



QUANTIFIED ASYMPTOTIC BEHAVIOUR OF BANACH SPACE OPERATORS AND APPLICATIONS TO ITERATIVE PROJECTION METHODS

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ABSTRACT. We present an extension of our earlier work [Ritt operators and convergence in the method of alternating projections, *J. Approx. Theory*, 205:133–148, 2016] by proving a general asymptotic result for orbits of an operator acting on a reflexive Banach space. This result is obtained under a condition involving the growth of the resolvent, and we also discuss conditions involving the location and the geometry of the numerical range of the operator. We then apply the general results to some classes of iterative projection methods in approximation theory, such as the Douglas-Rachford splitting method and, under suitable geometric conditions either on the ambient Banach space or on the projection operators, the method of alternating projections.

1. INTRODUCTION

Many problems in approximation theory can be formulated in terms of operators acting on suitable Banach or Hilbert spaces. Two particularly simple yet powerful methods in approximation theory are the Douglas-Rachford splitting method and the method of alternating projections (for two or more subspaces). These two iterative projection methods play important roles in convex optimisation, differential equations and signal processing; see for instance the references in [6]. In both cases one is led to consider a bounded linear operator T acting on a Hilbert space X , and one is interested in the asymptotic behaviour of the orbits $(T^n x)_{n \geq 0}$ as $n \rightarrow \infty$ for different initial vectors $x \in X$. It is now well known (see for instance the survey [18]) that in situations such as these one expects a dichotomy for the rate of convergence: either the sequence $(T^n x)_{n \geq 0}$ converges exponentially fast for all $x \in X$ or it converges arbitrarily slowly for suitably chosen initial vectors $x \in X$. It was recently shown by the authors in [3] that one can say more in the case of the method of alternating projections, namely that even when the convergence is arbitrarily slow there exists a rich supply of initial vectors $x \in X$ for which the sequence $(T^n x)_{n \geq 0}$ converges to a limit at a rate faster than any polynomial rate. The purpose of this paper is to present a general result in the Banach space situation which extends the approach in [3] so as to be applicable to a wider class of iterative

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schemes, including the Douglas-Rachford splitting method, and to more general ambient Banach spaces.

The paper is organised as follows. In Section 2 we present a general result, Theorem 2.1, describing the rate of convergence of orbits of certain operators acting on a Banach space under an assumption involving the growth of the resolvent. Then in Section 3 we obtain conditions on the location and the geometry of the numerical range of the operator which ensure that the required resolvent growth condition in Theorem 2.1 is satisfied. Finally, in Sections 4 and 5 we show how the general theory can be applied to particular methods in approximation theory, namely the Douglas-Rachford splitting method and the method of alternating projections. For the method of alternating projections we discuss, in particular, products of ortho-projections in uniformly convex and uniformly smooth Banach spaces, while for the Douglas-Rachford splitting method we discuss a variant of the original result for several subspaces of a given reflexive Banach space.

The notation we use is standard. Given a Banach space X , assumed to be complex throughout, we write $\mathcal{B}(X)$ for the algebra of bounded linear operators $T: X \rightarrow X$. The identity operator on X is denoted by I_X , or simply by I if the space X is clear from the context. Given an operator $T \in \mathcal{B}(X)$ we write $\text{Ker } T$ for the kernel of T and $\text{Ran } T$ for the range of T , and we let $\text{Fix } T = \text{Ker}(I - T)$. An operator T is said to be power-bounded if $\sup_{n \geq 1} \|T^n\| < \infty$. Moreover, given $T \in \mathcal{B}(X)$ we let $\sigma(T)$ denote the spectrum of T , $r(T)$ its spectral radius and we let $R(\lambda, T)$ be the resolvent operator $(\lambda I - T)^{-1}$ when $\lambda \in \mathbb{C} \setminus \sigma(T)$. The dual of X is denoted by X^* and we write $\langle x, \phi \rangle = \phi(x)$ for $\phi \in X^*$ and $x \in X$. If X is a Hilbert space and $T \in \mathcal{B}(X)$ we write $W(T) = \{(Tx, x) : x \in X, \|x\| = 1\}$ for the numerical range (field of values) of T . An extension of the numerical range to the Banach space situation will be introduced in Section 3. Furthermore, we let $\mathbb{D} = \{\lambda \in \mathbb{C} : |\lambda| < 1\}$ and $\mathbb{T} = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$, and we use ‘big O’ and ‘little o’ notation in the usual way. Other notation and definitions will be introduced when needed.

2. A GENERAL DICHOTOMY RESULT FOR THE RATE OF CONVERGENCE

Let X be a Banach space and suppose that $T_n \in \mathcal{B}(X)$, $n \geq 0$, are operators such that $\|T_n x\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in X$. We say that the convergence is *arbitrarily slow* if for every sequence $(r_n)_{n \geq 0}$ of non-negative scalars satisfying $r_n \rightarrow 0$ as $n \rightarrow \infty$ there exists $x \in X$ such that $\|T_n x\| \geq r_n$ for all $n \geq 0$. We say that the convergence is *weakly arbitrarily slow* if for every sequence $(r_n)_{n \geq 0}$ of non-negative scalars satisfying $r_n \rightarrow 0$ as $n \rightarrow \infty$ there exist $x \in X$ and $\phi \in X^*$ such that $\text{Re}\langle T_n x, \phi \rangle \geq r_n$ for all $n \geq 0$. We say that the convergence is *superpolynomially fast* for a particular $x \in X$ if $\|T_n x\| = o(n^{-k})$ as $n \rightarrow \infty$ for all $k \geq 1$.

Theorem 2.1. *Let X be a reflexive Banach space and suppose that $T \in \mathcal{B}(X)$ is a power-bounded operator such that $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$ and, for some $\alpha \geq 1$,*

$$(2.1) \quad \|R(e^{i\theta}, T)\| = O(|\theta|^{-\alpha}), \quad \theta \rightarrow 0.$$

Then $X = \text{Fix } T \oplus Z$, where Z denotes the closure of $\text{Ran}(I - T)$, and for all $x \in X$ we have

$$(2.2) \quad \|T^n x - P_T x\| \rightarrow 0, \quad n \rightarrow \infty,$$

where P_T denotes the projection onto $\text{Fix } T$ along Z . Moreover, there is a dichotomy for the rate of convergence. Indeed, if $\text{Ran}(I - T)$ is closed then there exist $C > 0$ and $r \in [0, 1)$ such that

$$(2.3) \quad \|T^n - P_T\| \leq Cr^n, \quad n \geq 0,$$

whereas if $\text{Ran}(I - T)$ is not closed then the convergence in (2.2) is arbitrarily slow and weakly arbitrarily slow. In either case there exists a dense subspace X_0 of X such that for all $x \in X_0$ the convergence in (2.2) is superpolynomially fast.

Proof. Since T is assumed to be power-bounded and X is reflexive it follows from classical ergodic theory that $X = \text{Fix } T \oplus Z$; see for instance [26, Section 2.1]. Let $Y = \text{Ran}(I - T)$. Since T is power-bounded and $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$, it follows from the Katznelson-Tzafriri theorem [25, Theorem 1] that $\|T^n x\| \rightarrow 0$ as $n \rightarrow \infty$ for all $x \in Y$. By a simple density argument the same is true for all $x \in Z$. Since $T^n x = x$ for all $x \in \text{Fix } T$ and $n \geq 0$, we may deduce (2.2).

Note that both of the spaces $\text{Fix } T$ and Z are invariant under T . Thus if we let S denote the restriction of T to Z , then it is easy to see that $\sigma(S) \subseteq \sigma(T) \subseteq \mathbb{D} \cup \{1\}$, and in particular $\sigma(S) \cap \mathbb{T} \subseteq \{1\}$. Note also that $I_Z - S$ maps Z bijectively onto Y . It follows from the Inverse Mapping Theorem that $1 \in \sigma(S)$ if and only if $Y \neq Z$, which is to say if and only if Y is not closed. Thus if Y is closed then $r(S) < 1$ and we may find, for each $r \in (r(S), 1)$ a suitable constant $C > 0$ such that

$$\|T^n - P_T\| \leq \|S^n\| \|I - P_T\| \leq Cr^n, \quad n \geq 0,$$

so (2.3) holds. On the other hand, if Y is not closed then $r(S) = 1$ and it follows from [29, Theorem 1] that the convergence in (2.2) is arbitrarily slow. Furthermore, the space X , being reflexive, does not contain an isomorphic copy of c_0 , so it follows from [30, Theorem 1] that the convergence in (2.2) is weakly arbitrarily slow. Now using [38, Theorem 2.5] or [39, Theorem 2.11] it follows from assumption (2.1) that

$$(2.4) \quad \|S^n x\| = O\left(\frac{(\log n)^{1/\alpha}}{n^{1/\alpha}}\right), \quad n \rightarrow \infty,$$

for all $x \in Y$. If we let $Y_k = \text{Ran}(I - T)^k$, $k \geq 1$, then iterating the estimate in (2.4) shows that for $x \in Y_k$, $k \geq 1$, we have

$$\|S^n x\| = O\left(\frac{(\log n)^{k/\alpha}}{n^{k/\alpha}}\right), \quad n \rightarrow \infty.$$

Let $X_k = \text{Fix } T \oplus Y_k$, $k \geq 1$. Then each X_k is dense in X and, for $k \geq 1$ and $x \in X_k$, we have

$$\|T^n x - P_T x\| = O\left(\frac{(\log n)^{k/\alpha}}{n^{k/\alpha}}\right), \quad n \rightarrow \infty.$$

Now let $X_0 = \bigcap_{k=1}^\infty X_k$. It follows from the Esterle-Mittag-Leffler theorem [22, Theorem 2.1] that X_0 is a dense subspace of X , and it is clear that for all $x \in X_0$ the convergence in (2.2) is superpolynomially fast. \square

Remark 2.2. Given a Banach space X and an operator $T \in \mathcal{B}(X)$ with $r(T) \leq 1$, the resolvent condition (2.1) is equivalent to having $\|R(\lambda, T)\| = O(|\lambda - 1|^{-\alpha})$ as $\lambda \rightarrow 1$ with $|\lambda| > 1$; see [14, Lemma 3.3] and [39, Lemma 3.9]. If these equivalent conditions are satisfied for $\alpha = 1$ then T is said to be a *Ritt operator*. It was shown in [28, 31] that T is a Ritt operator if and only if T is power-bounded and satisfies $\|T^n(I - T)\| = O(n^{-1})$ as $n \rightarrow \infty$. Thus for the case of Ritt operators the proof of Theorem 2.1 can be simplified, and in particular the logarithmic terms are not needed. By [39, Theorem 3.10] the logarithmic factors are also redundant if X is a Hilbert space.

3. THE NUMERICAL RANGE AND GENERALISED STOLZ DOMAINS

Theorem 2.1 gives a detailed description of the asymptotic behaviour of orbits of certain bounded linear operators T in terms of the growth of the resolvent. In general, when estimating the growth of the resolvent one needs precise spectral information about T . One important case in which such information is available is when the location and the geometry of the numerical range of T are known.

We begin by recalling the notion of numerical range for a bounded linear operator T acting on a Banach space X ; see [8] for more information. Let $J: X \rightarrow X^*$ be an isometry with the property that $\langle x, \phi_x \rangle = \|x\|^2$ for all $x \in X$, where $\phi_x = J(x)$; note that the existence of such maps is a straightforward consequence of the Hahn-Banach theorem. We define the *numerical range* of $T \in \mathcal{B}(X)$ as

$$W(T) = \{\langle Tx, \phi_x \rangle : \|x\| = 1\}.$$

Although the numerical range of T depends on the choice of the map J , its closed convex hull $\text{co } W(T)$ does not. Indeed, according to [43, Theorem 6] (see also [8]) we have $\text{co } W(T) = W_0(T)$, where

$$W_0(T) = \{\phi(T) : \phi \in \mathcal{B}(X)^*, \|\phi\| = \phi(I_X) = 1\}.$$

If X is a Hilbert space then $W_0(T)$ coincides with the closure of the usual numerical range $W(T)$ of T ; see [43, page 420].

Our main aim in this section is to obtain a version of Theorem 2.1 under conditions involving geometric assumptions on the numerical range of the operator T . We say that a non-empty closed set $\Omega \subseteq \mathbb{D} \cup \{1\}$ is a *generalised Stolz domain* if there exist constants $c, \varepsilon > 0$ and $\alpha \geq 1$ such that

$$(3.1) \quad 1 - |\lambda| \geq c|\lambda - 1|^\alpha$$

for all $\lambda \in \Omega$ with $|\lambda - 1| \leq \varepsilon$. In particular, any closed subset of a generalised Stolz domain is itself a generalised Stolz domain. If (3.1) holds for a set Ω then we say that Ω is a *generalised Stolz domain with (Stolz) parameter α* . Note that if Ω is a generalised Stolz domain with parameter α , then Ω is also a generalised Stolz domain with parameter β for any $\beta \geq \alpha$. If Ω is the convex hull of the set $\{\lambda \in \mathbb{C} : |\lambda| \leq r\} \cup \{1\}$ for some $r \in (0, 1)$, then Ω is said to be a *Stolz domain*; note, however, that there is some inconsistency in the use of this terminology throughout the literature. Any Stolz domain is a generalised Stolz domain with parameter $\alpha = 1$. Other important examples of generalised Stolz domains are sets of the form $\{\lambda \in \mathbb{C} : |\lambda - r| \leq 1 - r\}$ for some $r \in (0, 1)$. In the language of hyperbolic geometry such sets (or, more precisely, their boundaries) are examples of *horocycles*, and they

are generalised Stolz domains with parameter 2. Another class of regions closely related to generalised Stolz domains are so-called *quasi-Stolz* domains; see [14, 34] for details.

Remark 3.1. If Ω is a Stolz domain with parameter $\alpha \geq 1$, then in the terminology of [40, 41, 42] we have that 1 is a point of contact of type $\alpha - 1$ between Ω and the unit circle \mathbb{T} . More general subsets of the closed unit disc, even ones touching the unit circle in a finite number of points, have been considered in [40, 41, 42] in relation to stability of step-by-step methods for the numerical solution of differential equations. We refer to these three papers for more information and to [36, Ch.II.4] for the similar notion of order of contact between curves.

Proposition 3.2. *Let X be a Banach space and suppose that $T \in \mathcal{B}(X)$ is a power-bounded operator such that $W_0(T)$ is a generalised Stolz domain with parameter $\alpha \geq 1$. Then $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$ and (2.1) holds.*

Proof. Since $\sigma(T) \subseteq W_0(T)$ by [43, Theorem 1] we have that $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. It follows from [43, Lemma 1] that

$$(3.2) \quad \|R(e^{i\theta}, T)\| \leq \frac{1}{\text{dist}(e^{i\theta}, W_0(T))}, \quad 0 < |\theta| \leq \pi.$$

By [40, Lemma 5.1] there exists a constant $c > 0$ such that

$$\text{dist}(e^{i\theta}, W_0(T)) \geq c|\theta|^\alpha, \quad 0 < |\theta| \leq \pi,$$

which together with (3.2) immediately implies (2.1). □

Remark 3.3. If X is a Hilbert space then it follows from the inclusion of the numerical range in the closed unit disc that $\sup_{n \geq 0} \|T^n\| \leq 2$ ([32, Ch.1, §11]), so the condition of power-boundedness holds automatically in this case. Hilbert space contractions with numerical range included in a Stolz domain were called quasi-sectorial in [13]. It was proved in [3] that a Hilbert space operator is a so-called unconditional Ritt operator if and only if it is similar to an operator whose numerical range is contained in a Stolz domain. If $r \in (0, 1)$ and $\|T - rI\| \leq 1 - r$, then $W_0(T) \subseteq \{\lambda \in \mathbb{C} : |\lambda - r| \leq 1 - r\}$ and in particular (2.1) holds for $\alpha = 2$. Note also that $\|T - rI\| \leq 1 - r$ implies that $\|T\| \leq 1$. Hilbert space operators satisfying $\|T - rI\| \leq 1 - r$ for some $r \in (0, 1)$ are characterised in [20, 21].

The next result gives a sufficient condition in the Hilbert space setting for the resolvent growth condition in Theorem 2.1 to be satisfied.

Proposition 3.4. *Let X be a Hilbert space and let $T \in \mathcal{B}(X)$. Suppose that the closure Ω of the numerical range $W(T)$ of T is contained in the closed unit disc and that there exist constants $C > 0$ and $\beta \in (0, 1]$ such that*

$$\sup\{|\lambda^n(1 - \lambda)| : \lambda \in \Omega\} \leq Cn^{-\beta}, \quad n \geq 1.$$

Then T is power-bounded, $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$ and (2.1) holds for $\alpha = 1/\beta$.

Proof. Since Ω is contained in the closed unit disc, the operator T is power-bounded. Suppose that $\lambda \in \Omega \cap \mathbb{T}$. Then

$$|\lambda - 1| = |\lambda^n(\lambda - 1)| \leq Cn^{-\beta}$$

for all $n \geq 1$ and hence $\lambda = 1$. Thus $\Omega \cap \mathbb{T} \subseteq \{1\}$, and since $\sigma(T) \subseteq \Omega$ we deduce that $\sigma(T) \cap \mathbb{T} \subseteq \{1\}$. Now let $K = 1 + \sqrt{2}$. By [15] the set Ω is a K -spectral set for T , in the sense that for every rational function u with poles outside Ω we have

$$\|u(T)\| \leq K \sup\{|u(\lambda)| : \lambda \in \Omega\}.$$

In particular, choosing $u(\lambda) = \lambda^n(\lambda - 1)$ shows that

$$\|T^n(I - T)\| \leq CKn^{-\beta}, \quad n \geq 1.$$

The result now follows from [39, Theorem 3.10]. \square

Remark 3.5. The above argument shows that in the setting of Proposition 3.4 the proof of Theorem 2.1 can be simplified; see also Remark 2.2. Note furthermore that it is possible to use Proposition 3.4 to give an alternative proof of Proposition 3.2 for Hilbert space operators.

4. APPLICATIONS TO THE METHOD OF ALTERNATING PROJECTIONS

In this section we apply the general results of the previous sections to Banach space versions of the method of alternating projections. Recall that, given a Banach space X , a linear operator $P: X \rightarrow X$ is said to be a *projection* if $P^2 = P$ or, equivalently, if $\text{Ker } P = \text{Ran}(I - P)$. It is clear that if $P \in \mathcal{B}(X)$ is a bounded projection then either $P = 0$ or $\|P\| \geq 1$. A bounded projection $P \in \mathcal{B}(X)$ is said to be an *orthoprojection* if $\|P\| \leq 1$. If X is a Hilbert space then a projection $P \in \mathcal{B}(X)$ is an orthoprojection if and only if $\text{Ker } P$ and $\text{Ran } P$ are mutually orthogonal.

Remark 4.1. Note that the projection P_T appearing in Theorem 2.1 satisfies $\|P_T\| \leq \sup_{n \geq 0} \|T^n\|$, as can be seen from (2.2). In particular, P_T is an orthoprojection whenever T is a contraction.

Let X be a Hilbert space, $N \geq 2$, and suppose that M_1, \dots, M_N are closed subspaces of X . For $1 \leq k \leq N$ we let P_k denote the orthogonal projection onto M_k , and we write P_M for the orthogonal projection onto the intersection $M = M_1 \cap \dots \cap M_N$. In many applications one wishes to study sequences in X which are obtained by picking a starting vector $x \in X$ and then projecting x cyclically onto the subspaces M_1, \dots, M_N ; see for instance [16] and [17, Chapter 9]. One is naturally led therefore to consider the operator $T \in \mathcal{B}(X)$ given by $T = P_N \cdots P_1$, and it is a classical result due to Halperin [24] that

$$(4.1) \quad \|T^n x - P_M x\| \rightarrow 0, \quad n \rightarrow \infty,$$

for all $x \in X$; see [1, 3, 37] for a discussion of the rate of convergence in (4.1) and its dependence on the geometric relationship between the subspaces M_1, \dots, M_N .

Our main interest here is in obtaining quantified versions of Halperin's theorem for Banach spaces with special geometric properties. Recall that a Banach space X is said to be *uniformly convex* if for every $\varepsilon \in (0, 2]$ there exists $\delta > 0$ such that for any two vectors $x, y \in X$ with $\|x\| \leq 1$ and $\|y\| \leq 1$ the inequality $\|x + y\|/2 > 1 - \delta$ implies $\|x - y\| < \varepsilon$. Halperin's theorem was generalised to products of orthoprojections in uniformly convex Banach spaces by Bruck and Reich [12]; see also [2] and the references therein. Our aim now is to obtain a quantified result of

this type, and for this we require some further terminology. Recall therefore that, given a Banach space X , the increasing function $\delta_X : [0, 2] \rightarrow [0, \infty)$ given by

$$\delta_X(\varepsilon) = \inf \left\{ 1 - \frac{\|x + y\|}{2} : \|x\| \leq 1, \|y\| \leq 1, \|x - y\| \geq \varepsilon \right\}$$

is called the *modulus of convexity* of the space X . Thus a Banach space X is uniformly convex if and only if $\delta_X(\varepsilon) > 0$ for all $\varepsilon \in (0, 2]$. Any uniformly convex Banach space is reflexive and every Hilbert space is uniformly convex. Indeed, if X is a Hilbert space then $\delta_X(\varepsilon) = \delta(\varepsilon)$ for all $\varepsilon \in [0, 2]$, where

$$\delta(\varepsilon) = 1 - \left(1 - \frac{\varepsilon^2}{4}\right)^{1/2} \sim \frac{\varepsilon^2}{8}, \quad \varepsilon \rightarrow 0.$$

For any Banach space X we have $\delta_X(\varepsilon) \leq \delta(\varepsilon)$ for all $\varepsilon \in [0, 2]$; see [33]. For $q \geq 2$ we say that a Banach space X is *q-uniformly convex* if there exists a constant $c > 0$ such that $\delta_X(\varepsilon) \geq c\varepsilon^q$ for all $\varepsilon \in [0, 2]$. For instance, any L^p -space is q -uniformly convex for $q = \max\{2, p\}$ when $1 < p < \infty$. According to a result of Pisier every uniformly convex Banach space can be renormed to be q -uniformly convex for some suitable $q \geq 2$. We refer for instance to [27, Ch. 1.e] and [35, Ch. 10] for these results and for more information about uniformly convex Banach spaces. The first main result of this section is the following quantified version of Halperin’s theorem for products of projections acting on uniformly convex Banach spaces.

Theorem 4.2. *Let X be a Banach space which is q -uniformly convex for some $q \geq 2$ and suppose that $T = P_N \cdots P_1$ for certain orthoprojections $P_1, \dots, P_N \in \mathcal{B}(X)$. Furthermore, let $M = \text{Ran } P_1 \cap \dots \cap \text{Ran } P_N$. Then $X = M \oplus Z$, where Z denotes the closure of $\text{Ran}(I - T)$. Furthermore, for all $x \in X$ we have*

$$(4.2) \quad \|T^n x - Px\| \rightarrow 0, \quad n \rightarrow \infty,$$

where P denotes the orthoprojection onto M along Z . Moreover, there is a dichotomy for the rate of convergence. Indeed, if $\text{Ran}(I - T)$ is closed then there exist $C > 0$ and $r \in [0, 1)$ such that

$$\|T^n - P\| \leq Cr^n, \quad n \geq 0,$$

whereas if $\text{Ran}(I - T)$ is not closed then the convergence in (4.2) is arbitrarily slow and weakly arbitrarily slow. In either case there exists a dense subspace X_0 of X such that for all $x \in X_0$ the convergence in (4.2) is superpolynomially fast.

We begin with the following lemma.

Lemma 4.3. *In the setting of Theorem 4.2 there exists $C > 0$ such that*

$$(4.3) \quad \|x - Tx\| \leq C(1 - \|Tx\|)^{1/q^N}$$

for all $x \in X$ with $\|x\| = 1$.

Proof. The proof is by induction. Suppose first that $N = 1$ and that $T = P$ is an orthoprojection. Suppose now that $\|x\| = 1$ and let $\varepsilon = 1 - \|Px\|$. Then $\varepsilon \in [0, 1]$ and we have

$$(4.4) \quad \frac{1}{2}\|x + Px\| \geq \frac{1}{2}\|P(x + Px)\| = \|Px\| = 1 - \varepsilon.$$

Consider the function $\beta_X: [0, 1] \rightarrow [0, \infty)$ defined for $0 \leq s \leq 1$ by

$$\beta_X(s) = \sup \left\{ \|x - y\| : \|x\| \leq 1, \|y\| \leq 1, \frac{\|x + y\|}{2} \geq 1 - s \right\}.$$

Since X is uniformly convex, we have that β_X is continuous and increasing on $[0, 1]$, with $\beta_X(0) = 0$ and $\beta_X(1) = 2$; see [4]. We also have $\delta_X(\beta_X(s)) = s$ for all $s \in [0, 1]$ by [4, Theorem 3.4]. It follows in particular that $\beta_X(s) \lesssim s^{1/q}$ for $0 \leq s \leq 1$. Using (4.4) and the definition of β_X we obtain

$$(4.5) \quad \|x - Px\| \leq \beta_X(\varepsilon) \lesssim \varepsilon^{1/q}.$$

Here and in what follows we write $a \lesssim b$ if $a \leq Cb$ for some $C > 0$ which is independent of all parameters which are free to vary in the given situation. Thus (4.3) is proved for $N = 1$.

Now assume that (4.3) is true for a product $S \in \mathcal{B}(X)$ of $N \geq 1$ orthoprojections and let $P \in \mathcal{B}(X)$ be a further orthoprojection. Set $T = PS$. Suppose that $\|x\| = 1$ and let $\varepsilon = 1 - \|Tx\|$. Using the induction hypothesis, we have

$$(4.6) \quad \|x - Tx\| \leq \|x - Px\| + \|P(x - Sx)\| \lesssim \|x - Px\| + \varepsilon^{1/q^N},$$

using the fact that $1 - \|Sx\| \leq \varepsilon$. In order to estimate $\|x - Px\|$ we use the induction hypothesis to obtain

$$1 - \|Px\| \leq 1 - \|Tx\| + \|P(x - Sx)\| \lesssim \varepsilon + \varepsilon^{1/q^N} \lesssim \varepsilon^{1/q^N},$$

so that by (4.5) we have

$$\|x - Px\| \leq \beta_X(1 - \|Px\|) \lesssim \varepsilon^{1/q^{N+1}}.$$

Using this inequality in (4.6) we see that

$$\|x - Tx\| \lesssim \varepsilon^{1/q^{N+1}} + \varepsilon^{1/q^N} \lesssim \varepsilon^{1/q^{N+1}},$$

which completes the proof. □

Lemma 4.3 leads to the following result, which is a key ingredient in the proof of Theorem 4.2 but is also of independent interest.

Theorem 4.4. *Let X be a Banach space which is q -uniformly convex for some $q \geq 2$ and suppose that $T = P_N \cdots P_1$ for certain orthoprojections $P_1, \dots, P_N \in \mathcal{B}(X)$. Then the numerical range $W_0(T)$ of T is a generalised Stolz domain with parameter q^N .*

Proof. Fix an isometry $J: X \rightarrow X^*$ with the property that $\langle x, \phi_x \rangle = \|x\|^2$ for all $x \in X$. Here and subsequently we let $\phi_x := J(x)$ for $x \in X$. Consider an element $\lambda \in W(T)$ of the form $\lambda = \langle Tx, \phi_x \rangle$ with $\|x\| = 1$. Then $|\lambda| \leq \|Tx\| \leq 1$ and thus $W(T)$ is contained in the closed unit disc. By Lemma 4.3 we have

$$|\lambda - 1| = |\langle Tx - x, \phi_x \rangle| \leq \|x - Tx\| \lesssim (1 - \|Tx\|)^{1/q^N} \leq (1 - |\lambda|)^{1/q^N}.$$

Using the concavity and monotonicity properties of the functions $t \mapsto (1 - t)^{1/q^N}$ and $t \mapsto t^{1/q^N}$ we see that in fact

$$|\lambda - 1| \lesssim (1 - |\lambda|)^{1/q^N}$$

for all $\lambda \in W_0(T)$, the closed convex hull of $W(T)$. It follows that $W_0(T)$ is a generalised Stolz domain with parameter q^N . \square

Proof of Theorem 4.2. The result follows at once from Proposition 3.2 and Theorem 4.4, noting that $\text{Fix } T = M$ as a consequence of Lemma 4.3. The fact that P is an orthoprojection is clear by Remark 4.1. \square

A notion closely related to uniform convexity, and in some sense dual to it, is that of uniform smoothness. A Banach space X is said to be *uniformly smooth* if for every $\varepsilon > 0$ there exists $\delta > 0$ such that the inequality

$$\|x + y\| + \|x - y\| < 2 + \varepsilon\|y\|$$

holds for any two vectors $x, y \in X$ with $\|x\| = 1$ and $\|y\| \leq \delta$. An equivalent definition is that

$$\lim_{t \rightarrow 0} \frac{\rho_X(t)}{t} = 0,$$

where ρ_X is the *modulus of smoothness* of X defined for $t \geq 0$ by

$$\rho_X(t) = \sup \left\{ \frac{\|x + ty\| + \|x - ty\|}{2} - 1 : x, y \in X, \|x\| = 1, \|y\| = 1 \right\}.$$

A Banach space X is uniformly smooth if and only if its dual X^* is uniformly convex, and vice versa. In particular, any uniformly smooth space is reflexive and every Hilbert space is uniformly smooth. Indeed, if X is a Hilbert space then $\rho_X(t) = \rho(t)$ for all $t \geq 0$, where

$$\rho(t) = (1 + t^2)^{1/2} - 1 \sim \frac{t^2}{2}, \quad t \rightarrow 0.$$

For any Banach space X we have $\rho_X(t) \geq \rho(t)$ for all $t \geq 0$. For $p \in (1, 2]$ we say that X is *p-uniformly smooth* if there is a constant $C > 0$ such that $\rho_X(t) \leq Ct^p$ for all $t \geq 0$. For instance, any L^q -space is p -uniformly smooth for $p = \min\{2, q\}$ when $1 < q < \infty$. We refer again to [27, Ch. 1.e] and [35, Ch. 10] for more information. The next result is an analogue of Theorem 4.2 for uniformly smooth spaces.

Theorem 4.5. *Let X be a Banach space which is p -uniformly smooth for some $p \in (1, 2]$ and suppose that $T = P_N \cdots P_1$ for certain orthoprojections $P_1, \dots, P_N \in \mathcal{B}(X)$. Furthermore, let $M = \text{Ran } P_1 \cap \dots \cap \text{Ran } P_N$. Then $X = M \oplus Z$, where Z denotes the closure of $\text{Ran}(I - T)$. Furthermore, for all $x \in X$ we have*

$$(4.7) \quad \|T^n x - Px\| \rightarrow 0, \quad n \rightarrow \infty,$$

where P denotes the orthoprojection onto M along Z . Moreover, there is a dichotomy for the rate of convergence. Indeed, if $\text{Ran}(I - T)$ is closed then there exist $C > 0$ and $r \in [0, 1)$ such that

$$\|T^n - P\| \leq Cr^n, \quad n \geq 0,$$

whereas if $\text{Ran}(I - T)$ is not closed then the convergence in (4.7) is arbitrarily slow and weakly arbitrarily slow. In either case there exists a dense subspace X_0 of X such that for all $x \in X_0$ the convergence in (4.7) is superpolynomially fast.

Proof. Let $q \in [2, \infty)$ be the Hölder conjugate of p , so that $p^{-1} + q^{-1} = 1$. Since X is uniformly smooth, its dual space X^* is uniformly convex. Let

$$\gamma_{X^*}(\varepsilon) = \sup_{t \geq 0} \left(\frac{t\varepsilon}{2} - \rho_X(t) \right), \quad 0 \leq \varepsilon \leq 2.$$

Then by [35, Lemma 10.20] we have $\delta_{X^*}(\varepsilon) \geq \gamma_{X^*}(\varepsilon)$ for $0 \leq \varepsilon \leq 2$, and hence

$$\delta_{X^*}(\varepsilon) \geq \sup_{t \geq 0} \left(\frac{t\varepsilon}{2} - Ct^p \right) \geq c\varepsilon^q, \quad 0 \leq \varepsilon \leq 2,$$

for $c = 4^{-q}C^{1-q}$, as can be seen by considering $t = (\varepsilon/4C)^{q-1}$. Thus X^* is q -uniformly convex. By considering the dual operator T^* of T , which itself is a product of orthoprojections, and observing that $\|R(\lambda, T)\| = \|R(\lambda, T^*)\|$ for all $\lambda \in \mathbb{C} \setminus \sigma(T)$ we deduce from Theorem 4.4 and Proposition 3.2 that condition (2.1) of Theorem 2.1 is satisfied for $\alpha = q^N$. The result now follows as in the case of Theorem 4.2. □

We conclude this section with another application of Theorem 2.1 to products of orthoprojections, this time involving assumptions on the projections rather than the space. Given a Banach space X and a projection $P \in \mathcal{B}(X)$ we shall say, in loose accordance with the terminology of [2], that P is a *type-D projection* if $P \neq 0$ and there exists $r \in (0, 1)$ such that $\|P - rI\| \leq 1 - r$. In particular, any type-D projection is an orthoprojection. Loosely following [23], a type-D projection $P \in \mathcal{B}(X)$ will be called a *type-U projection* if $\|P - \frac{1}{2}I\| \leq \frac{1}{2}$. For instance, any orthogonal projection on a Hilbert space is a type-U projection. Note that by Remark 3.3 that for a type-D projection P we have $W_0(P) \subseteq \{\lambda \in \mathbb{C} : |\lambda - r| \leq 1 - r\}$ for some $r \in (0, 1)$, and hence $W_0(P)$ is a generalised Stolz domain with parameter $\alpha = 2$.

Theorem 4.6. *Let X be a reflexive Banach space and suppose that $T \in \mathcal{B}(X)$ is a convex combination of products of certain type-D projections P_1, \dots, P_N . Then the conclusions of Theorem 2.1 hold and the projection P_T onto $\text{Fix } T$ along the closure of $\text{Ran}(I - T)$ is an orthoprojection. Furthermore, if $\text{Ran } P_k = \{x \in X : \|P_k x\| = \|x\|\}$ for $1 \leq k \leq N$ and if all of the N projections P_1, \dots, P_N actually appear in the decomposition of T , then $\text{Fix } T = \text{Ran } P_1 \cap \dots \cap \text{Ran } P_N$.*

Proof. Note first that T , being a convex combination of products of orthoprojections, is a contraction. It was proved in [2, Lemma 3.5] that the set of contractions $Q \in \mathcal{B}(X)$ for which there exists $r \in (0, 1)$ such that $\|Q - rI\| \leq 1 - r$ is a convex multiplicative semigroup. It follows in particular that $\|T - rI\| \leq 1 - r$ for some $r \in (0, 1)$. Hence the numerical range $W_0(T)$ of T is contained a generalised Stolz domain. The first part of the result now follows from Theorem 2.1, and the rest is proved in [2]. □

Remark 4.7. If X is a Hilbert space and $T = P_N \cdots P_1$, where P_k is the orthogonal projection onto the closed subspace M_k of X , $1 \leq k \leq N$, then $\text{Ran}(I - T)$ is closed if and only if $M_1^\perp + \dots + M_N^\perp$ is closed, and moreover $\text{Fix } T = M$, where $M = M_1 \cap \dots \cap M_N$. By contractivity of T we deduce that P_T coincides with the orthogonal projection P_M onto M . It was shown in [3, Theorem 4.3] that it is possible to obtain explicit values for the numbers C and r appearing in (2.3) in terms

of the Friedrichs number of the subspaces M_1, \dots, M_N . In fact, [3, Theorem 4.3] goes beyond the present theorem in other ways too by exploiting the theory of (unconditional) Ritt operators. Indeed, it is known that in the Hilbert space case the numerical range $W(T)$ of T is contained in a Stolz domain, which by means of (3.2) leads to (2.1) with $\alpha = 1$ rather than $\alpha = 2$. For a closer analysis of the asymptotic behaviour in the method of alternating projections see for instance [1, 3, 37].

5. THE DOUGLAS-RACHFORD SPLITTING METHOD

Let X be a Hilbert space and suppose that M_1, M_2 are closed subspaces of X . If for $k = 1, 2$ we let P_k denote the orthogonal projection onto M_k then we may consider the operator $T \in \mathcal{B}(X)$ given by

$$T = P_2P_1 + (I - P_2)(I - P_1).$$

If we let $Q_k = 2P_k - I$, $k = 1, 2$, then we may write T as $T = \frac{1}{2}(I + Q_2Q_1)$. The operator T is known as the *Douglas-Rachford operator* and plays an important role in the Douglas-Rachford splitting method. Here one is usually interested in the asymptotic behaviour of sequences of the form $(P_1T^n x)_{n \geq 0}$ for different initial vectors $x \in X$, but in order to understand such sequences one needs first to understand the sequences $(T^n x)_{n \geq 0}$ with $x \in X$; see [5, 19] for further details. A generalisation to several subspaces (the *cyclic* Douglas-Rachford iteration scheme) has been proposed recently in [9, 10]. The following result is a Banach space version of the Douglas-Rachford splitting method with several reflection operators arising from type-U projections.

Theorem 5.1. *Let X be a reflexive Banach space and let $P_1, \dots, P_N \in \mathcal{B}(X)$ be type-U projections on X . For $1 \leq k, \ell \leq N$ let*

$$T_{k\ell} = P_kP_\ell + (I - P_k)(I - P_\ell)$$

and suppose that $T \in \mathcal{B}(X)$ is a convex combination of products of the operators $T_{k\ell}$, $1 \leq k, \ell \leq N$. Then the conclusions of Theorem 2.1 hold and the projection P_T onto $\text{Fix } T$ along the closure of $\text{Ran}(I - T)$ is an orthoprojection.

Proof. If we let $Q_k = 2P_k - I$, $1 \leq k \leq N$, then we may write $T_{k\ell}$ as $T_{k\ell} = \frac{1}{2}(I + Q_kQ_\ell)$ for $1 \leq k, \ell \leq N$. Now

$$\|Q_k\| = 2\left\|P_k - \frac{1}{2}I\right\| \leq 1$$

for $1 \leq k \leq N$ and hence

$$\left\|T_{k\ell} - \frac{1}{2}I\right\| = \frac{\|Q_kQ_\ell\|}{2} \leq \frac{1}{2}$$

for $1 \leq k, \ell \leq N$. Using [2, Lemma 3.5] as in the proof of Theorem 4.6 the result now follows from Theorem 2.1 and Remark 4.1. □

We conclude with the following new result on the Douglas-Rachford splitting method for the classical case of two subspaces of a Hilbert space.

Theorem 5.2. *Let X be a Hilbert space and let*

$$T = P_2P_1 + (I - P_2)(I - P_1)$$

be the Douglas-Rachford operator corresponding to the orthogonal projections P_1, P_2 onto two closed subspaces M_1, M_2 of X . Then for all $x \in X$ we have

$$(5.1) \quad \|T^n x - Px\| \rightarrow 0, \quad n \rightarrow \infty,$$

where P denotes the orthogonal projection onto $(M_1 \cap M_1) \oplus (M_1^\perp \cap M_2^\perp)$. Moreover, there is a dichotomy for the rate of convergence. Indeed, if $M_1 + M_2$ is closed then there exist $C > 0$ and $r \in [0, 1)$ such that

$$(5.2) \quad \|T^n - P\| \leq Cr^n, \quad n \geq 0,$$

whereas if $M_1 + M_2$ is not closed then the convergence in (5.1) is arbitrarily slow and weakly arbitrarily slow. In either case there exists a dense subspace X_0 of X such that for all $x \in X_0$ the convergence in (5.1) is superpolynomially fast.

Proof. It is shown in [5, Proposition 3.6] that

$$\text{Fix } T = (M_1 \cap M_1) \oplus (M_1^\perp \cap M_2^\perp).$$

The result now follows from Theorem 5.1 and the observation that (5.2) holds for some $C > 0$ and $r \in [0, 1)$ if and only if $M_1 + M_2$ is closed; see [5, Fact 2.3 and Theorem 4.1]. \square

Remark 5.3. (a) Note that by [5, Theorem 4.1] we may choose $C = 1$ and $r = c(M_1, M_2)$ in (5.2), where $c(M_1, M_2)$ is the Friedrichs number of the subspaces M_1, M_2 .

(b) We would like to mention that the Hilbert space results obtained in Sections 4 and 5 easily extend to closed affine subspaces. As a final remark, we note that the methods of this paper can be applied to other projection algorithms and that, on certain classes of problems, various iterative projection methods coincide with each other. For example, if the sets are closed affine subspaces, then the method of alternating projections coincides with Dykstra's method [11]. Applied to the phase retrieval problem, the method of alternating projections coincides with the error reduction method and the Douglas-Rachford method coincides with Fienup's hybrid input-output algorithm [7]. For other such coincidences (in the cases of hyperplanes and half-spaces), see [9].

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