Pure and Applied Functional Analysis
Volume 2, Number 4, 2017, 625–655

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

PROFILE DECOMPOSITION IN METRIC SPACES

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Abstract. The concentration compactness method for sequences of functions consists in isolating singular behavior of a sequence in elementary sequences called "bubbles" that results in refined convergence. Following [14] that studies concentration structure of sequences in Banach spaces, the present paper provides comparable results for a class of metric spaces, uniformly rotund spaces, that possess an analog of weak (Banach-Alaoglu) compactness property: sequential compactness of bounded sets with respect to polar convergence.

1. Introduction

The present paper develops a general method for analysis of convergence properties of bounded sequences in metric spaces, that originates in the concentration compactness method [8, 9, 16, 11]. This analysis originally developed for Sobolev spaces, shows that, even in absence of compactness, it is possible to extract, from any bounded sequence of the space, convergent subsequences up to a well-structured sum of concentration terms, called defect of compactness or profile decomposition. This structure is achieved by studying the sequence in relation to a group of scalings (isometries), and the concentration-compactness method has been recently formalized for general, uniformly convex and uniformly smooth Banach spaces (see [14]).

Bounded functional sequences have, generally, poor convergence properties. The fundamental compactness theorem, Banach-Alaoglu Theorem (see [10, Theorem 3.15 and Theorem 3.17]), assures (among its various formulations and corollaries) that every weakly-$*$ closed and bounded set of the dual space of a Banach space is weakly-$*$ compact, (or, when in addition the space itself is the dual of some separable Banach space, that any bounded sequence has a weakly-$*$ convergent subsequence). In the metric setting, a less known Delta-compactness theorem due to T. C. Lim (see [5, Theorem 3], assures that every bounded sequence has a Delta-convergent subsequence, provided that the metric space is asymptotically complete. A sufficient condition for asymptotic completeness of a metric space is its Staples rotundity (see [15] or condition (SR) below), which, in the case of Banach spaces, is equivalent to uniform convexity. In the case of Banach spaces Delta-convergence and weak-$*$ convergence do not generally coincide. The Opial condition (see [3, Definition 5.1]), which is a necessary and sufficient condition for the coincidence of the two

2010 *Mathematics Subject Classification.* 46B50, 46B10, 52A01, 49N99, 54D30, 54E99.

Key words and phrases. Concentration compactness, cocompactness, profile decomposition, Delta-convergence, polar convergence, rotund metric spaces.

Research of one of the authors (G.D.) was partly supported by a visitor fellowship from Wenner-Gren Foundations.

modes of convergence, can be achieved, in separable spaces, by passing to a suitable equivalent norm (see [17]). For details on convergence of weak type in metric spaces we refer the reader to the recent survey [3].

An important refinement of Banach-Alaoglu Theorem is profile decomposition. If a reflexive Banach space E is equipped with a suitable group G of isometries, for any bounded sequence one can find a subsequence that will converge better than weakly: namely, the remainder r_n between the element of the sequence and a suitable sum of scaled "bubbles" is G-vanishing, that is, $\forall (g_n)_{n\in\mathbb{N}} \subset G$, $g_n(r_n) \rightharpoonup 0$ (in applications it is a significantly better convergence than weak convergence). This remainder is attainable, however, only after subtraction of a series of decoupled "bubbles", i.e. terms of the form $g_n(\varphi)$, which in concrete cases are often shrinking the effective support of a function φ to a point. Existence of profile decompositions in Sobolev spaces, with the first proof given (for P.S. sequences of a suitable functional) by Struwe in [16, Proposition 2.1], following the more empiric concentration compactness method of (among others) Sacks-Uhlenbeck in [11] and Lions in [8, 9], and by one of the authors in [13] (for general bounded sequences), has been extended to Hilbert spaces in [12], and to general uniformly convex Banach spaces in [14]. Notably, already in the case of Banach spaces, the relevant mode of convergence, is polar convergence (a slightly stronger variant of Delta-convergence), rather than weak convergence, which is indicative that concentration compactness argument can be carried out, on the functional-analytic level, in metric spaces as well. This, of course, restricts consideration to metric spaces where bounded sequences have polar convergent subsequences, namely, to asymptotically complete metric spaces. As we have already said, asymptotic completeness, similarly to reflexivity in Banach spaces, can be inferred from a property of uniform rotundity given by Staples that, in metric spaces, corresponds to uniform convexity. In [2], functional-analytic notions concerning profile decompositions in Lebesgue spaces were clarified, allowing to formulate conjectures how the presence of a group of isometries can be used to improve the information about the convergence (modulo subsequences) of bounded sequences in metric spaces.

The key notions of concentration analysis in metric spaces are *profile system* with an associated *blowup system*, *null points*, and *profile reconstruction*. The notion of profile is the same as in Banach spaces. If $(u_n)_{n\in\mathbb{N}}$ is a bounded sequence in a metric space (E, d), equipped with a group of isometries G, and if there exists a sequence $(g_n)_{n\in\mathbb{N}} \subset G$, such that $(g_n^{-1}(u_n))_{n\in\mathbb{N}}$ has a polar limit w, then w is called a *profile* of the sequence $(u_n)_{n\in\mathbb{N}}$, and the sequence $(g_n)_{n\in\mathbb{N}}$ is called a *blowup* sequence of this profile (or a scale transitions sequence ($s.t.s$) in [2]), with understanding that the "bubble" sequence $(g_n(w))_{n\in\mathbb{N}}$ describes a somehow isolated formation of singularity of the original sequence $(u_n)_{n\in\mathbb{N}}$. In the linear case, one could speak about profile reconstruction of $(u_n)_{n\in\mathbb{N}}$ as a sum of bubbles, which differs from $(u_n)_{n\in\mathbb{N}}$ by a remainder close to zero. In the metric spaces, the notion of profile reconstruction as well as the notion of zero has to be of course reformulated.

Our counterpart of zero is the *null set* $Z_G \subset E$ (relative to the group G), defined as the set of profiles of constant sequences produced by *discrete* (see Definition 2.8 below) sequences of blowup. Since constant sequences are not supposed, in the intuitive sense, to form concentrations ("to bubble"), it is natural to mark such profiles as trivial (in the Banach space case one always has $Z_G = \{0\}$), and to associate profile decomposition only with non-trivial profiles.

Since sequences of the form $(g_n^{-1}(u_n))_{n\in\mathbb{N}}$ (even if they are still bounded) do not necessarily converge, but only have polar convergent subsequences, we prove the existence of a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ where each profile is given as polar limit of a sequence like $(g_n^{-1}(u_n))_{n\in\mathbb{N}}$, while no subsequence has additional profiles. This is, roughly speaking, the notion of *complete profile system*, used in [2] and, subsequently, here (see Definition 5.4 below). A property of decoupled (also called quasiorthogonal or mutually divergent) blowup sequences corresponding to distinct profiles (see Remark 3.3 below) still holds in the metric setting.

Given a family of profiles and an associated family of blowup sequences, we have no linear structure to reconstruct the singular portion of the original sequence from its bubbles. Following [2], however, we are able to identify a sequence, that has the same bubbles of the original sequence, and it is "optimal" since it has the minimal asymptotical distance from the "origin" (that is, from the null set Z_G).

The paper is organized as follows. In Section 2 we recall the notion of polar convergence and some related properties. Then we set axiomatic properties of the scaling group G and of the null set Z_G. In Section 3 we define profiles, blowup sequences, and profile systems (and blowup systems) of a bounded sequence, and prove some of their elementary properties. In Section 4 we introduce the notions related to profile systems and their quantitative characterization in terms of asymptotic radii and asymptotic distance from the null set. These notions allow to establish an important estimate in terms of a suitable additive energy of profiles (the energy of a profile is given trough the modulus of rotundity of the metric space evaluated at the distance between the profile and the null set, see Definition 4.15 below), which, in turn, allows to prove the "multiscale polar compactness" property (i.e. every bounded sequence has a subsequence with a complete profile system) which is discussed in detail in Section 5. In Section 6 we prove existence and essential uniqueness of profile reconstruction of a given family of (profile) elements and a given family of blowup sequences, under additional assumptions on the space (existence of "middle points" and Axiom E2) and on the null set (required to be Staples convex, see Axiom E3).

In Section 7 we discuss the notion of G-cocompactness, which expresses metrization of our improved convergence in terms of a second, generally weaker, metric.

The results of this paper have been presented by the first author at the "9$^{\text{th}}$ European conference on Elliptic and Parabolic Problems" held in May 2016 in Gaeta (Italy) in the particular case in which the null set Z_G reduces to a single point. The proceedings paper [1] can be seen as a more detailed introduction to this one.

2. Polar convergence and scalings group on a metric space

Let (E, d) be a metric space. On the space

$$\mathcal{E} = \{(u_n)_{n\in\mathbb{N}} \subset E \mid (u_n)_{n\in\mathbb{N}} \text{ is bounded } \}$$

of *bounded* sequences of elements of E we shall consider the following mode of convergence which we have introduced and surveyed (with respect to other modes of convergence of weak type in metric spaces) in [3]. The definition is very close to Δ-convergence of T. C. Lim [5] and to "almost convergence" of Kuczumow [4]. Weak convergence in Hadamard spaces (and in particular in Hilbert spaces) coincides with polar and Δ-convergence, while in Banach spaces this is generally not the case. For the reader's convenience we provide [3, Definition 2.7].

Definition 2.1 (Polar limit)**.** Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ and $u \in E$. We shall say that u is the *polar limit* of $(u_n)_{n\in\mathbb{N}}$ (or, equivalently, that $(u_n)_{n\in\mathbb{N}}$ is polar convergent to u) and we shall write $u_n \rightharpoondown u$, if for every $v \neq u$ there exists $M(v) \in \mathbb{N}$ such that

$$d(u_n, u) < d(u_n, v) \text{ for all } n \geq M(v). \tag{2.1}$$

In the following we shall assume that E is a complete Staples rotund (SR for short) metric space.

We recall (see [3, Definition 3.1]) that a metric space (E, d) is a (uniformly) SR ("Staples rotund") metric space (or satisfies (uniformly) property SR) if there exists a function $\delta : (\mathbb{R}_+)^2 \to \mathbb{R}_+$ (called *modulus of rotundity* of the space) such that for any r, $\overline{d} > 0$, for any $u, v \in E$ with $d(u, v) \geq \overline{d}$:

$$\operatorname{rad}(B_{r+\delta(r,\overline{d})}(u) \cap B_{r+\delta(r,\overline{d})}(v)) \leq r - \delta(r, \overline{d}), \tag{2.2}$$

where, for any subset $X \subset E$, $\operatorname{rad}(X) := \inf_{u\in E} \sup_{v\in X} d(u, v)$ is the Chebyshev radius of the set X. As pointed out in [3, Remark 3.2], it suffices only to require that for any r, $\overline{d} > 0$, for any $u, v \in E$ with $d(u, v) \geq \overline{d}$:

$$\operatorname{rad}(B_r(u) \cap B_r(v)) \leq r - \delta(r, \overline{d}), \tag{SR}$$

when the modulus of rotundity of the space is a continuous function (as will be assumed throughout the paper).

We take the opportunity to recall the notion of Chebyshev center of a set $X \subset E$, denoted by $\operatorname{cen}(X)$, as one of the points (if they exist) $u \in E$ such that such infimum is achieved, i.e. such that $\sup_{v\in X} d(u, v) = \operatorname{rad}(X)$.

Note that, given $r > 0$ and $\overline{d} > \overline{d}' > 0$, since if $d(u, v) \geq \overline{d}$ then also $d(u, v) > \overline{d}'$ we get, by (SR), that $\operatorname{rad}(B_r(u) \cap B_r(v))$ not only is bounded by $r - \delta(r, \overline{d})$ but also by $r - \delta(r, \overline{d}')$. Therefore we deduce that $\delta(r, \overline{d}')$ cannot be bigger than $\delta(r, \overline{d})$ and so we shall always assume that the modulus of rotundity is also monotone increasing with respect to $\overline{d}$.

Complete SR metric spaces have a remarkable compactness property with respect to polar convergence (see [3, Theorem 3.4], based on results of Lim [5] and Staples [15]) reported below for the reader's convenience.

Theorem 2.2. *Let (E, d) be a complete SR metric space. Then every sequence in $\mathcal{E}$ has a polar convergent subsequence.*

Since (see [3, Remark 3.2]) a SR Banach space is a uniformly convex Banach space (and therefore a reflexive space, see [7, Proposition 1.e.3]), Theorem 2.2 is an analog of the Banach-Alaoglu Theorem for weak convergence, and coincides with it in Hilbert spaces or, more generally, whenever polar convergence coincides with weak convergence.

Of course, in what follows we do not assume that the space E is compact, otherwise polar convergence would coincide with distance convergence and most of the statements in this paper would become tautological or trivial.

The following easy lemma, concerning sequences which are polarly but not strongly convergent, will be helpful in the proof of Proposition 6.4 below.

Lemma 2.3. *Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$, $u \in E$ such that $u_n \rightharpoonup u$ and such that there exists a constant $\overline{d} > 0$ such that*

$$\liminf_{n\to+\infty} d(u_n, u) > \overline{d}.$$

Then, there exists a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ such that

$$d(u_n, u_{k_n}) \geq \overline{d} \quad \forall n \in \mathbb{N}.$$

Proof. Since

$$\forall n \in \mathbb{N} \quad d(u_n, u_m) \geq d(u_n, u) > \overline{d} \text{ for large } m,$$

we can take as k_n the first integer $m > k_{n-1}$ for which the above inequality holds true. ☐

The following lemma gives a certain kind of "stability property" of the polar limit.

Lemma 2.4. *Let E be a SR metric space, let $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$. If for each $\varepsilon > 0$ there exists $(w_n)_{n\in\mathbb{N}} \in \mathcal{E}$ such that*

$$d(v_n, w_n) \leq \varepsilon \text{ for large } n \qquad \text{and} \qquad w_n \rightharpoonup v,$$

then $v_n \rightharpoonup v$ too.

Proof. Let $w \neq v$ and $\varepsilon > 0$ be given. Then, by choosing w_n as in the statement, we have, for large n, $d(v_n, v) \leq d(v_n, w_n) + d(w_n, v) \leq d(v_n, w_n) + d(w_n, w) \leq 2d(v_n, w_n) + d(v_n, w) \leq d(v_n, w) + 2\varepsilon$. The thesis then follows by applying [3, Lemma 3.5]. ☐

We consider E equipped with a group G of isometries, and we would like to study the question, which was successfully resolved for uniformly convex Banach spaces in [14], of finding (for bounded sequences) a convergence which is better than polar convergence, by accounting for "concentration bubbles" created by the action of an isometries group G. We shall denote by $\mathcal{G}$ the space of sequences $(g_n)_{n\in\mathbb{N}} \subset G$.

Note that, since G consists of bijective isometries on E, any element $g \in G$ is (sequentially) continuous with respect to polar convergence, i.e.

$$\forall (u_n)_{n\in\mathbb{N}} \in \mathcal{E} \text{ s.t. } u_n \rightharpoonup u \in E: \quad g(u_n) \rightharpoonup g(u) \quad \forall g \in G. \tag{2.3}$$

Definition 2.5 (convergent sequences of scalings)**.** We shall say that $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ is *convergent* (strongly pointwise) if there exists $g : E \to E$ such that, $d(g_n(u), g(u)) \to 0$ for all $u \in E$. In such a case we shall write $g_n \to g$.

Note that the strong (pointwise) limit of a sequence in $\mathcal{G}$ is an isometry, even if it is not in general bijective as the following example shows.

Example 2.6. Let E be an infinite dimensional Hilbert space. Let $(e_k)_{k\in\mathbb{N}} \subset E$ be a sequence of orthogonal (unitary) vectors and let F be its closed span (so that $E = F \oplus F^\perp$). Given $n \in \mathbb{N}$ let $g_n : E \to E$ be the linear map which is equal to the identity map on $F^\perp$ and is defined on the basis $(e_k)_{k\in\mathbb{N}}$ of F as follows

$$g_n(e_k) := \begin{cases} e_{k+1} & \text{if} \quad k \in \{0,\dots,n-1\} \\ e_0 & \text{if} \quad k = n \\ e_k & \text{if} \quad k > n. \end{cases}$$

For any n g_n is a bijective isometry on E equipped with the Euclidean metric, but the strong pointwise limit g of $(g_n)_{n\in\mathbb{N}}$ is not bijective, indeed the function g is (the identity map on $F^\perp$ and is) defined on the elements $(e_k)_{k\in\mathbb{N}}$ by setting $g(e_k) = e_{k+1}$.

So, by eventually replacing G with its closure (with respect to the above introduced convergence) we shall assume the following axiom.

Axiom G1. The group G is closed with respect to the strong pointwise convergence.

Lemma 2.7. *If G satisfies Axiom G1 then the following equivalence holds true for any sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$:*

$$g_n \to g \quad \textit{if and only if} \quad g_n^{-1} \to g^{-1}. \tag{2.4}$$

Proof. For any $v \in E$ we have $g_n(g^{-1}(v)) \to g(g^{-1}(v)) = v = g_n(g_n^{-1})(v)$ for all $n \in \mathbb{N}$, so, since each g_n is an isometry, we deduce that $d(g^{-1}(v), g_n^{-1}(v)) \to 0$. ☐

Definition 2.8 (discrete sequences)**.** We shall say that $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ is a *discrete sequence* if the sequence $(g_n)_{n\in\mathbb{N}}$ has no convergent subsequence.

The set

$$\mathcal{G}_\infty := \{(g_n)_{n\in\mathbb{N}} \in \mathcal{G} \mid (g_n)_{n\in\mathbb{N}} \text{ is discrete } \}$$

has many nice stability properties pointed out in the following remark.

Remark 2.9. Let G satisfy Axiom G1. Then for all $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$ we have that, for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, $(g_{k_n})_{n\in\mathbb{N}} \in \mathcal{G}_\infty$. Moreover it is easy to prove, by using (2.4), that

i) $(g_n^{-1})_{n\in\mathbb{N}} \in \mathcal{G}_\infty$;
ii) $\forall g \in G$, $(g \circ g_n)_{n\in\mathbb{N}}$ and $(g_n \circ g)_{n\in\mathbb{N}}$ are still in $\mathcal{G}_\infty$.

As in the linear case addressed in [14], we identify the group G as a "concentration mechanism" in the space E, i.e. our purpose is to identify "concentration bubbles" $(g_n(\varphi))_{n\in\mathbb{N}}$, $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$, $\varphi \in E$, of a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ whenever $g_n^{-1}(u_n) \rightharpoonup \varphi$. Moreover, in the linear case, the element φ, when $\varphi \neq 0$, is called a concentration profile while $\varphi = 0$, describes rather an absence of concentration relative to scalings $(g_n)_{n\in\mathbb{N}}$. In the more general setting of metric spaces, since it is still natural to think that constant sequences do not "bubble", we define the *null set* as the set of all polar limits of discrete blowup sequences of a constant sequence. The null set is a "surrogate zero", relative to the group G, for the metric space E.

More precisely, we give the following definitions.

Definition 2.10 (null points - null set - polar infinitesimal sequences)**.** We shall say that $z \in E$ is a *null point* of E (relative to G) if there exist a (discrete) sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$ and a point $u \in E$ such that $g_n(u) \rightharpoondown z$. The set $Z_G \subset E$ of all null points will be called the *null set*.

Finally, we shall say that a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ is *polar infinitesimal* if all polar limits of its (polar convergent) subsequences belong to the null set Z_G.

From definitions 2.8 and 2.10 it follows that if $u \in E$, $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ and $g_n(u) \rightharpoondown w \notin Z_G$, then $(g_n)_{n\in\mathbb{N}}$ has a convergent subsequence $(g_{k_n})_{n\in\mathbb{N}}$. Indeed, otherwise the sequence $(g_n)_{n\in\mathbb{N}}$ would be discrete and then w would belong to Z_G. Note also that, if for some $c \in E$ and $R > 0$, $(u_n)_{n\in\mathbb{N}} \subset B_R(c)$ and $u_n \rightharpoondown u \in E$ then, since $u \in B_{2R}(c)$,

$$d(u, Z_G) \leq d(u, z) \leq d(z, c) + 2R, \quad \forall z \in Z_G.$$

Proposition 2.11. *Let G satisfy Axiom G1. The null set is stable with respect to the action of the group G, namely $G(Z_G) := \{g(z) \mid z \in Z_G,\ g \in G\} = Z_G$.*

Proof. Since G contains the identity, it suffices to show that for any $z \in Z_G$ and any $g \in G$, one has $g(z) \in Z_G$. Assume, by the definition of null point, that $g_n(u) \rightharpoondown z$ where $u \in E$ and $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ is a discrete sequence. Then, by (2.3) and Remark 2.9 - item ii), we deduce that $g(z)$ is a null point (since $g(z)$ is the polar limit of the constant sequence $(u)_{n\in\mathbb{N}}$ scaled by the discrete sequence $(g \circ g_n)_{n\in\mathbb{N}}$). ☐

Taking into account that the maps $g \in G$ are isometries one also has the following corollary.

Corollary 2.12. *Let G satisfy Axiom G1. Then*

$$d(u,\, Z_G) = d(g(u),\, Z_G) \quad \forall u \in E, \forall g \in G.$$

Remark 2.13. Note that Axiom G1 does not guarantee that the null set $Z_G \neq \emptyset$. Indeed, if the group G is sequentially compact (with respect to the strong pointwise convergence) then no sequence of scalings can be discrete, i.e. $\mathcal{G}_\infty = \emptyset$ and therefore $Z_G = \emptyset$.

Of course we can give a sufficient condition, which immediately follows from Definition 2.10, to have $Z_G \neq \emptyset$.

Proposition 2.14. *If $\mathcal{G}_\infty \neq \emptyset$ and there exists $z \in E$ which is fixed by a (discrete) sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$ (i.e. such that $g_n(z) = z$ for all n), then $z \in Z_G$.*

Corollary 2.15. *If $\mathcal{G}_\infty \neq \emptyset$ and there exists $z \in E$ which is fixed by G (i.e. such that $g(z) = z$ for all $g \in G$), then $z \in Z_G$.*

In the following we shall require the following axiom.

Axiom G2. The null set Z_G is not empty and satisfies the Heine-Borel property, i.e. its closed and bounded subsets are compact.

Note that, see Remark 2.13, Axiom G2 implies that the group G is not sequentially compact.

Remark 2.16. If G satisfies Axiom G2 then the null set Z_G is closed.

Of course, by taking into account Remark 2.13 one gets examples where Axiom G1 holds while Axiom G2 does not.

A less trivial example in which Axiom G1 holds but Axiom G2 fails in the following.

Example 2.17. Let $E = L^\infty(\mathbb{R})$ and let G be the group of integer shifts. (Note that G is closed and so Axiom G1 holds true). Since any function $u \in E$ which is 1-periodic is fixed by G and so, since $\mathcal{G}_\infty \neq \emptyset$, by Corollary 2.15, it must belong to Z_G, we deduce that Z_G contains an infinite dimensional space and so Axiom G2 fails.

Note that Example 2.17 deals with the metric space $E = L^\infty(\mathbb{R})$ which is not SR, but, even under property SR, Axiom G1 does not imply Axiom G2 as shown by the following example.

Example 2.18. Let $B := \{x \in \mathbb{C} \,|\, |x| < 1\}$ and let us consider the Hilbert (and therefore SR) space $E = L^2(B) \times L^2(\mathbb{R})$. For all $t \in \mathbb{R}$ we denote by g_t the scaling which maps any $u = (v, w) \in E$ into a pair of functions $g_t(u)$ defined by setting for all $(x, y) \in B \times \mathbb{R}$ $g_t(u)(x, y) = (v(xe^{-2\pi it}), w(y - t))$. It is easy to verify that the group $G := \{g_t \,|\, t \in \mathbb{R}\}$ satisfies Axiom G1 and that $\mathcal{G}_\infty \neq \emptyset$, indeed the sequence $(g_n)_{n\in\mathbb{N}}$ is discrete. Since for all $n \in \mathbb{N}$ and for any $v \in L^2(B)$ we have $g_n(v, 0) = (v, 0)$ (where 0 denotes the null function), we deduce that any pair $(v, 0) \in E$ is fixed by the discrete sequence $(g_n)_{n\in\mathbb{N}}$ and so, by Proposition 2.14, that $L^2(B) \times \{0\} \subset Z_G$ getting, as a consequence, that Axiom G2 does not hold.

When axioms G1 and G2 hold true we can prove the following proposition.

Proposition 2.19. *Let G satisfy axioms G1 and G2. Let $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ and assume that there exists $u \in E \setminus Z_G$, such that $g_n(u) \rightharpoonup z \in Z_G$, then:*

i) $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$;
ii) *for any $v \in E$, the sequence $(g_n(v))_{n\in\mathbb{N}}$ is polar infinitesimal (see Definition 2.10).*

Proof.

i) Assume that $(g_n)_{n\in\mathbb{N}}$ is not discrete. Then there exists $g \in G$ such that, modulo a renamed subsequence, $g_n(u) \to g(u) = z$. Thus, by Proposition 2.11, we get the contradiction $u \in Z_G$.
ii) The statement follows from the definition of Z_G since every subsequence of $(g_n)_{n\in\mathbb{N}}$ is still discrete.

□

3. Profiles and related notions

In this section we shall give some definitions which generalize some concepts introduced in [2] for Lebesgue spaces and shall give some remarks which hold true when the group G of scalings satisfies axioms G1 and G2.

Definition 3.1 (Profiles, blowup sequences and bubbles)**.** Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ be given. We shall say that $\varphi \in E \setminus Z_G$ is a *profile* of the sequence $(u_n)_{n\in\mathbb{N}}$ if there exists $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$ such that

$$\rho_n^{-1}(u_n) \rightharpoonup \varphi. \tag{3.1}$$

In such a case we shall call $\boldsymbol{\rho}$ a *blowup sequence* (or *scale transitions sequence* (*s.t.s.*) as in [2]) for the profile φ, while we shall refer to the sequence $(\rho_n(\varphi))_{n\in\mathbb{N}}$ as to a *bubble* of the sequence $(u_n)_{n\in\mathbb{N}}$.

If (3.1) is satisfied with $\varphi \in Z_G$, we shall call φ a *null profile*.

Definition 3.2 (Scale equivalence, mutual divergence)**.** Let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$, $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be two sequences of scalings. We shall say that $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are *scale equivalent* if the sequence $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}}$ converges (strongly pointwise) to the identity mapping $\boldsymbol{i_d}$. While we shall say that $\boldsymbol{\rho}$ and $\boldsymbol{\sigma}$ are *mutually divergent* or *quasiorthogonal* if the sequence $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$ (i.e. is discrete).

Note that the first above defined relation is an equivalence relation on the set $\mathcal{G}$ of the sequences of scalings and we denote by $[\boldsymbol{\rho}]_S$ the *scale equivalence class* containing $\boldsymbol{\rho}$.

Remark 3.3. Note that if φ is a profile (resp. null profile) of the sequence $(u_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is a blowup sequence of φ, then any $\boldsymbol{\sigma} \in [\boldsymbol{\rho}]_S$ is still a blowup sequence of φ, while for all $g \in G$, by (2.3), $g(\varphi)$ is still a profile (resp. null profile) of the sequence $(u_n)_{n\in\mathbb{N}}$ and $(\rho_n \circ g^{-1})_{n\in\mathbb{N}}$ is a blowup of the profile (resp. null profile) $g(\varphi)$.

Therefore we give the following definitions.

Definition 3.4 (profile copies, distinct profiles, orbit of copies)**.** We shall say that two profiles φ and ψ of a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ are *copies* if there exists $g \in G$ such that $\psi = g(\varphi)$. On the contrary we shall say that φ and ψ are *distinct profiles* if $\psi \neq g(\varphi)$ for all $g \in G$. So any profile, or null profile, φ can be thought as an *orbit of copies* $G(\varphi) := (g(\varphi))_{g\in G}$ which have, by Corollary 2.12, the same distance $d(\varphi, Z_G)$ from the null set.

Lemma 3.5. *The orbit $G(\varphi)$ of every profile $\varphi \in E \setminus Z_G$ is closed.*

Proof. Let $u \in E$ be the (strong) limit of a sequence in $G(\varphi)$, i.e. let $u = \lim_n g_n(\varphi)$ with $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$. If $(g_n)_{n\in\mathbb{N}}$ is discrete then, since strong convergence implies polar convergence, $u \in Z_G$ and, therefore, by Proposition 2.11, $(g_n^{-1}(u))_{n\in\mathbb{N}} \subset Z_G$. Then, since $d(g_n^{-1}(u), \varphi) = d(u, g_n(\varphi)) \to 0$, by Remark 2.16, we get the contradiction $\varphi \in Z_G$. So we can assume that $(g_n)_{n\in\mathbb{N}}$ is not discrete and therefore there exists a subsequence $(g_{k_n})_{n\in\mathbb{N}}$ and $g \in G$ such that $g_{k_n} \to g$. Therefore $u = \lim_n g_{k_n}(\varphi) = g(\varphi) \in G(\varphi)$. □

We also remark that if there exists $u \in E$ with a bounded orbit $G(u)$ (in particular, if Axiom G3 below holds true), then the orbit of every element in E is bounded too.

Lemma 3.6. *Blowup sequences related to distinct profiles are quasiorthogonal (mutually divergent).*

Proof. Assume, by contradiction, that φ and ψ are distinct profiles of a bounded sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ with related blowup sequences $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\sigma} = (\sigma_n)_{n\in\mathbb{N}}$ which are not mutually divergent. Then, see Definition 3.2, $(\sigma_n^{-1} \circ \rho_n)_{n\in\mathbb{N}}$ is not discrete, i.e. , see Definition 2.8, there exists $g \in G$ and an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $\sigma_{k_n}^{-1} \circ \rho_{k_n} \to g$, i.e. $\sigma_{k_n}^{-1} \circ \rho_{k_n} \circ g^{-1} \to i_d$. So $(\sigma_{k_n})_{n\in\mathbb{N}}$ and $(\rho_{k_n} \circ g^{-1})_{n\in\mathbb{N}}$ are scale equivalent. Since $(\sigma_{k_n})_{n\in\mathbb{N}}$ is a blowup sequence related to ψ (as profile of the subsequence $(u_{k_n})_{n\in\mathbb{N}}$) and $(\rho_{k_n})_{n\in\mathbb{N}}$ is a blowup sequence related to φ (as profile of the subsequence $(u_{k_n})_{n\in\mathbb{N}}$) and therefore, by Remark 3.3, $(\rho_{k_n} \circ g^{-1})_{n\in\mathbb{N}}$ is a blowup sequence related to φ, we deduce the contradiction $\psi = g(\varphi)$. ☐

Definition 3.7 (Multiplicity)**.** Let φ be a profile of a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$. We shall call *multiplicity* of the profile φ the supremum $m(\varphi)$ of the cardinality of the set of mutually divergent blowup sequences of φ. If $m(\varphi) = 1$ we shall say that φ is a *simple profile* while, if $m(\varphi) \geq 2$, we shall say that φ is a *multiple profile.*

We shall introduce an axiom (see Axiom G3 below) which will guarantee that the multiplicity of any (non-null) profile of a bounded sequence is finite (see Remark 5.1 below).

Remark 3.8. Every subsequence inherits any profile φ of the whole sequence, with at least the same multiplicity $m(\varphi)$. Indeed, if φ is a profile of a sequence $(u_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}}$ is a related blowup sequence, then for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, φ is a profile of the subsequence $(u_{k_n})_{n\in\mathbb{N}}$ and $(\rho_{k_n})_{n\in\mathbb{N}}$ is a related blowup (i.e. $\rho_{k_n}^{-1}(u_{k_n}) \rightharpoonup \varphi$) see [3, Remark 22.9-item (iii)].

Definition 3.9 (Profile system)**.** Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ be given. A family $\Phi = (\varphi_i)_{i\in I}$ of profiles of the sequence $(u_n)_{n\in\mathbb{N}}$ is said to be a *profile system* (in E) of the sequence $(u_n)_{n\in\mathbb{N}}$ if, for any profile φ, all elements φ_i which are copies of φ are equal and their number is finite and does not exceed $m(\varphi)$.

By combining Remark 3.3 with Definition 3.7 we can assume that, in relation to a profile system $(\varphi_i)_{i\in I}$ of the sequence $(u_n)_{n\in\mathbb{N}}$, there exists a family $(\boldsymbol{\rho}_i)_{i\in I}$ such that

i) for all $i \in I$, $\boldsymbol{\rho}_i = (\rho_n^i)_{n\in\mathbb{N}}$ is a blowup sequence of the profile φ_i;
ii) for all $i, j \in I$, $i \neq j$, $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j$ are quasiorthogonal (mutually divergent).

So we can give the following definition.

Definition 3.10 (blowup system - concentration system)**.** Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ be given and let $\Phi = (\varphi_i)_{i\in I}$ be a profile system of $(u_n)_{n\in\mathbb{N}}$. A family $\boldsymbol{P} = (\boldsymbol{\rho}_i)_{i\in I}$ is a *blowup system* related to the profile system $(\varphi_i)_{i\in I}$ of $(u_n)_{n\in\mathbb{N}}$ if items i) and ii) above hold true. The pair $(\Phi, \boldsymbol{P})$ will be called a *concentration system* of $(u_n)_{n\in\mathbb{N}}$.

Remark 3.11. Note that, if for all $i \in I$ $\boldsymbol{\sigma}_i \in [\boldsymbol{\rho}_i]_S$, then, by Remark 3.3, also the family $(\boldsymbol{\sigma}_i)_{i\in I}$ is a blowup system of the profile system, $(\varphi_i)_{i\in I}$.

Definition 3.12 (extracted blowup system - extracted concentration system)**.** Let $(\Phi, \boldsymbol{P})$ as in Definition 3.10. Let $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}_i)_{i\in I} = ((\sigma_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a *blowup system* related to the profile system $\Phi = (\varphi_i)_{i\in I}$. We will say that $\boldsymbol{\Sigma}$ is *extracted* from $\boldsymbol{P}$ (by the extraction law $(k_n)_{n\in\mathbb{N}}$) if there exists an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that

$$\boldsymbol{\sigma}_i = (\sigma_n^i)_{n\in\mathbb{N}} = (\rho_{k_n}^i)_{n\in\mathbb{N}} \quad \forall i \in I.$$

In such a case we shall say that $(\Phi, \mathbf{\Sigma})$ is the *concentration system extracted* from $(\Phi, \boldsymbol{P})$ (by the extraction law $(k_n)_{n\in\mathbb{N}}$).

With a clear abuse of terminology which does not give rise to possible misunderstandings, we shall say that a profile system Φ' (resp. a concentration system $(\Phi', \boldsymbol{P}')$) *is included in* - or *is a subsystem of* - Φ (resp. $(\Phi, \boldsymbol{P})$) and we shall write $\Phi' \subset \Phi$ (resp. $(\Phi', \boldsymbol{P}') \subset (\Phi, \boldsymbol{P})$) if there exists $J \subset I$ such that $\Phi' = (\varphi_i)_{i\in J}$ (and, in the respective case, $\boldsymbol{P}' = (\boldsymbol{\rho}_i)_{i\in J}$). Finally, we shall say that Φ' (resp. $(\Phi', \boldsymbol{P}')$) is a *maximal subsystem* of Φ (resp. $(\Phi, \boldsymbol{P})$)) if the set $I \setminus J$ reduces to single point.

Remark 3.13. Given a profile system $\Phi = (\varphi_i)_{i\in I}$ of a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$, in relation to any profile $\overline{\varphi} \notin Z_G$ of $(u_n)_{n\in\mathbb{N}}$ we can build a "richer" profile system of $(u_n)_{n\in\mathbb{N}}$, denoted by $\Phi \cup \{\overline{\varphi}\}$, if one of the following alternatives holds true:

a) $\overline{\varphi} \neq g(\varphi_i)$ for all $i \in I$, and for all $g \in G$ (i.e. when we are really adding a new profile);

b) if there exists $\overline{\imath} \in I$ and $\overline{g} \in G$ such that $\overline{\varphi} = \overline{g}(\varphi_{\overline{\imath}})$ (and in such a case we have to replace $\overline{\varphi}$ by $\varphi_{\overline{\imath}}$) then $\mathrm{card}(I_{\overline{\varphi}}) < m(\varphi_{\overline{\imath}})$ where $I_{\overline{\varphi}} := \{i \in I \mid \exists g \in G, \overline{\varphi} = g(\varphi_i)\}$.

If a) or b) holds true we shall say that $\Phi \cup \{\overline{\varphi}\}$ is the profile system of $(u_n)_{n\in\mathbb{N}}$ obtained by "adding" $\overline{\varphi}$ to Φ.

4. Asymptotic radius notions and energy estimates

Here, given $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$, we use the definition of asymptotic center (denoted by $\mathrm{cen}_{as}((u_n)_{n\in\mathbb{N}})$ or by $\mathrm{cen}_{n\to\infty} u_n$) and asymptotic radius (denoted by $\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}})$ or by $\mathrm{rad}_{n\to\infty} u_n$) of the sequence $(u_n)_{n\in\mathbb{N}}$ as minimum points and, respectively, infimum value of the following functional (depending on $(u_n)_{n\in\mathbb{N}}$) defined on E by setting for all $v \in E$

$$I_{as}(v) = \limsup_n d(u_n, v).$$

So,

$$\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) := \inf_{v\in E} I_{as}(v) = \inf_{v\in E} \limsup_n d(u_n, v). \tag{4.1}$$

Remark 4.1. We recall that [15, Theorem 2.5 and Theorem 3.3] give respectively existence and uniqueness of the asymptotic center of a bounded sequence of a SR metric space, moreover by taking into account [3, Statement a) in Section 3 and Remark 2.4] we deduce that the polar limit of a (polar convergent) sequence coincides with its asymptotic center. Therefore

$$\mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) = \limsup_n d(u_n, u) \quad \text{for all} \quad u_n \rightharpoonup u. \tag{4.2}$$

We introduce also *"asymptotic seminorms"* (denoted by $\mathrm{rad}_z((u_n)_{n\in\mathbb{N}})$, $z \in Z_G$, and $\mathrm{rad}_Z((u_n)_{n\in\mathbb{N}})$) and the *multiscale asymptotic radius* (denoted by $\mathrm{rad}_G((u_n)_{n\in\mathbb{N}})$) of the sequence $(u_n)_{n\in\mathbb{N}}$ by setting, respectively,

$$\mathrm{rad}_z((u_n)_{n\in\mathbb{N}}) := \limsup_n d(u_n, z), \; z \in Z_G, \tag{4.3}$$

$$\mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) := \limsup_n d(u_n, Z_G) \tag{4.4}$$

and

$$\mathrm{rad}_G((u_n)_{n\in\mathbb{N}}) := \inf_{(g_n)_{n\in\mathbb{N}}\in\mathcal{G}} \mathrm{rad}_{as}((g_n(u_n))_{n\in\mathbb{N}}). \tag{4.5}$$

Of course since $d(u_n, Z_G) := \inf_{z\in Z_G} d(u_n, z)$, we deduce that

$$\mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) \leq \inf_{z\in Z_G} \mathrm{rad}_z((u_n)_{n\in\mathbb{N}}).$$

Remark 4.2. Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ and assume Axiom G2. Then there exists a renamed subsequence $(u_n)_{n\in\mathbb{N}}$ such that

$$\mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) = \min_{z\in Z_G} \mathrm{rad}_z((u_n)_{n\in\mathbb{N}}).$$

Remark 4.3. Let $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$, let $\overline{r} > \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}})$ and set

$$\mathcal{N}_{\overline{r}}(Z_G) := \{u \in E \mid \exists z \in Z_G \text{ s.t. } d(u, z) < \overline{r}\}.$$

Then

$$u_n \in \mathcal{N}_{\overline{r}}(Z_G) \text{ for large } n.$$

As a consequence

$$\liminf_n d(u_n, u) \geq d(u, Z_G) - \overline{r} \quad \forall u \in E.$$

Remark 4.4. For any sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ we have

$$\mathrm{rad}_G((u_n)_{n\in\mathbb{N}}) \leq \mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) \leq \mathrm{rad}_z((u_n)_{n\in\mathbb{N}}), \quad \forall z \in Z_G. \tag{4.6}$$

We will extend the above notions to the set of all sequences which admit a given profile system (and a related blowup system). Therefore, given $I \subset \mathbb{N}$, $\Phi = (\varphi_i)_{i\in I}$ a family in $E \setminus Z_G$, and $\boldsymbol{P} = (\boldsymbol{\rho_i})_{i\in I} = ((\rho^i_n)_{n\in\mathbb{N}})_{i\in I}$ a family in $\mathcal{G}$ we introduce the sets

$$U(\Phi) := \{(u_n)_{n\in\mathbb{N}} \in \mathcal{E} \mid \Phi \text{ is a profile system of } (u_n)_{n\in\mathbb{N}}\},$$

and

$$\begin{aligned} U(\Phi, \boldsymbol{P}) &:= \{(u_n)_{n\in\mathbb{N}} \in U(\Phi) \mid \boldsymbol{P} \text{ is a blowup system related to } \Phi\} \\ &= \{(u_n)_{n\in\mathbb{N}} \in \mathcal{E} \mid (\Phi, \boldsymbol{P}) \text{ is a concentration system of } (u_n)_{n\in\mathbb{N}}\}. \end{aligned}$$

Since when $U(\Phi) \neq \emptyset$ there exists a sequence of which Φ is a profile system, in the remaining part of the paper we shall abuse Definition 3.9 by saying that $\Phi = (\varphi_i)_{i\in I} \subset E \setminus Z_G$ is a "profile system" when $U(\Phi) \neq \emptyset$. Analogously, given $\boldsymbol{P} = (\boldsymbol{\rho}_i)_{i\in I} \subset \mathcal{G}$ we shall abuse Definition 3.10 by saying that $\boldsymbol{P}$ is a "blowup system" related to Φ if $U(\Phi, \boldsymbol{P}) \neq \emptyset$. In such a case we shall also say that $\boldsymbol{P}$ is *compatible* with Φ or that $(\Phi, \boldsymbol{P})$ is a "concentration system". Note that

$$U(\Phi) = \bigcup_{\boldsymbol{P}} U(\Phi, \boldsymbol{P}),$$

where the union is restricted to those $\boldsymbol{P}$ which are compatible with Φ.

Remark 4.5. When $\boldsymbol{P} = (\boldsymbol{\rho}_i)_{i\in I} = ((\rho^i_n)_{n\in\mathbb{N}})_{i\in I}$ is compatible with $\Phi = (\varphi_i)_{i\in I}$ (i.e. when $(\Phi, \boldsymbol{P})$ is a concentration system) the following properties hold true.

- i) If $\exists g \in G$ and $\exists i, j \in I$ such that $\varphi_j = g(\varphi_i)$ then g is the identity map i_d on E;
- ii) $\forall i, j \in I$, $i \neq j$, $\boldsymbol{\rho}_i$ and $\boldsymbol{\rho}_j$ are quasiorthogonal.

In the remaining part of the paper we shall reserve the notation $U(\Phi)$ (resp. $U(\Phi, \boldsymbol{P})$) to profile systems Φ (resp. concentration systems $(\Phi, \boldsymbol{P})$).

Note that, with the above introduced sets, we can restate Remark 3.8 as follows.

Proposition 4.6. *Let $(\Phi, \boldsymbol{P})$ be a concentration system, then for every $(u_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$ and for every extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, the subsequence $(u_{k_n})_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{\Sigma})$ where $(\Phi, \boldsymbol{\Sigma})$ is the concentration system extracted from $(\Phi, \boldsymbol{P})$ by the extraction law $(k_n)_{n\in\mathbb{N}}$ (see Definition 3.12).*

Denoting by $\mathcal{U}$ one of the sets $U(\Phi)$ or $U(\Phi, \boldsymbol{P})$, we define the following *asymptotic radius notions* of the set $\mathcal{U}$.

$$\operatorname{rad}_{as}(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \operatorname{rad}_{as}((u_n)_{n\in\mathbb{N}}), \tag{4.7}$$

$$\operatorname{rad}_{z}(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \operatorname{rad}_{z}((u_n)_{n\in\mathbb{N}}),\ z \in Z_G, \tag{4.8}$$

and

$$\operatorname{rad}_{Z}(\mathcal{U}) := \inf_{(u_n)_{n\in\mathbb{N}}\in\mathcal{U}} \operatorname{rad}_{Z}((u_n)_{n\in\mathbb{N}}). \tag{4.9}$$

Note that

$$\operatorname{rad}_{as}(U(\Phi)) = \inf_{(u_n)_{n\in\mathbb{N}}\in U(\Phi)} \operatorname{rad}_{G}((u_n)_{n\in\mathbb{N}}). \tag{4.10}$$

This is the reason for which we have not extended the notion of the multiscale asymptotic radius rad_G to the set $U(\Phi)$ while one can define

$$\operatorname{rad}_{G}(U(\Phi, \boldsymbol{P})) = \inf_{(u_n)_{n\in\mathbb{N}}\in U(\Phi, \boldsymbol{P})} \operatorname{rad}_{G}((u_n)_{n\in\mathbb{N}}). \tag{4.11}$$

Moreover, by (4.6), we have

$$\operatorname{rad}_{as}(U(\Phi)) \le \operatorname{rad}_{z}(U(\Phi)) \quad \forall z \in Z_G.$$

When a profile system Φ is given, by taking into account Remark 3.8 and the definition of rad_z and rad_Z (see (4.3) and (4.4)), we have that, given $(u_n)_{n\in\mathbb{N}} \in U(\Phi)$, for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$,

$$\operatorname{rad}_{z}(U(\Phi)) \le \operatorname{rad}_{z}((u_{k_n})_{n\in\mathbb{N}}) \le \operatorname{rad}_{z}((u_n)_{n\in\mathbb{N}}), \quad \forall z \in Z_G, \tag{4.12}$$

and

$$\operatorname{rad}_{Z}(U(\Phi)) \le \operatorname{rad}_{Z}((u_{k_n})_{n\in\mathbb{N}}) \le \operatorname{rad}_{Z}((u_n)_{n\in\mathbb{N}}). \tag{4.13}$$

When a concentration system $(\Phi, \boldsymbol{P})$ is given, by taking into account Proposition 4.6 (and the definition of rad_z and rad_Z) we have that, given $(u_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$, for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$,

$$\operatorname{rad}_{z}(U(\Phi, \boldsymbol{\Sigma})) \le \operatorname{rad}_{z}((u_{k_n})_{n\in\mathbb{N}}) \le \operatorname{rad}_{z}((u_n)_{n\in\mathbb{N}}), \quad \forall z \in Z_G, \tag{4.14}$$

and

$$\operatorname{rad}_{Z}(U(\Phi, \boldsymbol{\Sigma})) \le \operatorname{rad}_{Z}((u_{k_n})_{n\in\mathbb{N}}) \le \operatorname{rad}_{Z}((u_n)_{n\in\mathbb{N}}), \tag{4.15}$$

where $(\Phi, \boldsymbol{\Sigma})$ is the concentration system extracted from $(\Phi, \boldsymbol{P})$ by the extraction law $(k_n)_{n\in\mathbb{N}}$. In particular, for any concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$ we have

$$\operatorname{rad}_Z(U(\Phi, \boldsymbol{\Sigma})) \leq \operatorname{rad}_Z(U(\Phi, \boldsymbol{P})). \tag{4.16}$$

The above inequality leads to the following definition.

Definition 4.7 (Optimal concentration system)**.** Let $(\Phi, \boldsymbol{P})$ be a concentration system. We shall say that $(\Phi, \boldsymbol{P})$ is *optimal* if

$$\operatorname{rad}_Z(U(\Phi, \boldsymbol{P})) = \operatorname{rad}_Z(U(\Phi, \boldsymbol{\Sigma}))$$

for any concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$ (see Definition 3.12).

Lemma 4.8. *From any concentration system it is possible to extract an optimal concentration system.*

The proof is rather easy and technically it can be reached by a maximality argument on the (partial) ordering $\preccurlyeq$ introduced below the set $\mathcal{C}$ of concentration systems.

Definition 4.9. Let $(\Phi, \boldsymbol{\Sigma})$ and $(\Phi, \boldsymbol{P})$ be concentration systems having the same profile system $\Phi = (\varphi_i)_{i\in I}$ and let $\boldsymbol{\Sigma} = (\boldsymbol{\sigma}_i)_{i\in I} = ((\sigma^i_n)_{n\in\mathbb{N}})_{i\in I}$, $\boldsymbol{P} = (\boldsymbol{\rho}_i)_{i\in I} = ((\rho^i_n)_{n\in\mathbb{N}})_{i\in I}$. We shall say that $(\Phi, \boldsymbol{\Sigma})$ is better than $(\Phi, \boldsymbol{P})$ and we shall write $(\Phi, \boldsymbol{P}) \preccurlyeq (\Phi, \boldsymbol{\Sigma})$ if $\boldsymbol{\Sigma} = \boldsymbol{P}$ or, if $(\Phi, \boldsymbol{\Sigma})$ is extracted from $(\Phi, \boldsymbol{P})$ with the possible exception of finitely many terms (i.e. there exists $\nu \in \mathbb{N}$ and an extraction law $(k_n)_{n\in\mathbb{N}}$ such that for all $n \geq \nu$, $\sigma^i_n = \rho^i_{k_n}$ for all $i \in I$), and, in this case, strict inequality holds in (4.16).

Proof of Lemma 4.8. Since a concentration system is optimal if and only if it is maximal with respect to $\preccurlyeq$, it is enough to prove that the (partially) ordered set $(\mathcal{C}, \preccurlyeq)$ is countably inductive (in the sense of [6, Appendix A], i.e. every increasing sequence has an upper bound). Indeed the presence of the real valued, strictly increasing function f, which maps any concentration system $(\Phi, \boldsymbol{P})$ into $-\operatorname{rad}_Z(U(\Phi, \boldsymbol{P}))$, allows to deduce that $(\mathcal{C}, \preccurlyeq)$ is also inductive and so the assertion follows as a consequence of Zorn Lemma.

Alternatively the reader can use a well known simple argument, see [6, Theorem A.1], to get that any element of a countably inductive ordered set is "less or equal to" a maximal element.

So, we fix an increasing sequence with respect to $\preccurlyeq$. Note that, if it is constant for large n, then of course it has an upper bound. Otherwise, after removing a finite number of terms from each element, we have a sequence of sequences which are all extracted from the previous one. Then, we take the diagonal selection and use the monotonicity of the function f in order to conclude that it is an upper bound of the whole sequence. □

Finally, by a straightforward application of Definition 3.9, we note the following monotonicity properties of the above introduced sets (with respect to subsystems). For any concentration systems $(\Phi', \boldsymbol{P}')$, $(\Phi, \boldsymbol{P})$, we have

$$U(\Phi) \subset U(\Phi') \quad \text{if} \quad \Phi' \subset \Phi$$

and

$$U(\Phi,\boldsymbol{P})\subset U(\Phi',\boldsymbol{P}') \quad \text{if} \quad (\Phi',\boldsymbol{P}')\subset(\Phi,\boldsymbol{P}).$$

Therefore, we deduce that the above defined radii are monotone increasing functions with respect to inclusion. In particular, if rad denotes one of the radii defined by (4.7), (4.8), (4.9) and (4.11), then

$$\operatorname{rad}(U(\Phi',\boldsymbol{P}'))\le\operatorname{rad}(U(\Phi,\boldsymbol{P})) \qquad \forall(\Phi',\boldsymbol{P}')\subset(\Phi,\boldsymbol{P}). \tag{4.17}$$

When strict inequality holds in (4.17) whenever $\Phi'\neq\Phi$, we shall say that rad is *strictly increasing* with respect to inclusion.

Proposition 4.10. *Let $(u_n)_{n\in\mathbb{N}}\in\mathcal{E}$. Then, for all $(g_n)_{n\in\mathbb{N}}\in\mathcal{G}$*

$$\operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}})=\operatorname{rad}_Z((u_n)_{n\in\mathbb{N}}). \tag{4.18}$$

Proof. By symmetry it suffices to show that $\operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}})\le\operatorname{rad}_Z((u_n)_{n\in\mathbb{N}})$. Let $z_n\in Z_G$ be such that $d(u_n,z_n)\to\operatorname{rad}_Z((u_n)_{n\in\mathbb{N}})$. Then,

$$\begin{aligned}\operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}})&\le\limsup_n d(g_n(u_n),Z_G)\le\limsup_n d(g_n(u_n),g_n(z_n))\\&=\limsup_n d(u_n,z_n)=\operatorname{rad}_Z((u_n)_{n\in\mathbb{N}}).\end{aligned}$$

□

Proposition 4.11. *Let Φ be a profile system in a complete SR metric space E. Then*

$$\operatorname{rad}_{as}(U(\Phi))\le\operatorname{rad}_Z(U(\Phi)).$$

Proof. By taking into account Remark 3.8, (and since the passage to a subsequence does not increase the lim sup) thanks to Remark 4.2 the assertion follows by the second inequality in (4.6). □

Now we are able to state the last axiom required to the group G.

Axiom G3. The function rad_Z is strictly increasing with respect to inclusion. Moreover, any profile system Φ consisting of $n\in\mathbb{N}$ elements admits a subsystem Φ' of $n-1$ elements (i.e. a maximal subsystem) such that

$$\operatorname{rad}_Z(U(\Phi'))\le\operatorname{rad}_{as}(U(\Phi)). \tag{4.19}$$

Axiom G3 is required since we are working without any additive structure. Moreover, it is obviously true in the case in which $\operatorname{card}(I)=1$ (indeed with the unique possible choice of Φ' in (4.19) (the empty family), the infimum in the definition of $\operatorname{rad}_Z(U(\Phi'))$ is achieved and is equal to zero on $u_n=z$ for any $z\in Z_G$). On the other hand, whenever the space is equipped with a subtraction law, which is compatible with polar convergence and is such that the distance between two points is a function of their difference, Axiom G3 is trivially satisfied with the equality sign in (4.19). Indeed, it is easy to prove that the asymptotic center (which is unique in a SR metric space, see [3, Corollary 3.9]) of any sequence $(u_n)_{n\in\mathbb{N}}\in U(\Phi)$, which, roughly speaking, gives the right hand side of (4.19), is one of the elements of $\Phi=(\varphi_i)_{i\in I}$ (and can be denoted by $\varphi_{\bar\imath}$). It can be assumed, without any restriction, to be the polar limit of the nonrescaled sequence and by subtracting $\varphi_{\bar\imath}$ from each element of the sequence $(u_n)_{n\in\mathbb{N}}$ one easily deduces (4.19) with $\Phi'=(\varphi_i)_{i\in J}$ where $J=I\setminus\{\bar\imath\}$.

Definition 4.12 (Admissible group)**.** Let G be a group of isometries on a metric space E. We shall say that G is an *admissible* group (or a gauge group or a group of scalings) if axioms G1, G2 and G3 hold true.

From now on we shall assume that G is an admissible group of scalings on E.

Now we build some functions which will allow us to associate a suitable "energy" to any profile of a (bounded) sequence.

Given $R > 0$ we define the function $\boldsymbol{\delta}_R :]0, 2R] \to \mathbb{R}_+$ (by means of the modulus of rotundity δ of the space, see condition (SR)) by setting for all $0 < \overline{d} \leq 2R$

$$\boldsymbol{\delta}_R(\overline{d}) := \min_{\frac{\overline{d}}{2} \leq r \leq R} \delta(r, \overline{d}). \tag{4.20}$$

Note that, by applying (SR) we deduce that for any $R > 0$, and for any $\overline{d} \leq 2R$:

(4.21)

$$\operatorname{rad}(B_r(u) \cap B_r(v)) \leq r - \boldsymbol{\delta}_R(\overline{d}), \quad \forall r \in \left[2^{-1}\,\overline{d},\, R\right],\ \forall u, v \in E \text{ s.t. } d(u, v) \geq \overline{d}.$$

Remark 4.13. Note that, fixed $R > 0$, since the modulus of rotundity δ is monotone increasing with respect to the variable $\overline{d}$ and since the interval $[2^{-1}\,\overline{d}, R]$ reduces when $\overline{d}$ increases, we deduce that the function $\boldsymbol{\delta}_R$ is monotone increasing in the variable $\overline{d}$. On the contrary, fixed $\overline{d} > 0$, the value $\boldsymbol{\delta}_R(\overline{d})$ decreases if $R \geq 2^{-1}\,\overline{d}$ increases. Finally, since $\boldsymbol{\delta}_R(\overline{d}) \leq \delta(2^{-1}\,\overline{d}, \overline{d}) \leq 2^{-1}\,\overline{d}$, we can extend $\boldsymbol{\delta}_R$ to 0 by setting $\boldsymbol{\delta}_R(0) = 0$.

Lemma 4.14. *Let $\Phi = (\varphi_i)_{i \in I}$ be a profile system. If $R \geq \operatorname{rad}_Z(U(\Phi))$ then*

$$d(\varphi_i, Z_G) \leq 2R \quad \forall i \in I.$$

Proof. Let us assume, by contradiction, that there exist $(u_n)_{n\in\mathbb{N}} \in U(\Phi)$, $\bar{\imath} \in I$, and $R' > R$ such that

$$\operatorname{rad}_Z((u_n)_{n\in\mathbb{N}}) < R' \qquad \text{and} \qquad d(\varphi_{\bar{\imath}}, Z_G) > 2R'. \tag{4.22}$$

Let $\boldsymbol{\rho}_{\bar{\imath}} = (\rho_n^{\bar{\imath}})_{n\in\mathbb{N}}$ be a blowup related to $\varphi_{\bar{\imath}}$. Then, by applying (4.18) to $g_n = (\rho_n^{\bar{\imath}})^{-1}$ we deduce, respectively from the first inequality in (4.22) and (3.1), that

$$\operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}}) = \operatorname{rad}_Z((u_n)_{n\in\mathbb{N}}) < R' \qquad \text{and} \qquad g_n(u_n) \rightharpoonup \varphi_{\bar{\imath}}. \tag{4.23}$$

Now, by the triangle inequality, we deduce, from the second inequality in (4.22), that $2R' < d(\varphi_{\bar{\imath}}, Z_G) \leq d(\varphi_{\bar{\imath}}, g_n(u_n)) + d(g_n(u_n), Z_G)$. Then, by taking the $\limsup_n$, we deduce by (4.2), (4.6) and (4.23) that

$$2R' < \operatorname{rad}_{as}((g_n(u_n))_{n\in\mathbb{N}}) + \operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}}) \leq 2\operatorname{rad}_Z((g_n(u_n))_{n\in\mathbb{N}}) < 2R',$$

a contradiction. □

Definition 4.15 (Profile energy - profile system energy)**.** Given $\varphi \in E$, and $R > d(\varphi, Z_G)$, we shall call *profile energy* of φ the number $V_R(\varphi)$, depending on R, defined by setting

$$V_R(\varphi) := \boldsymbol{\delta}_R(d(\varphi, Z_G)).$$

Analogously if $\Phi = (\varphi_i)_{i\in I}$ is a profile system, for any $R > \sup_{i \in I} d(\varphi_i, Z_G)$, we shall call *profile system energy* of (the profile system) Φ the number $V_R(\Phi)$, depending on R, defined by setting

$$V_R(\Phi) := \sum_{i\in I} V_R(\varphi_i) = \sum_{i\in I} \boldsymbol{\delta}_R(d(\varphi_i, Z_G)). \tag{4.24}$$

Note that trivial profiles have null (profile) energy (for any $R > 0$) and, by Corollary 2.12, copies $g(\varphi)$ of the same profile φ have the same (profile) energy. Therefore, the orbit $G(\varphi)$ is in some sense an "equipotential surface".

The definition above does presume that values of the supremum and the sum in it are finite, but this will be the case in the following *energy estimate*, similar to (2.8) in [2, Lemma 2.12], in which we use the just introduced energy.

Lemma 4.16 (Energy estimate)**.** *Let $R > 0$ be given. Then for any profile system $\Phi = (\varphi_i)_{i\in I}$ such that* $\mathrm{rad}_Z(U(\Phi)) < R$ *we have*

$$V_R(\Phi) = \sum_{i\in I} V_R(\varphi_i) \le \mathrm{rad}_Z(U(\Phi)) < R. \tag{4.25}$$

Proof. Of course it is enough to prove the statement for finite sets I, so we can proceed by induction in cardinality. The statement is trivial when $I = \emptyset$ and therefore the sum in the left hand side in (4.25) is 0. Given $\overline{n} \in \mathbb{N}$, we assume the statement true for any family $\Psi = (\psi_j)_{j\in J}$ such that $\mathrm{card}(J) = \overline{n}$ and $\mathrm{rad}_Z(U(\Psi)) < R$ and we shall prove the statement for $\Phi = (\varphi_i)_{i\in I}$ with $\mathrm{card}(I) = \overline{n} + 1$. By Axiom G3 there exists $\overline{\imath} \in I$ such that, setting $\Phi' = (\varphi_i)_{i\in I\setminus\{\overline{\imath}\}}$ we have (4.19). Fix $\varepsilon > 0$ and let $(u_n)_{n\in\mathbb{N}} \in U(\Phi)$ be such that

$$\mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) < \mathrm{rad}_Z(U(\Phi)) + \varepsilon < R. \tag{4.26}$$

Thanks to (4.10) and Proposition 4.10 we can assume that the blowup sequence $\boldsymbol{\rho}_{\overline{\imath}}$ is the identity mapping $\boldsymbol{i_d}$, and so (see (3.1)) that $u_n \rightharpoonup \varphi_{\overline{\imath}}$. Then, taking into account Remark 4.2, there exists $\overline{z} \in Z_G$ such that we have, on a renamed subsequence, for large m, and modulo an infinitesimal term,

$$d(u_m, \varphi_{\overline{\imath}}) \le \mathrm{rad}_{\overline{z}}((u_n)_{n\in\mathbb{N}}) = \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}). \tag{4.27}$$

In particular, set $r = \mathrm{rad}_Z(U(\Phi)) + \varepsilon$, we have by (4.27) and (4.26), that

$$u_m \in B_r(\varphi_{\overline{\imath}}) \cap B_r(\overline{z}) \qquad \text{for large } m, \tag{4.28}$$

and therefore, that $d(\varphi_{\overline{\imath}}, \overline{z}) \le 2r \le 2R$ (by (4.26)). Then, by taking $\overline{d} = d(\varphi_{\overline{\imath}}, \overline{z})$ (which is strictly positive since $\varphi_{\overline{\imath}} \notin Z_G$) in (4.21), we get that

$$\mathrm{rad}(B_r(\varphi_{\overline{\imath}}) \cap B_r(\overline{z})) \le r - \boldsymbol{\delta}_R(d(\varphi_{\overline{\imath}}, \overline{z})) \le \mathrm{rad}_Z(U(\Phi)) - V_R(\varphi_{\overline{\imath}}) + \varepsilon, \tag{4.29}$$

where the last inequality holds true since (see Remark 4.13)

$$\boldsymbol{\delta}_R(d(\varphi_{\overline{\imath}}, \overline{z})) \ge \boldsymbol{\delta}_R(d(\varphi_{\overline{\imath}}, Z_G)) = V_R(\varphi_{\overline{\imath}}).$$

On the other hand, by (4.19), (4.7) and (4.28) we have

$$\mathrm{rad}_Z(U(\Phi')) \le \mathrm{rad}_{as}(U(\Phi)) \le \mathrm{rad}_{as}((u_n)_{n\in\mathbb{N}}) \le \mathrm{rad}(B_r(\varphi_{\overline{\imath}}) \cap B_r(\overline{z})). \tag{4.30}$$

So, by linking inequalities (4.29) and (4.30), we get

$$\mathrm{rad}_Z(U(\Phi)) \ge \mathrm{rad}_Z(U(\Phi')) + V_R(\varphi_{\overline{\imath}}) - \varepsilon,$$

and so (since by (4.17) $\mathrm{rad}_Z(U(\Phi')) \le \mathrm{rad}_Z(U(\Phi)) < R$)) the assertion follows by using induction hypothesis and letting ε go to zero. □

By taking into account Remark 4.13 we get that the energy bound (4.25) still holds true if one replaces R by any larger real number, indeed the left hand side decreases while the right hand side increases (the central term is independent on R). Therefore the smaller is R more significant is (4.25).

It is easy to verify that if E is a Hilbert space, the modulus of rotundity δ is given, for all $r, \overline{d} > 0$, by

$$\delta(r,\overline{d}) = \begin{cases} \sqrt{r^2 - \left(\frac{\overline{d}}{2}\right)^2} & \text{if} \quad 0 < \overline{d} \leq 2r \\ r & \text{if} \quad \overline{d} > 2r, \end{cases}$$

and so satisfies the following bounds

$$\frac{1}{2r}\left(\frac{\overline{d}}{2}\right)^2 \leq \delta(r,\overline{d}) \leq \frac{1}{r}\left(\frac{\overline{d}}{2}\right)^2, \quad \forall r > 0,\ 0 < \overline{d} \leq 2r, \tag{4.31}$$

moreover, since the function δ is decreasing with respect to r, we have that, given $R > 0$, $\boldsymbol{\delta}_R(\overline{d}) = \delta(R,\overline{d})$ for all $0 \leq \overline{d} \leq 2R$. Therefore we have the following bounds

$$\frac{1}{2R}\left(\frac{\overline{d}}{2}\right)^2 \leq \boldsymbol{\delta}_R(\overline{d}) \leq \frac{1}{R}\left(\frac{\overline{d}}{2}\right)^2, \quad \forall 0 \leq \overline{d} \leq 2R. \tag{4.32}$$

So the energy estimate (4.25) amounts, modulo constants, to [2, Formula (2.6)]. The energy estimate (4.25) will be used in next section to prove a "multiscale weak compactness" property (see [2, Theorem 3.1]) through a maximality argument.

5. Profile convergent sequences and multiscale polar compactness

For any real number $R > 0$ let $\mathcal{E}_R$ denote the space of bounded sequences in E whose asymptotic "seminorm" rad_Z has a value less than or equal to R, i.e.

$$\mathcal{E}_R := \{(u_n)_{n\in\mathbb{N}} \in \mathcal{E} \mid \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) \leq R\}.$$

Note that, by (4.13), $\mathcal{E}_R$ contains all the subsequences extracted from its elements.

We will evaluate the "profile bulk" of the sequence $(u_n)_{n\in\mathbb{N}}$ by introducing the following function defined by setting for all $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}_R$

$$\mathcal{S}_R((u_n)_{n\in\mathbb{N}}) := \sup\{V_R(\Phi) \mid \Phi \text{ is a profile system of } (u_n)_{n\in\mathbb{N}}\}. \tag{5.1}$$

Remark 5.1. In other terms $\mathcal{S}_R((u_n)_{n\in\mathbb{N}})$ can be defined as the value of the sum in (4.24) extended to all possible profiles counted with their multiplicity. Moreover, since for any $\Phi = (\varphi_i)_{i\in I}$ such that $(u_n)_{n\in\mathbb{N}} \in U(\Phi)$, by (4.13) we have $\mathrm{rad}_Z(U(\Phi)) \leq \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}) \leq R$, and so, as a consequence of Lemma 4.16, $\mathcal{S}_R((u_n)_{n\in\mathbb{N}}) \leq R$, we deduce that the multiplicity $m(\varphi)$ of any profile φ of the sequence $(u_n)_{n\in\mathbb{N}}$ is finite.

Recalling that, by (4.13), $\mathcal{E}_R$ contains all the subsequences extracted from its elements, we deduce from Remark 3.8 that for any $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}_R$

$$\mathcal{S}_R((u_n)_{n\in\mathbb{N}}) \leq \mathcal{S}_R((u_{k_n})_{n\in\mathbb{N}}) \quad \forall \text{ extraction law } (k_n)_{n\in\mathbb{N}}.$$

Moreover we can give the following definition.

Definition 5.2. Given $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}} \in \mathcal{E}_R$, we shall say that $(w_n)_{n\in\mathbb{N}}$ is *better profiled* than $(v_n)_{n\in\mathbb{N}}$, and we shall write $(v_n)_{n\in\mathbb{N}} \preceq (w_n)_{n\in\mathbb{N}}$, if $(w_n)_{n\in\mathbb{N}} = (v_n)_{n\in\mathbb{N}}$ or if $(w_n)_{n\in\mathbb{N}}$ is a subsequence of $(v_n)_{n\in\mathbb{N}}$ with the possible exception of finitely many terms (i.e. there exists $\nu \in \mathbb{N}$ and an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that, for all $n \geq \nu$, $w_n = v_{k_n}$) and $\mathcal{S}_R((v_n)_{n\in\mathbb{N}}) < \mathcal{S}_R((w_n)_{n\in\mathbb{N}})$.

Remark 5.3. The binary relation $\preceq$ is an ordering on $\mathcal{E}_R$ and the function $\mathcal{S}_R$ is increasing with respect to $\preceq$.

Definition 5.4 (Profile-convergent sequence, complete profile system)**.** We shall say that a bounded sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ is *profile convergent* if none of its subsequences is strictly better profiled (i.e. $\mathcal{S}_R((u_{k_n})_{n\in\mathbb{N}}) = \mathcal{S}_R((u_n)_{n\in\mathbb{N}})$ for any extraction law and for any $R \geq \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}})$).

We shall say that a profile system $\Phi = (\varphi_i)_{i\in I}$ of a profile convergent sequence $(u_n)_{n\in\mathbb{N}}$ is a *complete profile system* if

$$\mathcal{S}_R((u_n)_{n\in\mathbb{N}}) = V_R(\Phi) = \sum_{i\in I} V_R(\varphi_i) \qquad \forall R > \mathrm{rad}_Z((u_n)_{n\in\mathbb{N}}).$$

In other words $(u_n)_{n\in\mathbb{N}}$ is *profile convergent* if $(u_n)_{n\in\mathbb{N}}$ does not admit any subsequence with a larger number of profiles, or with profiles of a higher multiplicity. This happens of course when $(u_n)_{n\in\mathbb{N}}$ is a maximal element for $\preceq$. The following proposition clarifies the term "complete". (We remind the reader that profile systems do not include null profiles.)

Proposition 5.5. *Assume that $(u_n)_{n\in\mathbb{N}}$ is profile convergent and let $\Phi = (\varphi_i)_{i\in I}$ be a related complete profile system and let $\boldsymbol{P} = (\boldsymbol{\rho_i})_{i\in I} = ((\rho_n^i)_{n\in\mathbb{N}})_{i\in I}$ be a corresponding blowup system. Then, for any $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$ which is quasiorthogonal to any blowup sequence $\boldsymbol{\rho}_i$, we have that the sequence $(\rho_n^{-1}(u_n))_{n\in\mathbb{N}}$ is polar infinitesimal (see Definition 2.10).*

Proof. Let $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be quasiorthogonal to any blowup sequence $\boldsymbol{\rho}_i$ (note that the quasiorthogonality condition is required just in the light of Remark 3.3). Consider any renamed subsequence of $(\rho_n^{-1}(u_n))_{n\in\mathbb{N}}$ that has a polar limit φ. If, by contradiction $\varphi \notin Z_G$, the corresponding subsequence of $(u_n)_{n\in\mathbb{N}}$ would belong to $\mathcal{U}(\Phi \cup \{\varphi\})$ (see Remark 3.13) and so would be better profiled. Therefore $(u_n)_{n\in\mathbb{N}}$ would not be profile convergent. $\square$

The aim of the remaining part of this section is to prove the following result.

Theorem 5.6 (Multiscale polar compactness)**.** *Any bounded sequence in E admits a profile convergent subsequence.*

The proof is rather easy and technically it can be reached by a maximality argument on the ordering $\preceq$ introduced above on $\mathcal{E}_R$ (see Definition 5.2).

Lemma 5.7. *Let $R > 0$ and $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}_R$. Then $(u_n)_{n\in\mathbb{N}}$ is profile convergent if and only if it is maximal with respect to $\preceq$.*

Proof. If $(u_n)_{n\in\mathbb{N}}$ is not profile convergent there exists a subsequence $(u_{k_n})_{n\in\mathbb{N}}$ which admits a "new profile" $\overline{\varphi}$ (in the sense that $(u_{k_n})_{n\in\mathbb{N}}$ has a larger number of profiles or that some of the profiles of $(u_n)_{n\in\mathbb{N}}$ increases its multiplicity). Fix $\varepsilon \in]0, V_R(\overline{\varphi})[$

and take a profile system Φ of $(u_n)_{n\in\mathbb{N}}$ such that $V_R(\Phi) > \mathcal{S}_R((u_n)_{n\in\mathbb{N}}) - \varepsilon$. By assumption $(u_{k_n})_{n\in\mathbb{N}} \in U(\Phi \cup \{\overline{\varphi}\})$ in the sense of Remark 3.13. So $V_R(\Phi \cup \{\overline{\varphi}\})) = V_R(\Phi) + V_R(\overline{\varphi}) > \mathcal{S}_R((u_n)_{n\in\mathbb{N}}) - \varepsilon + V_R(\overline{\varphi}) > \mathcal{S}_R((u_n)_{n\in\mathbb{N}})$ getting in contradiction with the maximality of $(u_n)_{n\in\mathbb{N}}$. The converse implication is trivial. □

Proof of Theorem 5.6. It is enough to prove that for any $R > 0$ the ordered set $(\mathcal{E}_R, \preceq)$ is countably inductive (in the sense of [6, Appendix A], i.e. every increasing sequence has an upper bound). Indeed the presence of the real valued, strictly increasing function $\mathcal{S}_R$ allows to deduce that $(\mathcal{E}_R, \preceq)$ is also inductive and so the thesis follows as a consequence of Zorn Lemma.

Alternatively the reader can follow the proof of Lemma 4.8 by taking $f = \mathcal{S}_R$. □

6. Polar profile reconstruction

The notion of profile reconstruction determined by a family of functions (profiles) and a family of sequences of scalings (blowup sequences, or scale transition sequences) has been introduced in [2] as the sum of the elementary concentrations (bubbles, see Definition 3.1), defined as blowup sequences acting on profiles. Since in general, in a metric space, we have not any algebraic structure we shall use instead a suitable counterpart of the characterizing formula, given for L^p spaces, in [2, Formula (4.18)].

Definition 6.1 ((Polar) profile reconstruction)**.** Given an optimal concentration system $(\Phi, \boldsymbol{P})$ (see Definition 4.7) we shall say that $(v_n)_{n\in\mathbb{N}} \subset E$ is a *(polar) profile reconstruction* determined by $(\Phi, \boldsymbol{P})$ if

$$(v_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P}) \text{ and } \mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P})). \tag{6.1}$$

Since (6.1) is satisfied by any other sequence $(w_n)_{n\in\mathbb{N}} \subset E$ such that $d(v_n, w_n) \to 0$, one cannot expect that a given optimal concentration system $(\Phi, \boldsymbol{P})$ can determine a unique profile reconstruction. For this reason we will consider a profile reconstruction unique if its distance from any other profile reconstruction converges to zero.

Since the only mode of convergence, other than distance convergence, studied in this paper, is polar convergence, we will use the term "profile reconstruction" without the qualifier "polar". In a Banach space one can also speak of weak reconstruction, with profiles of a sequence defined as weak, rather than polar, limits of "deflations" $(\rho_n^{-1}(u_n))_{n\in\mathbb{N}}$.

Proposition 6.2. *Let E be a complete SR metric space, equipped with an admissible group G of scalings. Let $(\Phi, \boldsymbol{P})$ be an optimal concentration system and let $(v_n)_{n\in\mathbb{N}} \subset E$ be a profile reconstruction determined by $(\Phi, \boldsymbol{P})$. Then the following properties hold true.*

a) *Any concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$ is optimal.*
b) *For any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, the sequence $(v_{k_n})_{n\in\mathbb{N}}$ is a profile reconstruction determined by the concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$ by the same extraction law.*
c) *The sequence $(d(v_n, Z_G))_{n\in\mathbb{N}}$ converges and*

$$\lim_n d(v_n, Z_G) = \mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}).$$

As a consequence, for any extraction law $(k_n)_{n\in\mathbb{N}}$*,*

$$\mathrm{rad}_Z((v_{k_n})_{n\in\mathbb{N}}) = \mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P})).$$

d) *The sequence* $(v_n)_{n\in\mathbb{N}}$ *is profile convergent (see Definition 5.4) and* $(\Phi, \boldsymbol{P})$ *is a related complete concentration system.*

Proof.

- Item a) is trivial since any concentration system extracted from $(\Phi, \boldsymbol{\Sigma})$ is actually a concentration system extracted from $(\Phi, \boldsymbol{P})$ and $(\Phi, \boldsymbol{P})$ is optimal by assumption.
- To prove item b) we fix an extraction law $(k_n)_{n\in\mathbb{N}}$ and the corresponding concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$. By Proposition 4.6 we have that $(v_{k_n})_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{\Sigma})$, moreover, by using (4.14) and since $(\Phi, \boldsymbol{P})$ is optimal, we deduce $\mathrm{rad}_Z((v_{k_n})_{n\in\mathbb{N}}) \leq \mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P})) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{\Sigma}))$, and, as a consequence, that $\mathrm{rad}_Z((v_{k_n})_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{\Sigma}))$.
- Item c) follows from the optimality of the concentration system $(\Phi, \boldsymbol{P})$ and by the equality in (6.1). Indeed, if by contradiction there exists an extraction law $(k_n)_{n\in\mathbb{N}}$ such that $\limsup_n d(v_{k_n}, Z_G) < \limsup_n d(v_n, Z_G)$ we would get the existence of a concentration system $(\Phi, \boldsymbol{\Sigma})$ extracted from $(\Phi, \boldsymbol{P})$ by $(k_n)_{n\in\mathbb{N}}$ such that $\mathrm{rad}_Z(U(\Phi, \boldsymbol{\Sigma})) \leq \mathrm{rad}_Z((v_{k_n})_{n\in\mathbb{N}}) < \mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P}))$.
- Item d) follows from the strict monotonicity assumption of the function rad_Z required by Axiom G3. Indeed, if by contradiction there exists an extraction law $(k_n)_{n\in\mathbb{N}}$ and a (profile) $\overline{\varphi} \in E\setminus Z_G$ (with corresponding blowup sequence $\overline{\rho}$) such that, see Remark 3.13, $(v_{k_n})_{n\in\mathbb{N}} \in U(\Phi \cup \{\overline{\varphi}\}, \boldsymbol{P} \cup \{\overline{\rho}\})$, we would deduce that $\mathrm{rad}_Z(U(\Phi\cup\{\overline{\varphi}\}, \boldsymbol{P}\cup\{\overline{\rho}\})) \leq \mathrm{rad}_Z((v_{k_n})_{n\in\mathbb{N}}) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P}))$.

□

We shall show that this definition of profile reconstruction does not always correspond, in linear spaces, to the sum of the scaled profiles. This happens, in particular, when closed balls are not closed with respect to polar convergence, for instance, for $E = L^p(\mathbb{R})$, $p \in (1, \infty)$, $p \neq 2$ (see [3]). The following example deals with the intuitive case of a sequence with a single profile, where no algebraic structure is required to define the "sum". In such a case, i.e. when $\Phi = \{\varphi\}$ and $\boldsymbol{P} = \{\boldsymbol{\rho}\}$, to shorten notation, we shall write $(\varphi, \boldsymbol{\rho})$ and $U(\varphi, \boldsymbol{\rho})$ instead of respectively $(\{\varphi\}, \{\boldsymbol{\rho}\})$ and $U(\{\varphi\}, \{\boldsymbol{\rho}\})$.

Example 6.3. Let $E = L^p(\mathbb{R})$, $p \in (1, \infty)$, $p \neq 2$, and let G be the group of shifts $u \mapsto u(\cdot - y)$, $y \in \mathbb{R}$, for which we have $Z_G = \{0\}$. Then there exists a sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$, $\varphi \in E$ such that $u_n \rightharpoonup \varphi$ and $\limsup_n \|u_n\|_p < \|\varphi\|_p$. Then, by taking $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} = \boldsymbol{i_d}$, the sequence $(v_n)_{n\in\mathbb{N}} = (\rho_n(\varphi))_{n\in\mathbb{N}} = (\varphi)_{n\in\mathbb{N}}$ (of the "sum" of scaled profiles (bubbles)) is not the profile reconstruction determined by $(\varphi, \boldsymbol{\rho})$. Indeed $\mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}) = \|\varphi\|_p > \limsup_n \|u_n\|_p$ with $(u_n)_{n\in\mathbb{N}} \in U(\varphi, \boldsymbol{\rho})$.

As shown in Example 6.3, the reconstruction according to [2, Definition 4.3] (i.e. as the "sum" of the single scaled profile) doesn't match the polar profile reconstruction given by Definition 6.1. Thus we restrict our consideration to those SR metric spaces whose closed (geodesically) convex sets are closed with respect to the polar convergence. (In particular, this property can be regained in L^p spaces by replacing the standard norm with an equivalent, still scaling invariant, norm defined by the Littlewood-Paley decomposition, see [14]).

Furthermore we require that the radius of the intersection of the balls $B_r(u)$ and $B_r(v)$ in (SR) can be attained at the Chebyshev center $\operatorname{cen}\{u,v\}$ independently of r (it is easy to see that in complete SR spaces Chebyshev center always exists and is unique). Obviously, when we have the linear structure $\operatorname{cen}\{u,v\}$ corresponds to the middle point of the segment joining u and v. So we shall use this terminology also in general complete SR metric spaces by referring to $\operatorname{cen}\{u,v\}$ as to the Chebyshev center of the set $\{u,v\}$, whose definition does not require any linear structure.

Axiom E1. For any r, $\overline{d} > 0$, and for any $u, v \in E$ with $d(u,v) \geq \overline{d}$:

$$B_{r+\delta(r,\overline{d})}(u) \cap B_{r+\delta(r,\overline{d})}(v) \subset B_{r-\delta(r,\overline{d})}(\operatorname{cen}\{u,v\}). \tag{6.2}$$

As already remarked on condition (SR), when the modulus of rotundity δ is continuous, condition (6.2) can be replaced by

$$B_r(u) \cap B_r(v) \subset B_{r-\delta(r,\overline{d})}(\operatorname{cen}\{u,v\}). \tag{SR$'$}$$

Note that (SR) is weaker that (SR$'$), but it is necessary (if not replaced by some other suitable conditions) in order to give meaning to the term $\operatorname{cen}\{u,v\}$ appearing in (SR$'$). In the following, when Axiom E1 holds true, we shall say that E is a SR$'$-metric space. Obviously a uniformly convex Banach space is SR$'$.

Next axiom provides a partial "squeeze property" for polar convergence.

Axiom E2. If $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in \mathcal{E}$ are (bounded) sequences that have a common polar limit φ, then $(\operatorname{cen}\{v_n, w_n\})_{n\in\mathbb{N}} \rightharpoondown \varphi$.

In the Proposition 6.4 below we shall check that the counterintuitive phenomenon which has been pointed out in Example 6.3 disappears by assuming Axioms E1-E2, and that the present definition of profile reconstruction agrees with [2, Definition 4.3].

Proposition 6.4. *Let E be a complete SR metric space, equipped with an admissible group G of scalings, which satisfies axioms E1-E2. Let $\varphi \in E \setminus Z_G$ and $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \mathcal{G}$. Then the sequence $(\rho_n(\varphi))_{n\in\mathbb{N}}$ is a profile reconstruction determined by $(\varphi, \boldsymbol{\rho})$.*

Proof. Without loss of generality we can assume $\boldsymbol{\rho} = \boldsymbol{i_d}$. Thus we have to prove that the constant sequence $(\varphi)_{n\in\mathbb{N}} \in \mathcal{E}$ is a (unique) profile reconstruction of $(\varphi, \boldsymbol{i_d})$. Note that, by the definition of Z_G, the polar limit of a convergent subsequence of $(\sigma_n^{-1}(\varphi))_{n\in\mathbb{N}}$ with a discrete sequence $(\sigma_n)_{n\in\mathbb{N}}$ is an element of Z_G. In order to show that $(\varphi)_{n\in\mathbb{N}}$ is a profile reconstruction determined by $(\varphi, \boldsymbol{i_d})$ we only have to show that $(\varphi, \boldsymbol{i_d})$ is optimal (which is immediate) and that $\operatorname{rad}_Z(U(\varphi, \boldsymbol{i_d}))$ cannot be smaller than $\operatorname{rad}_Z((\varphi)_{n\in\mathbb{N}}) = d(\varphi, Z_G)$.

Assume, by contradiction, that there exists $\overline{d} > 0$ such that for some $\varepsilon > 0$ there exists a bounded sequence $(v_n)_{n\in\mathbb{N}} \in U(\varphi, \boldsymbol{i_d})$ and $z \in Z_G$ (see Remark 4.2), such that,

$$\overline{r} := \operatorname{rad}_Z((v_n)_{n\in\mathbb{N}}) = \operatorname{rad}_z((v_n)_{n\in\mathbb{N}}) < \operatorname{rad}_Z(U(\varphi, \boldsymbol{i_d})) = d(\varphi, Z_G) - \varepsilon \tag{6.3}$$

and

$$\liminf_{n\to+\infty} d(v_n,\varphi) > \overline{d}.$$

Then, by Lemma 2.3, we get the existence of a subsequence $(v_{k_n})_{n\in\mathbb{N}}$ of $(v_n)_{n\in\mathbb{N}}$ such that for all n, $d(v_n, v_{k_n}) \geq \overline{d}$.

Moreover, since $\overline{r}+\varepsilon > \mathrm{rad}_z((v_n)_{n\in\mathbb{N}})$, we have

$$z \in B_{\overline{r}+\varepsilon+\delta(\overline{r}+\varepsilon,\overline{d})}(v_n) \cap B_{\overline{r}+\varepsilon+\delta(\overline{r}+\varepsilon,\overline{d})}(v_{k_n}) \quad \text{for large } n,$$

and so, set for any n $w_n := \mathrm{cen}\{v_n, v_{k_n}\}$, we get, by the (SR$'$) condition (see also (6.2)), that

$$d(w_n, z) \leq \overline{r}+\varepsilon-\delta(\overline{r}+\varepsilon,\overline{d}). \tag{6.4}$$

On the other hand, since, by Proposition 4.6, $(v_{k_n})_{n\in\mathbb{N}}$ is still in $U(\varphi,\boldsymbol{i_d})$, by Axiom E2, we get $w_n = \mathrm{cen}\{v_n, v_{k_n}\} \rightharpoonup \varphi$ and so $(w_n)_{n\in\mathbb{N}} \in U(\varphi,\boldsymbol{i_d})$. Then, by combining (6.4) with the last inequality in (6.3) we immediately deduce that

$$\mathrm{rad}_Z((w_n)_{n\in\mathbb{N}}) < \mathrm{rad}_Z(U(\varphi,\boldsymbol{i_d}))$$

which, by (4.12), leads to a contradiction since $(w_n)_{n\in\mathbb{N}} \in U(\varphi,\boldsymbol{i_d})$. □

Example 6.5. Profile reconstruction is generally not unique. Let H be a Hilbert space and let $E = H\times\mathbb{R}$ supplied with a standard Euclidean metric for a product space. Let G_0 be an admissible group of isometries on H, let $i_{\mathbb{R}}$ denote the identity map on $\mathbb{R}$ and take $G = G_0 \times \{i_{\mathbb{R}}\}$ (so that for all $(u,p) \in H\times\mathbb{R}$, and for all $g\in G$, there exists $g_0 \in G_0$ such that $g(u,p) = (g_0(u),p)$). Then $Z_G = \{0\}\times\mathbb{R}$. In particular, the profile system $\{(\varphi,q)\}$ with the corresponding blowup system $\{\boldsymbol{i}_d\}$ will have infinitely many profile reconstructions $((\varphi,p))_{n\in\mathbb{N}}$ for every $p\in\mathbb{R}$, since for any sequence $((u_n,p_n))_{n\in\mathbb{N}} \in \mathcal{E}$, $\mathrm{rad}_Z((u_n,p_n)_{n\in\mathbb{N}}) = \limsup_n \|u_n\|_H$.

Intuitively, if a profile reconstruction is given by an expression that involves an element of Z_G, uniqueness will require that the geometry of Z_G will penalize, in terms of the value of rad_Z, any substitution of this element by another.

We shall introduce a convexity condition for a subset $Z\subset E$ of a SR metric space which we shall call *Staples convexity*.

Definition 6.6. Let E be an SR metric space. A subset $Z\subset E$ is Staples convex if there exists $\delta_Z : (\mathbb{R}_+)^2 \to \mathbb{R}_+$ (called *modulus of convexity) such that for any $r,\overline{d}>0$, and for any $u,v\in E$ such that*

$$d(u,v) \geq \overline{d} \quad \textit{and} \quad \max\{d(u,Z), d(v,Z)\} \leq r+\delta_Z(r,\overline{d}) \tag{6.5}$$

one has

$$d(\mathrm{cen}(\{u,v\}), Z) \leq r-\delta_Z(r,\overline{d}). \tag{SC}$$

Note that when δ_Z is a continuous function one can replace $r+\delta_Z(r,\overline{d})$ by r in condition (6.5).

The following axiom is a requirement of Staples convexity of the null set Z_G and will lead to both existence and uniqueness of a profile reconstruction.

Axiom E3. The null set Z_G is Staples convex and its modulus of convexity δ_{Z_G} is continuous.

When Axiom E3 holds true we shall rename δ_{Z_G} with δ_Z. Note that if Z_G consists of one point, then it fulfills Axiom E3 with $\delta_Z = \delta$.

Lemma 6.7. *Let E be a complete SR metric space, equipped with an admissible group G of scalings, which satisfies axioms E1, E2 and E3. Let $(\Phi, \boldsymbol{P})$ be an optimal concentration system (see Definition 4.7) and set $\overline{r} := \mathrm{rad}_Z(U(\Phi, \boldsymbol{P}))$. Then, for any $\overline{d} > 0$ and for any pair of sequences $(v_n)_{n\in\mathbb{N}}$, $(w_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$ such that*

$$\max(\mathrm{rad}_Z((v_n)_{n\in\mathbb{N}}), \mathrm{rad}_Z((w_n)_{n\in\mathbb{N}})) < \overline{r} + \delta_Z(\overline{r}, \overline{d}), \tag{6.6}$$

we have

$$d(v_n, w_n) \leq \overline{d} \ \textit{ for large } n. \tag{6.7}$$

Proof. Assume by contradiction that on a renamed subsequence $d(v_n, w_n) > \overline{d}$ holds for all n. Then, by (6.6) and (SC) we deduce that $\limsup_n d(\mathrm{cen}\{v_n, w_n\}, Z_G) < \overline{r}$ getting a contradiction since, by Proposition 4.6 and Axiom E2, $(\mathrm{cen}\{v_n, w_n\})_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{\Sigma})$ (where $(\Phi, \boldsymbol{\Sigma})$ is the corresponding concentration system extracted from $(\Phi, \boldsymbol{P})$, see Definition 3.12) and $\overline{r} := \mathrm{rad}_Z(U(\Phi, \boldsymbol{P})) = \mathrm{rad}_Z(U(\Phi, \boldsymbol{\Sigma}))$ since $(\Phi, \boldsymbol{P})$ is optimal. □

Theorem 6.8 (Uniqueness of profile reconstruction)**.** *Let E be a complete SR metric space, equipped with an admissible group G of scalings, which satisfies axioms E1, E2 and E3. Then any optimal concentration system $(\Phi, \boldsymbol{P})$ determines at most a unique (modulo subsequences) profile reconstruction.*

Proof. Assume that $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ are two profile reconstructions of $(\Phi, \boldsymbol{P})$. Since for any $\overline{d} > 0$ $(v_n)_{n\in\mathbb{N}}$ and $(w_n)_{n\in\mathbb{N}}$ satisfy (6.6), by Lemma 6.7 we have (6.7), so the assertion follows by letting $\overline{d}$ go to zero. □

Theorem 6.9 (Existence of a profile reconstruction)**.** *Let E be a complete SR metric space, equipped with an admissible group G of scalings, which satisfies axioms E1, E2 and E3. Then, from any concentration system it is possible to extract an optimal concentration system which determines a (unique) profile reconstruction.*

Proof. Given a concentration system, let (according to Lemma 4.8) $(\Phi, \boldsymbol{P})$ be an optimal concentration system extracted from it. Set $\overline{r} = \mathrm{rad}_Z(U(\Phi, \boldsymbol{P}))$. For any $k \in \mathbb{N}$ (since $\delta_Z(\overline{r}, 2^{-(k+1)}) > 0$), we fix a sequence $(v_n^k)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$ such that

$$\mathrm{rad}_Z((v_n^k)_{n\in\mathbb{N}}) \leq \overline{r} + \delta_Z(\overline{r}, 2^{-(k+1)}). \tag{6.8}$$

Note that, by Lemma 6.7, for any other sequence $(w_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$ such that $\mathrm{rad}_Z((w_n)_{n\in\mathbb{N}}) \leq \overline{r} + \delta_Z(\overline{r}, 2^{-(k+1)})$ we have $d(w_n, v_n^k) < 2^{-(k+1)}$ for large n.

We shall build the elements of the profile reconstruction as limit values of suitable sequences. Let the sequence $(w_n^0)_{n\in\mathbb{N}}$ be defined by setting

$$w_n^0 = \begin{cases} v_n^0 & \text{if} \quad n \leq n_0 \\ v_n^1 & \text{if} \quad n > n_0 \end{cases}$$

where n_0 is chosen large enough to have

$$d(v_n^0, w_n^0) \leq 2^{-1} \quad \forall n \geq n_0.$$

(Roughly speaking w_n^0 follows v_n^0 until v_n^1 becomes sufficiently close). Then, we define the sequence $(w_n^1)_{n\in\mathbb{N}}$ by setting

$$w_n^1 = \begin{cases} w_n^0 & \text{if} \quad n \le n_1 \\ v_n^2 & \text{if} \quad n > n_1 \end{cases}$$

where $n_1 \ge n_0$ has been chosen so that

$$d(w_n^0, w_n^1) \le 2^{-2} \quad \forall n \ge n_1.$$

Recursively, given $k \ge 1$ and once the sequence $(w_n^{k-1})_{n\in\mathbb{N}}$ has been defined, we fix a natural number $n_k > n_{k-1}$ such that

$$d(w_n^{k-1}, w_n^k) \le 2^{-(k+1)} \quad \forall n \ge n_k \tag{6.9}$$

where

$$w_n^k = \begin{cases} w_n^{k-1} & \text{if} \quad n \le n_k \\ v_n^{k+1} & \text{if} \quad n > n_k. \end{cases}$$

Note that for any $k \in \mathbb{N}$, $w_n^k = v_n^{k+1}$ for n large enough and that, by (6.9), $d(w_n^k, w_n^{k+1}) \le 2^{-(k+2)}$. Therefore for any $n \in \mathbb{N}$ and for any $h \ge k$ we have

$$d(w_n^k, w_n^h) \le 2^{-k}.$$

So for any $n \in \mathbb{N}$ we can set $w_n := \lim_k w_n^k$ and, letting h go to infinity in the above inequality, we deduce that

$$\forall k \in \mathbb{N}, \quad d(w_n, w_n^k) < 2^{-k} \text{ for large } n. \tag{6.10}$$

Let us show that the sequence $(w_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$. Indeed, fixed $i \in I$, since each element ρ_n^i of $\boldsymbol{\rho}_i$ is an isometry, from (6.10), we deduce that

$$\forall k \in \mathbb{N}, \quad d((\rho_n^i)^{-1}(w_n), (\rho_n^i)^{-1}(w_n^k)) < 2^{-k} \text{ for large } n. \tag{6.11}$$

By taking into account that, for any fixed k, (by definition) $w_n^k = v_n^{k+1}$ for large n, we deduce, by (4.4) and (6.8) that

$$\limsup_n d(w_n^k, Z_G) = \limsup_n d(v_n^{k+1}, Z_G) \le \overline{r} + \delta_Z(\overline{r}, 2^{-(k+2)}), \tag{6.12}$$

and, since $v_n^{k+1} \rightharpoondown \varphi_i$ (as n goes to infinity), we deduce by Lemma 2.4 and (6.11) that $(\rho_n^i)^{-1}(w_n) \rightharpoondown \varphi_i$. (i.e. each φ_i is a profile of $(w_n)_{n\in\mathbb{N}}$ and $\boldsymbol{\rho}_i = (\rho_n^i)_{n\in\mathbb{N}}$ is a related blowup sequence and so $(u_n)_{n\in\mathbb{N}} \in U(\Phi, \boldsymbol{P})$). Moroever, by the triangle inequality, we deduce from (6.10) and (6.12) that

$$\limsup_n d(w_n, Z_G) \le \limsup_n d(w_n^k, Z_G) + 2^{-k} \le \overline{r} + \delta(\overline{r}, 2^{-(k+2)}) + 2^{-k}.$$

Since k has been arbitrarily fixed we also deduce that $\limsup_n d(w_n, Z_G) \le \overline{r}$, i.e. $\mathrm{rad}_Z((w_n)_{n\in\mathbb{N}}) \le \mathrm{rad}_Z(U(\Phi, \boldsymbol{P}))$ and therefore equality holds (i.e. $(w_n)_{n\in\mathbb{N}}$ is a profile reconstruction determined by $(\Phi, \boldsymbol{P})$). ☐

Under the requirements (axioms E1, E2 and E3) which guarantee both existence and uniqueness (modulo subsequences) of the (polar) profile reconstruction (determined by (an optimal subsystem of) any concentration system) we can give the following definition.

Definition 6.10 (profile reconstruction of a sequence)**.** We shall say that $(\hat{u}_n)_{n\in\mathbb{N}}$ is the profile reconstruction of $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ if $(u_n)_{n\in\mathbb{N}}$ is profile convergent and $(\hat{u}_n)_{n\in\mathbb{N}}$ is the profile reconstruction determined by any optimal complete concentration system of $(u_n)_{n\in\mathbb{N}}$.

7. Cocompactness and profile decomposition

In this section we shall always assume that (E, d) is a complete SR metric space equipped with an admissible group G of scalings. On the set $\mathcal{E}$ of bounded sequences of E we shall introduce some equivalence relations which will be used to define related notions of cocompactness that generalize the corresponding ones given for the linear case in [14, 3]. Furthermore we shall prove, when also axioms E1, E2 and E3 hold true, that every profile convergent sequence is equivalent to its profile reconstruction. When, in addition, all the bounded sets in E will be assumed to be G-cocompact (see Definition 7.8 below) then we will get the profile decomposition given in Corollary 7.11, i.e. every profile convergent sequence $(u_n)_{n\in\mathbb{N}}$ is strongly approximated by its profile reconstruction $(\hat{u}_n)_{n\in\mathbb{N}}$.

Definition 7.1 (G-equivalence relations)**.** Given two sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$ we shall say that

- $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are *strongly G-equivalent*, and we shall write $(u_n)_{n\in\mathbb{N}} \doteq_G (v_n)_{n\in\mathbb{N}}$, if for any sequence $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ and for any polar convergent subsequence $(g_{k_n}(u_{k_n}))_{n\in\mathbb{N}}$ of $(g_n(u_n))_{n\in\mathbb{N}}$ with a polar limit $w \in E$ the sequence $(g_{k_n}(v_{k_n}))_{n\in\mathbb{N}}$ is also polar convergent to w;
- $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are *G-equivalent*, and we shall write $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$, if the above requirement is reduced to $w \in E \setminus Z_G$ and, furthermore, for any $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ if there exists an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $g_{k_n}(u_{k_n}) \rightharpoonup z \in Z_G$ then the sequence $(g_{k_n}(v_{k_n}))_{n\in\mathbb{N}}$ must be polar infinitesimal.

We shall reserve the notation $\simeq_G^\infty$ and $\doteq_G^\infty$ for the cases in which the respective above requirements are only posed for discrete sequences of scalings $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$.

Remark 7.2. Of course relations $\doteq_G$, $\simeq_G$, $\doteq_G^\infty$ and $\simeq_G^\infty$ carry over subsequences (i.e. denoted by $\simeq$ any of the above relations, if $(u_n)_{n\in\mathbb{N}} \simeq (v_n)_{n\in\mathbb{N}}$ then, for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$, $(u_{k_n})_{n\in\mathbb{N}} \simeq (v_{k_n})_{n\in\mathbb{N}}$).

Lemma 7.3. *Relations $\doteq_G$, $\simeq_G$, $\doteq_G^\infty$ and $\simeq_G^\infty$ are equivalence relations.*

Proof. Only symmetry, which actually is a consequence of Theorem 2.2, deserves some explanation and, since the argument is similar for the four relations, we just deal with $\simeq_G$. Let us fix $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$ such that $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$ and let $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ be given. Let $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ be any extraction law such that $g_{k_n}(v_{k_n}) \rightharpoonup w \in E$.

If $w \notin Z_G$ we have to prove that also $g_{k_n}(u_{k_n}) \rightharpoonup w$. Even if, as remarked in [3, Section 4], polar convergence is not, in general, induced by a topology, polar convergent sequences admit the characterization given in [3, Remark 2.9-item (iv)] so what we have to prove is that any subsequence of $(g_{k_n}(u_{k_n}))_{n\in\mathbb{N}}$ admits a subsequence which is polar convergent to w. Since any subsequence of $(g_{k_n}(u_{k_n}))_{n\in\mathbb{N}}$ is a bounded sequence, by Theorem 2.2, there exists a renamed subsequence which is polar convergent to some $w' \in E$. Note that, by Remark 7.2, $(u_{k_n})_{n\in\mathbb{N}} \simeq_G (v_{k_n})_{n\in\mathbb{N}}$ and so if $w' \in Z_G$ then $(g_{k_n}(v_{k_n}))_{n\in\mathbb{N}}$ must be polar infinitesimal, in contradiction to the assumption that $w \notin Z_G$. So $w' \notin Z_G$ and by definition of $\simeq_G$ we have $w' = w$.

Finally, if $w \in Z_G$ we also have to prove that $(g_{k_n}(u_{k_n}))_{n\in\mathbb{N}}$ is polar infinitesimal, but this is contained in the definition of G-equivalence. □

The above lemma allows us to reformulate Definition 7.1 by stating, for instance, that two sequences $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ are strongly G-equivalent if for all $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ and for any extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ the subsequence $g_{k_n}(u_{k_n})_{n\in\mathbb{N}}$ is polar convergent if and only if $g_{k_n}(v_{k_n})_{n\in\mathbb{N}}$ is polar convergent and, in such a case, the respective polar limits coincide. Roughly speaking, $(u_n)_{n\in\mathbb{N}} \doteq_G (v_n)_{n\in\mathbb{N}}$ if $(u_n)_{n\in\mathbb{N}}$ and $(v_n)_{n\in\mathbb{N}}$ have the same "polar behavior" with respect to scalings.

Of course $\dot{\simeq}_G$ is a stronger relation than $\simeq_G$, i.e. for any $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \in \mathcal{E}$

$$(7.1) \qquad (u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}} \quad \Rightarrow \quad (u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}},$$

and the two notions agree in the case in which the null set Z_G reduces to a unique point. Note that the following implication holds true

$$(7.2) \qquad d(u_n, v_n) \to 0 \quad \Rightarrow \quad (u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}}.$$

We can recognize the meaning of the equivalence relations, introduced above, in the linear setting of [14] where, since $Z_G = \{0\}$, relations $\simeq_G$ and $\dot{\simeq}_G$ (as well as $\simeq_G^\infty$ and $\dot{\simeq}_G^\infty$) coincide.

- $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$ means, in the linear case, that $(u_n - v_n)_{n\in\mathbb{N}}$ is G-vanishing, (i.e. $\forall (g_n)_{n\in\mathbb{N}} \in \mathcal{G}\ g_n(u_n - v_n) \rightharpoonup 0$).
- $(u_n)_{n\in\mathbb{N}} \simeq_G^\infty (v_n)_{n\in\mathbb{N}}$ means that $(u_n - v_n)_{n\in\mathbb{N}}$ has no concentrations.
- $(u_n)_{n\in\mathbb{N}}$ is equivalent under $\simeq_G^\infty$ to a constant sequence $(u)_{n\in\mathbb{N}}$ means in the linear case that $(u_n)_{n\in\mathbb{N}}$ has no concentrations and is G-convergent to u (i.e. $(u_n - u)_{n\in\mathbb{N}}$ is G-vanishing).

This leads us to introduce the following definition.

Definition 7.4 (G-boundedness)**.** A bounded subset $B \subset E$ is called *G-bounded* if for any sequence $(u_n)_{n\in\mathbb{N}} \subset B$ and for any $u \in E$ the following implication holds true

$$(7.3) \qquad u_n \rightharpoondown u \quad \Rightarrow \quad (u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (u)_{n\in\mathbb{N}}.$$

When in the above implication the consequent statement $(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (u)_{n\in\mathbb{N}}$ is replaced by $(u_n)_{n\in\mathbb{N}} \simeq_G (u)_{n\in\mathbb{N}}$ in (7.3), in the light of (7.1), the attribute "G-bounded" will be replaced by "weakly G-bounded".

Note that a bounded subset $B \subset E$ is weakly G-bounded if for any sequence $(u_n)_{n\in\mathbb{N}} \subset B$ and for any discrete sequence of scalings $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}_\infty$, the scaled sequence $(g_n(u_n))_{n\in\mathbb{N}}$ is polar infinitesimal (see Definition 2.10). In other terms a bounded subset $B \subset E$ is weakly G-bounded if, each sequence $(u_n)_{n\in\mathbb{N}} \subset B$ behaves, with respect to discrete sequences of scalings, as if it would be constant.

Lemma 7.5. *Any compact set $K \subset E$ is G-bounded.*

Proof. Given $(u_n)_{n\in\mathbb{N}} \subset K$ and $u \in K$ such that $u_n \rightharpoondown u$, since K is compact $u_n \to u$, so $(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (u)_{n\in\mathbb{N}}$ by (7.2). □

Theorem 7.6. *Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$ be such that $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$. Then $(u_n)_{n\in\mathbb{N}}$ is profile convergent if and only if $(v_n)_{n\in\mathbb{N}}$ is profile convergent. Moreover, any (complete) profile system (resp. (complete) concentration system) of $(u_n)_{n\in\mathbb{N}}$ is a (complete) profile system (resp. (complete) concentration system) of $(v_n)_{n\in\mathbb{N}}$. Conversely, two profile convergent sequences which have a common complete concentration system are G-equivalent.*

Proof. We shall only prove the second part of the statement since the first one is easy. Let $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$ be two profile convergent sequences which admit a common complete concentration system $(\Phi, \boldsymbol{P})$. Let $(g_n)_{n\in\mathbb{N}} \in \mathcal{G}$ and let $(k_n)_{n\in\mathbb{N}}$ be any extraction law such that $g_{k_n}(u_{k_n}) \rightharpoonup w \in E$. Note that this means (see Definition 3.1) that w is a (eventually null) profile of the subsequence $(u_{k_n})_{n\in\mathbb{N}}$ and that $\overline{\boldsymbol{\rho}} = (\overline{\rho}_n)_{n\in\mathbb{N}} = (g_{k_n}^{-1})_{n\in\mathbb{N}}$ is a related blowup sequence. So, since $(\Phi, \boldsymbol{P})$ is a complete concentration system of $(u_{k_n})_{n\in\mathbb{N}}$, the following alternative holds true (modulo subsequences): or $\overline{\boldsymbol{\rho}}$ is quasiorthogonal to any blowup sequence in $\boldsymbol{P}$ and then, by Proposition 5.5, $w \in Z_G$ or, otherwise, w is a copy of a (nontrivial) profile of Φ.

In the first case (when in particular $w \in Z_G$), since $(\Phi, \boldsymbol{P})$ is also a complete concentration system of $(v_{k_n})_{n\in\mathbb{N}}$, the sequence $(g_{k_n}(v_{k_n}))_{n\in\mathbb{N}}$ must be polar infinitesimal by Proposition 5.5. In the second case (when $w \notin Z_G$), there exist $g \in G$, $\varphi \in \Phi$ (which is a profile of both $(u_{k_n})_{n\in\mathbb{N}}$ and $(v_{k_n})_{n\in\mathbb{N}}$), with a related blowup sequence $\boldsymbol{\rho} = (\rho_n)_{n\in\mathbb{N}} \in \boldsymbol{P}$, such that $w = g(\varphi)$ and, see Remark 3.3, $\overline{\boldsymbol{\rho}} \circ g := (\overline{\rho}_n \circ g)_{n\in\mathbb{N}} \in [\boldsymbol{\rho}]_S$. Since $(\Phi, \boldsymbol{P})$ is also a complete concentration system of $(v_{k_n})_{n\in\mathbb{N}}$ we deduce that $(\overline{\rho}_n \circ g)^{-1}(v_{k_n}) \rightharpoonup \varphi$ and so, by (2.3), that $\overline{\rho}_n^{-1}(v_{k_n}) \rightharpoonup g(\varphi) = w$, i.e. $g_{k_n}(v_{k_n}) \rightharpoonup w$. ☐

We can in particular deduce the following result.

Corollary 7.7. *Any profile convergent sequence is G-equivalent to a profile reconstruction determined by any related complete concentration system.*

Definition 7.8 (G-cocompactness)**.** Let E be a complete SR metric space equipped with an admissible group G of scalings. Let (F, d') be a metric space and let $J : E \to F$ be a function. We shall say that the function J is *G-cocompact* if the implication

$$(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}} \quad \Rightarrow \quad d'(J(u_n), J(v_n)) \to 0$$

holds true for all (bounded) sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \in \mathcal{E}$.

A bounded set $K \subset E$ is called *G-cocompact* if the canonical injection of K into E is G-cocompact, i.e. for all (bounded) sequences $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subset K$

$$(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}} \;\Rightarrow\; d(u_n, v_n) \to 0. \tag{7.4}$$

When the requirement $(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}}$ is replaced by $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$ in any of the properties above, at the light of (7.1), the attribute "G-cocompact" will be replaced by "strongly G-cocompact".

Lemma 7.9. *Any compact set $K \subset E$ is G-cocompact.*

Proof. Let $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}} \subset K$ be such that $(u_n)_{n\in\mathbb{N}} \dot{\simeq}_G (v_n)_{n\in\mathbb{N}}$. Since K is compact, there exists $u \in E$ such that $u_n \to u$ (modulo a subsequence). Since $\mathbf{i_d} \in \mathcal{G}$ we deduce that $v_n \to u$, and, as a consequence, that $d(u_n, v_n) \to 0$. ☐

Note that if Z_G consists of one point, then G-cocompactness and strong G-cocompactness agree. From (7.2) and (7.4) we deduce that a G-cocompact function is a continuous function (uniformly on bounded subsets) and that, of course, the identity map on E is strongly G-cocompact (G-cocompact) if and only if all its bounded sets are strongly G-cocompact (G-cocompact).

The following proposition is a criterion for G-cocompactness of bounded sets.

Proposition 7.10. *A bounded subset $K \subset E$ is strongly G-cocompact if and only if all profile convergent sequences $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \subset K$ which have a common complete concentration system satisfy $d(u_n, v_n) \to 0$.*

Proof. The direct implication is a consequence of the last statement in Theorem 7.6 and of (7.4). To prove the converse implication we fix $(u_n)_{n\in\mathbb{N}}$, $(v_n)_{n\in\mathbb{N}} \subset K$ such that $(u_n)_{n\in\mathbb{N}} \simeq_G (v_n)_{n\in\mathbb{N}}$ and we shall prove that $d(u_n, v_n) \to 0$ by proving that (each subsequence of) $(d(u_n, v_n))_{n\in\mathbb{N}}$ admits a subsequence which goes to zero. Since, by Theorem 5.6, there exists an extraction law $(k_n)_{n\in\mathbb{N}} \subset \mathbb{N}$ such that $(u_{k_n})_{n\in\mathbb{N}}$ is profile convergent and since, by Remark 7.2, $(u_{k_n})_{n\in\mathbb{N}} \simeq_G (v_{k_n})_{n\in\mathbb{N}}$, we deduce by Theorem 7.6 that also $(v_{k_n})_{n\in\mathbb{N}}$ is profile convergent and admits the same concentration system, therefore, by assumptions, $d(u_{k_n}, v_{k_n}) \to 0$. □

Under the additional requirements ensuring the existence of a profile reconstruction determined by a concentration system we can give the following result which immediately follows by Corollary 7.7 and property (7.4).

Corollary 7.11 (Profile decomposition)**.** *Assume axioms E1, E2 and E3. If E is strongly G-cocompact then any profile convergent sequence $(u_n)_{n\in\mathbb{N}}$ is strongly approximated by its profile reconstruction $(\hat{u}_n)_{n\in\mathbb{N}}$, see Definition 6.10.*

Corollary 7.12. *Assume axioms E1, E2 and E3. If a map $J : E \to F$ is strongly G-cocompact, then every profile convergent sequence $(u_n)_{n\in\mathbb{N}} \in \mathcal{E}$ with a complete profile system given by $\{\varphi\}$, ($\varphi \notin Z_G$), satisfies $J(u_n) \to J(\varphi)$ in F.*

Proposition 7.13. *If the finite subsets of Z_G are strongly G-cocompact, then the set Z_G reduces to a unique fixed point of G (i.e. an element which is fixed by any $g \in G$).*

Proof. Fixed $z_1, z_2 \in Z_G$, the two corresponding constant sequences $(z_1)_{n\in\mathbb{N}}$ and $(z_2)_{n\in\mathbb{N}}$ are G-equivalent, and thus, by (7.4) (where $\dot{\simeq}_G$ is replaced by $\simeq_G$), $d(z_1, z_2) = 0$, i.e. $z_1 = z_2 =: z$. So $Z_G = \{z\}$ and since Z_G is stable with respect to G (see Proposition 2.11), $g(z) = z$ for all $g \in G$. □

We have the following immediate criterion for compactness of closed sets.

Proposition 7.14. *Let E be a complete SR metric space equipped with an admissible group G of scalings. Then for any closed bounded set $K \subset E$ the following propositions are equivalent*

a) *K is compact;*
b) *K is G-bounded and G-cocompact.*

Proof. The implication a) ⇒ b) follows by applying lemmas 7.5 and 7.9. To prove the converse implication, we fix $(u_n)_{n\in\mathbb{N}} \subset K$. Since K is bounded, thanks to

Theorem 2.2, there exists an extraction law $(k_n)_{n\in\mathbb{N}}$ and $u \in E$ such that $u_{k_n} \rightharpoonup u$. Then, since K is G-bounded, we deduce from (7.3) that $(u_{k_n})_{n\in\mathbb{N}} \dot{\simeq}_G (u)_{n\in\mathbb{N}}$ and, since K is G-cocompact, we deduce from (7.4) that $d(u_{k_n}, u) \to 0$. $\square$

Note that by Proposition 7.13 compact sets are not in general strongly G-cocompact unless Z_G is not reduced to a single point. In such a case, as we have already observed, the notions of G-cocompactness and strong G-cocompactness as well as those of G-boundedness and weak G-boundedness agree.

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Manuscript received March 19 2017
revised August 18 2017

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