii) See [5, Example 4] for an example of a positively α-far set D and a set-valued map F whose intersection with the Bouligand tangent cone is nonempty but the solution map $\mathcal{S}$ does not have R_δ-values.

Now we can state an existence result for (6.1).

Theorem 7.3. *Let H be a finite-dimensional Hilbert space. Assume that $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_4^F)$ hold. Let M be a Lipschitz map such that there exists a closed, contractible, positively α-far and bounded set D, satisfying* (7.2)*, such that $M(C([T_0, T]; D)) \subseteq D$. Then, there exists at least one solution of* (6.1)*. Moreover, $x(t) \in D$ for all $t \in [T_0, T]$.*

Proof. It is enough to apply Proposition 2.2 with $X := D$, $\Phi := \mathcal{S}$ and $f := M$. $\square$

The following result gives a characterization of the tangential condition (7.2) for convex sets.

Proposition 7.4 ([20]). *Let $S \neq H$ be a closed convex set and $t \in [T_0, T]$. Then, the following conditions are equivalent:*

i) $F(t,x) \cap T^C(S;x) \neq \emptyset$ *for all* $x \in S$.
ii) $\inf\limits_{v \in F(t,x)} \langle v, \zeta \rangle \leq 0$ *for all* $\zeta \in N(S;x)$ *and* $x \in S$
iii) $\inf\limits_{v \in F(t,x)} \langle v, \zeta \rangle \leq 0$ *for all* $\zeta \in \partial d_S(x)$ *and* $x \in S$.
iv) $\inf\limits_{v \in F(t,x)} \langle v, \zeta \rangle \leq 0$ *for all* $\zeta \in \partial \Delta_S(x)$ *and* $x \in \operatorname{bd} S$, *where* $\Delta_S(x) = d_S(x) - d_{S^c}(x)$.

Example 7.5. Let us consider $S := R_0 \mathbb{B}$. Then, the condition $F(t,x) \cap T^C(S;x) \neq \emptyset$ is equivalent to

$$\inf_{v \in F(t,x)} \langle v, x \rangle \leq 0 \quad \text{for all } x \text{ with } \|x\| = R_0. \tag{7.3}$$

Inequality (7.3) is known in the literature as Hartman's condition and was first used by Hartman in the context of second order systems (see [19]). Since then, it has been used to deal with periodic problems (see, e.g., [2, 6]).

Example 7.6. Let $V : H \to \mathbb{R}$ be a convex function such that $S := \{x \in H : V(x) \leq 0\}$ is bounded with nonempty interior. Then, the condition (7.2) is equivalent to

$$\inf_{v \in F(t,x)} \langle v, \zeta \rangle \leq 0 \quad \text{for all } \zeta \in \partial V(x) \text{ and } x \in S \text{ with } V(x) = 0.$$

Example 7.7. Let $V : H \to \mathbb{R}$ be a C^1 function. Define $S := \{x \in H : V(x) \leq 0\}$ and assume that S is bounded, $\operatorname{bd} S = \{x \in H : V(x) = 0\}$ and that $\nabla V(x) \neq 0$ for all $x \in \partial S$. Then, S is positively α-far and condition (7.2) is equivalent to

$$\inf_{v \in F(t,x)} \langle v, \nabla V(x) \rangle \leq 0 \quad \text{for all } x \in \operatorname{bd} S. \tag{7.4}$$

If (7.4) holds for all $x \in H$, it is said that V is a weak Lyapunov function for F (see [14]). Therefore, the existence of a weak Lyapunov function for F, with bounded level sets, implies the existence of solutions for (6.1).

8. An application to nonlocal differential complementarity systems

Let K be a closed convex cone in $\mathbb{R}^m$ and K^* be its dual cone. In this section, we consider differential complementarity systems (CDSs), which are differential equations coupled with complementarity conditions (see [37] for more details). More specifically,

$$\begin{cases} \dot{x}(t) = f(t, x(t), u(t)) & \text{a.e. } t \in [T_0, T], \\ K \ni u(t) \perp (G(t, x(t)) + F(u(t))) \in K^* & \text{a.e. } t \in [T_0, T], \\ x(T_0) = Mx, \end{cases} \tag{8.1}$$

where $f : [T_0, T] \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, $G : [T_0, T] \times \mathbb{R}^n \to \mathbb{R}^m$ and $F : \mathbb{R}^m \to \mathbb{R}^m$ are continuous mappings; $M : C([T_0, T]; \mathbb{R}^n) \to \mathbb{R}^n$ is a (possible nonlinear) operator.

In order to give sufficient conditions for the existence of solutions of (8.1), we consider the following hypotheses:

($\mathcal{A}_1$) for each $(t,z) \in [T_0,T] \times \mathbb{R}^n$ the set

$$f(t,z,\Omega) := \{f(t,z,y)\colon y \in \Omega\}$$

is convex for every convex subset $\Omega \subseteq \mathbb{R}^m$.

($\mathcal{A}_2$) for every bounded subset $Z \subseteq \mathbb{R}^n \times \mathbb{R}^m$ there exists $\alpha_Z > 0$ such that $\|f(t,z,w)\| \le \alpha_Z$ for $(t,z,w) \in [T_0,T] \times Z$.

($\mathcal{A}_3$) for every bounded subset $\Omega \subseteq \mathbb{R}^n$there exists $\gamma_\Omega > 0$ such that

$$\|G(t,z)\| \le \gamma_\Omega \quad \text{for } (t,z) \in [T_0,T] \times \Omega.$$

($\mathcal{A}_4$) F is monotone and there exists $a_* > 0$ such that

$$\langle x, F(x)\rangle \ge a_* |x\|^2 \quad \text{for all } x \in K.$$

Remark 8.1. A common example is $f(t,x,y) \equiv \tilde{f}(t,x) + B(t,x)y$ (see [37] for more details).

Let $U\colon [T_0,T] \times \mathbb{R}^n \rightrightarrows K$ be the set-valued map defined as

$$U(t,z) := \mathrm{SOL}\,(K, G(t,z) + F) = \{w \in K\colon \langle w, G(t,z) + F(w)\rangle = 0\}.$$

According to [28, Lemma 9], under ($\mathcal{A}_4$), for every $z \in \mathbb{R}^n$, the set $U(t,z)$ is nonempty, convex and closed. Consider $\Phi\colon [T_0,T] \times \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ the set-valued map

$$\Phi(t,x) := \{f(t,x,w)\colon w \in U(t,x)\}.$$

Then, according to [28, Lemma 10], under ($\mathcal{A}_1$)-($\mathcal{A}_4$), Φ satisfies ($\mathcal{H}_1^F$), ($\mathcal{H}_2^F$) and ($\mathcal{H}_4^F$). Thus, the existence of solutions for (8.1) can be obtained from the following differential inclusion:

$$\begin{cases} \dot{x}(t) \in \Phi(t,x(t)) & \text{a.e. } t \in [T_0,T], \\ x(T_0) = Mx. \end{cases}$$

Therefore, by virtue of Theorems 4.4 and 6.3, we obtain the following result, which improves [28, Theorem 12] by allowing to M be a nonlinear map. Moreover, the statements (i) and (ii), in the following theorem, are new.

Theorem 8.2. *Assume, in addition to* ($\mathcal{A}_1$)-($\mathcal{A}_4$), *that one of the following conditions is verified:*

(i) ($\mathcal{H}_1^M$) *and* ($\mathcal{H}_3^M$) *hold.*

(ii) ($\mathcal{H}_2^M$) *holds and there exists a weak bounding function V for Φ.*

(iii) ($\mathcal{H}_2^M$) *holds and there exists a strong bounding function V for Φ.*

Then, there exists at least one solution of (8.1).

9. An application to hysteresis

In this section, we illustrate our results by giving an application to the existence of periodic solutions for the Play operator (see [26] for more details). We denote by $W^{1,1}_{\mathrm{per}}([T_0,T];H)$ the space of periodic absolutely continuous functions. Let $g\colon [T_0,T] \times H \times H \times H \to H$ be a continuous function such that

(9.1) $$\|g(t,x,y,w)\| \le \beta(t) \quad \text{a.e. } (t,x,y,w) \in [T_0,T] \times H \times H \times H,$$

for some $\beta \in L^1(T_0, T)$. Given $y \in W^{1,1}_{\text{per}}([T_0, T]; H)$ we consider the following differential inclusion:

$$(9.2) \qquad \begin{cases} \dot{x}(t) \in -N\left(K(y(t)); x(t)\right) + g(t, x(t), y(t), \dot{y}(t)) & \text{a.e. } t \in [T_0, T], \\ x(T_0) = x(T), \end{cases}$$

where $K \colon H \rightrightarrows H$ is a κ-Lipschitz set-valued map with nonempty, compact and convex values satisfying $K(y) \subseteq R\,\mathbb{B}$ for some $R > 0$.

Proposition 9.1. *Under the above conditions, for all $y \in W^{1,1}_{per}([T_0, T]; H)$ there exists at least one solution $x \in W^{1,1}_{per}([T_0, T]; H)$ of* (9.2). *Moreover,*

$$\|\dot{x}(t)\| \leq \kappa \|\dot{y}(t)\| + 2\beta(t) \quad \text{a.e. } t \in [T_0, T].$$

Proof. Define $C(t) := K(y(t))$ and $F(t, x) = g(t, x, y(t), \dot{y}(t))$. To show existence of solutions, we verify the hypotheses of Corollary 4.8.

- $(\mathcal{H}_1)$ holds: Let $x \in H$ and $t, s \in [T_0, T]$. Then
$$\begin{aligned} |d(x, C(t)) - d(x, C(s))| &= |d(x, C(y(t))) - d(x, C(y(s)))| \\ &\leq \kappa \|y(t) - y(s)\| \\ &\leq \kappa \left| \int_{T_0}^{t} \|\dot{y}(\tau)\| d\tau - \int_{T_0}^{s} \|\dot{y}(\tau)\| d\tau \right|, \end{aligned}$$
which shows that $(\mathcal{H}_1)$ holds with $\zeta(t) := \kappa \int_{T_0}^{t} \|\dot{y}(\tau)\| d\tau$.
- $(\mathcal{H}_1^F)$, $(\mathcal{H}_2^F)$ and $(\mathcal{H}_3^F)$ hold: It follows from the continuity of g, the fact that $y \in W^{1,1}_{\text{per}}([T_0, T]; H)$ and (9.1).
- $C(T) \subseteq D \subseteq C(T_0)$ for some set D convex and bounded: Indeed, if $D := C(T_0) \cap R\,\mathbb{B}$ then,
$$C(T) = K(y(T)) = K(y(T_0)) = C(T_0) \cap R\mathbb{B},$$
where we have used that $y \in W^{1,1}_{\text{per}}([T_0, T]; H)$.

Thus, the existence for (9.2) follows from Corollary 4.8. □

Remark 9.2. Proposition 9.1 allows us to define the set-valued Play operator

$$P \colon W^{1,1}_{\text{per}}([T_0, T]; H) \rightrightarrows W^{1,1}_{\text{per}}([T_0, T]; H),$$

which to every function y associates the set of solutions of (9.2). Thus, the Play operator is well defined for inputs in $W^{1,1}_{\text{per}}([T_0, T]; H)$.

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