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CREATING MATERIALS IN WHICH HEAT PROPAGATES ALONG A LINE: THEORY AND NUMERICAL RESULTS

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Abstract. In this paper the theory is developed for creating a material in which the heat is transmitted along a given line. This gives a possibility to transfer information using heat signals. This seems to be a novel idea. The technical part of the theory is the construction of the potential $q(x)$. This potential describes the heat equation $u_t = \Delta u - q(x)u$ in the limiting medium which is obtained after the small impedance particles are distributed in a given domain. A numerical method is also established to construct numerically such a potential.

1. Introduction

To create materials in which heat propagates along a line, one needs to create a medium in which the heat transfer is governed by the equation

$$u_t = \Delta u - q(x)u \quad \text{in } D, \quad u|_S = 0, \quad u|_{t=0} = f(x), \tag{1.1}$$

where D is a bounded domain with a piecewise-smooth boundary S, $D = D_0 \times [0, L]$, $D_0 \subset \mathbb{R}^2$ is a disc of radius R orthogonal to the axis x_1, $x = (x_1, x_2, x_3)$, $x_2, x_3 \in D_0$, $0 \le x_1 \le L$.
Such a medium is created by embedding many small impedance particles D_m, $1 \le m \le M$, into a given domain D filled with a homogeneous material. Let us assume that the distribution of small particles is:

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx(1 + o(1)), \quad a \to 0, \tag{1.2}$$

where $\mathcal{N}(\Delta)$ is the number of small particles in an arbitrary open subset $\Delta \in D$, $\kappa \in [0, 1)$ is a number that can be chosen by an experimenter as desired, $a = \frac{1}{2} \max_{1 \le m \le M} \max_{x,y \in D_m} |x - y|$ and $N(x) \ge 0$ is a continuous in D function that can be chosen by an experimenter as desired. As $a \to 0$, it is proved (see [**?**], [**?**]) that the solution $u(x, t, a)$ to the problem

$$u_t = \Delta u \quad \text{in } D \setminus \bigcup_{m=1}^{M} D_m, \; u_N = \zeta_m u \quad \text{on } S_m, \; 1 \le m \le M, \tag{1.3}$$

$$u|_{t=0} = f(x), \; \zeta_m := \frac{h(x_m)}{a^\kappa}, \; \mathrm{Re}\zeta_m \ge 0, \tag{1.4}$$

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where $x_m \in \mathbb{R}^3$ is an arbitrary point in D_m, has a limit, $u(x,t) = \lim_{a\to 0} u(x,t,a)$ which solves problem (1.1) with

$$q(x) = c_S N(x) h(x), \quad c_S := \frac{|S_m|}{a^2} = const, \tag{1.5}$$

so c_S does not depend on m, but the shape of S_m may depend on m. Therefore, given a potential $q(x)$ (which makes heat propagate along a line), one can choose an arbitrary continuous function $N(x) > 0$, can construct a continuous function $h(x) = \frac{q(x)}{c_S N(x)}$, and can distribute the small bodies according to the rule (1.2) to obtain the medium in which heat propagates along a line.

Suppose that

$$(-\Delta + q(x))\phi_n(x) = \lambda_n \phi_n, \quad \phi_n|_S = 0, \quad ||\phi_n||_{L^2(D)} = ||\phi_n|| = 1, \tag{1.6}$$

where $\{\phi_n\}$ is an orthonormal basis of $L^2(D) := H$, and $(f, \phi_n) := \int_D f(x)\overline{\phi_n(x)}dx$. Then the unique solution to (1.1) is

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\lambda_n t}(f, \phi_n)\phi_n(x). \tag{1.7}$$

If $q(x)$ is such that $\lambda_1 = 0$, $\lambda_2 \gg 1$, and $\lambda_2 \leq \lambda_3 \leq \dots$, then, as $t \to \infty$, the series (1.7) is well approximated by its first term (see [6]):

$$u(x,t) = (f, \phi_1)\phi_1 + O(e^{-10t}), \quad t \to \infty. \tag{1.8}$$

Thus, our problem is solved if $q(x)$ has the following property:

$$|\phi_1(x)| \text{ decays as } \rho \text{ grows}, \quad \rho = (x_2^2 + x_3^2)^{1/2}. \tag{1.9}$$

Since the eigenfunction is normalized, $||\phi_1|| = 1$, this function will not tend to zero in a neighborhood of the line $\rho = 0$, so information can be transformed by the heat signals along the line $\rho = 0$, that is, along $s-$axis. Here we use the cylindrical coordinates: $x = (x_1, x_2, x_3) = (s, \rho, \theta)$, $s = x_1$, $\rho = (x_2^2 + x_3^2)^{1/2}$. In Section 2 the potential $q(x)$ will not depend on θ.

The technical part of the solution is the construction of $q(x) = c_S N(x) h(x)$ such that

$$\lambda_1 = 0, \quad \lambda_2 \gg 1; \quad |\phi_1(x)| \text{ decays as } \rho \text{ grows}. \tag{1.10}$$

Since the function $N(x) > 0$ and $h(x), \mathrm{Re} h \geq 0$ are at our disposal, any desirable $q, \mathrm{Re}\, q \geq 0$, can be obtained by embedding many small impedance particles in a given domain D.

In section 2, the method for finding such a potential $q(x)$ is presented. In section 3, the numerical method is presented for finding this q, and in section 4, the numerical results are presented. In section 5, another numerical method is presented based on the method described in section 4.

2. Construction of $q(x)$

Let $q(x) = p(\rho) + Q(s)$, where $s := x_1$, $\rho := (x_2^2 + x_3^2)^{1/2}$. Then the solution to (1.1) is $u = v(\rho)w(s)$, where

$$-v_m'' - \rho^{-1}v_m' + p(\rho)v_m = \mu_m v_m,\ 0 \le \rho \le R,\ |v_m(0)| < \infty,\ v_m(R) = 0; \tag{2.1}$$

$$-w_j'' + Q(s)w_j = \nu_j w_j,\quad 0 \le s \le L,\ w_j(0) = w_j(L) = 0. \tag{2.2}$$

Our task is to find $Q(s)$ such that $\nu_1 = 0$, $\nu_2 \gg 1$ and $p(\rho)$ such that $\mu_1 = 0, \mu_2 \gg 1$, and $|v_m(\rho)|$ decays as ρ grows.

We use the solution to inverse spectral problem for finding a potential which have the desired properties. The spectral function $\varrho(\lambda)$ of the Dirichlet differential operator $\ell w = -w'' + Q(s)w$ (see formula (2.2)) is defined by the formula

$$\varrho(\lambda) = \sum_{\nu_j < \lambda} \frac{1}{\alpha_j},$$

where α_j are normalizing constants. If $Q = 0$, then the eigenvalues of the corresponding operator ℓ are $\nu_{j0} = \left(\frac{\pi j}{L}\right)^2$, $j = 1, 2, \dots$, the corresponding normalized eigenfunctions are $\sqrt{\frac{2}{L}}\sin\frac{j\pi x}{L}$, and the normalizing constants $\alpha_{j0} = \sqrt{\int_0^L \sin^2\frac{j\pi x}{L}dx} = \sqrt{\frac{L}{2}}$. If Q is unknown, then the corresponding eigenfunctions are unknown and the role of the normalizing constants can play arbitrary positive numbers which have the right asymptotic. If $L = \pi$ then $\alpha_j = \sqrt{\frac{\pi}{2}}j\big(1 + O(\frac{1}{j})\big)$, $\sqrt{\nu_j} = j + \frac{c_1}{j} + O(j^{-2})$, and $w_j = \frac{\sin(jx)}{j} + O(j^{-2})$ as $j \to \infty$.

Let us recall the procedure, due to Gel'fand and Levitan (see [1], [2]) for finding Q from the known spectral function. One defines the kernel

$$L(x, y) = \int_{-\infty}^{\infty} \frac{\sin(\sqrt{\lambda}x)}{\sqrt{\lambda}} \frac{\sin(\sqrt{\lambda}y)}{\sqrt{\lambda}} d(\varrho(\lambda) - \varrho_0(\lambda)),$$

where $\varrho(\lambda)$ is the spectral function of the operator ℓ with the potential $Q = Q(s)$, and $\varrho_0(\lambda)$ is the spectral function of the operator ℓ with the potential $Q = 0$ and the same boundary conditions.
Consider the Gel'fand-Levitan (GL) integral equation for the kernel $K(x, y)$:

$$K(x, y) + \int_0^x K(x, s)L(s, y)ds = -L(x, y), \qquad 0 \le y \le x.$$

The solution to this equation allows one to calculate the potential:

$$Q(x) = 2\frac{dK(x, x)}{dx}.$$

From now on we set $L=\pi$. Then $\nu_{l0}=j^2$. Let $\nu_1=0,\nu_2=11,\nu_3=14,\nu_j=\nu_{j0}$ for $j>3$. Then the kernel $L(x,y)$ in the GL equation is defined as follows

$$L(x,y)=\frac{3xy}{\pi^3}+\frac{2}{\pi}\Big(\frac{\sin(\sqrt{\nu_2}x)}{\sqrt{\nu_2}}\frac{\sin(\sqrt{\nu_2}y)}{\sqrt{\nu_2}}+\frac{\sin(\sqrt{\nu_3}x)}{\sqrt{\nu_3}}\frac{\sin(\sqrt{\nu_3}y)}{\sqrt{\nu_3}}\Big)- \\ -\frac{2}{\pi}\Big(\sin x\sin y+\sin(2x)\sin(2y)+\sin(3x)\sin(3y)\Big), \tag{2.3}$$

where we set the normalizing constants $\alpha_j=\frac{\pi}{2},j>3,\alpha_1=\frac{\pi^3}{3}$. The term xy is the value of the function $\frac{\sin\nu x}{\nu}\frac{\sin\nu y}{\nu}$ at $\nu=0$, and $\frac{\pi^3}{3}=||x||^2=\int_0^\pi x^2dx$.

Solve the GL equation:

$$K(s,\tau)+\int_0^s K(s,s')L(s',\tau)ds'=-L(s,\tau),\quad 0\le\tau\le s, \tag{2.4}$$

which is uniquely solvable (see [1], [2]). Equation (2.4) has finite-rank kernel and therefore can be solved analytically, being equivalent to a linear algebraic system. If $K(s,\tau)$ is found, then

$$Q(s)=2\frac{dK(s,s)}{ds}, \tag{2.5}$$

and this $Q(s)$ has the required properties: $\nu_1=0,\nu_2\gg 1,\nu_j\le\nu_{j+1}$.

Consider now the operator (2.1) for $v(\rho)$. We want to calculate $p(\rho)$ such that $\mu_1=0,\mu_2\gg 1,\mu_m\le\mu_{m+1},|v_m(\rho)|$ decays as ρ grows.

We reduce this problem to the previous one that was solved. To do this, set $v=\frac{\psi}{\sqrt{\rho}}$. Then equation $-v''-\frac{1}{\rho}v'+p(\rho)v=\mu v$, is transformed to the equation

$$-\psi''-\frac{1}{4\rho^2}\psi+p(\rho)\psi=\mu\psi. \tag{2.6}$$

Let $p(\rho)=\frac{1}{4\rho^2}+Q(\rho)$, where $Q(\rho)$ is constructed above. Then equation (2.6) becomes

$$-\psi''+Q(\rho)\psi=\mu\psi,\quad \psi(R)=0,\quad \psi(0)=0. \tag{2.7}$$

It has the desired eigenvalues $\mu_1=0,\mu_2\gg 1,\mu_m\le\mu_{m+1}$.

The eigenfunction $\phi_1(x)=v_1(\rho)w_1(s)$, where $v_1(\rho)=\frac{\psi_1(\rho)}{\sqrt{\rho}}$, decays as ρ grows, and the eigenvalues $\lambda_n=\mu_m+\nu_l$. Since $\mu_1=\nu_1=0$ one has $\lambda_1=0$. Since $\nu_2=11,\mu_2=11,\lambda_2=11\gg 1$. Thus, the desired potential is constructed: $q(x)=Q(s)+(\frac{1}{4\rho^2}+Q(\rho))$, where $Q(s)$ is given by formula (2.5).

This concludes the description of our procedure for the construction of q.

3. Numerical procedure

In section 3.1, a numerical method to construct $q(x)$ is presented. In section 3.2, a procedure is presented to check whether the constructed potential $q(x)$ is valid, by finding the eigenvalues of $q(x)$.

3.1. Numerical construction of q(x).

From the construction of $q(x)$ in Section 2, if one can construct $Q(s)$ then one gets $q(x) = Q(s) + \left(\frac{1}{4\rho^2} + Q(\rho)\right)$.

To construct $Q(s)$, one can use equation (2.5) and rewrite it as

$$Q(s) = 2\frac{dK(s,s)}{ds} = 2\left(\frac{\partial K(s,\tau)}{\partial s}\bigg|_{\tau=s} + \frac{\partial K(s,\tau)}{\partial \tau}\bigg|_{\tau=s}\right), \tag{3.1}$$

One can get $K_s := \frac{\partial K(s,\tau)}{\partial s}$ and $K_\tau := \frac{\partial K(s,\tau)}{\partial \tau}$ numerically by the following procedure.

The function $L(x,y)$ in (2.3) can be written as

$$L(x,y) = \sum_{j=1}^{6} a_j(x)b_j(y), \tag{3.2}$$

where $a_1(x) = \frac{3x}{\pi^3}$, $a_2(x) = \frac{2}{\pi}\frac{\sin(\sqrt{\nu_2}x)}{\nu_2}$, $a_3(x) = \frac{2}{\pi}\frac{\sin(\sqrt{\nu_3}x)}{\nu_3}$, $a_4(x) = -\frac{2}{\pi}\sin(x)$, $a_5(x) = -\frac{2}{\pi}\sin(2x)$, $a_6(x) = -\frac{2}{\pi}\sin(3x)$ and $b_1(x) = x$, $b_2(x) = \sin(\sqrt{\nu_2}x)$, $b_3(x) = \sin(\sqrt{\nu_3}x)$, $b_4(x) = \sin(x)$, $b_5(x) = \sin(2x)$, $b_6(x) = \sin(3x)$.

Then equation (2.4) becomes

$$K(s,\tau) + \sum_{j=1}^{6} b_j(\tau)\int_0^s K(s,s')a_j(s')ds' = -\sum_{j=1}^{6} a_j(s)b_j(\tau), \quad 0 \le \tau \le s. \tag{3.3}$$

Let $\psi_j(s) := \int_0^s K(s,s')a_j(s')ds'$, then equation (3.3) becomes

$$K(s,\tau) + \sum_{j=1}^{6} b_j(\tau)\psi_j(s) = -\sum_{j=1}^{6} a_j(s)b_j(\tau), \quad 0 \le \tau \le s. \tag{3.4}$$

Multiply (3.4) with $a_m(\tau), 1 \le m \le 6$ and integrate it with respect to τ to get

$$\psi_m(s) + \sum_{j=1}^{6}\left(\int_0^s b_j(\tau)a_m(\tau)d\tau\right)\psi_j(s) = -\sum_{j=1}^{6} a_j(s)\left(\int_0^s b_j(\tau)a_m(\tau)d\tau\right). \tag{3.5}$$

For a fix $s = s_0$, equation (3.5) is a 6×6 linear system which can be solved for $\psi_j(s_0), 1 \le j \le 6$. So, we can solve equation (3.5) to get $\psi_j(s)$ analytically and numerically. Differentiating equation (3.5), one can get a similar linear system to

(3.5) and can get $\psi_j'(s)$ analytically and numerically from the linear system:

$$\psi_m'(s)+\sum_{j=1}^{6}\left(\int_0^s b_j(\tau)a_m(\tau)d\tau\right)\psi_j'(s)= \tag{3.6}$$
$$=-\sum_{j=1}^{6}\left(a_j(s)\left(\int_0^s b_j(\tau)a_m(\tau)d\tau\right)+(a_j(s)+\psi_j(s))b_j(s)a_m(s)\right).$$

After finding $\psi_j(s)$ and $\psi_j'(s)$, one can find K_s and K_τ by differentiating equation (3.3) with respect to s and τ

$$K_s(s,\tau)=-\sum_{j=1}^{6}\left(a_j'(s)b_j(\tau)+b_j(\tau)\psi_j'(s)\right), \tag{3.7}$$

$$K_\tau(s,\tau)=-\sum_{j=1}^{6}\left(a_j(s)b_j'(\tau)+b_j'(\tau)\psi_j(s)\right). \tag{3.8}$$

From equation (3.7) - (3.8), one finds $K_s(s,\tau)|_{\tau=s}$ and $K_\tau(s,\tau)|_{\tau=s}$, and then finds $Q(s)$ numerically using equation (3.1).

3.2. **Checking the eigenvalues of q(x).** To check whether the constructed potential $q(x)$ is the correct potential, one has to check whether the eigenvalues generated by the constructed potential $q(x)$ satisfy the conditions formulated in Section 2. It is sufficient to check the eigenvalues of $Q(s)$ are $\nu_1=0,\nu_2=11,\nu_3=14,\nu_j=j^2,j\geq 4$.

One can find numerically the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2}+Q(s)$ on the interval $[0,\pi]$ by minimizing the following functional where u is taken from an N-dimensional subspace of functions u_N, N is a large integer:

$$\frac{\int_0^\pi(|u'|^2+Q|u|^2)dx}{\int_0^\pi|u|^2dx}. \tag{3.9}$$

We take

$$u_N=\sum_{n=1}^{N}c_n\varphi_n(x),$$

and $\varphi_n(x):=\sqrt{\frac{2}{\pi}}\sin(nx),1\leq n\leq N$. The minimization is taken over parameters c_n. The eigenvalues of the resulting matrix approximate the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2}+Q(s)$ on the interval $[0,\pi]$. Finding minima of the functional (3.9) is equivalent to finding the minima of the quadratic form

$$\sum_{n=1}^{N}n^2|c_n|^2+\sum_{n,m=1}^{N}c_nc_mq_{nm},\ \text{under the restriction}\ \sum_{n=1}^{N}|c_n|^2=1, \tag{3.10}$$

where $q_{nm} := \frac{2}{\pi}\int_0^\pi Q(x)\sin(nx)\sin(mx)dx$, $q_{nm} = q_{mn}$. Minimizing (3.10) is equivalent to minimizing

$$f(c_1,\dots,c_N) = \sum_{n=1}^{N} n^2|c_n|^2 + \sum_{n,m=1}^{N} c_n c_m q_{nm} - \nu\left(\sum_{n=1}^{N} c_n^2 - 1\right). \tag{3.11}$$

A necessary condition for a smooth function $f(c_1,\dots,c_N)$ to have minima is $\frac{\partial f}{\partial c_n} = 0$. This leads to a linear system with respect to $c_1,\dots,c_N$:

$$n^2 c_n + \sum_{m=1}^{N} q_{nm}c_m - \nu c_n = 0. \tag{3.12}$$

Linear system (3.12) can be written as

$$PC = \nu C, \tag{3.13}$$

where P is a symmetric matrix with entries $P_{nm} := n^2\delta_{nm} + q_{nm}$, and C is a column vector $C = (c_1,\dots,c_N)$. Then the eigenvalues of the Dirichlet operator $-\frac{d^2}{dx^2} + Q(s)$ on the interval $[0,\pi]$ are approximated by the eigenvalues of the matrix P. The approximation is the better the larger is N.

3.3. **Calculating q_{nm}.** In section 3.2, one needs to calculate

$$q_{nm} = \frac{2}{\pi}\int_0^\pi Q(x)\sin(nx)\sin(mx)dx$$

to construct the matrix P. One can calculate the matrix P by using the formula $2\sin A\sin B = \cos(A-B) - \cos(A+B)$. So, one first calculates

$$\tilde{q}(k) := \frac{1}{\pi}\int_0^\pi Q(x)\cos(kx)dx, \quad 0 \le k \le 2N,$$

and then calculates q_{nm} by the formula

$$q_{nm} = \tilde{q}(|n-m|) - \tilde{q}(n+m).$$

4. Numerical results

Based on the numerical procedure in section 3, a computer algorithm/program is developed with the following main steps:

1. Partition the interval $[0,\pi]$ into $\mathcal{M}$ equal-distanced intervals with the endpoints x_i, $1 \le i \le \mathcal{M}+1$.
2. For each $x_i, 1 \le i \le \mathcal{M}+1$, one solves the linear systems (3.5) and (3.6) for $\psi_j(x_i)$ and $\psi_j'(x_i)$, $1 \le j \le 6$.
3. Find $K_s(x_i,x_i)$ and $K_\tau(x_i,x_i)$ by using equations (3.7) and (3.8).
4. Find $Q(x_i), 1 \le i \le \mathcal{M}+1$.
5. Construct the matrix P in equation (3.13) by calculating q_{nm} using the procedure in section 3.3, where $\tilde{q}(k)$ is calculated using the Riemann sum

$$\tilde{q}(k) = \frac{1}{\pi}\sum_{i=1}^{\mathcal{M}} \frac{Q(x_i)\cos(kx_i) + Q(x_{i+1})\cos(kx_{i+1})}{2}(x_{i+1} - x_i).$$

6. Find the eigenvalues of P using the Jacobi eigenvalue algorithm see, for example [7].

The above algorithm is run five times with $\mathcal{M} = 100, 150, 200, 250$, and 300. The constructed potentials $Q(s)$ are as in the following graph.

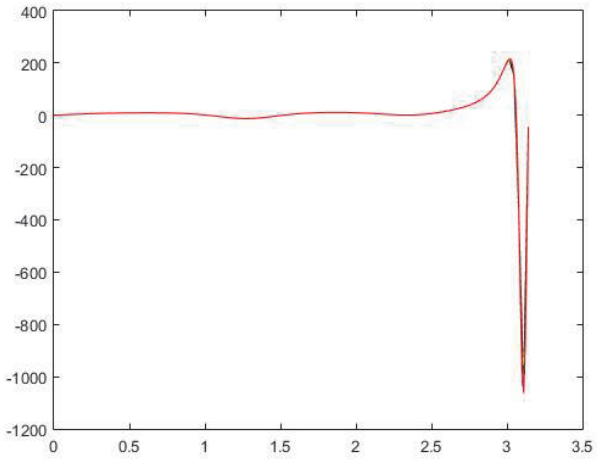

FIGURE 1. Numerically constructed potentials $Q(s)$.

The relative error of the eigenvalues is calculated by the following formula:

$$\delta_{\mathcal{M}} = \max_{1 \le j \le \mathcal{M}} \frac{|\nu_j^{(\mathcal{M})} - \nu_j|}{\nu_j}.$$

The following table gives the relative errors of the eigenvalues of the constructed potentials $Q(s)$ for $\mathcal{M} = 100, 150, 200, 250$, and 300.

$\mathcal{M}$	Relative errors $\delta_{\mathcal{M}}$
100	57.12%
150	5.01%
200	1.65%
250	0.67%
300	0.32%

From the above table, one can construct the potential $Q(s)$ with $\mathcal{M} = 250$ equal-distance small intervals and gets the relative error of the eigenvalues less than 1%. The above result also shows that the constructed $Q(s)$ is valid.

5. Another method to calculate eigenvalues

In the numerical results in section 4, one needs $\mathcal{M} = 250$ equal-distance small intervals to get the relative error less than 1%. From the graph of the constructed $Q(s)$, since $Q(s)$ is pretty steep close to π, one can improve the method in section 4 by distributing more equal-distanced intervals $\mathcal{M}_2$ in the interval $[\frac{9\pi}{10}, \pi]$ and less

equal-distanced intervals $\mathcal{M}_1$ in the interval $[0, \frac{9\pi}{10}]$.

The following result is obtained

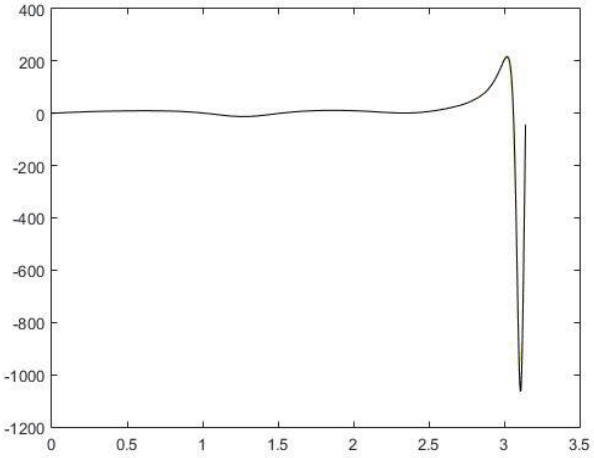

FIGURE 2. Numerically constructed potential $Q(s)$ for non-uniform small intervals.

$\mathcal{M}_1$	$\mathcal{M}_2$	Relative errors
50	50	4.68%
50	75	0.94%
50	100	0.23%

Remark 5.1. About calculating $\tilde{q}(k)$

In the constructed potential $Q(s)$ in Figure 2, let $Q_{max} := \max_{1\le i\le \mathcal{M}+1} Q(x_i)$, $x_{max} := \{x_i : Q(x_i) = Q_{max}\}$, $Q_{min} := \min_{1\le i\le \mathcal{M}+1} Q(x_i)$, and $x_{min} := \{x_i : Q(x_i) = Q_{min}\}$. The parts of the potential from x_{max} to x_{min} and from x_{min} to π look like straight lines and one may try to calculate $\tilde{q}(k)$ by

$$\tilde{q}(k) = \frac{1}{\pi}\left(\int_0^{x_{max}} Q(x)\cos(kx)dx + \right.$$
$$\left. + \int_{x_{max}}^{x_{min}} Q(x)\cos(kx)dx + \int_{x_{min}}^{\pi} Q(x)\cos(kx)dx\right) \tag{5.1}$$
$$:= I_1 + I_2 + I_3.$$

If $Q(x)$ is a straight line from x_{max} to x_{min} and from x_{min} to π, one can calculate I_2 and I_3 analytically. However, this does not provide the desired numerical accuracy as the following numerical experiment shows.

In this experiment, P_1 is the numerical matrix in equation (3.13) obtained in the experiment described in Section 5 for $\mathcal{M}_1 = 50$ and $\mathcal{M}_2 = 75$, and P_2 is the numerical matrix obtained by considering $Q(x)$ as a straight line from x_{max} to

x_{min} and from x_{min} to π. The relative error matrix E is calculated by $E_{nm} = \frac{|P_{1,nm}-P_{2,nm}|}{|P_{1,nm}|}$. Then $\min(E) = 0.37$ but $\max(E) = 510.17$. So, although the parts of the potential look like straight lines from x_{max} to x_{min} and from x_{min} to π, one cannot consider them as straight lines in numerical calculations.

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