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EXTENSIONS OF THE DUGUNDJI–GRANAS AND NADLER'S THEOREMS ON THE CONTINUITY OF FIXED POINTS

JACEK JACHYMSKI

To the memory of Professor Jonathan Michael Borwein

Abstract. We discuss a problem concerning a continuous dependence of fixed points on parameters. We establish extensions of the Dugundji–Granas theorem on families of weakly contractive mappings and Nadler's theorem on sequences of contractive selfmaps of a locally compact metric space. In particular, our argument shows that the well known parametrized version of the Banach fixed point theorem can be deduced directly from the contraction principle applied to a Nemyckiĭ operator defined on a certain space of continuous functions equipped with the supremum metric.

1. Introduction

Let (X, d) be a metric space, $F\colon X \to X$ and $F_n\colon X \to X$ for $n \in \mathbb{N}$. We denote by $\operatorname{Fix} F$ the set of all fixed points of F and by $L(F)$ the Lipschitz constant of F. We say that F is *contractive* if $L(F) < 1$. One of the basic problems of the metric fixed point theory is the following [16, Problem 3.4.4c]: assume that $\operatorname{Fix} F = \{x_0\}$, $\operatorname{Fix} F_n = \{x_n\}$ for $n \in \mathbb{N}$ and (F_n) converges (pointwise or uniformly) to F. When does (x_n) converge to x_0? Probably, the first answer to this question was given by Bonsall [1, p. 6] in 1962:

Theorem 1.1 (Bonsall)**.** *Let (X, d) be complete and F_n be contractive for each $n \in \mathbb{N}$. If $\sup_{n\in\mathbb{N}} L(F_n) < 1$ and (F_n) converges pointwise to F, then F is contractive and $x_n \to x_0$.*

Theorem 1.1 can be formulated in the following equivalent form, which is known as the parametrized version of the Banach contraction principle (see, e.g., [9, p. 18]).

Theorem 1.2. *Let (X, d) be complete, (Λ, ρ) be a metric space and $\{F_\lambda : \lambda \in \Lambda\}$ be a family of selfmaps of X. Set $F(\lambda, x) := F_\lambda x$ for $\lambda \in \Lambda$ and $x \in X$, and assume that F is continuous in the first variable and contractive in the second one. Set*

$$f(\lambda) := x_\lambda \quad \text{for} \quad \lambda \in \Lambda,$$

where x_λ is the unique fixed point of F_λ. If $\sup_{\lambda\in\Lambda} L(F_\lambda) < 1$, then f is continuous.

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In 1968 Nadler [13] showed that the assumption '$\sup_{n\in\mathbb{N}} L(F_n) < 1$' in Theorem 1.1 could be omitted in some cases:

Theorem 1.3 (Nadler)**.** *Let (X,d) be locally compact (not necessarily complete) and for $n \in \mathbb{N}$, F_n be contractive with fixed point x_n. If (F_n) converges pointwise to F with fixed point x_0 and F is contractive, then $x_n \to x_0$.*

On the other hand, a partial generalization of Theorem 1.2 was given in 1978 by Dugundji and Granas [6], who considered families of weakly contractive mappings. Recall that F is *weakly contractive* if

$$d(Fx, Fy) \le d(x,y) - \Theta(x,y) \quad \text{for any} \quad x, y \in X,$$

where $\Theta\colon X \times X \to [0,\infty)$ is *compactly positive*, i.e., for any $a, b > 0$ with $a < b$,

$$\inf\{\Theta(x,y) : a \le d(x,y) \le b\} > 0.$$

It was proved in [6] that every weakly contractive selfmap of a complete metric space has a unique fixed point. For families of such mappings, we have the following

Theorem 1.4 (Dugundji–Granas)**.** *Let (X,d) be complete, (Λ,ρ) be a metric space and $\{F_\lambda : \lambda \in \Lambda\}$ be a family of selfmaps of X. Under the notations of Theorem 1.2 concerning mappings F and f, assume that F is continuous in the first variable and weakly contractive in the second one with function Θ independent of λ, i.e.,*

$$d(F_\lambda x, F_\lambda y) \le d(x,y) - \Theta(x,y) \quad \text{for} \quad x, y \in X \quad \text{and} \quad \lambda \in \Lambda. \tag{1.1}$$

Then f is continuous at each point $\lambda \in \Lambda$, where it is locally bounded.

The last property means that the restriction of f to some neighbourhood of λ is bounded. This condition seems to be rather hardly verifiable in practice and it was not known whether it is essential in Theorem 1.4.

Applying [11, Lemma 3] to the set

$$D := \{(d(x,y), d(F_\lambda x, F_\lambda y)) : x, y \in X,\ \lambda \in \Lambda\}$$

we may infer that condition (1.1) is equivalent to the following: there exists a continuous nondecreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for $t > 0$, and

$$d(F_\lambda x, F_\lambda y) \le \varphi(d(x,y)) \quad \text{for} \quad x, y \in X \quad \text{and} \quad \lambda \in \Lambda, \tag{1.2}$$

i.e., all F_λ are *φ-contractive in Browder's sense* [3] with φ independent of λ. Hence a sequential version of Theorem 1.4 may be formulated as follows.

Theorem 1.5. *Let (X,d) be complete and for $n \in \mathbb{N}$, $F_n\colon X \to X$ be φ-contractive in Browder's sense with φ independent of n. Let $x_n = F_n x_n$ for $n \in \mathbb{N}$. If (F_n) converges pointwise to F, then F has a unique fixed point x_0, and if (x_n) is bounded, then $x_n \to x_0$.*

It is easily seen that Theorems 1.4 and 1.5 are equivalent in the sense that Theorem 1.5 is an immediate consequence of Theorem 1.4 and vice versa.

In [11] it is shown that the local boundedness of f in Theorem 1.4 can be omitted if we assume that all F_λ are φ-contractive in Browder's sense with φ satisfying an extra condition that

$$\liminf_{t\to\infty}(t - \varphi(t)) > 0. \tag{1.3}$$

(See [11, Theorem 11 and Remark 11]. Of course, the last condition is satisfied if $\varphi(t) = \alpha t$ with $\alpha \in [0,1)$, so [11, Theorem 11] generalizes Theorem 1.2.) Such φ-*contractions in Dugundji's sense* were studied in [5] (under different but equivalent definition) and a comprehensive characterization of them was given in [11, Theorem 8]. In Section 2 we extend [11, Theorem 11] by considering families of nonself mappings. We give a sufficient condition for f to have a closed graph and we also discuss, when f is continuous. Moreover, we provide an example illustrating the role of the continuity assumption in the first part of our Theorem 2.1.

In Section 3, with the help of a completely different method than that used in Section 2, we show that the assumption on local boundedness of f in the Dugundji–Granas Theorem 1.4 is redundant. As a consequence, we get that also the assumption of Theorem 1.5 that (x_n) be bounded is superflous. Moreover, it follows from our proof of Theorem 3.2 that Theorem 1.2 can be obtained *directly* from the Banach contraction principle applied to a Nemyckiĭ operator defined on a certain space of continuous functions equipped with the supremum metric. We also give a generalization of Nadler's Theorem 1.3 by relaxing the assumption that mappings be contractive: instead, we consider a family of Edelstein [7] contractions (cf. Theorem 3.5). At last, we establish a theorem characterizing finite dimensional normed linear spaces by the property of continuity of fixed points of contractive mappings (cf. Theorem 3.8).

Finally, in Section 4 we give two partial extensions of Nadler's Theorem 1.3 by considering a sequence (F_n) of continuous (possibly noncontractive) selfmaps of a nonempty closed convex (not necessarily bounded) subset of a finite dimensional normed linear space. In particular, our Theorem 4.2 implies that if each F_n is nonexpansive and (F_n) is pointwise convergent to a contraction with a fixed point x_0, then almost all F_n have fixed points and $d(x_0, \operatorname{Fix} F_n) \to 0$.

2. Continuity of fixed points of nonself mappings

We may extend the definition of a Dugundji contraction to nonself mappings in the following natural way. Let (X,d) be a metric space and A be a nonempty subset of X. Let $\varphi\colon [0,\infty) \to [0,\infty)$ be a continuous and nondecreasing function such that $\varphi(t) < t$ for $t > 0$ and (1.3) holds. Then we say that a mapping $T\colon A \to X$ is a *Dugundji φ-contraction* if $d(Tx,Ty) \le \varphi(d(x,y))$ for any $x,y \in A$. The main result of this section is the following

Theorem 2.1. *Let (X,d) and (Λ,ρ) be metric spaces, $\emptyset \ne A \subseteq X$, and $F\colon \Lambda\times A \to X$ be continuous. Set $F_\lambda := F(\lambda,\cdot)$ for $\lambda \in \Lambda$ and*

$$\Lambda_0 := \{\lambda \in \Lambda : \operatorname{Fix} F_\lambda \ne \emptyset\}.$$

For $\lambda \in \Lambda_0$, let $\operatorname{Fix} F_\lambda = \{x_\lambda\}$ and define $f(\lambda) := x_\lambda$. Then the following statements hold.

1. *f has a closed graph in $\Lambda \times A$.*
2. *If A is compact and $\Lambda_0 \ne \emptyset$, then f is continuous.*
3. *If each F_λ is a Dugundji φ-contraction (with φ independent of λ) and $\Lambda_0 \ne \emptyset$, then f is continuous.*

Proof. We show that f has a closed graph in $\Lambda \times A$. So assume that $\lambda_n \in \Lambda_0$ for $n \in \mathbb{N}$, $\lambda_n \to \lambda_0$ and $f(\lambda_n) \to x_0$ for some $\lambda_0 \in \Lambda$ and $x_0 \in A$. We are to show that $\lambda_0 \in \Lambda_0$ and $x_0 = f(\lambda_0)$. For any $n \in \mathbb{N}$, we have that $f(\lambda_n) = F_{\lambda_n} f(\lambda_n) = F(\lambda_n, f(\lambda_n))$, so

$$d(x_0, F_{\lambda_0} x_0) \leq d(x_0, f(\lambda_n)) + d(F(\lambda_n, f(\lambda_n)), F(\lambda_0, x_0)). \tag{2.1}$$

By continuity of F, we get that $d(F(\lambda_n, f(\lambda_n)), F(\lambda_0, x_0)) \to 0$. Thus letting n tend to ∞ in (2.1), we obtain that $x_0 = F_{\lambda_0} x_0$, so $\lambda_0 \in \Lambda_0$ and $x_0 = f(\lambda_0)$, which completes the proof of point 1.

Statement 2 follows immediately from point 1 and [8, Exercise 3.1.D, p. 179].

Now assume that each F_λ is a Dugundji φ-contraction. Let $\lambda_n \in \Lambda_0$ for $n \in \mathbb{N} \cup \{0\}$ and $\lambda_n \to \lambda_0$. Then we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} d(f(\lambda_n), f(\lambda_0)) &= d(F_{\lambda_n} f(\lambda_n), F_{\lambda_0} f(\lambda_0)) \leq d(F_{\lambda_n} f(\lambda_n), F_{\lambda_n} f(\lambda_0)) \\ &\quad + d(F_{\lambda_n} f(\lambda_0), F_{\lambda_0} f(\lambda_0)) \leq \varphi(d(f(\lambda_n), f(\lambda_0))) \\ &\quad + d(F(\lambda_n, f(\lambda_0)), F(\lambda_0, f(\lambda_0))). \end{aligned}$$

Set $a_n := d(f(\lambda_n), f(\lambda_0))$ for $n \in \mathbb{N}$. By the above inequality,

$$a_n - \varphi(a_n) \leq d(F(\lambda_n, f(\lambda_0)), F(\lambda_0, f(\lambda_0))) \to 0$$

as $n \to \infty$ because of continuity of $F(\cdot, f(\lambda_0))$. Hence $a_n - \varphi(a_n) \to 0$ since $a_n - \varphi(a_n) \geq 0$. Suppose that (a_n) is unbounded. Then there is a subsequence (a_{k_n}) such that $a_{k_n} \to \infty$ and $a_{k_n} - \varphi(a_{k_n}) \to 0$, which gives that $\liminf_{t\to\infty}(t - \varphi(t)) = 0$, a contradiction. Thus (a_n) is bounded. We show that $a_n \to 0$. Consider any subsequence (a_{k_n}) of (a_n). Then (a_{k_n}) contains a subsequence $(a_{k_{m_n}})$ convergent to some a, $a \geq 0$. By continuity of φ,

$$a_{k_{m_n}} - \varphi(a_{k_{m_n}}) \to a - \varphi(a) \quad \text{as} \quad n \to \infty,$$

so $a - \varphi(a) = 0$ and hence $a = 0$ since $\varphi(t) < t$ for $t > 0$. Thus $a_n \to 0$, i.e., $f(\lambda_n) \to f(\lambda_0)$, which completes the proof. □

The following example shows that the assumption of Theorem 2.1 that F be continuous cannot be weakened by assuming only that F is continuous in each variable separately: the latter condition does not guarantee that f has a closed graph even if both Λ and A are compact.

Example 2.2. Set $X := \mathbb{R}$, $\Lambda := \{1/n : n \in \mathbb{N}\} \cup \{0\}$, $A := [0,1]$, and endow X and Λ with the Euclidean metric. Set $F(0,x) := 0$ for $x \in A$. Consider any two sequences (a_n) and (b_n) of reals such that $a_n \to 0$, $b_n \to 1$, (a_n) is strictly decreasing, (b_n) is strictly increasing and $a_1 < b_1$. For $n \in \mathbb{N}$, define $F(1/n, \cdot)$ as the polygonal line with nodes $(0, -a_n)$, $(a_n, 0)$, $(b_n, 0)$, (b_{n+1}, b_{n+1}) and $(1, 0)$, i.e.,

$$F(1/n, x) := x - a_n \quad \text{for} \quad x \in [0, a_n];$$
$$F(1/n, x) := 0 \quad \text{for} \quad x \in (a_n, b_n);$$

$$F(1/n, x) := b_{n+1} \frac{x - b_n}{b_{n+1} - b_n} \quad \text{for} \quad x \in [b_n, b_{n+1}];$$

$$F(1/n, x) := b_{n+1} \frac{1 - x}{1 - b_{n+1}} \quad \text{for} \quad x \in (b_{n+1}, 1].$$

It is easily seen that F is continuous in each variable separately. Moreover, $\Lambda_0 = \Lambda$, $f(0) = 0$ and $f(1/n) = b_{n+1} \to 1$ for $n \in \mathbb{N}$, so the graph of f is not closed. Observe that F is not continuous at $(0, 1)$ since $(1/n, b_{n+1}) \to (0, 1)$, but

$$F(1/n, b_{n+1}) = b_{n+1} \to 1 \neq 0 = F(0, 1).$$

Let us note that Theorem 2.1 focuses only on the problem of continuity of fixed points and does not give any sufficient conditions for the existence of fixed points of mappings F_λ. However, such conditions for nonself φ-contractions are known in the literature and were given, e.g., by Reich and Zaslavski [14, 15].

3. Continuity of fixed points of self mappings

In this section first we generalize the Dugundji–Granas Theorem 1.4 by removing an inconvenient assumption that a mapping f is locally bounded. Here we replace condition (1.1) by its equivalent form (1.2). In the proof we will use the following obvious lemma.

Lemma 3.1. *Let (X, d) and (Y, ρ) be metric spaces, and $f\colon X \to Y$. Then f is continuous if and only if for any nonempty compact subset A of X, the restriction $f|_A$ is continuous.*

Theorem 3.2. *Let (X, d) be complete, (Λ, ρ) be a metric space and $\{F_\lambda : \lambda \in \Lambda\}$ be a family of selfmaps of X such that*

$$d(F_\lambda x, F_\lambda y) \leq \varphi(d(x, y)) \quad \text{for} \quad x, y \in X \quad \text{and} \quad \lambda \in \Lambda,$$

where $\varphi\colon [0, \infty) \to [0, \infty)$ is continuous nondecreasing and such that $\varphi(t) < t$ for $t > 0$. Assume that for any $x \in X$, the mapping $\Lambda \ni \lambda \mapsto F_\lambda x$ is continuous. Then each F_λ has a unique fixed point x_λ and the mapping $\Lambda \ni \lambda \mapsto x_\lambda$ is continuous.

Proof. By Browder's (see [3] or [9, p. 19]) theorem, each F_λ has a unique fixed point x_λ. For $\lambda \in \Lambda$, set $f(\lambda) := x_\lambda$. Let Ω be a nonempty compact subset of Λ and $C(\Omega, X)$ denote the family of all continuous functions from Ω to X, endowed with the supremum metric σ. Since (X, d) is complete, so is $(C(\Omega, X), \sigma)$ (see, e.g., [12, Theorem 3, p. 90]). Consider the following Nemyckiĭ operator:

$$(Tg)(\lambda) := F(\lambda, g(\lambda)) \quad \text{for} \quad g \in C(\Omega, X) \quad \text{and} \quad \lambda \in \Omega,$$

where $F(\lambda, x) := F_\lambda x$ for $\lambda \in \Lambda$ and $x \in X$. By hypothesis, F is continuous in the first variable and nonexpansive in the second one, which easily yields the continuity of F. Hence we may infer that T maps $C(\Omega, X)$ into itself. We show that T is a φ-contraction with respect to σ. Let $g, h \in C(\Omega, X)$ and $\lambda \in \Omega$. Then, by monotonicity of φ, we have

$$d((Tg)(\lambda), (Th)(\lambda)) = d(F_\lambda g(\lambda), F_\lambda h(\lambda)) \leq \varphi(d(g(\lambda), h(\lambda))) \leq \varphi(\sigma(g, h)).$$

Since λ is arbitrary, we may infer that $\sigma(Tg, Th) \leq \varphi(\sigma(g, h))$, i.e., T is Browder's φ-contraction on $C(\Omega, X)$. Hence T has a fixed point $g_* \in C(\Omega, X)$, so for any $\lambda \in \Lambda$,

$$g_*(\lambda) = F(\lambda, g_*(\lambda)) = F_\lambda g_*(\lambda),$$

i.e., $g_*(\lambda) \in \operatorname{Fix} F_\lambda = \{f(\lambda)\}$. Thus $g_* = f|_\Omega$, so the restriction of f to Ω is continuous. By Lemma 3.1, we get that f is continuous on Λ. $\square$

As an immediate consequence, we get the following extension of Theorem 1.5.

Corollary 3.3. *Let (X,d) be complete and for $n \in \mathbb{N}$, $F_n\colon X \to X$ be φ-contractions in Browder's sense with φ independent of n. Let $x_n = F_n x_n$ for $n \in \mathbb{N}$. If (F_n) converges pointwise to F, then F has a unique fixed point x_0 and $x_n \to x_0$.*

Remark 3.4. The proof of Theorem 3.2 shows that the result on continuity of fixed points for a family of Browder's contractions can be deduced directly from Browder's fixed point theorem. Consequently, the latter theorem is in some sense equivalent to Theorem 3.2. The same remark concerns — as already mentioned in Section 1 — relations between Theorem 1.2 and the Banach contraction principle.

Now we present a generalization of Nadler's Theorem 1.3.

Theorem 3.5. *Let (X,d) be locally compact (not necessarily complete), (Λ,ρ) be a metric space and $\{F_\lambda : \lambda \in \Lambda\}$ be a family of selfmaps of X such that*

$$d(F_\lambda x, F_\lambda y) < d(x,y) \quad \text{for} \quad x,y \in X \quad \text{with} \quad x \neq y, \quad \text{and} \quad \lambda \in \Lambda. \tag{3.1}$$

Assume that for any $x \in X$, the mapping $\Lambda \ni \lambda \mapsto F_\lambda x$ is continuous. Set

$$\Lambda_0 := \{\lambda \in \Lambda : \operatorname{Fix} F_\lambda \neq \emptyset\}.$$

Then Λ_0 is open. Moreover, if $\Lambda_0 \neq \emptyset$, then $\operatorname{Fix} F_\lambda = \{x_\lambda\}$ for $\lambda \in \Lambda_0$, and the mapping $\Lambda_0 \ni \lambda \mapsto x_\lambda$ is continuous.

Proof. Clearly, (3.1) implies the uniqueness of fixed points of mappings F_λ. Assume that $\Lambda_0 \neq \emptyset$. Then we may define $f(\lambda) := x_\lambda$ for $\lambda \in \Lambda_0$, where x_λ is the unique fixed point of F_λ. Let $\lambda_0 \in \Lambda_0$ and set $x_0 := f(\lambda_0)$ so that $x_0 = F_{\lambda_0} x_0$. By hypothesis, there is $r > 0$ such that the closed ball $B(x_0, r)$ is compact. Fix $\varepsilon \in (0, r]$ and set $B := B(x_0, \varepsilon)$. Clearly, B is compact and for $x \in B$,

$$d(F_{\lambda_0} x, x_0) = d(F_{\lambda_0} x, F_{\lambda_0} x_0) \leq d(x, x_0) \leq \varepsilon,$$

so $F_{\lambda_0}(B) \subseteq B$. By (3.1) and [10, Proposition 1], there exists a nondecreasing function $\varphi\colon [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for $t > 0$ and

$$d(F_{\lambda_0} x, F_{\lambda_0} y) \leq \varphi(d(x,y)) \quad \text{for} \quad x, y \in B.$$

Now assume that $\lambda_n \in \Lambda$ for $n \in \mathbb{N}$ and $\lambda_n \to \lambda_0$. Since $\{F_{\lambda_n} : n \in \mathbb{N}\}$ is equicontinuous and $F_{\lambda_n} x \to F_{\lambda_0} x$ for $x \in B$, we may infer by compactness of B that $(F_{\lambda_n}|_B)$ converges uniformly to $F_{\lambda_0}|_B$. Hence, since $\varepsilon - \varphi(\varepsilon) > 0$, there exists $k \in \mathbb{N}$ such that for $n \geq k$ and $x \in B$,

$$d(F_{\lambda_n} x, F_{\lambda_0} x) \leq \varepsilon - \varphi(\varepsilon).$$

Then for such n and x, we get that

$$\begin{aligned} d(F_{\lambda_n} x, x_0) &\leq d(F_{\lambda_n} x, F_{\lambda_0} x) + d(F_{\lambda_0} x, F_{\lambda_0} x_0) \leq \varepsilon - \varphi(\varepsilon) + \varphi(d(x, x_0)) \\ &\leq \varepsilon - \varphi(\varepsilon) + \varphi(\varepsilon) = \varepsilon, \end{aligned}$$

which means that $F_{\lambda_n}(B) \subseteq B$ for $n \geq k$. By (3.1) and Edelstein's [7] fixed point theorem, $\operatorname{Fix} F_{\lambda_n} \cap B \neq \emptyset$, so $\lambda_n \in \Lambda_0$ and $f(\lambda_n) \in B$, i.e., $d(f(\lambda_n), f(\lambda_0)) \leq \varepsilon$ for any $n \geq k$. Thus we have shown that if $\lambda_n \in \Lambda$ for $n \in \mathbb{N}$ and $\lambda_n \to \lambda_0$, then $\lambda_n \in \Lambda_0$ for sufficiently large n, which implies that $\lambda_0 \in \operatorname{int} \Lambda_0$. Hence Λ_0 is open. Moreover, the above argument shows that the sequence $(f(\lambda_n))$ is well defined for

n large enough and it converges to $f(\lambda_0)$. Hence, in particular, if $\lambda_n \in \Lambda_0$ for $n \in \mathbb{N}$ and $\lambda_n \to \lambda_0$, then $f(\lambda_n) \to f(\lambda_0)$, which completes the proof. $\square$

Now Theorem 3.5 and Browder's [3] fixed point theorem imply the following

Corollary 3.6. *Let (X, d) be locally compact and complete, (Λ, ρ) be a metric space and for any $\lambda \in \Lambda$, $F_\lambda \colon X \to X$ be Browder's φ_λ-contraction. Assume that for any $x \in X$, the mapping $\Lambda \ni \lambda \mapsto F_\lambda x$ is continuous. Then each F_λ has a unique fixed point x_λ and the mapping $\Lambda \ni \lambda \mapsto x_\lambda$ is continuous.*

The following example shows that under the assumptions of Theorem 3.5, a set Λ_0 need not be closed.

Example 3.7. Let $X := \mathbb{R}$ and $\Lambda := [-1, 1]$ be endowed with the Euclidean metric. For $\lambda \in \Lambda$ and $x \in \mathbb{R}$, set

$$F_\lambda x := \lambda \ln(1 + e^x).$$

It can easily be verified that all the assumptions of Theorem 3.5 are satisfied. In particular, by the Lagrange mean theorem, we get that $L(F_\lambda) = |\lambda|$ for $\lambda \in \Lambda$, so by the contraction principle, $\operatorname{Fix} F_\lambda \neq \emptyset$ for $\lambda \in (-1, 1)$, whereas

$$\operatorname{Fix} F_1 = \emptyset \quad \text{and} \quad \operatorname{Fix} F_{-1} = \left\{ \ln \left((\sqrt{5} - 1)/2 \right) \right\}.$$

Consequently, $\Lambda_0 = [-1, 1)$, so Λ_0 is open in Λ, but is not closed.

It is also worth noticing that the assumption on the local compactness in Theorem 3.5 cannot be omitted. Namely, as shown by Nadler [13, Theorem 3], for any infinite dimensional separable or reflexive Banach space X, there exists a sequence of contractive selfmaps of X pointwise convergent to the zero mapping, but such that the sequence of fixed points of these mappings does not converge to 0. In fact, Nadler's construction can be done in any Banach space X with the property that there exists a sequence (x_n^*) of linear functionals on X of norm one such that (x_n^*) is weak* convergent to the zero functional. But now it is known that such a sequence exists for any infinite dimensional Banach space. This deep result is known as the Josefson–Nissenzweig theorem (see, e.g., [4, p. 219]). Consequently, the following theorem holds in which, differently from [13], we assume at the beginning that X is only a normed linear space (not necessarily complete).

Theorem 3.8. *Let X be a normed linear space. The following statements are equivalent:*

(i) *X is finite dimensional;*
(ii) *for any metric space (Λ, ρ) and any mapping $F \colon \Lambda \times X \to X$ which is continuous in the first variable and contractive in the second one, we have that every F_λ has a fixed point x_λ and the mapping $\Lambda \ni \lambda \mapsto x_\lambda$ is continuous.*

Proof. (i)$\Rightarrow$(ii): Let (Λ, ρ) and F be as in (ii). By (i), X is a Banach space, so by the contraction principle, every F_λ has a unique fixed point x_λ. The continuity of the mapping $\Lambda \ni \lambda \mapsto x_\lambda$ follows from Theorem 3.5 since X is locally compact as a finite dimensional Banach space.

(ii)$\Leftarrow$(i): By (ii), we may infer that in particular, taking a singleton as Λ, every contractive selfmap of X has a fixed point. Hence, by Borwein's [2] theorem, X is a Banach space. Suppose, on the contrary, that $\dim X = \infty$. Then by Nadler's construction supported by the Josefson–Nissenzweig theorem, there exists a sequence (F_n) of contractive selfmaps of X which converges pointwise to the zero mapping, and such that $\|x_n\| = 1$, where $x_n = F_n x_n$ for $n \in \mathbb{N}$. Define $\Lambda := \{1/n : n \in \mathbb{N}\} \cup \{0\}$ and set

$$F(0,x) := 0 \quad \text{and} \quad F(1/n,x) := F_n x \quad \text{for} \quad n \in \mathbb{N} \quad \text{and} \quad x \in X.$$

Then F is continuous in the first variable and contractive in the second one, but $\Lambda \ni \lambda \mapsto x_\lambda$ is not continuous at 0, which yields a contradiction. $\square$

4. Two partial extensions of Nadler's theorem

Clearly, Nadler's Theorem 1.3 implies the following

Corollary 4.1. *Let X be a finite dimensional normed linear space and A be a nonempty closed subset of X. If $F_n \colon A \to A$ are contractive for $n \in \mathbb{N}$ and (F_n) converges pointwise to a contractive mapping F, then $x_n \to x_0$, where $x_0 = Fx_0$ and $x_n = F_n x_n$ for $n \in \mathbb{N}$.*

In this section we give two partial extensions of Corollary 4.1 by weakening the assumption that F_n be contractive. However, we require A to be convex, which is unnecessary in Corollary 4.1. For $x \in X$ and $B \subseteq X$, we denote by $d(x,B)$ the distance between x and B, with the convention that $d(x,\emptyset) = \infty$.

Theorem 4.2. *Let X be a finite dimensional normed linear space and A be a nonempty closed convex (not necessarily bounded) subset of X. If $F_n \colon A \to A$ for $n \in \mathbb{N}$ are such that the family $\{F_n : n \in \mathbb{N}\}$ is equicontinuous and (F_n) converges pointwise to a contractive mapping F, then $d(x_0, \operatorname{Fix} F_n) \to 0$, where $x_0 = Fx_0$. In particular,* $\operatorname{Fix} F_n \neq \emptyset$ *for sufficiently large n.*

Proof. Fix $\varepsilon > 0$ and consider the closed ball $B(x_0,\varepsilon)$. Set $C := B(x_0,\varepsilon) \cap A$. Since $\dim X < \infty$, C is compact. Hence and by hypothesis, $(F_n|_C)$ converges uniformly to $F|_C$, so there exists $k \in \mathbb{N}$ such that

$$\|F_n x - Fx\| \le (1-\alpha)\varepsilon \quad \text{for} \quad n \ge k \quad \text{and} \quad x \in C,$$

where $\alpha := L(F)$. Then for such n and x, we get that

$$\|F_n x - x_0\| \le \|F_n x - Fx\| + \|Fx - Fx_0\| \le (1-\alpha)\varepsilon + \alpha\|x - x_0\| \le \varepsilon,$$

which implies that $F_n(C) \subseteq C$ for $n \ge k$. Since C is nonempty compact and convex, Brouwer's fixed point theorem yields the existence of a fixed point of F_n in C. In particular, $\operatorname{Fix} F_n \cap B(x_0,\varepsilon) \neq \emptyset$, so

$$d(x_0, \operatorname{Fix} F_n) \le \varepsilon \quad \text{for} \quad n \ge k.$$

Thus we may infer that $d(x_0, \operatorname{Fix} F_n) \to 0$. $\square$

Clearly, if (F_n) is a sequence of nonexpansive mappings, then the family $\{F_n n \in \mathbb{N}\}$ is equicontinuous. However, for nonexpansive mappings Theorem 4.2 can further be generalized. To do that, we need the following definition: a sequence (F_n) is said to be *locally uniformly convergent* to a mapping F if for any $x \in X$, there exists a neighbourhood U of x such that $(F_n|_U)$ converges uniformly to $F|_U$. Observe that under the assumptions of Theorem 4.2, (F_n) converges locally uniformly to F

because of local compactness of X. Therefore, the following result is a generalization of Theorem 4.2 restricted to sequences of nonexpansive mappings.

Theorem 4.3. *Let X be a uniformly convex Banach space and A be a nonempty closed convex (not necessarily bounded) subset of X. If $F_n\colon A \to A$ are nonexpansive for $n \in \mathbb{N}$ and (F_n) converges locally uniformly to a contractive mapping F with fixed point x_0, then almost all F_n have fixed points and $d(x_0, \operatorname{Fix} F_n) \to 0$.*

Proof. By hypothesis, there exists $r > 0$ such that $(F_n|_{B(x_0,r)\cap A})$ is uniformly convergent to $F|_{B(x_0,r)\cap A}$. Fix $\varepsilon \in (0, r]$ and set $C := B(x_0, \varepsilon) \cap A$. Then the same argument as in the proof of Theorem 4.2 shows that there is $k \in \mathbb{N}$ such that $F_n(C) \subseteq C$ for $n \geq k$. C is nonempty closed bounded and convex, so by the Browder-Göhde-Kirk theorem (see, e.g., [9, p. 76]), $\operatorname{Fix} F_n \cap C \neq \emptyset$, which yields that $d(x_0, \operatorname{Fix} F_n) \leq \varepsilon$ for $n \geq k$. Thus we get that $d(x_0, \operatorname{Fix} F_n) \to 0$. $\square$

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Jacek Jachymski
Institute of Mathematics, Łódź University of Technology, Wólczańska 215, 93-005 Łódź, Poland
E-mail address: jacek.jachymski@p.lodz.pl