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THE SPLIT COMMON FIXED POINT PROBLEM AND THE SHRINKING PROJECTION METHOD FOR NEW NONLINEAR MAPPINGS IN TWO BANACH SPACES

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Abstract. In this paper, we consider the split common fixed point problem in two Banach spaces. Using the shrinking projection method and a new nonlinear operator which generalizes strict pseudo-contractions in Hilbert spaces and the generalized projections and the generalized resolvents of maximal monotone operators in Banach spaces, we prove a strong convergence theorem for finding a solution of the common fixed point problem in two Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and Banach spaces.

1. Introduction

Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called inverse strongly monotone if there exists $\alpha > 0$ such that

$$\langle x - y, Ux - Uy \rangle \geq \alpha \|Ux - Uy\|^2, \quad \forall x, y \in C.$$

Such a mapping U is called α-inverse strongly monotone. Let H_1 and H_2 be two Hilbert spaces. Given mappings $S : H_1 \to H_1$ and $T : H_2 \to H_2$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common fixed point problem* is to find a point $z \in H_1$ such that $z \in F(S) \cap A^{-1}F(T)$, where $F(S)$ and $F(T)$ are fixed point sets of S and T, respectively. Let D and Q be nonempty, closed and convex subsets of H_1 and H_2, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator. Then the *split feasibility problem* [8] is to find $z \in H_1$ such that $z \in D \cap A^{-1}Q$. Defining $S = P_D$ and $T = P_Q$, where P_D and P_Q are the metric projections of H_1 onto D and H_2 onto Q, respectively, we have that

$$z \in D \cap A^{-1}Q \iff z \in F(S) \cap A^{-1}F(T).$$

Furthermore, given maximal monotone mappings $G : H_1 \to 2^{H_1}$ and $B : H_2 \to 2^{H_2}$, respectively, and a bounded linear operator $A : H_1 \to H_2$, the *split common null point problem* [7] is to find a point $z \in H_1$ such that $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $G^{-1}0$ and $B^{-1}0$ are null point sets of G and B, respectively. Defining $S = J_\lambda^G$ and

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$T = J_\mu^B$, where J_λ^G and J_μ^B are the resolvents of G for $\lambda > 0$ and B for $\mu > 0$, respectively, we have that

$$z \in G^{-1}0 \cap A^{-1}(B^{-1}0) \quad \Longleftrightarrow \quad z \in F(S) \cap A^{-1}F(T).$$

Thus, the split common fixed point problem generalizes the split feasibility problem and the split common null point problem. Defining $U = A^*(I - P_Q)A$ in the split feasibility problem, we have that $U : H_1 \to H_1$ is an inverse strongly monotone operator [3], where A^* is the adjoint operator of A. If $D \cap A^{-1}Q$ is nonempty, then $z \in D \cap A^{-1}Q$ is equivalent to

$$z = P_D(I - \lambda A^*(I - P_Q)A)z, \tag{1.1}$$

where $\lambda > 0$ and P_D is the metric projection of H_1 onto D. Furthermore, if $G^{-1}0 \cap A^{-1}(B^{-1}0)$ is nonempty, then $z \in G^{-1}0 \cap A^{-1}(B^{-1}0)$ is equivalent to

$$z = J_\lambda^G(I - \gamma A^*(I - J_\mu^B)A)z, \tag{1.2}$$

where $\lambda, \mu > 0$ and $\gamma > 0$. There are many results for the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces; see, for instance, [9, 10, 19, 21, 24, 39]. Recently, Takahashi [33–35] and Hojo and Takahashi [12] extended the results of (1.1) and (1.2) in Hilbert spaces to Banach spaces. Furthermore, by using the shrinking projection method, metric projections and generalized projections in Banach spaces, Takahashi and Takahashi [27, 29] proved strong convergence theorems for metric resolvents and generalized resolvents of maximal monotone operators in two Banach spaces; see also [28]. These theorems solve the split common null point problem in two Banach spaces.

In this paper, motivated by the split common null point problem in Banach spaces, we consider the split common fixed point problem in two Banach spaces. Then using the shrinking projection method and a new nonlinear operator which generalizes strict pseudo-contractions in Hilbert spaces and the generalized projections and the generalized resolvents of maximal monotone operators in Banach spaces, we prove a strong convergence theorem for finding a solution of the split common fixed point problem for such new nonlinear operators in two Banach spaces. Using this result, we get well-known and new results which are connected with the split feasibility problem, the split common null point problem and the split common fixed point problem in Hilbert spaces and Banach spaces.

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let E be a real Banach space with norm $\|\cdot\|$ and let E^* be the dual space of E. We denote the value of $y^* \in E^*$ at $x \in E$ by $\langle x, y^* \rangle$. When $\{x_n\}$ is a sequence in E, we denote the strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and the weak convergence by $x_n \rightharpoonup x$. The modulus δ of convexity of E is defined by

$$\delta(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} : \|x\| \le 1, \|y\| \le 1, \|x - y\| \ge \epsilon\right\}$$

for all ϵ with $0 \leq \epsilon \leq 2$. A Banach space E is said to be uniformly convex if $\delta(\epsilon) > 0$ for every $\epsilon > 0$. A uniformly convex Banach space is strictly convex and reflexive. We also know that a uniformly convex Banach space has the Kadec-Klee property, i.e., $x_n \rightharpoonup u$ and $\|x_n\| \to \|u\|$ imply $x_n \to u$; see [11, 25].

The duality mapping J from E into 2^{E^*} is defined by

$$Jx = \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2 = \|x^*\|^2\}$$

for all $x \in E$. Let $U = \{x \in E : \|x\| = 1\}$. The norm of E is said to be Gâteaux differentiable if for each $x, y \in U$, the limit

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.1}$$

exists. In this case, E is called smooth. We know that E is smooth if and only if J is a single-valued mapping of E into E^*. The norm of E is said to be Fréchet differentiable if for each $x \in U$, the limit (2.1) is attained uniformly for $y \in U$. The norm of E is said to be uniformly smooth if the limit (2.1) is attained uniformly for $x, y \in U$. We also know that E is reflexive if and only if J is surjective, and E is strictly convex if and only if J is one-to-one. Therefore, if E is a smooth, strictly convex and reflexive Banach space, then J is a single-valued bijection and in this case, the inverse mapping J^{-1} coincides with the duality mapping J_* on E^*. For more details, see [30] and [31]. We know the following result.

Lemma 2.1 ([30])**.** *Let E be a smooth Banach space and let J be the duality mapping on E. Then, $\langle x - y, Jx - Jy \rangle \geq 0$ for all $x, y \in E$. Furthermore, if E is strictly convex and $\langle x - y, Jx - Jy \rangle = 0$, then $x = y$.*

Let E be a smooth Banach space and let J be the duality mapping on E. Define a function $\phi : E \times E \to \mathbb{R}$ by

$$\phi_E(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2, \quad \forall x, y \in E. \tag{2.2}$$

In the case when E is clear, ϕ_E is simply denoted by ϕ. Observe that, in a Hilbert space H, $\phi(x, y) = \|x - y\|^2$ for all $x, y \in H$. Furthermore, we know that for each $x, y, z, w \in E$,

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2; \tag{2.3}$$

$$\phi(x, y) = \phi(x, z) + \phi(z, y) + 2\langle x - z, Jz - Jy \rangle; \tag{2.4}$$

$$2\langle x - y, Jz - Jw \rangle = \phi(x, w) + \phi(y, z) - \phi(x, z) - \phi(y, w). \tag{2.5}$$

If E is additionally assumed to be strictly convex, then

$$\phi(x, y) = 0 \quad \text{if and only if} \quad x = y. \tag{2.6}$$

The following lemma was proved by Kamimura and Takahashi [15].

Lemma 2.2 ([15])**.** *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}$, $\{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$ and either $\{y_n\}$ or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty, closed and convex subset of a smooth, strictly convex and reflexive Banach space E. Then, for any $x \in E$, there exists a unique element $z \in C$ such that

$$\phi(z, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $z = \Pi_C x$ is called the generalized projection of E onto C. For example, see [1, 2, 15].

Lemma 2.3 ([1, 2, 15])**.** *Let E be a smooth, strictly convex and reflexive Banach space. Let C be a nonempty, closed and convex subset of E and let $x \in E$ and $z \in C$. Then, the following conditions are equivalent:*

(1) $z = \Pi_C x$;

(2) $\langle z - y, Jx - Jz \rangle \geq 0, \quad \forall y \in C$.

Let E be a Banach space and let B be a mapping of E into 2^{E^*}. The effective domain of B is denoted by $\mathrm{dom}(B)$, that is, $\mathrm{dom}(B) = \{x \in E : Bx \neq \emptyset\}$. A multi-valued mapping B on E is said to be monotone if $\langle x - y, u^* - v^* \rangle \geq 0$ for all $x, y \in \mathrm{dom}(B)$, $u^* \in Bx$, and $v^* \in By$. A monotone operator B on E is said to be maximal if its graph is not properly contained in the graph of any other monotone operator on E. The following theorem is due to [5, 26]; see also [31, Theorem 3.5.4].

Theorem 2.4 ([5, 26])**.** *Let E be a uniformly convex and smooth Banach space and let J be the duality mapping of E into E^*. Let B be a monotone operator of E into 2^{E^*}. Then B is maximal if and only if for any $r > 0$,*

$$R(J + rB) = E^*,$$

where $R(J + rB)$ is the range of $J + rB$.

Let E be a uniformly convex Banach space with a Gâteaux differentiable norm and let B be a maximal monotone operator of E into 2^{E^*}. For all $x \in E$ and $r > 0$, we consider the following equation

$$Jx \in Jx_r + rBx_r.$$

This equation has a unique solution x_r. We define J_r^B by $x_r = J_r^B x$. Such $J_r^B, r > 0$ are called the generalized resolvents of B. The set of null points of B is defined by $B^{-1}0 = \{z \in E : 0 \in Bz\}$. We know that $B^{-1}0$ is closed and convex; see [31].

For a sequence $\{C_n\}$ of nonempty, closed and convex subsets of a Banach space E, define $\text{s-Li}_n C_n$ and $\text{w-Ls}_n C_n$ as follows: $x \in \text{s-Li}_n C_n$ if and only if there exists $\{x_n\} \subset E$ such that $\{x_n\}$ converges strongly to x and $x_n \in C_n$ for all $n \in \mathbb{N}$. Similarly, $y \in \text{w-Ls}_n C_n$ if and only if there exist a subsequence $\{C_{n_i}\}$ of $\{C_n\}$ and a sequence $\{y_i\} \subset E$ such that $\{y_i\}$ converges weakly to y and $y_i \in C_{n_i}$ for all $i \in \mathbb{N}$. If C_0 satisfies

$$C_0 = \operatorname*{s-Li}_n C_n = \operatorname*{w-Ls}_n C_n, \tag{2.7}$$

it is said that $\{C_n\}$ converges to C_0 in the sense of Mosco [23] and we write $C_0 = \text{M-lim}_{n\to\infty} C_n$. It is easy to show that if $\{C_n\}$ is nonincreasing with respect to inclusion, then $\{C_n\}$ converges to $\bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco. For more details, see [23]. The following lemma was proved by Ibaraki, Kimura and Takahashi [13].

Lemma 2.5 ([13]). *Let E be a smooth Banach space such that E^* has a Fréchet differentiable norm. Let $\{C_n\}$ be a sequence of nonempty, closed and convex subsets of E. If $C_0 = M\text{-}\lim_{n\to\infty} C_n$ exists and nonempty, then for each $x \in E$, $\{\Pi_{C_n} x\}$ converges strongly to $\Pi_{C_0} x$, where Π_{C_n} and Π_{C_0} are the generalized projections of E onto C_n and C_0, respectively.*

3. Main result

In this section, using the shrinking projection method introduced by Takahashi, Takeuchi and Kubota [36], we first prove a strong convergence theorem for finding a solution of the split common fixed point problem in two Banach spaces. Before proving the theorem, we need a few definitions and lemmas. Let E be a smooth and strictly convex Banach space and let J be the duality mapping on E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectively. Then a mapping $U : C \to E$ with $F(U) \neq \emptyset$ is called (η, s)-demigeneralized [22, 37] if, for any $x \in C$ and $q \in F(U)$,

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux) + s\phi(Ux, x), \tag{3.1}$$

where $F(U)$ is the set of fixed points of U. In particular, if $s = 0$ in (3.1), then the mapping U is as follows:

$$2\langle x - q, Jx - JUx\rangle \geq (1 - \eta)\phi(x, Ux)$$

for all $x \in C$ and $q \in F(U)$. Especially, such $(\eta, 0)$-demigeneralized mappings in the class of demigeneralized mappings are important.

Examples. (1) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \leq k < 1$. A mapping $U : C \to H$ is called a k-strict pseudo-contraction [6] if

$$\|Ux - Uy\|^2 \leq \|x - y\|^2 + k\|x - Ux - (y - Uy)\|^2$$

for all $x, y \in C$. If U is a k-strict pseudo-contraction and $F(U) \neq \emptyset$, then U is $(k, 0)$-demigeneralized [37].

(2) Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. A mapping $U : C \to H$ is called generalized hybrid [16] if there exist $\alpha, \beta \in \mathbb{R}$ such that

$$\alpha\|Ux - Uy\|^2 + (1 - \alpha)\|x - Uy\|^2 \leq \beta\|Ux - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping U is called (α, β)-generalized hybrid. If U is generalized hybrid and $F(U) \neq \emptyset$, then U is $(0, 0)$-demigeneralized [37], i.e.,

$$2\langle x - u, x - Ux\rangle \geq \|x - Ux\|^2, \quad \forall x \in C,\ u \in F(U).$$

Notice that the class of generalized hybrid mappings covers several well-known mappings. For example, a (1,0)-generalized hybrid mapping is nonexpansive. It is nonspreading [17, 18] for $\alpha = 2$ and $\beta = 1$, i.e.,

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

It is also hybrid [32] for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$, i.e.,

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2, \quad \forall x, y \in C.$$

In general, nonspreading and hybrid mappings are not continuous; see [14].

(3) Let E be a smooth, strictly convex and reflexive Banach space and let C be a nonempty, closed and convex subset of E. Let Π_C be the generalized projection of E onto C. Then Π_C is $(0,1)$-demigeneralized. In fact, since Π_C is the generalized projection of E onto C, we have that, for any $x \in E$ and $q \in C$,

$$2\langle \Pi_C x - q, Jx - J\Pi_C x\rangle \geq 0.$$

Then we get

$$2\langle \Pi_C x - x + x - q, Jx - J\Pi_C x\rangle \geq 0$$

and hence

$$\begin{aligned} 2\langle x - q, Jx - J\Pi_C x\rangle &\geq 2\langle x - \Pi_C x, Jx - J\Pi_C x\rangle \\ &= \phi(x, \Pi_C x) + \phi(\Pi_C x, x). \end{aligned}$$

This means that Π_C is $(0,1)$-demigeneralized. Furthermore, since

$$\phi(x, \Pi_C x) + \phi(\Pi_C x, x) \geq \phi(x, \Pi_C x),$$

Π_C is also $(0,0)$-demigeneralized.

(4) Let E be a uniformly convex and smooth Banach space and let B be a maximal monotone operator with $B^{-1}0 \neq \emptyset$. Let $\lambda > 0$. Then the generalized resolvent J_λ^B is $(0,1)$-demigeneralized. In fact, since J_λ^B is the generalized resolvent of B for $\lambda > 0$, we have that, for any $x \in E$ and $q \in B^{-1}0$,

$$2\langle J_\lambda^B x - q, Jx - JJ_\lambda^B x\rangle \geq 0.$$

Then we get

$$2\langle J_\lambda^B x - x + x - q, Jx - JJ_\lambda^B x\rangle \geq 0$$

and hence

$$\begin{aligned} 2\langle x - q, Jx - JJ_\lambda^B x\rangle &\geq 2\langle x - J_\lambda^B x, Jx - JJ_\lambda^B x\rangle \\ &= \phi(x, J_\lambda^B x) + \phi(J_\lambda^B x, x). \end{aligned}$$

This means that J_λ^B is $(0,1)$-demigeneralized. Furthermore, since

$$\phi(x, J_\lambda^B x) + \phi(J_\lambda^B x, x). \geq \phi(x, J_\lambda^B x),$$

J_λ^B is also $(0,0)$-demigeneralized.

The following lemma is important and crucial in the proof of our main result which was proved in [37]. For the sake of completeness, we give the proof.

Lemma 3.1 ([37])**.** *Let E be a smooth and strictly convex Banach space and let C be a nonempty, closed and convex subset of E. Let η and s be real numbers with $\eta \in (-\infty, 1)$ and $s \in [0, \infty)$, respectivelt. Let U be an $(\eta, 0)$-demigeneralized mapping of C into E. Then $F(U)$ is closed and convex. In particular, if U is (η, s)-demigeneralized, then $F(U)$ is closed and convex.*

Proof. Assume that U is an $(\eta, 0)$-demigeneralized. Let us show that $F(U)$ is closed. For a sequence $\{q_n\}$ such that $q_n \to q$ and $q_n \in F(U)$, we have from the definition of U that

$$\langle q - q_n, Jq - JUq\rangle \geq \frac{1-\eta}{2}\phi(q, Uq).$$

From $q_n \to q$, we have $0 \geq \frac{1-\eta}{2}\phi(q, Uq)$. From $1-\eta > 0$, we get $0 \geq \phi(q, Uq)$ and hence $q = Uq$. This implies that $F(U)$ is closed. Let us prove that $F(U)$ is convex. Let $p, q \in F(U)$ and set $x = \alpha p + (1-\alpha)q$, where $\alpha \in [0, 1]$. From the definition of U, we have that, for $x \in C$ and $u \in F(U)$,

$$\langle x - u, Jx - JUx \rangle \geq \frac{1-\eta}{2}\phi(x, Ux).$$

This implies from (2.5) that

$$\phi(x, Ux) + \phi(u, x) - \phi(u, Ux) \geq (1-\eta)\phi(x, Ux)$$

and hence

$$\phi(u, x) + \eta\phi(x, Ux) \geq \phi(u, Ux).$$

Using this, we have that, for $x = \alpha p + (1-\alpha)q$ and $p, q \in F(U)$,

$$\begin{aligned}
\phi(x, Ux) &= \|x\|^2 - 2\langle x, JUx \rangle + \|Ux\|^2 \\
&= \|x\|^2 - 2\langle \alpha p + (1-\alpha)q, JUx \rangle + \|Ux\|^2 \\
&= \|x\|^2 - 2\alpha\langle p, JUx \rangle - 2(1-\alpha)\langle q, JUx \rangle + \|Ux\|^2 \\
&= \|x\|^2 + \alpha\phi(p, Ux) + (1-\alpha)\phi(q, Ux) - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&\leq \|x\|^2 + \alpha(\phi(p, x) + \eta\phi(x, Ux)) \\
&\qquad + (1-\alpha)(\phi(q, x) + \eta\phi(x, Ux)) - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&= \|x\|^2 + \alpha(\|p\|^2 - 2\langle p, Jx \rangle + \|x\|^2 + \eta\phi(x, Ux)) \\
&\qquad + (1-\alpha)(\|q\|^2 - 2\langle q, Jx \rangle + \|x\|^2 + \eta\phi(x, Ux)) - \alpha\|p\|^2 - (1-\alpha)\|q\|^2 \\
&= 2\|x\|^2 - 2\langle \alpha p + (1-\alpha)q, Jx \rangle + \eta\phi(x, Ux) \\
&= 2\|x\|^2 - 2\langle x, Jx \rangle + \eta\phi(x, Ux) \\
&= \eta\phi(x, Ux)
\end{aligned}$$

and hence

$$0 \leq (\eta - 1)\phi(x, Ux).$$

We have from $0 > \eta - 1$ that $\phi(x, Ux) = 0$. Since E is strictly convex, we have $x = Ux$. This means that $F(U)$ is convex. If U is (η, s)-demigeneralized, then U is $(\eta, 0)$-demigeneralized and hence $F(U)$ is closed and convex. □

Let E be a Banach space and let C be a nonempty, closed and convex subset of E. A mapping $U : C \to E$ is called demiclosed if for a sequence $\{x_n\}$ in C such that $x_n \rightharpoonup p$ and $x_n - Ux_n \to 0$, $p = Up$ holds.

Theorem 3.2. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let τ and η be real numbers with $\tau, \eta \in (-\infty, 1)$. Let $T : E \to E$ be a $(\tau, 0)$-demigeneralized and demiclosed mapping with $F(T) \neq \emptyset$ and let $U : F \to F$ be an $(\eta, 0)$-demigeneralized and demiclosed mapping with $F(U) \neq \emptyset$. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of*

A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. *For* $x_1 \in E$ *and* $C_1 = E$, *let* $\{x_n\}$ *be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F U A x_n)\big), \\ y_n = T z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n(1-\eta)\phi_F(Ax_n, UAx_n) \\ \qquad\qquad \text{and} \quad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq (1-\tau)\phi_E(z_n, y_n)\}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0,\infty)$ *and* $a \in \mathbb{R}$ *satisfy the following inequalities:*

$$0 < a \leq r_n \leq \frac{1-\eta}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ *converges strongly to a point* $z_0 \in F(T) \cap A^{-1}F(U)$, *where* $z_0 = \Pi_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. It is obvious that C_n are closed and convex for all $n \in \mathbb{N}$. We show that $F(T) \cap A^{-1}F(U) \subset C_n$ for all $n \in \mathbb{N}$. It is easy that $F(T) \cap A^{-1}F(U) \subset C_1 = E$. Suppose that $F(T) \cap A^{-1}F(U) \subset C_k$ for some $k \in \mathbb{N}$. Using this, let us show that $2\langle x_k - z, J_E x_k - J_E z_k\rangle \geq r_k(1-\eta)\phi_F(Ax_k, UAx_k)$ and

$$2\langle z_k - z, J_E z_k - J_E y_k\rangle \geq (1-\tau)\phi_E(z_k, y_k)$$

for all $z \in F(T) \cap A^{-1}F(U)$. In fact, we have that, for all $z \in F(T) \cap A^{-1}F(U)$,

$$\begin{aligned} 2\langle x_k - z, J_E x_k - J_E z_k\rangle &= 2\langle x_k - z, r_k A^*(J_F A x_k - J_F U A x_k)\rangle \\ &= 2r_k\langle Ax_k - Az, J_F A x_k - J_F U A x_k\rangle \\ &\geq r_k(1-\eta)\phi_F(Ax_k, UAx_k). \end{aligned}$$

Furthermore, we have that, for all $z \in F(T) \cap A^{-1}F(U)$,

$$\begin{aligned} &2\langle z_k - z, J_E z_k - J_E y_k\rangle - (1-\tau)\phi_E(z_k, y_k) \\ &\quad = 2\langle z_k - z, J_E z_k - J_E T z_k\rangle - (1-\tau)\phi_E(z_k, Tz_k) \\ &\quad \geq (1-\tau)\phi_E(z_k, Tz_k) - (1-\tau)\phi_E(z_k, Tz_k) \\ &\quad = 0. \end{aligned}$$

Then, we have $F(T) \cap A^{-1}F(U) \subset C_{k+1}$. By mathematical induction, we have that $F(T) \cap A^{-1}F(U) \subset C_n$ for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well defined.

Since $F(T) \cap A^{-1}F(U)$ is a nonempty, closed and convex subset of E, there exists $z_0 \in F(T) \cap A^{-1}F(U)$ such that $z_0 = \Pi_{F(T)\cap A^{-1}F(U)} x_1$. We have from $x_n = \Pi_{C_n} x_1$ that

$$\phi_E(x_n, x_1) \leq \phi_E(y, x_1), \quad \forall y \in C_n.$$

Since $z_0 \in F(T) \cap A^{-1}F(U) \subset C_n$, we have that

$$\phi_E(x_n, x_1) \leq \phi_E(z_0, x_1), \quad \forall n \in \mathbb{N}. \tag{3.2}$$

Let $C_0 = \bigcap_{n=1}^{\infty} C_n$. Since $C_0 \supset F(T) \cap A^{-1}F(U) \neq \emptyset$, C_0 is also nonempty. Since $\{C_n\}$ converges to $C_0 = \bigcap_{n=1}^{\infty} C_n$ in the sense of Mosco and $x_n = \Pi_{C_n} x_1$ for every $n \in \mathbb{N}$, by Lemma 2.5, we have

$$x_n \to w = \Pi_{C_0} x_1. \tag{3.3}$$

We have from $x_{n+1} \in C_{n+1}$ that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \geq r_n(1-\eta)\phi_F(Ax_n, UAx_n). \tag{3.4}$$

Furthermore, we claim that $\{J_E x_n - J_E z_n\}$ is bounded. That $\{J_E x_n - J_E z_n\}$ is bounded is proved as follows. We first have that

$$\|J_E x_n - J_E z_n\| = \|r_n A^*(J_F Ax_n - J_F UAx_n)\|.$$

Furthermore, we have that

$$\|J_F Ax_n\| = \|Ax_n\| \leq \|A\|\|x_n\|.$$

Since $2\langle Ax_n - Az, J_F Ax_n - J_F UAx_n\rangle \geq (1-\eta)\phi_F(Ax_n, UAx_n)$ for $z \in A^{-1}F(U)$, we have from (2.5) that

$$\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) - \phi_F(Az, UAx_n) \geq (1-\eta)\phi_F(Ax_n, UAx_n)$$

and hence

$$\eta\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) \geq \phi_F(Az, UAx_n).$$

In the case of $\eta \leq 0$, we have $\phi_F(Az, Ax_n) \geq \phi_F(Az, UAx_n)$. So, we have that, for $z \in A^{-1}F(U)$,

$$\begin{aligned}(\|Az\| - \|UAx_n\|)^2 &\leq \phi_F(Az, UAx_n) \\ &\leq \phi_F(Az, Ax_n) \leq (\|Az\| + \|Ax_n\|)^2 \\ &\leq \|A\|^2(\|z\| + \|x_n\|)^2.\end{aligned}$$

Using this, we have that

$$\|UAx_n\| \leq \|A\|(\|z\| + \|x_n\|) + \|Az\| \leq \|A\|(\|z\| + \|x_n\|) + \|A\|\|z\|.$$

Then, we have that

$$\|J_F UAx_n\| = \|UAx_n\| \leq \|A\|(2\|z\| + \|x_n\|).$$

Hence, we have that

$$\begin{aligned}\|J_E x_n - J_E z_n\| &= \|r_n A^*(J_F Ax_n - J_F UAx_n)\| \\ &\leq \frac{1-\eta}{\|A\|^2}\|A\| \left(\|J_F Ax_n\| + \|J_F UAx_n)\|\right) \\ &\leq \frac{1-\eta}{\|A\|^2}\|A\| \left(\|A\|\|x_n\| + \|A\|(2\|z\| + \|x_n\|)\right) \\ &\leq 2(1-\eta)(\|x_n\| + \|z\|).\end{aligned}$$

This implies that $\{J_E x_n - J_E z_n\}$ is bounded. In the case of η withh $0 < \eta < 1$, we have

$$\eta\phi_F(Ax_n, UAx_n) + \phi_F(Az, Ax_n) \geq \phi_F(Az, UAx_n).$$

So, we have that, for $z \in A^{-1}F(U)$,

$$\begin{aligned}(\|Az\| - \|UAx_n\|)^2 &\leq \phi_F(Az, UAx_n) \\ &\leq \phi_F(Az, Ax_n) + \eta\phi_F(Ax_n, UAx_n) \\ &\leq (\|Az\| + \|Ax_n\|)^2 + \eta(\|Ax_n\| + \|UAx_n\|)^2 \\ &\leq \left(\|Az\| + \|Ax_n\| + \sqrt{\eta}(\|Ax_n\| + \|UAx_n\|)\right)^2.\end{aligned}$$

From this, we have that

$$\big|\|Az\| - \|UAx_n\|\big| \le \|Az\| + \|Ax_n\| + \sqrt{\eta}(\|Ax_n\| + \|UAx_n\|)$$

and hence

$$(1-\sqrt{\eta})\|UAx_n\| \le (1+\sqrt{\eta})\|Ax_n\| + 2\|Az\| \le (1+\sqrt{\eta})\|A\|\|x_n\| + 2\|A\|\|z\|.$$

Then, we have that

$$\|J_F UAx_n\| = \|UAx_n\| \le \|A\|\Big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\Big).$$

Hence, we have that

$$\begin{aligned}
\|J_E x_n - J_E z_n\| &= \|r_n A^*(J_F Ax_n - J_F UAx_n)\| \\
&\le \frac{1-\eta}{\|A\|^2}\|A\| \ \big(\|J_F Ax_n\| + \|J_F UAx_n)\|\big) \\
&\le \frac{1-\eta}{\|A\|^2}\|A\| \ \Big(\|A\|\|x_n\| + \|A\|\big(\frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\big)\Big) \\
&\le (1-\eta)\Big(\|x_n\| + \frac{1+\sqrt{\eta}}{1-\sqrt{\eta}}\|x_n\| + \frac{2}{1-\sqrt{\eta}}\|z\|\Big).
\end{aligned}$$

This implies that $\{J_E x_n - J_E z_n\}$ is bounded. Since $r_n \ge a > 0$ for all $n \in \mathbb{N}$, we have from (3.4) that

$$2\langle x_n - x_{n+1}, J_E x_n - J_E z_n\rangle \ge a(1-\eta)\phi_F(Ax_n, UAx_n). \tag{3.5}$$

Since $\|x_n - x_{n+1}\| \to 0$ from (3.3) and $\{J_E x_n - J_E z_n\}$ is bounded, we get that

$$\lim_{n\to\infty} \phi_F(Ax_n, UAx_n) = 0. \tag{3.6}$$

Therefore, we get from Lemma 2.2 that

$$\lim_{n\to\infty} \|Ax_n - UAx_n\| = 0. \tag{3.7}$$

Furthermore, since F is uniformly smooth, we have from (3.7) that

$$\lim_{n\to\infty} \|J_F Ax_n - J_F UAx_n\| = 0. \tag{3.8}$$

Since $\|J_E x_n - J_E z_n\| = \|r_n A^*(J_F Ax_n - J_F UAx_n)\|$ and $\{r_n\}$ is bounded, we get from (3.8) that

$$\lim_{n\to\infty} \|J_E x_n - J_E z_n\| = 0. \tag{3.9}$$

Since E^* is uniformly smooth, we have from (3.9) that

$$\lim_{n\to\infty} \|x_n - z_n\| = 0. \tag{3.10}$$

Furthermore, we have from $x_{n+1} \in C_{n+1}$ that

$$2\langle z_n - x_{n+1}, J_E z_n - J_E y_n\rangle \ge (1-\tau)\phi_E(z_n, y_n)$$

and hence

$$2\langle z_n - x_n + x_n - x_{n+1}, J_E z_n - J_E y_n\rangle \ge (1-\tau)\phi_E(z_n, y_n).$$

Since $\{J_E z_n - J_E y_n\}$ is bounded, from $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - z_n\| \to 0$, we have that $\lim_{n\to\infty} \phi_E(z_n, y_n) = 0$. Using Lemma 2.2 and $y_n = Tz_n$, we have that

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.11}$$

Since $x_n \to w$ and A is bounded and linear, we have that $\{Ax_n\}$ converges strongly to Aw and hence $\{Ax_n\}$ converges weakly to Aw. Since U is demiclosed, we have from (3.7) that $Aw = UAw$. From $x_n \to w$ we also have that $\{x_n\}$ converges weakly to w. Since T is demiclosed, we have from (3.11) tat $w = Tw$. This implies that $w \in F(T) \cap A^{-1}F(U)$.

From $z_0 = \Pi_{F(T)\cap A^{-1}F(U)} x_1$, $w \in F(T) \cap A^{-1}F(U)$ and (3.2), we have that

$$\phi_E(z_0, x_1) \le \phi_E(w, x_1) = \lim_{n\to\infty} \phi_E(x_n, x_1) \le \phi_E(z_0, x_1).$$

Then we get $z_0 = w$. Therefore, we have $x_n \to z_0$. This completes the proof. □

4. Applications

In this section, using Theorem 3.2, we get well-known and new strong convergence theorems which are connected with the split common fixed point problems in Banach spaces. We know the following result obtained by Marino and Xu [20]; see also [38].

Lemma 4.1 ([20, 38]). *Let H be a Hilbert space and let C be a nonempty, closed and convex subset of H. Let k be a real number with $0 \le k < 1$ and let $U : C \to H$ be a k-strict pseudo-contraction. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

We also know the following result from Kocourek, Takahashi and Yao [16]; see also [40].

Lemma 4.2 ([16, 40]). *Let H be a Hilbert space, let C be a nonempty, closed and convex subset of H and let $U : C \to H$ be generalized hybrid. If $x_n \rightharpoonup z$ and $x_n - Ux_n \to 0$, then $z \in F(U)$.*

Theorem 4.3. *Let H_1 and H_2 be Hilbert spaces. Let k be a real number with $k \in [0, 1)$. Let $T : H_1 \to H_1$ be a nonexpansive mapping and let $U : H_2 \to H_2$ be a k-strict pseudo-contraction such that $F(U) \ne \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \ne 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \ne \emptyset$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - UAx_n), \\ y_n = Tz_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, x_n - z_n\rangle \ge r_n(1-k)\|Ax_n - UAx_n\|^2 \\ \qquad\qquad and \quad 2\langle z_n - z, z_n - y_n\rangle \ge \|z_n - y_n\|^2 \}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \le r_n \le \frac{1-k}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. Since U is a k-strict pseudo-contraction of H_2 into itself such that $F(U) \neq \emptyset$, from (1) in Examples, U is $(k, 0)$-demigeneralized. Furthermore, from Lemma 4.1, U is demiclosed. We also have that a nonexpansive maping T is $(0, 0)$-demigeneralized and demiclosed. Therefore, we have the desired result from Theorem 3.2. □

Theorem 4.4. *Let H_1 and H_2 be Hilbert spaces. Let $T : H_1 \to H_1$ be a nonexpansive mapping with $F(T) \neq \emptyset$ and let $U : H_2 \to H_2$ be a generalized hybrid mapping with $F(U) \neq \emptyset$. Let $A : H_1 \to H_2$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $F(T) \cap A^{-1}F(U) \neq \emptyset$. For $x_1 \in H_1$ and $C_1 = H_1$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = x_n - r_n A^*(Ax_n - UAx_n), \\ y_n = Tz_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, x_n - z_n\rangle \geq r_n \|Ax_n - UAx_n\|^2 \\ \qquad\qquad\qquad\qquad \text{and} \quad 2\langle z_n - z, z_n - y_n\rangle \geq \|z_n - y_n\|^2 \ \}, \\ x_{n+1} = P_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in F(T) \cap A^{-1}F(U)$, where $z_0 = P_{F(T)\cap A^{-1}F(U)} x_1$.

Proof. Since U is a generalized hybrid mapping of H_2 into itself such that $F(U) \neq \emptyset$, from (2) in Examples, U is $(0, 0)$-demigeneralized. Furthermore, from Lemma 4.2, U is demiclosed. We also have that a nonexpansive maping T is $(0, 0)$-demigeneralized and demiclosed. Therefore, we have the desired result from Theorem 3.2. □

Theorem 4.5. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let C and D be nonempty, closed and convex subsets of E and F, respectively. Let Π_C and Π_D be the generalized projections of E onto C and F onto D, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $C \cap A^{-1}D \neq \emptyset$. For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F Ax_n - J_F \Pi_D Ax_n)\big), \\ y_n = \Pi_C z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Ax_n, \Pi_D Ax_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) \ \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then $\{x_n\}$ converges strongly to a point $z_0 \in C \cap A^{-1}D$, where $z_0 = \Pi_{C\cap A^{-1}D} x_1$.

Proof. Since Π_C is the generalized projection, Π_C is $(0,0)$-demigeneralized from (3) in Examples. Furthermore, since Π_D is the generalized projection of F onto D, from (3) in Examples, Π_D is $(0,0)$-demigeneralized. We also have that if $\{u_n\}$ is a sequence in F such that $u_n \rightharpoonup p$ and $u_n - \Pi_D u_n \to 0$, then $p = \Pi_D p$. In fact, assume that $u_n \rightharpoonup p$ and $u_n - \Pi_D u_n \to 0$. It is clear that $\Pi_D x_n \rightharpoonup p$. Furthermore, since F is uniformly smooth, we have that $\|J_F u_n - J_F \Pi_D u_n\| \to 0$. Since Π_D is the generalized projection of F onto D, we have that

$$\langle \Pi_D u_n - \Pi_D p, J_F u_n - J_F \Pi_D u_n - (J_F p - J_F \Pi_D p)\rangle \geq 0.$$

Therefore, $\langle p - \Pi_D p, -(J_F p - J_F \Pi_D p)\rangle \geq 0$. This implies that

$$\phi_F(p, \Pi_D p) + \phi_F(\Pi_D p, p) \leq 0$$

and hence $p = \Pi_D p$. Therefore, Π_D is demiclosed. Similarly, Π_C is demiclosed. Therefore, we have the desired result from Theorem 3.2. □

Theorem 4.6. *Let E and F be uniformly convex and uniformly smooth Banach spaces and let J_E and J_F be the duality mappings on E and F, respectively. Let G and B be maximal monotone operators of E into E^* and F into F^*, respectively. Let J_λ^G be the generalized resolvent of G for $\lambda > 0$ and let J_μ^B be the generalized resolvent of B for $\mu > 0$, respectively. Let $A : E \to F$ be a bounded linear operator such that $A \neq 0$ and let A^* be the adjoint operator of A. Suppose that $G^{-1}0 \cap A^{-1}(B^{-1}0) \neq \emptyset$. Let For $x_1 \in E$ and $C_1 = E$, let $\{x_n\}$ be a sequence generated by*

$$\begin{cases} z_n = J_E^{-1}\big(J_E x_n - r_n A^*(J_F A x_n - J_F J_\mu^B A x_n)\big), \\ y_n = J_\lambda^G z_n, \\ C_{n+1} = \{z \in C_n : 2\langle x_n - z, J_E x_n - J_E z_n\rangle \geq r_n \phi_F(Ax_n, J_\mu^B A x_n) \\ \qquad\qquad\qquad\qquad \text{and} \quad 2\langle z_n - z, J_E z_n - J_E y_n\rangle \geq \phi_E(z_n, y_n) \ \}, \\ x_{n+1} = \Pi_{C_{n+1}} x_1, \quad \forall n \in \mathbb{N}, \end{cases}$$

where $\{r_n\} \subset (0, \infty)$ and $a \in \mathbb{R}$ satisfy the following inequalities:

$$0 < a \leq r_n \leq \frac{1}{\|A\|^2}, \quad \forall n \in \mathbb{N}.$$

Then the sequence $\{x_n\}$ converges strongly to a point $z_0 \in G^{-1}0 \cap A^{-1}(B^{-1}0)$, where $z_0 = \Pi_{G^{-1}0 \cap A^{-1}(B^{-1}0)} x_1$.

Proof. Since J_λ^G is the generalized resolvent of G on E, J_λ^G is $(0,0)$-demigeneralized from (4) in Examples. Furthemore, since J_μ^B is the generalized resolvent of B on F, from (4) in Examples, J_μ^B is $(0,0)$-demigeneralized. We also have that if $\{u_n\}$ is a sequence in F such that $u_n \rightharpoonup p$ and $u_n - J_\mu^B u_n \to 0$, then $p = J_\mu^B p$. In fact, assume that $u_n \rightharpoonup p$ and $u_n - J_\mu^B u_n \to 0$. It is clear that $J_\mu^B u_n \rightharpoonup p$. Furthermore, since F is uniformly smooth, we have that $\|J_F u_n - J_F J_\mu^B u_n\| \to 0$. Since J_μ^B is the generalized resolvent of B, we have from [4] that

$$\langle J_\mu^B u_n - J_\mu^B p, J_F u_n - J_F J_\mu^B u_n - (J_F p - J_F J_\mu^B p)\rangle \geq 0.$$

Therefore, $\langle p - J_\mu^B p, -(J_F p - J_F J_\mu^B p)\rangle \geq 0$. This implies that

$$\phi_F(p, J_\mu^B p) + \phi_F(J_\mu^B p, p) \leq 0$$

and hence $p = J_\mu^B p$. Therefore, J_μ^B is demiclosed. Similarly, J_λ^G is demiclosed. Therefore, we have the desired result from Theorem 3.2. □

References

[1] Y. I. Alber, *Metric and generalized projections in Banach spaces: Properties and applications*, in Theory and Applications of Nonlinear Operators of Accretive and Monotone Type (A. G. Kartsatos Ed.), Marcel Dekker, New York, 1996, pp. 15–50.

[2] Y. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator in Banach spaces*, Panamer. Math. J. **4** (1994), 39–54.

[3] S. M. Alsulami and W. Takahashi, *The split common null point problem for maximal monotone mappings in Hilbert spaces and applications*, J. Nonlinear Convex Anal. **15** (2014), 793–808.

[4] K. Aoyama, F. Kohsaka and W. Takahashi, *Three generalizations of firmly nonexpansive mappings: Their relations and continuous properties*, J. Nonlinear Convex Anal. **10** (2009), 131–147.

[5] F. E. Browder, *Nonlinear maximal monotone operators in Banach spaces*, Math. Ann. **175** (1968), 89–113.

[6] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.

[7] C. Byrne, Y. Censor, A. Gibali and S. Reich, *The split common null point problem*, J. Nonlinear Convex Anal. **13** (2012), 759–775.

[8] Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.

[9] Y. Censor, A. Gibali and S. Reich, *Algorithms for the split variational inequality problem*, Numer. Algorithms **59** (2012), 301–323.

[10] Y. Censor and A. Segal, *The split common fixed-point problem for directed operators*, J. Convex Anal. **16** (2009), 587–600.

[11] I. Cioranescu, *Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Kluwer, Dordrecht, 1990.

[12] M. Hojo and W. Takahashi, *A strong convegence theorem by shrinking projection method for the split common null point problem in Banach spaces*, Numer. Funct. Anal. Optim. **37** (2016), 541–553.

[13] T. Ibaraki, Y. Kimura and W. Takahashi, *Convegence theorems for generalized projections and maximal monotone operators in Banach spaces*, Abstr. Appl. Anal. 2003:10 (2003), 621–629.

[14] T. Igarashi, W. Takahashi and K. Tanaka, *Weak convergence theorems for nonspreading mappings and equilibrium problems*, in Nonlinear Analysis and Optimization (S. Akashi, W. Takahashi and T. Tanaka Eds.), Yokohama Publishers, Yokohama, 2008, pp. 75–85.

[15] S. Kamimura and W. Takahashi, *Strong convergence of a proximal-type algorithm in a Banach space*, SIAM J. Optim. **13** (2002), 938–945.

[16] P. Kocourek, W. Takahashi and J.-C. Yao, *Fixed point theorems and weak convergence theorems for generalized hybrid mappings in Hilbert spaces*, Taiwanese J. Math. **14** (2010), 2497–2511.

[17] F. Kohsaka and W. Takahashi, *Existence and approximation of fixed points of firmly nonexpansive-type mappings in Banach spaces*, SIAM J. Optim. **19** (2008), 824–835.

[18] F. Kohsaka and W. Takahashi, *Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces.*, Arch. Math. (Basel) **91** (2008), 166–177.

[19] P. E. Maingé, *A viscosity method with no spectral radius requirements for the split common fixed point problem*, Eur. J. Oper. Res. **235** (2014), 17–27.

[20] G. Marino and H.-K. Xu, *Weak and strong convergence theorems for strict pseudo-contractions in Hilbert spaces*, J. Math. Anal. Appl. **329** (2007), 336–346.

[21] E. Masad and S. Reich, *A note on the multiple-set split convex feasibility problem in Hilbert space*, J. Nonlinear Convex Anal. **8** (2007), 367–371.

[22] S. Matsushita, K. Nakajo and W. Takahashi, *Strong convergence theorems obtained by a generalized projections hybrid method for families of mappings in Banach spaces*, Nonlinear Appl. **73** (2010), 1466–1480.

[23] U. Mosco, *Convergence of convex sets and of solutions of variational inequalities*, Adv. Math. **3** (1969), 510–585.
[24] A. Moudafi, *The split common fixed point problem for demicontractive mappings*, Inverse Problems **26** (2010), 055007, 6 pp.
[25] S. Reich, *Book Review: Geometry of Banach Spaces, Duality Mappings and Nonlinear Problems*, Bull. Amer. Math. Soc. **26** (1992), 367–370.
[26] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.
[27] S. Takahashi and W. Takahashi, *The split common null point problem and the shrinking projection method in two Banach spaces*, Linear Nonlinear Anal. **1** (2015), 297–304.
[28] S. Takahashi and W. Takahashi, *The split common null point problem and the shrinking projection method in Banach spaces*, Optimization **65** (2016), 281–287.
[29] S. Takahashi and W. Takahashi, *Split common null point problem and shrinking projection method for generalized resolvents in two Banach spaces*, J. Nonlinear Convex Anal. **17** (2016), 2171–2182.
[30] W. Takahashi, *Nonlinear Functional Analysis*, Yokohama Publishers, Yokohama, 2000.
[31] W. Takahashi, *Convex Analysis and Approximation of Fixed Points (Japanese)*, Yokohama Publishers, Yokohama, 2000.
[32] W. Takahashi, *Fixed point theorems for new nonlinear mappings in a Hilbert space*, J. Nonlinear Convex Anal. **11** (2010), 79–88.
[33] W. Takahashi, *The split feasibility problem in Banach spaces*, J. Nonlinear Convex Anal. **15** (2014), 1349–1355.
[34] W. Takahashi, *The split feasibility problem and the shrinking projection method in Banach spaces*, J. Nonlinear Convex Anal. **16** (2015), 1449–1459.
[35] W. Takahashi, *The split common null point problem in Banach spaces*, Arch. Math. **104** (2015), 357–365.
[36] W. Takahashi, Y. Takeuchi and R. Kubota, *Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert spaces*, J. Math. Anal. Appl. **341** (2008), 276–286.
[37] W. Takahashi, C.-F. Wen and J.-C. Yao, *Strong convergence theorem by shrinking projection method for new nonlinear mappings in Banach spaces and applications*, Optimization **66** (2017), 609–621.
[38] W. Takahashi, N.-C. Wong and J.-C. Yao, *Weak and strong mean convergence theorems for extended hybrid mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **12** (2011), 553–575.
[39] W. Takahashi, H.-K. Xu and J.-C. Yao, *Iterative methods for generalized split feasibility problems in Hilbert spaces*, Set-Valued Var. Anal. **23** (2015), 205–221.
[40] W. Takahashi, J.-C. Yao and P. Kocourek, *Weak and strong convergence theorems for generalized hybrid nonself-mappings in Hilbert spaces*, J. Nonlinear Convex Anal. **11** (2010), 567–586.

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