Pure and Applied Functional Analysis
Volume 2, Number 4, 2017, 599–623

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

AN IMPROVED NONLINEAR MODEL FOR IMAGE RESTORATION

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Abstract. The image denoising is a key step in image processing. This step can be treated by non-linear diffusive filters requiring solving evolving partial differential equations. In this work, a nonlinear diffusive filter for image denoising and edge detection based on a nonlinear partial differential equation is studied analytically and tested numerically. Existence, uniqueness and regularity of the solution for the proposed mathematical model are established in an Hilbert space. The discretization of the partial differential equation of the proposed model is performed using finite element method. A result of convergence of this approximation is established under suitable hypotheses.
The efficiency of this model has been tested numerically. Signal noise ratio (SNR) is used to estimate the quality of the restored images.

1. Introduction

In recent decades, several models for image restoration in image processing have been proposed in the literature, see [1, 2, 9, 10, 14, 17, 19, 20, 21]. Those models are based on nonlinear diffusive filters requiring solving nonlinear evolutionary partial differential equations.
Nonlinear diffusion filters are used in image processing to simultaneously smoothen noisy images and enhance sharp contrasts in brightness. This approach was initiated by P. Perona and J. Malik [19] by means of the following nonlinear PDE problem

$$\left\{\begin{array}{c} \frac{\partial v}{\partial t} - div(\mu_1(|\nabla v|)\nabla v) = 0, \;\text{ in } Q, \\ v(x,0) = v_0(x), \forall x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, \forall x \in \partial\Omega, \forall t \in [0,T], \end{array}\right. \tag{1.1}$$

where v_0 is the grey level distribution of a given (distorted) image occupying a bounded domain Ω in $\mathbb{R}^d$ (with $d \leq 3$ in most applications) for which boundary is $\partial\Omega$. Q is defined by
$Q = \Omega \times [0,T]$, for some given $T > 0$, and n is the unit normal vector to the domain boundary.

2010 *Mathematics Subject Classification.* 35K55, 65M60, 68U10, 76M10, 94A08.
Key words and phrases. edge detection, finite element analysis, image restoration, nonlinear diffusion, Hilbert space.
This work was funded by the project research PHC Volubilis MA/11/246 managed by Campus France,Agency French for the promotion of higher education and international mobility and by LIRIMA. The authors would like to thank those institutions for material and financial support.

Starting from the initial image $v_0(x)$ and by running (1.1) we construct a family of functions (i.e images) $\{v(t,x)\}_{t>0}$ representing restored versions of $v_0(x)$. The diffusion coefficient $\mu_1(|\nabla v|)$ is designed with this choice:

- Inside the regions where the magnitude of the gradient of v is weak, equation (1.1) acts like the heat equation, resulting in isotropic smoothing.
- Near the boundaries where the magnitude of the gradient is large, the regularization is stopped and the edges are preserved.

The assumptions imposed on μ_1 are usually

$$\begin{cases} \mu_1 : [0,+\infty) \to [0,+\infty) \ \textit{decreasing}, \\ \mu_1(0) = 1, \ \lim_{s\to+\infty} \mu_1(s) = 0, \\ \mu_1(s) + 2s\mu_1'(s) > 0. \end{cases} \tag{1.2}$$

Typical example for an edge stopping function μ_1 which, in fact, have been used by Perona and Malik, is

$$\mu_1(s) = \frac{1}{1+s^2/k^2} \ \ (k>0). \tag{1.3}$$

The Parameter k is a mesure for the steepness of an edge to be preserved. Unfortunately, with such a choice of the edge stopping function, it is not possible to prove that the operator A_1 defined by $A_1(u) = -div(\mu_1(|\nabla u|)\nabla u)$ in Perona-Malik problem (1.1) is monotone. And then the Faedo-Galerkin method cannot be used to prove that this problem is well posed. Apart of this inconvenient, numerical approximations of (1.1) do not exhibit significant instabilities. This numerical performance triggered many attempts to replace the Perona-Malik model by nearby versions which, on one hand side, admit solid analysis in terms of existence and uniqueness theorems, and, on the other hand side, possess essentially the same numerical properties as (1.1). The first, and widely used approach is due to Catt and al. [9] who employ a space regularization. In this model ∇v is replaced by ∇v_σ where $v_\sigma = G_\sigma * v$ and $*$ is denoting convolution with respect to the space variable and G_σ is the Gaussian with variance $\sigma > 0$. In [9] existence, uniqueness and regularity of a solution has been established. At the same time Alvarez, Lions, and Morel [4] investigated the diffusion equation

$$\frac{\partial v}{\partial t} - \mu_1(|\nabla v_\sigma|)|\nabla v| div\left(\frac{\nabla v}{|\nabla v|}\right) = 0. \tag{1.4}$$

It is shown in [4] that (1.4) possesses a unique global viscosity solution. Other spatial regularizations of Perona Malik equations type have been proposed by Weickert in [20]. Kichenssamy [16] has demonstrated in one dimension that any weak solution of (1.1) must possess an infinitely differentiable initial condition for $|\nabla v| > k$. He noticed that, even if v_0 is smooth, there are minor perturbations of the initial value problem for which weak solutions do not exist, thus the Perona-Malik model is ill-posed in the sense of Hadamard. Zhang and al. [22] established that the Perona-Malik equation in one dimension admits infinitely many weak solutions. Always in the one dimensional case of (1.1) Gobbino and al. [13] exhibited that every C^1 solution on $\mathbb{R}$ is a function of the form $v(x,t) = ax + b$. Taheri and al. [15] and Chen and al. [11] established that there exist infinitely many Young measure solutions [8] of (1.1) in one and two dimension. Recently Calder et al. [7] examined a perturbed Perona-Malik equation, their perturbation technique is to consider the diffusion equation as L^2 gradient flows on integral functionals and then modify the inner product from L^2 to a Sobolev inner product. He establish a very general existence and uniqueness result which applies to a family of high order diffusion equations which are generalizations of the Perona-Malik equation.

Our approach consists to replace the Perona-Malik (1.1) model by the following problem

$$\begin{cases} \frac{\partial v}{\partial t} - div(\mu_2(|\nabla v|)\nabla v) = 0, \text{ in } Q, \\ v(x,0) = v_0(x), \forall x \in \Omega, \\ \frac{\partial v}{\partial n} = 0, \forall x \in \partial\Omega, \forall t \in [0,T]. \end{cases} \tag{1.5}$$

Where $\mu_2 = \mu_1 + \alpha$ and $\alpha \in \mathbb{R}_+^*$.
In this case if μ_1 verifies the assumptions (1.2) then μ_2 verifies:

$$\begin{cases} \mu_2 : [0,+\infty) \to [0,+\infty) \ \textit{decreasing}, \\ \mu_2(0) = 1+\alpha, \ \lim_{s\to+\infty} \mu_2(s) = \alpha, \\ \mu_2(s) + 2s\mu_2'(s) > 0. \end{cases} \tag{1.6}$$

With such a choice of μ_2 and by imposing additional conditions on μ_2 the problem of the monotony of the operator is surmounted. In this work we establish that the problem (1.5) is well posed in the Hadamard sense and admits an unique weak solution in $L^2(0,T,H^1(\Omega))$ under suitable hypotheses on μ_2. The proof is based on Faedo-Galerkin method and the monotony of non linear differential operator. A result of the convergence for finite element methods (FEM) that is very few investigated compared to the extensively discussed finite difference schemes is demonstrated. Finally the non linear filter (1.5) is tested numerically and the obtained results are sensibly the same to those obtained by the Perona-Malik model.

2. Existence and Uniqueness

We put $H = H^1(\Omega)$ and $V = L^2(\Omega)$. H is equipped with the scalar product $((u,v)) = \int_\Omega uvdx + \int_\Omega \nabla u \nabla v dx$ and its associated norm is $\|.\|$. On the other hand, the space V is provided with the scalar product $(u,v) = \int_\Omega uvdx$ and its associated norm $|.|$.
We first begin by giving some results that will be useful in the existence and uniqueness proof of weak solution of the problem (1.5).

2.1. **Preliminary results.**

Lemma 2.1. *Let δ be a function which verifies the following assumptions*

i) $\delta : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$
ii) δ *is continuous function*
iii) $\lim_{s\to+\infty} [\delta(s)] = \delta_0$, *with* $\delta_0 > 0$
iv) δ *is differential continuous*
v) $s\left|\delta'(s)\right| \leq \delta(s) \ \forall s \in \mathbb{R}^+$

Then the operateur A defined by:

$$(A(v),w)=\int_\Omega \delta\left(|\nabla v|\right)\nabla v.\nabla w dx \quad for \quad v,w\in H, \tag{2.1}$$

is an operator monotone hemicontinuous, satisfying for all $u,v\in H$:

$$(A(u)-A(v),u-v)\geq \inf_{s\in\mathbb{R}^+}\delta(s)|\nabla u-\nabla v|^2. \tag{2.2}$$

Proof. see [1]. □

Remark 2.2. (1) The conditions (iii) and (iv) of Lemma 2.1 are not imposed on μ_1 in the Perona Malik problem, the choice of $\mu_2=\mu_1+\alpha$ allows to check the condition (iii) to μ_2, imposing more the condition (iv) to the fucntion μ_2, the monotony of the operator A defined in (2.1) for $\delta=\mu_2$ is ensured by Lemma 2.1.
(2) If a function δ verifying hypotheses of lemma 2.1 is decreasing then the condition (v) can be written $\delta(s)+2s\delta'(s)\geq 0$. Furthermore, due to (iii), $\inf_{s\in\mathbb{R}^+}(\delta(s))=\delta_0$.

Definition 2.3. A weak solution of (1.5) is a function $v\in L^2(0,T,H^1(\Omega))$, $\frac{dv}{dt}\in L^2(0,T,H^{-1})$ that verifies

$$\textstyle\int_\Omega \frac{\partial v(t)}{\partial t}w dx+\int_\Omega \mu_2\left(|\nabla v|\right)\nabla v.\nabla w dx=0 \quad \forall w\in H^1(\Omega) \tag{2.3}$$

Before proving existence and uniqueness of weak solution of the problem (1.5), we first study the existence and uniqueness of weak solution of the general following problem

$$\begin{cases} \frac{\partial u}{\partial t}-div(\mu(|\nabla u|)\nabla u)+\alpha u=0, \text{ in } Q,\\ u(x,0)=u_0(x),\forall x\in\Omega,\\ \frac{\partial u}{\partial n}=0,\forall x\in\partial\Omega,\forall t\in[0,T], \end{cases} \tag{2.4}$$

where the real $\alpha>0$ and the function μ are given.

Definition 2.4. A weak solution of (2.4) is a function $u\in L^2(0,T,H^1(\Omega))$, $\frac{du}{dt}\in L^2(0,T,H^{-1})$ that verifies

$$\begin{array}{c}\int_\Omega \frac{\partial u(t)}{\partial t}w dx+\int_\Omega \mu\left(|\nabla u|\right)\nabla u.\nabla w dx+\alpha\int_\Omega u.w dx=0\\ \forall w\in H^1(\Omega).\end{array} \tag{2.5}$$

Theorem 2.5. *Let $u_0\in V$ and μ a decreasing function satisfying the hypotheses of lemma 2.1 then, for all given real $\alpha>0$, there exists an unique global weak solution u for problem (2.4) such that $u\in L^2(0,T,H)\cap L^{+\infty}(0,T,V)$ and $u^{'}=\frac{du}{dt}\in L^2(0,T,H^{'})$. With $H^{'}=H^{-1}$.*

Proof. 1.<u>Existence</u>
The hypothesis $(i-iii)$ involve that μ is bounded.
Notice that $a=\sup_{s\in\mathbb{R}^+}\mu(s)$ and $b=\inf_{s\in\mathbb{R}^+}\mu(s)$ exist and $b=\delta_0$.
By using lemma 2.1 we have for all $u,w\in H$:

$$(A(u)-A(w),u-w)\geq b|\nabla u-\nabla w|^2, \tag{2.6}$$

where the operateur A is defined by:

$$(A(u), w) = \int_\Omega \mu\left(|\nabla u|\right) \nabla u . \nabla w dx \quad \text{for} \quad u, w \in H. \tag{2.7}$$

The existence demonstration is based on Faedo-Galerkin method. We consider the spectral problem

$$((w, \nu)) = \lambda(w, \nu) \qquad \forall \nu \in H. \tag{2.8}$$

Since the injection of H in V is compact, the problem (2.8) admits a sequence of eigenvalues λ_j associated of eigenvectors w_j such that

$$((w_j, \nu)) = \lambda_j(w_j, \nu) \qquad \forall \nu \in H, \tag{2.9}$$

and $(w_j)_{j\in\mathbb{N}}$ is orthonormal in V and orthogonal in H. We denote $u_N(t)$ an approximate solution of (2.5) defined by

$$\begin{aligned} u_N(x,t) &= u_N(t)(x) \in [w_1, ..., w_N], \\ u_N(x,t) &= \sum_{j=1}^{N} C_j^N(t) w_j(x). \end{aligned} \tag{2.10}$$

We have then

$$\begin{cases} (u_N'(t), w_j) + (\mu(|\nabla u_N(t)|)\nabla u_N(t), \nabla w_j) \\ +\alpha(u_N(t), w_j) = 0, \qquad 1 \le j \le N, \quad t \in [0,T], \\ with\ u_N(.,0) = u_{0N}(.) \in [w_1, ..., w_N] \\ and\ u_{0N} \to u_0 \in H. \end{cases} \tag{2.11}$$

With $u_{0N} = \sum_{j=0}^{N} (u_0, w_j) w_j$.

Each $C_j^N(t)$ verifies $\frac{dC_j^N(t)}{dt} = G_j(t, C_1^N(t), ..., C_N^N(t))$ where G_j is a continuous function, then by using the Cauchy theorem we deduce that there exist a local solution $u_N(t)$ of (2.11) on $[0, T_N]$. By multiplying (2.11) by $C_j^N(t)$ and by adding, we deduct that:

$$\int_\Omega \frac{\partial u_N(t)}{\partial t} u_N(t) dx + \int_\Omega \mu\left(|\nabla u_N(t)|\right) (\nabla u_N(t))^2 dx + \alpha \int_\Omega \left(u_N(t)\right)^2 dx = 0.$$

Then

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\left|u_N(t)\right|^2 &+ \int_\Omega \mu\left(|\nabla u_N(t)|\right) (\nabla u_N(t))^2 dx \\ &+ \alpha \int_\Omega \left(u_N(t)\right)^2 dx = 0. \end{aligned} \tag{2.12}$$

Due to the remark 1, $b > 0$ then

$$\int_\Omega \mu\left(|\nabla u_N(t)|\right) (\nabla u_N(t))^2 dx \ge b \int_\Omega \left(\nabla u_N(t)\right)^2 dx.$$

From (2.12) we have

$$\frac{1}{2}\frac{d}{dt}\left|u_N(t)\right|^2 + b\int_\Omega \left(\nabla u_N(t)\right)^2 dx + \alpha \int_\Omega \left(u_N(t)\right)^2 dx \le 0. \tag{2.13}$$

Then

$$\frac{1}{2}\frac{d}{dt}|u_N(t)|^2 + \min(\alpha, b)\|u_N(t)\|^2 \le 0. \tag{2.14}$$

There exists thus a constant $C_1 = |u_0| > 0$ and a constant $C_2 = \frac{|u_0|^2}{2\min(\alpha,b)} > 0$ depending only on b, α and u_0 such that

$$\left\{ \begin{array}{c} |u_N(t)| \le C_1 \quad \text{and} \quad \int_0^t \|u_N(\tau)\|^2 \, d\tau \le C_2, \\ \forall t \in [0, T_N], \forall N \in \mathbb{N}. \end{array} \right. \tag{2.15}$$

We deduce that $T_N = T$ and that for all $N \in \mathbb{N}$, $u_N \in L^{+\infty}(0, T, V) \cap L^2(0, T, H)$. Furthermore the sequence u_N is bounded.
We will prove now that $u_N^{'} \in L^1(0, T, H^{'})$ for all $N \in \mathbb{N}$ and that the sequence $u_N^{'}$ is bounded in $L^1(0, T, H^{'})$ where $H^{'} = H^{-1}$.
Let P_N be the projector of V on $[w_1, ..., w_N]$ thus $P_N h = \sum_{i=1}^{N}(h, w_i)w_i$. Therefore (2.11) is written:

$$u_N^{'} = -P_N A u_N - \alpha P_N u_N. \tag{2.16}$$

We deduce through to the choice of w_j that:

$$\|P_N\|_{\mathcal{L}(H,H)} \le 1$$

hence by transposition (and since $P_N^* = P_N$)

$$\|P_N\|_{\mathcal{L}(H',H')} \le 1. \tag{2.17}$$

Otherwise, let $\nu \in H$, thus

$$(A(u_N(t)), \nu) \le a\|u_N(t)\|\,\|\nu\|.$$

then using (2.15), and by duality, we deduce that for all $N \in \mathbb{N}$ and for all $t \in [0, T]$,

$$\|A(u_N(t))\| \le a||u_N(t)|| \tag{2.18}$$

accordingly $A(u_N(t)) \in L^2(0, T, H^{'})$ and $A(u_N(t))$ is bounded. Finally, we deduce from (2.16), (2.17), (2.18) and assumptions of the existence theorem that

$$u_N^{'} \in L^2(0, T, H^{'}) \text{ and } u_N^{'} \text{ is bounded in } H' \tag{2.19}$$

due to (2.15),(2.19) and by using compactness theorem (see [18]) we deduce that we can extract a subsequence $(u_m)_{m\in\mathbb{N}}$ such that

a) $u_m \rightharpoonup u$ in $L^2(0, T, H)$.
b) $u_m \to u$ weakly-* in $L^{+\infty}(0, T, V)$.
c) $u_m \to u$ in $L^2(0, T, V)$ and a.e in $\Omega \times [0, T]$

Furthermore for all $m \in \mathbb{N}$, u_m satisfies (2.11) and $A(u_m) \rightharpoonup \chi$ in $L^2(0, T, H^{'})$.
Let ψ be a continuously differentiable function on $[0, T]$ such that $\psi(T) = 0$.

By multiplying (2.11) by ψ and integrating by parts it follows

$$\begin{aligned} -\int_0^T (u_m(t), \psi'(t) w_j) dt + \int_0^T (A(u_m(t)), w_j \psi(t)) dt \\ + \alpha \int_0^T (u_m, w_j \psi(t)) dt \\ = (u_m(0), w_j)\psi(0) = (u_{0m}, w_j)\psi(0), \end{aligned} \tag{2.20}$$

passing to the limit, we obtain

$$\begin{aligned} -\int_0^T (u(t), \psi'(t) w_j) dt + \int_0^T (\chi(t), w_j \psi(t)) dt \\ + \alpha \int_0^T (u, w_j \psi(t)) dt \\ = (u_0, w_j)\psi(0), \quad \forall w_j \end{aligned} \tag{2.21}$$

and then, by density, (2.21) holds for all $\nu \in H$. So (2.21) is especially true for all $\nu \in H$ and $\psi \in D(0,T)$. And we deduce that

$$(u'(t), \nu) + (\chi(t), \nu) + \alpha(u(t), \nu) = 0, \quad \forall \nu \in H, \tag{2.22}$$

within the meaning of distributions.
Otherwise, by multiplying (2.22) by ψ continuously differentiable ($\psi(T) = 0$) and integrating by parts, we obtain

$$\begin{aligned} -\int_0^T (u(t), \psi'(t)\nu) dt + \int_0^T (\chi(t), \psi(t)\nu) dt \\ + \alpha \int_0^T (u, \psi(t)\nu) dt = (u(0), \nu)\psi(0). \end{aligned} \tag{2.23}$$

Comparing (2.23) and (2.21) was written for ν was:

$$(u(0) - u_0, \nu)\psi(0) = 0,$$

ψ can be chosen such that $\psi(0) = 1$, thus

$$(u(0) - u_0, \nu)\nu = 0, \qquad \forall v \in H$$

We will prove, in what follows, that

$$(\chi(t), \nu) = (A(u(t)), \nu) \qquad \forall \nu \in H \qquad \forall t \in [0,T].$$

For $m \in \mathbb{N}$ and $s \in [0,T]$, we put

$$X_m = \int_0^s (A(u_m(t)) - A(\nu), u_m(t) - \nu) dt \qquad \forall \nu \in H.$$

Since A is monotone, we deduce that

$$X_m \geq 0, \qquad \forall m \in \mathbb{R}, \qquad \forall s \in [0,T].$$

In the other hand, we have

$$X_m = \int_0^s (A(u_m(t)), u_m(t)) dt - \int_0^s (A(u_m(t)), \nu) dt - \int_0^s (A(\nu), u_m(t) - \nu) dt,$$

then

$$\begin{aligned} X_m = \frac{1}{2}|u_{0m}|^2 - \frac{1}{2}|u_m(s)|^2 - \alpha \int_0^s (u_m(t), u_m(t))dt \\ - \int_0^s (A(u_m(t)), \nu)dt - \int_0^s (A(\nu), u_m(t) - \nu)dt. \end{aligned} \tag{2.24}$$

Since $u_m(t) \to u(t)$ in $L^2(\Omega)$ for all $t \in [0,T]$ then $\limsup |u_m(t)| \geq |u(t)|$. It follows that

$$\begin{aligned} X_m \leq \limsup X_m \leq \tfrac{1}{2}|u_0|^2 - \tfrac{1}{2}|u(s)|^2 - \alpha \textstyle\int_0^s |u(t)|^2 dt \\ - \textstyle\int_0^s (\chi(t), \nu)dt - \int_0^s (A(\nu), u(t) - \nu)dt. \end{aligned} \tag{2.25}$$

Furthermore

$$\frac{|u(s)|^2 - |u(0)|^2}{2} + \alpha \int_0^s |u(t)|^2 dt + \int_0^s (\chi(t), u(t))dt = 0,$$

(obtained by taking $v = u(t)$ in (2.22) and by integrating on $[0,s]$) thus

$$\int_0^s (\chi(t) - A(\nu), u(t) - \nu)\, dt \geq 0, \quad \forall \nu \in H, \quad \forall s \in [0,t].$$

On the other hand, let $\lambda > 0$ and $w \in H$, we put $\nu = u(t) - \lambda w$ for $t \in [0,s]$ then $\nu \in H$ and we have

$$\lambda \int_0^s (\chi(t) - A((u(t) - \lambda w)), w)\, dt \geq 0,\ \forall w \in H,\ \forall s \in [0,T],$$

and

$$\int_0^s (\chi(t) - A((u(t) - \lambda w)), w)\, dt \geq 0,\ \forall w \in H,\ \forall s \in [0,T].$$

Using hemicontinuity of A, we deduce, for $\lambda \to 0$, that

$$\int_0^s (\chi(t) - A(u(t)), w)\, dt \geq 0, \qquad \forall w \in H, \qquad \forall s \in [0,T]$$

whence

$$\int_0^s (\chi(t) - A(u(t)), w)\, dt = 0 \qquad \forall w \in H \qquad \forall s \in [0,T]$$

and

$$(\chi(t) - A(u(t)), w)\, dt = 0 \qquad \forall w \in H \qquad \forall s \in [0,T].$$

In conclusion $u(t)$ is a solution of the equation

$$\left(\frac{\partial u}{\partial t}(t), w\right) + (A(u(t)), w) + \alpha(u(t), w) = 0 \quad w \in H, \tag{2.26}$$

such that

$$u(.,0) = u_0(.),$$

and

$$u \in L^{+\infty}(0,T,V) \cap L^2(0,T,H).$$

2. Uniqueness

Let u_1 and u_2 two solutions of the problem (2.4), we have then for all $w \in H$ and all $t \in [0,T]$

$$\left(\frac{\partial u_1}{\partial t}(t) - \frac{\partial u_2}{\partial t}(t), w\right) + (A(u_1(t)) - A(u_2(t)), w) + \alpha\,(u_1(t) - u_2(t), w) = 0, \tag{2.27}$$

taking $u_1 - u_2 = w$ and $v = w(t)$, we can write:

$$\left(\frac{\partial w}{\partial t}(t), w(t)\right) + (A(u_1(t)) - A(u_2(t)), w(t)) + \alpha\,(w(t), w(t)) = 0,$$

then

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 + (A(u_1(t)) - A(u_2(t)), w(t)) + \alpha|w(t)|^2 = 0,$$

knowing, from (2.6) that

$$(A(u_1(t)) - A(u_2(t)), w(t)) \geq b\,\|w\|^2,$$

we deduce that

$$\frac{1}{2}\frac{d}{dt}|w(t)|^2 = -(A(u_1(t)) - A(u_2(t)), w(t)) - \alpha|w(t)|^2 \leq 0,$$

and

$$|w(t)|^2 \leq |w(0)|^2 = 0.$$

Thus $w(t) = u_1(t) - u_2(t) = 0$ and the uniqueness is established.
Besides, taking into account (2.4) and (2.7), and knowing that $u^{'}(t) = -A(u(t)) - \alpha u(t)$ in $H^{'}$, we have

$$(A(u(t)), w) \leq a\,\|u(t)\|\,\|w\|.$$

Then $\|A(u(t))\|_{H'} \leq a\,\|u(t)\|$ for all $t \in [0,T]$. Moreover $A(u(t)) \in L^2(0,T,H^{'})$ and $u^{'} \in L^2(0,T,H^{'})$. □

We state, in the following, our main theorem.

Theorem 2.6. *Let $v_0 \in V$ and μ_2 a decreasing function which satisfies the assumptions of lemma 2.1 then there exists an unique global weak solution v for problem (1.5) such that $v \in L^2(0,T,H) \cap L^{+\infty}(0,T,V)$ and $v^{'} = \frac{dv}{dt} \in L^2(0,T,H^{'})$. Furthermore*

$$|v(t)| \leq |v_0|, \quad \forall t \in [0,T].$$

Proof. Let $\mu(s) = \mu_2(e^{\alpha t}s)$ and u the weak solution of problem (2.4) corresponding to this choise of μ. Then

$$v(x,t) = e^{\alpha t}u(x,t)$$

is a weak solution of (1.5). In fact, since u is a weak solution of problem (2.4) then

$$\left(\frac{\partial u}{\partial t}, w\right) + (A(u), w) + \alpha(u, w) = 0.$$

Multiplying by $e^{\alpha t}$ and making the sum of the first and third term we have

$$\left(e^{\alpha t}\frac{\partial u}{\partial t} + \alpha e^{\alpha t}u, w\right) + (e^{\alpha t}A(u), w) = 0.$$

Whence

$$\left(e^{\alpha t}\frac{\partial u}{\partial t}+\alpha v,w\right)+\left(e^{\alpha t}A(u),w\right)=0,$$

and

$$\left(\frac{\partial v}{\partial t},w\right)+\int_\Omega \mu\left(\left|e^{-\alpha t}\nabla v\right|\right)e^{\alpha t}\nabla\left(e^{-\alpha t}v\right)\nabla w dx=0.$$

Thus

$$\left(\frac{\partial v}{\partial t},w\right)+\int_\Omega \mu\left(e^{-\alpha t}|\nabla v|\right)\nabla v\nabla w dx=0,$$

or also

$$\left(\frac{\partial v}{\partial t},w\right)+\int_\Omega \mu_2\left(|\nabla v|\right)\nabla v\nabla w dx=0.$$

We deduce then that v is a weak solution of problem (1.5).
The uniqueness of the weak solution of the problem (2.4) implies the uniqueness of the weak solution of the problem (1.5).
By theorem 1 we have $u\in L^2(0,T,H)\cap L^{+\infty}(0,T,V)$ then $v\in L^2(0,T,H)\cap L^{+\infty}(0,T,V)$. On the other side we have $\frac{dv}{dt}=\alpha e^{\alpha t}u+e^{\alpha t}\frac{du}{dt}$ and $u^{'}\in L^2(0,T,H^{'})$ then $v^{'}\in L^2(0,T,H^{'})$.
Otherwise taking $w=v$ in

$$\left(\frac{\partial v}{\partial t},w\right)+\int_\Omega \mu_2\left(|\nabla v|\right)\nabla v\nabla w dx=0$$

we obtain

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2+\int_\Omega \mu_2\left(|\nabla v|\right)\nabla v\nabla v dx=0.$$

Then

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2=-\int_\Omega \mu_2\left(|\nabla v|\right)|\nabla v|^2 dx.$$

Now using the non negativity of μ_2, we deduce

$$\frac{1}{2}\frac{d}{dt}|v(t)|^2\leq 0$$

and

$$|v(t)|\leq|v_0|,\quad \forall t\in[0,T].$$

□

Theorem 2.7. *Let μ_2 a decreasing function which satisfies the assumptions of lemma 2.1 and v (respectively w) the weak solution of (1.5) corresponding to the initial condition v_0 (respectively the weak solution of (1.5) corresponding to the initial condition w_0) then,*

$$|v(t)-w(t)|\leq|v_0-w_0|,\quad \forall t\in[0,T].$$

Proof. Since v is the weak solution of (1.5) corresponding to the initial condition v_0 and w is the weak solution of (1.5) corresponding to the initial condition w_0 then, for all $\varphi\in H$ and all $t\in[0,T]$, we have

$$\left(\frac{\partial v}{\partial t}(t)-\frac{\partial w}{\partial t}(t),\varphi\right)+\left(A(v(t))-A(w(t)),\varphi\right)=0, \tag{2.28}$$

where A is the operator defined in the lemma 2.1 corresponding to $\delta = \mu_2$. Taking $v - w = \psi$ and $\varphi = \psi(t)$, we can write:

$$\left(\frac{\partial \psi}{\partial t}(t), \psi(t)\right) + (A(v(t)) - A(w(t)), \psi(t)) = 0,$$

then

$$\frac{1}{2}\frac{d}{dt}|\psi(t)|^2 + (A(v(t)) - A(w(t)), \psi(t)) = 0,$$

knowing, from (2.2) that

$$(A(v(t)) - A(w(t)), \psi(t)) \geq \inf_{s \in \mathbb{R}} \mu_2(s) \|\psi\|^2,$$

we deduce that

$$\frac{1}{2}\frac{d}{dt}|\psi(t)|^2 = -(A(v(t)) - A(w(t)), \psi(t)) \leq 0,$$

and

$$|\psi(t)|^2 \leq |\psi(0)|^2.$$

Thus

$$|v(t) - w(t)| \leq |v_0 - w_0|, \quad \forall t \in [0, T].$$

□

3. Error Estimation

Let $\mathcal{T}_{h>0}$ be a shape-regular uniform triangulation of Ω. For all $h > 0$, we denote by V_h the finite dimensional approached sub-space of V by a Lagrange first order finite element method. Q_h designates the L^2 interpolation operator from H into $V_h \subset V$ then, thanks to the stability of the $L^2(\Omega)$ projection on $H^1(\Omega)$ ([6], [5]), there exists a constant c independant of h such that

$$|Q_h v| \leq c|v|, \quad \forall v \in V. \tag{3.1}$$

For $s \geq 1$, we introduce the semi norm defined in H^s by

$$|v|_{s,\Omega} = \left(\sum_{|\xi|=s} |\partial^\xi v|^2\right)^{1/2}.$$

Theorem 3.1 (global interpolation error)**.** *Let $s > 1$, for all $h > 0$ and for any function v in H^s*

$$|v - Q_h v| + \sum_{m=1}^{2} h^m \left(\sum_{K \in \mathcal{T}_h} |v - Q_h v|^2_{m,K}\right)^{1/2} \leq Ch^2 |v|_{2,\Omega}.$$

If h is sufficiently small and V_h is H^1-conforme, we have

$$\forall h,\ \forall v \in H^s(\Omega),\ \ \|v - Q_h v\| \leq Ch|v|_{2,\Omega}.$$

.

Proof. see ([12]). □

Theorem 3.2. *Let $h > 0$ and μ_2 a decreasing function verifying hypotheses of lemma 2.1. Let v be the weak solution of (1.5) and v_h be the solution of the approximated weak formulation of (1.5) defined on V_h. If $v \in \mathcal{C}^1(0, T, H^2(\Omega))$, then we have the following error estimation*

$$|v(t) - v_h(t)| \leq \kappa C h \left(h|v|_{2,\Omega} + |v|_{2,\Omega} + h|\frac{\partial v}{\partial t}|_{2,\Omega} \right) + e^{t/2}|e_h(0)|,$$

where C is the interpolation constant, $a_2 = \sup_{s\in\mathbb{R}} \mu_2(s)$, $\kappa = max(\frac{\sqrt{e^T-1}}{\sqrt{2b_2}} a_2(1 + max(1, c)), \sqrt{e^T - 1}, 1)$ and $\theta \in]0, 1[$

Proof. We have

$$\left(\frac{\partial v}{\partial t}, w_h\right) + (\mu_2(|\nabla v|)\nabla v, \nabla w_h) = 0, \tag{3.2}$$

and

$$\left(\frac{\partial v_h}{\partial t}, w_h\right) + (\mu_2(|\nabla v_h|)\nabla v_h, \nabla w_h) = 0, \tag{3.3}$$

then

$$\left(\frac{\partial v}{\partial t} - \frac{\partial v_h}{\partial t}, w_h\right) + (A(v) - A(v_h), w_h) = 0.$$

Adding and subtracting $\frac{\partial Q_h v}{\partial t}$ to the first term and $A(Q_h v)$ to the second term we obtain

$$\left(\frac{\partial v}{\partial t} - \frac{\partial Q_h v}{\partial t} + \frac{\partial Q_h v}{\partial t} - \frac{\partial v_h}{\partial t}, w_h\right) + (A(v) - A(Q_h v) + A(Q_h v) - A(v_h), w_h) = 0,$$

but we have

$$\left(\frac{\partial Q_h v}{\partial t} - \frac{\partial v_h}{\partial t}, w_h\right) + (A(Q_h v) - A(v_h), w_h)$$
$$= -(\frac{\partial v}{\partial t} - \frac{\partial Q_h v}{\partial t}, w_h) - (A(v) - A(Q_h v), w_h).$$

Then by setting $\eta = v - Q_h v$, $e_h = Q_h v - v_h$ and taking $w_h = e_h$, the previous equality becomes:

$$\frac{1}{2}\frac{d}{dt}|e_h|^2 + b_2|\nabla e_h|^2 \leq |\frac{\partial}{\partial t}\eta||e_h| + (A(v) - A(Q_h v), e_h), \tag{3.4}$$

where $b_2 = \inf_{s\in\mathbb{R}} \mu_2(s)$. On the other hand

$$\begin{aligned}
&(A(v) - A(Q_h v), e_h) = (A(v), e_h) - (A(Q_h v), e_h) \\
&= \int_\Omega [\mu(|\nabla v|)\nabla v - \mu(|\nabla Q_h v|)\nabla Q_h v] \nabla e_h dx \\
&= \int_\Omega [\mu(|\nabla v|)\nabla v - \mu(|\nabla v|)\nabla Q_h v + \mu(|\nabla v|)\nabla Q_h v - \mu(|\nabla Q_h v|)\nabla Q_h v] \nabla e_h dx \\
&= \int_\Omega \mu(|\nabla v|)[\nabla v - \nabla Q_h v]\nabla e_h dx + \int_\Omega [\mu(|\nabla v|) - \mu(|\nabla Q_h v|)]\nabla Q_h v \nabla e_h dx
\end{aligned}$$

Applying the mean value theorem, there exists $\theta \in]0, 1[$ such that

$$(A(v) - A(Q_h v), e_h) \leq a_2|\nabla v - \nabla Q_h v||\nabla e_h|$$

$$+\int_{\Omega}\lfloor\mu'(\theta|\nabla v|+(1-\theta)|\nabla Q_h v|)\rfloor\lfloor\nabla Q_h v\nabla e_h\rfloor\lfloor|\nabla v|-|\nabla Q_h v|\rfloor\,dx$$

where $a_2=\sup_{s\in\mathbb{R}}\mu_2(s)$ and $\lfloor.\rfloor$ denotes the absolute value.
Let $\beta=\theta|\nabla v|+(1-\theta)|\nabla Q_h v|$ and $Y=\int_\Omega\lfloor\mu'(\beta)\rfloor\lfloor\nabla Q_h v\nabla e_h\rfloor\lfloor|\nabla v|-|\nabla Q_h v|\rfloor\,dx$.
We have

$$Y=\int_\Omega\beta\lfloor\mu'(\beta)\rfloor\,\frac{\lfloor\nabla Q_h v\nabla e_h\rfloor|\nabla v-\nabla Q_h v|}{\beta}dx\le a_2\left[\int_\Omega\frac{|\nabla v-\nabla Q_h v|}{\beta}\lfloor\nabla Q_h v\nabla e_h\rfloor\,dx\right]$$

We know that

$$|\nabla Q_h v|\le c|\nabla v|$$

Case 1: if $c<1$ then $|\nabla Q_h v|\le|\nabla v|$ so

$$\theta|\nabla v|+(1-\theta)|\nabla Q_h v|\ge\theta|\nabla Q_h v|+(1-\theta)|\nabla Q_h v|=|\nabla Q_h v|$$

Thus

$$\begin{aligned}Y&\le a_2\int_\Omega\frac{\lfloor\nabla Q_h v\nabla e_h\rfloor}{|\nabla Q_h v|}dx\\&\le\frac{a_2}{|\nabla Q_h v|}\int_\Omega\lfloor\nabla Q_h v\nabla e_h\rfloor\,dx\\&\le\frac{a_2}{|\nabla Q_h v|}|\nabla Q_h v||\nabla e_h|\\&\le a_2|\nabla e_h|\end{aligned}$$

Case 2: if $c\ge1$ then we distinguish two cases

- if $|\nabla v|<|\nabla Q_h v|<c|\nabla v|$
  we have

$$\theta|\nabla v|+(1-\theta)|\nabla Q_h v|\ge|\nabla v|$$

  then

$$\begin{aligned}Y&\le a_2\int_\Omega\frac{\lfloor\nabla Q_h v\nabla e_h\rfloor}{|\nabla v|}dx\\&\le\frac{a_2}{|\nabla v|}|\nabla Q_h v||\nabla e_h|\\&\le a_2c|\nabla e_h|\end{aligned}$$

- if $|\nabla Q_h v|<|\nabla v|<c|\nabla v|$ we have

$$\theta|\nabla v|+(1-\theta)|\nabla Q_h v|\ge\theta|\nabla Q_v|+(1-\theta)|\nabla Q_h v|\ge|\nabla Q_h v|$$

  then

$$Y\le a_2\int_\Omega\frac{\lfloor\nabla Q_h v\nabla e_h\rfloor}{|\nabla Q_h v|}dx\le\frac{a_2}{|\nabla Q_h v|}|\nabla Q_h v||\nabla e_h|\le a_2|\nabla e_h|$$

Thus

$$\begin{aligned}(A(v)-A(Q_h v),e_h)&\le a_2|\nabla v-\nabla Q_h v||\nabla e_h|+\sup(1,c)a_2|\nabla v-\nabla Q_h v||\nabla e_h|\\&\le a_2(1+\sup(1,c))|\nabla v-\nabla Q_h v||\nabla e_h|\\&\le a_2\left\|\eta\right\|(1+\sup(1,c))|\nabla e_h|\end{aligned}$$

and the inequality (3.4) becomes

$$\frac{1}{2}\frac{d}{dt}|e_h|^2 + b_2|\nabla e_h|^2 \le \frac{1}{2}|\frac{\partial}{\partial t}\eta|^2 + \frac{1}{2}|e_h|^2 + \frac{1}{4b_2}\left(a_2(1+\sup(1,c))\right)^2 ||\eta||^2 + b_2\,|\nabla e_h|^2$$

Then

$$\frac{d}{dt}|e_h|^2 \le |\frac{\partial}{\partial t}\eta|^2 + \frac{1}{2b_2}\left(a_2(1+\sup(1,c))\right)^2 ||\eta||^2 + |e_h|^2.$$

By applying Gronwall's lemma we obtain

$$|e_h|^2 \le e^t|e_h(0)|^2 + \int_0^t e^{(t-s)} f(s)ds, \;\; \forall t \in [0,T]$$

with $f(s) = |\frac{\partial}{\partial t}\eta(s)|^2 + \frac{1}{2b_2}\left(a_2(1+\sup(1,c))\right)^2 ||\eta(s)||^2$.
Then

$$|e_h|^2 \le \sup_{0\le\tau\le t} |\frac{\partial \eta(\tau)}{\partial t}|^2(e^T-1) + e^t|e_h(0)|^2 + \frac{1}{2b_2}\left(a_2(1+\sup(1,c))\right)^2 \sup_{0\le\tau\le t} ||\eta(\tau)||^2(e^T-1)$$

Hence

$$|e_h| \le \sup_{0\le\tau\le t}\left(|\frac{\partial \eta(\tau)}{\partial t}|\right)\sqrt{e^T-1} + e^{t/2}|e_h(0)| + \frac{1}{\sqrt{2b_2}}\left(a_2(1+\sup(1,c))\right) \sup_{0\le\tau\le t} ||\eta(\tau)||\sqrt{e^T-1}$$

We put $\epsilon(t) = v(t) - v_h(t)$ then, from the triangle inequality, we have $|v - v_h| \le |e_h| + |\eta|$ and we deduce

$$|\epsilon(t)| \le \sup_{0\le\tau\le t}\left(|\frac{\partial \eta(\tau)}{\partial t}|\right)\sqrt{e^T-1} + e^{t/2}|e_h(0)| + \sup_{0\le\tau\le t}|\eta(\tau)| + \frac{\sqrt{e^T-1}}{\sqrt{2b_2}}\left(a_2(1+\sup(1,c))\right) \sup_{0\le\tau\le t} ||\eta(\tau)||,$$

Therefore

$$|\epsilon(t)| \le \kappa\left[\sup_{0\le\tau\le t}\left(|\eta(\tau)| + |\frac{\partial \eta(\tau)}{\partial t}| + ||\eta(\tau)||\right)\right] + e^{t/2}|e_h(0)|.$$

With $\kappa = max(\frac{\sqrt{e^T-1}}{\sqrt{2b_2}} a_2(1+max(1,c)), \sqrt{e^T-1}, 1)$.
Using interpolation theorem (see [12]) we deduce

$$|\epsilon(t)| \le \kappa Ch\left(h|v|_{2,\Omega} + |v|_{2,\Omega} + h|\tfrac{\partial v}{\partial t}|_{2,\Omega}\right) + e^{t/2}|e_h(0)|.$$

□

4. Numerical approximation and simulations

4.1. **Discretization.** We consider the Galerkin finite element method for the discretization of (1.5).
Let $V_h = \left\{ w \in C^0(\Omega), w|_\Sigma \in P^1 \right\}$ denotes the approximation space where Σ is a partition of Ω. The Galerkin finite element formulation consists in finding a function $v_h \in V_h$ such that:

$$\int_\Omega \frac{\partial v_h(t)}{\partial t} w dx + \int_\Omega \mu\left(|\nabla v_h|\right) \nabla v_h . \nabla w dx = 0, \tag{4.1}$$

for all $w \in V_h$ and $t \in [0, T]$.
Let $0 = t_0 < t_1 < t_2 < ... < t_{n+1} = T$ be a subdivision of $[0, T]$ with uniform time step $\Delta t = t_n - t_{n-1}$ for some $n > 0$.
The backward Euler scheme is considered for (4.1) in the time discritization and we formulate the nonlinear cofficient $\mu\left(|\nabla v_h|\right)$ by using the previous scale step value v_h^n. Thus the discrete equation is

$$\frac{1}{\Delta t} \int_\Omega (v_h^{n+1} - v_h^n) w dx + \int_\Omega \mu\left(|\nabla v_h^n|\right) \nabla v_h^{n+1} \nabla w dx = 0, \tag{4.2}$$

for all $w \in V_h$ and $n > 0$. The stepsize Δt should be chosen less then 0.25 in orther to result in a stable solution scheme [3].
Assume a basis of the finite-dimensional space V_h is $(\varphi_1, ..., \varphi_p)$, $p > 0$. Taking $w = \varphi_j(x)$ for $j = 1, 2, .., p$ and using the representation $v_h^{n+1}(x) = \sum_{i=0}^{p} v_i^{n+1} \varphi_i(x)$ where v_i^{n+1} are unknown, the system (4.2) can be trasformed into the discrete linear equation expressed as

$$(A + \Delta t B) U^{n+1} = A U^n$$

Where

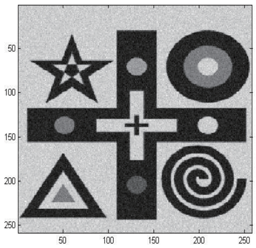
$$U^n = \begin{pmatrix} v_1^n \\ \vdots \\ v_p^n \end{pmatrix}, \quad A_{ij} = \int_\Omega \varphi_i \varphi_j dx, \quad \text{and } B_{ij} = \int_\Omega \mu\left(|\nabla v^n|\right) \nabla \varphi_i . \nabla \varphi_j dx.$$

A is so called mass matrix and B is the stifness matrix. Thus the discrete solution can be found efficiently by preconditioned conjugate gradient methods.

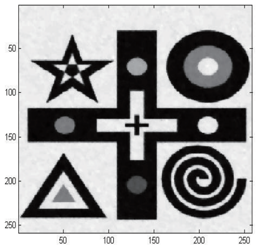
4.2. **Numerical simulations and interpretation.** In this section we present the results of numerical experiments to show the perfermance of our model and compare it with a nonlinear Perona-Malik diffusion filter with Galerkin discretization. All testing problems were performed on a PC with Intel(R) Core(TM) 2 Duo Processor=2.00GHz.
The discret scaling step is selected to be $\Delta t = 10^{-3}$ for boths models. We set the constant $\alpha = 10^{-6}$ and the nonlinear diffusion coefficient $\mu(s) = 1/\sqrt{1 + s^2} + \alpha$.
Table 1 shows the sensitivity analysis with respect to the values of the parameter α on an image of size 70×70 affected by different type of noise.
In all these calculations a continuous piecewise affine finite element method has been applied for space discretization. For this each side of the picture has been subdivided into equal intervals and the resulting rectangular net has been triangulated by the first diagonal. We use the popular measurement Signal-to-noise ratio (SNR)

for the gray scale image and the peak Signal-to-noise ratio (PSNR) for color image.
In figure 1, top left image shows a noisy image affected by gaussian noise (15 %). Middle top image presents the restored image obtained by the proposed non linear diffusion model. It well preserves the details and edges while effectively removing noise. The restoration result indicates that the proposed model can improve the visual of images. Top right image shows the result obtained by using the Perona Malik diffusion model. We note that we obtaine almost the same SNR but with less computation time than Perona-Malik model.
In figure 2, left image depicts a noisy image affected by gaussian noise (30 %). Middle image presents the restored image obtained from the proposed diffusion model. The diffusion result improves the mottled background, but also retains the edges. Right image shows the result obtained from the Perona Malik diffusion model.
In figure 3, top left image is a noisy image affected by salt-pepper noise. middle top image presents the result obtained by the proposed non linear diffusion model, The proposed method can eliminate the speckled background and preserve the edges. top right image shows the result from the Perona Malik diffusion model.
In figure 4, we smooth an initial 128×128 pixel image affected by speckle noise. (a) noisy image, (b) restored image obtained by using the proposed non linear model, (c) restored image obtained by Perona Malik
Figure 5 presents the results obtained on image (220×220) affected by poisson noise from the proposed diffusion and from the Perona Malik diffusion. We can notice an increase in SNR (0.06dB).
Figure 6 shows the results on a color image (171×171) affected by Gaussian noise. Again, we note the reduction of noise effects.
In figure 7, right image is a noisy color image (Lena image) affected by poisson noise. Middle image presents the result from the proposed diffusion model. Right image shows the result from the Perona Malik diffusion model. We can notice an increase in PSNR (0.02 dB).
In figure 8, (a) initial signal. (b) noisy signal affected by gaussian noise. (c) presents the restored signal obtained by the proposed non linear diffusion model. (d) shows the result obtained by using the Perona Malik diffusion model. (e) presents both (c) and (d) figures.
In figures 9 and 10, (a) initial image (degraded). (b) presents the result from the proposed diffusion model, we can observe that it well preserves the details and edges while effectively removing noise and improves the quality of the degraded image. (c) shows the magnitude ($|\nabla v|$) of the initial image. Note that the resulting image is blurred and the edges is not clear. (d) magnitude ($|\nabla v|$) of the resored image (b), contours are now more clearer.

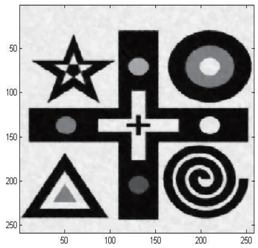
example 1: Gaussian noise 15%



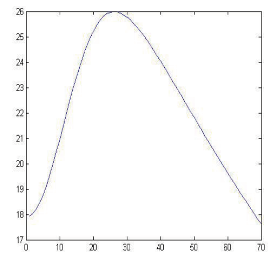
noisy image (258×258 pixels): Gaussian additive noise 15%.



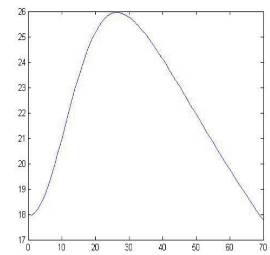
restored image by own model, SNR=26,03 CPU=2621 s.



restored image by P-M model, SNR=26,04 CPU=2702 s.



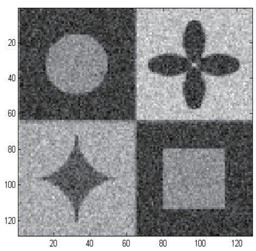
(a) SNR (own model)



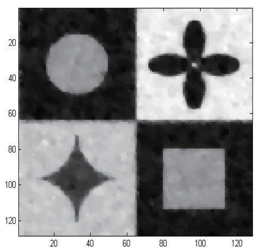
(b) SNR (Perona-Malik)

FIGURE 1. Restoration results after 27 scales. (a) behavior of SNR with our model; (b) behavior of SNR with Perona-Malik model.

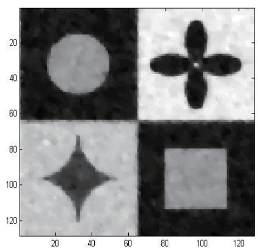
example 2: Gaussian noise 30%



noisy image (128×128 pixels): Gaussian noise 30%.



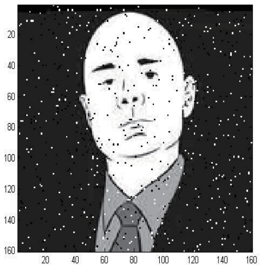
restored image by own model, SNR=18,30 CPU=557 s.



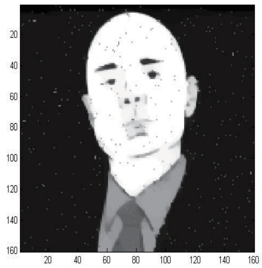
restored image by P-M model, SNR=18,31 CPU=514 s.

FIGURE 2. Restoration results after 52 scales.

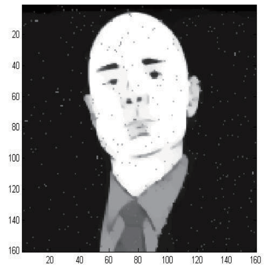
example 3: Salt and Pepper noise



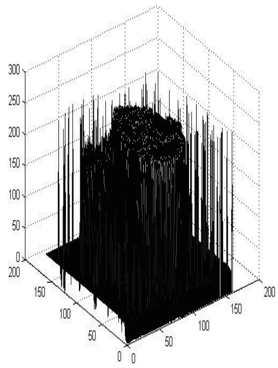
noisy image (160×160 pixels): Salt-Pepper noise.



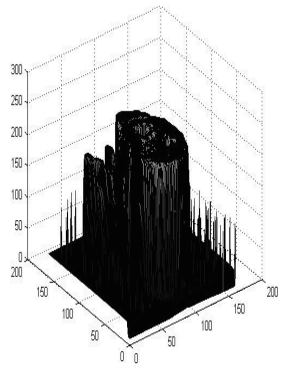
restored image using own model, SNR=18,07 CPU=1084 s.



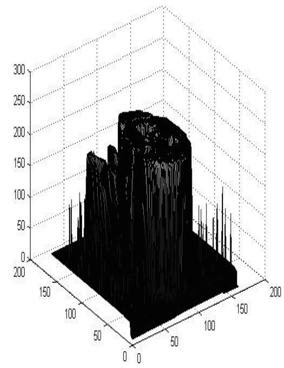
restored image using Perona_Malik model, SNR=18,11 CPU=1078 s.



Corresponding surface to the noisy image



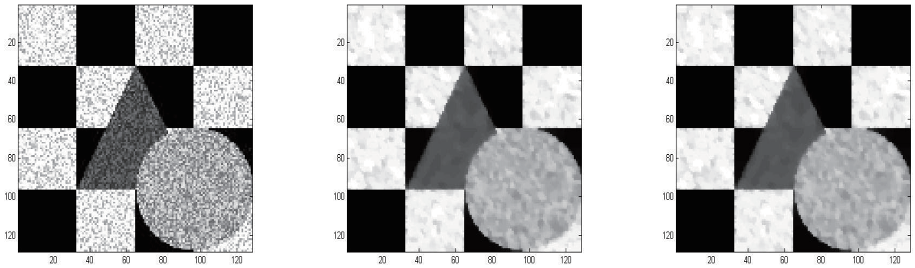
Corresponding surface to the restored image obtained by using own model



Corresponding surface to the restored image obtained by using Perona-Malik model

FIGURE 3. In this experiment Salt and Pepper noise is added to image. In the middle we show the smoothed image obtained after 49 scales with model (2.4). Right image is the smoothed image obtained with Perona-malik equation after 49 scales.

example 4: Speckle noise



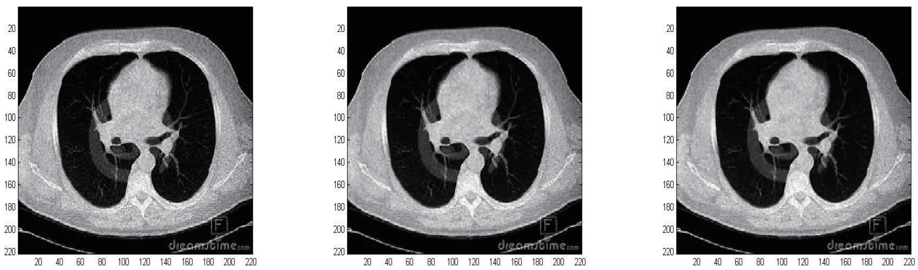
(a) noisy image (128×128 pixels): Speckle noise.

(b) restored image using own model, SNR=16,27 CPU=599 s.

(c) restored image using Perona_malik model, SNR=16,41 CPU=637 s.

FIGURE 4. results of the noisy image (128×128 pixels) affected by Speckle noise using model (2.4) and Perona-Malik equation after 47 scales.

example 5: Poisson noise



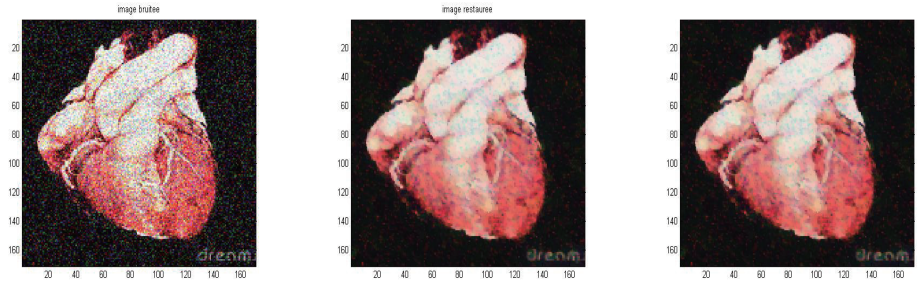
noisy image (220×220 pixels): Poisson noise.

restored image using own model, SNR=20,66 and CPU=1427 s.

restored image using Perona_Malik model, SNR=20,63 and CPU=1557 s.

FIGURE 5. numerical experiment for filtering the noisy image (220×220 pixels) corrupted by Poisson noise using model (2.4) and Perona-malik equation after 11 scales.

example 6: (color image) Gaussian noise



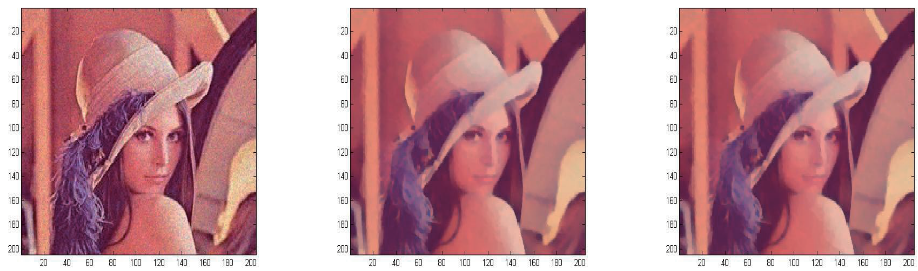
noisy image (171×171 pixels): Gaussian noise(mean 0 and variance 0.03).

restored image using own model, PSNR=21,04 CPU=1475 s.

restored image using Perona_Malik model, PSNR=21,04 CPU=1491 s.

FIGURE 6. The denoising effect obtained by (2.4) and Perona-malik equation after 23 scales. The initial condition is a image (171×171 pixels) affected by a Gaussian additive white noise.

example 7: (color image) Poisson noise



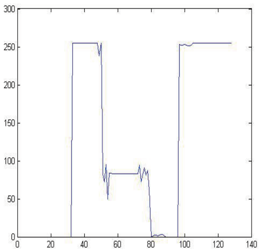
noisy image (204×204 pixels): Poisson noise.

restored image using own model, PSNR=25,91 CPU=5020 s.

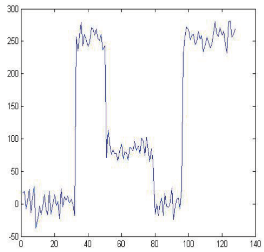
restored image using Perona_Malik model, PSNR=25,89 CPU=5123 s.

FIGURE 7. Left to right: initial image (171×171 pixels) affected by a Gaussian additive white noise, image after 25 scales with (2.4), result with Perona-malik equation.

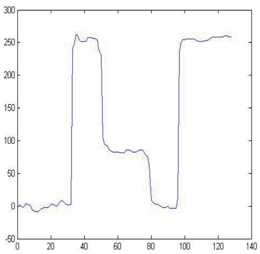
example 8: Signal 1D



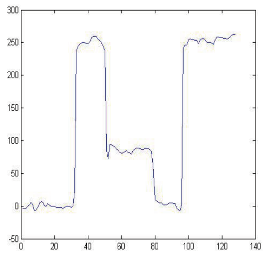
(a) original signal



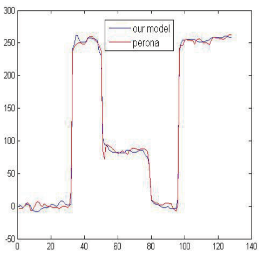
(b) noisy signal.



(c) restored signal (our model)



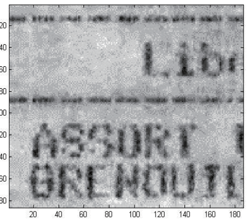
(d) restored signal (Perona).



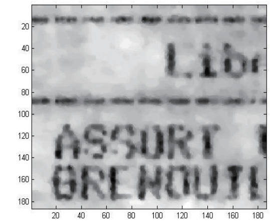
(e) comparison

Figure 8. Restauration results of the noisy signal 1D. (a) initial image; (b) noisy image affected by gaussian noise; (c) restored image obtained by using model (1.5); (d) restored image obtained by using model (1.1); (e) comparison of (c) and (d).

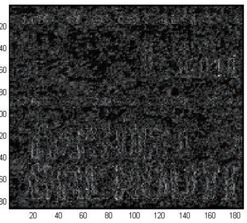
example 9: Restauration and edge detection



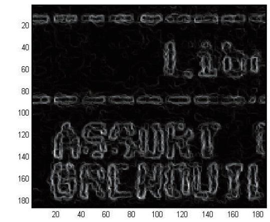
(a) initial image



(b) restored image.



(c) edge of initial image



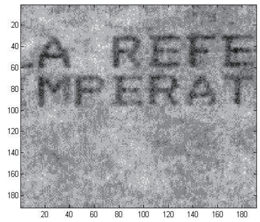
(d) edge of restored image.

FIGURE 9. Restauration and segmentation results of the noisy image using model (2.4). (a) initial image; (b) restored image obtained by using model (1.5); (c) magnitude $|\nabla u|$ of the initial image; (d) magnitude $|\nabla u|$ of the restored image

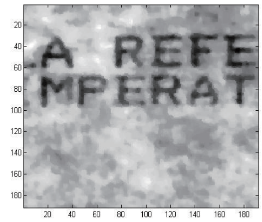
Gaussian noise 15%		Salt & Pepper noise		Speckle noise		Poisson noise	
α	SNR	α	SNR	α	SNR	α	SNR
10^{-6}	20.4317	10^{-6}	17.5082	10^{-6}	**19.5440**	10^{-6}	**23.8519**
10^{-5}	**20.8667**	10^{-5}	**18.1482**	10^{-5}	19.1032	10^{-5}	23.7871
10^{-4}	20.5691	10^{-4}	17.8226	10^{-4}	16.9765	10^{-4}	22.9705
10^{-3}	16.7612	10^{-3}	12.8298	10^{-3}	12.3240	10^{-3}	17.3813
10^{-2}	12.0910	10^{-2}	09.7034	10^{-2}	09.3051	10^{-2}	11.7947
10^{-1}	08.4894	10^{-1}	05.3651	10^{-1}	05.2718	10^{-1}	08.3184
1	04.1993	1	00.9195	1	00.1940	1	04.0967

TABLE 1. Influence of the parameter α on the SNR of the image denoising result by the proposed model.

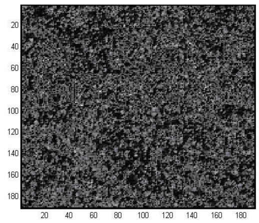
example 10: Restauration and edge detection



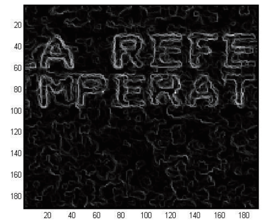
(a) initial image



(b) restored image.



(c) edge of initial image



(d) edge of restored image.

FIGURE 10. (a) initial image; (b) restored image obtained by using model (1.5); (c) magnitude $|\nabla u|$ of the initial image; (d) magnitude $|\nabla u|$ of the restored image

5. Conclusion

We propose, in this work, a modified version of the Perona-Malik model for edge detection and image restoration. The particularity of this model lies in the fact that we consider new diffusion functions other than those proposed by Perona-Malik. With such choice, we overcome the inherent difficulty of the monotony of the differential operator associated with the model. This new version keeps all the advantages of the original model and avoids its drawbacks. Indeed the mathematical analysis of the PDE problem permits to establish an existence and uniqueness of the solution in an Hilbert space. And this study is completed by a numerical analysis for the finite element method conducted by means of an error estimation under suitable hypotheses.
Furthermore, numerical simulations on various images are compared to those obtained using Perona-Malik model. Their interpretation proves the effectiveness of the proposed model. Our future work will be consecrated to the development of a finite element method algorithm with less computation time. On the other hand we want to apply this nonlinear mathematical model for image-based 3D reconstruction.

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Manuscript received January 25 2016
revised July 24 2016

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