Pure and Applied Functional Analysis
Volume 2, Number 4, 2017, 575–584

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

ON A THEOREM OF LOBANOVA AND SADOVSKIJ

JÜRGEN APPELL, NELSON MERENTES, AND JOSÉ LUIS SÁNCHEZ

ABSTRACT. We show how so-called diode operators occur in the theory of biological predator-prey systems with constraint. This makes it possible to apply a theorem of Lobanova and Sadovskij to prove the existence of a closed limit cycle for such systems.

1. INTRODUCTION

In a recent paper [3] Lobanova and Sadovskij studied a system of the form

$$\dot{x} = \tau_x(f(x)) \tag{1.1}$$

and proved, under some suitable hypotheses, the existence of a unique closed trajectory for (1.1). Here f is some continuous function on a closed convex set $Q \subset \mathbb{R}^2$, and $\tau_x(y)$ denotes the metric projection of a point $y \in \mathbb{R}^2$ onto the tangent cone of Q at x. (Precise definitions will be recalled below.) The usual existence, uniqueness and stability theorems for ODE's do not apply to the system (1.1), because it contains a projection operator τ_x which varies for x running over Q.

Systems like (1.1) occur in surprisingly many fields of the natural sciences. Thus, (1.1) is a very useful tool for describing oscillations in electrical circuits with diode converter. Moreover, some problems in linear programming also lead to the system (1.1). In this paper we show how a biological system with constraints may also be formulated in the form (1.1). This makes it possible to apply the Lobanova-Sadovskij theorem in order to prove the existence and uniqueness of a closed limit cycle.

2. NORMAL CONES AND DN-OPERATORS.

Throughout this paper, we consider the Euclidean space $\mathbb{R}^m$ with the usual scalar product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. We have to recall some definitions.

Given a nonempty set $Q \subset \mathbb{R}$ and an element $y \in \mathbb{R}^m$, an element $x \in Q$ is called *point of best approximation to y in Q* and denoted by $x = P(y, Q)$ if

$$\|y - x\| = \text{dist}(y, Q) := \inf \{\|y - z\| : z \in Q\}. \tag{2.1}$$

Such a point need not exist, and even if it exists it need not be unique. If $x = P(y, Q)$ exists and is unique for any $y \in \mathbb{R}^m$, then the map $P(\cdot, Q) : \mathbb{R}^m \to Q$

2010 *Mathematics Subject Classification.* 47H07, 47J04, 34C07, 34C15, 92D25.
Key words and phrases. Diode operator, normal cone, tangent cone, closed limit cycle, auto-oscillation, predator-prey system.

is called *metric projection* onto Q. For example, this is fulfilled if Q is closed and convex (e.g., a closed cone).

Let $Q \subset \mathbb{R}$ be convex and $x \in Q$. Recall that the *normal cone* to Q at x is defined by

$$N_Q(x) := \{y \in \mathbb{R}^m : \langle y, z - x \rangle \leq 0 \text{ for all } z \in Q\}, \tag{2.2}$$

while the *tangent cone* to Q at x is defined as dual cone to $N_Q(x)$, i.e.,

$$T_Q(x) := \{z \in \mathbb{R}^m : \langle z, y \rangle \leq 0 \text{ for all } y \in N_Q(x)\}. \tag{2.3}$$

By $\nu_x : \mathbb{R}^m \to N_Q(x)$ we denote the metric projection onto the normal cone, and by $\tau_x : \mathbb{R}^m \to T_Q(x)$ the metric projection onto the tangent cone; so we have

$$\nu_x(y) = P(y, N_Q(x)), \quad \tau_x(y) = P(y, T_Q(x)) \qquad (x \in Q, y \in \mathbb{R}^m). \tag{2.4}$$

The orthogonal decomposition

$$y = \nu_x(y) + \tau_x(y), \quad \langle \nu_x(y), \tau_x(y) \rangle = 0 \tag{2.5}$$

holds which means that every element $y \in \mathbb{R}^m$ is the orthogonal sum of its projections onto a normal and tangent cone. A certain converse of this is also true: if $K \subset \mathbb{R}^m$ is a cone with dual cone K^*, and an element $y \in \mathbb{R}^m$ is the orthogonal sum $y = x + z$ of two elements $x \in K$ and $z \in K^*$, then $x = P(y, K)$ and $z = P(y, K^*)$.

Observe that the cones (2.2) and (2.3) are interesting only for boundary points $x \in \partial Q$. In fact, if x is an interior point of Q, then $N_Q(x) = \{0\}$ and $T_Q(x) = \mathbb{R}^m$.

The multivaled map $N_Q : x \mapsto N_Q(x)$ has some interesting analytical properties. For example, it is *upper semicontinuous* in the sense that $x_k \to x$, $y_k \in N_Q(x_k)$ and $y_k \to y$ implies that $y \in N_Q(x)$, and also *maximal monotone* which means that $y \in N_Q(x)$ and $\hat{y} \in N_Q(\hat{x})$ implies that $\langle y - \hat{y}, x - \hat{x} \rangle \geq 0$, and there is no proper extension of N_Q with this property. We point out that the multivalued map $T_Q : x \mapsto T_Q(x)$ is neither upper semicontinuous (take as Q the closed unit disc in $\mathbb{R}^2$, $x_k := (1 - 1/k, 0)$, and $x := (1, 0)$) nor monotone (for the same Q, take $x = \hat{y} := (1, 0)$ and $\hat{x} = y := (0, 0)$).

Now let $Q \subset \mathbb{R}^m$ be closed and convex as before, and suppose that $f : Q \to \mathbb{R}^m$ is continuous. A differential inclusion of the form

$$\dot{x} \in f(x) - N_Q(x) \tag{2.6}$$

involving the normal cone (2.2) with respect to Q is called *diode nonlinearity system* (or *DN-system*, for short) in the literature, while the multivalued nonlinear operator $x \mapsto f(x) - N_Q(x)$ occurring on the right-hand side of (2.6) is called *diode operator*. A *solution* of (2.6) is, by definition, an absolutely continuous function x which satisfies $\dot{x}(t) \in f(x(t)) - N_Q(x(t))$ almost everywhere on some interval.

The name is motivated by the fact that such systems have been encountered first in the study of electric circuits containing so-called *ideal diodes*, see [4].

3. The Lobanova-Sadovskij theorem.

Given a closed convex set $Q \subseteq \mathbb{R}^2$ and a continuous function $f : Q \to \mathbb{R}^2$, Lobanova and Sadovskj [3] proved in 2007 a theorem on the existence of a unique closed trajectory for the system (1.1) involving the projection (2.4). A *solution* of (1.1) is, by definition, an absolutely continuous function x which satisfies $\dot{x}(t) = \tau_{x(t)} f(x(t))$ almost everywhere on some interval.

The system (1.1) is more complicated than an ordinary differential equation, because it contains the projection operator $\tau_{x(t)}$ onto the tangent cone $T_Q(x(t))$ which depends on t. We point out that (1.1) is in fact *equivalent* to the differential inclusion with diode operator (2.6). To see this, suppose first that x is an absolutely continuous function on some interval J which solves (1.1) almost everywhere. Applying (2.5) to $y = f(x(t))$ we obtain

$$\tau_{x(t)} f(x(t)) = f(x(t)) - \nu_{x(t)} f(x(t)) \in f(x(t)) - N_Q(x(t)),$$

which shows that x satisfies the differential inclusion (2.6) almost everywhere on J.

Conversely, suppose that x is an absolutely continuous function on some interval J which solves (2.6) almost everywhere. Choose a point $u \in N_Q(x(t))$ such that $\dot{x}(t) = f(x(t)) - u$. We claim that $\dot{x}(t)$ and u are orthogonal, i.e., $\langle \dot{x}(t), u \rangle = 0$. In fact, choosing $\varepsilon > 0$ so small that $x(t+s) \in Q$ for $0 \leq s < \varepsilon$ we have

$$\frac{1}{s} \langle x(t+s) - x(t), u \rangle \leq 0 \qquad (s > 0),$$

by the definition (2.2) of the cone $N_Q(x(t))$. Passing in this inequality to the limit $s \to 0+$ we conclude that $\langle \dot{x}(t), u \rangle \leq 0$, and so $\dot{x}(t) \in T_Q(x(t))$. In the same way one may prove, by considering points $x(t+s)$ for small negative s, that $\langle \dot{x}(t), u \rangle \geq 0$, and so $-\dot{x}(t) \in T_Q(x(t))$. Now we may apply the remark after (2.5) to the choice

$$K := N_Q(x(t)), \ K^* := T_Q(x(t)), \ x := u, \ y := \dot{x}(t)$$

and obtain $u = \nu_{x(t)} f(x(t))$ and $\dot{x}(t) = \tau_{x(t)} f(x(t))$, and the second equality means precisely that x solves (1.1) almost everywhere.

In view of the equivalence of (1.1) and (2.6), in what follows we retain the name DN-system also for (1.1). For stating the Lobanova-Sadovskij theorem we collect the following 5 hypotheses:

(A) The set $Q \subset \mathbb{R}^2$ is closed and convex and contains 0 as interior point.
(B) The function $f : Q \to \mathbb{R}^2$ is locally Lipschitz continuous.
(C) The condition $f(x) \notin N_Q(x)$ holds for $x \neq 0$.
(D) There exist a symmetric positive definite matrix B such that $\langle Bx, f(x) \rangle \geq \mu(\|x\|)$, where $\mu : [0, \infty) \to [0, \infty)$ is a continuous function satisfying $\mu(t) > 0$ for $t > 0$.
(E) There exists a constant $\nu > 0$ such that $\langle Cx, f(x) \rangle \geq \nu \|x\|^2$, where $C(x_1, x_2) = (-x_2, x_1)$ denotes the counterclockwise rotation by $\pi/2$ in the plane.

Theorem 3.1 (Lobanova-Sadovskij)**.** Suppose that the hypotheses *(A) – (E)* are satisfied. Then the DN-system (1.1) has a unique closed trajectory which attracts, for $t \to \infty$, all nonconstant solutions.

Recall that a trajectory Σ is said to attract all nonconstant solutions [1] if the following holds: denoting by $x(\cdot; x_0)$ a nonconstant solution of (1.1) which satisfies the initial condition $x(0) = x_0$, for every $x_0 \in Q$ we find some $t_0 \geq 0$ such that $x(t; x_0) \in \Sigma$ for all $t \geq t_0$.

4. Auto-oscillations in a predator-prey model.

In this section we sketch an application of the Lobanova-Sadovskij theorem to problems arising in biology and life sciences.

Recall that a *predator-prey system* (also called *Lotka-Volterra model* in the literature, a model which originated in the study of fish populations of the Mediterranean) is a pair of first order nonlinear differential equations describing the dynamics of biological systems, or eco-systems, in which two species of animals interact: the predator and the prey. The populations change through time according to the pair of equations

$$\begin{cases} \dot{N}_1 = \varepsilon_1 N_1 - \gamma_1 N_1 N_2, \\ \dot{N}_2 = \gamma_2 N_1 N_2 - \varepsilon_2 N_2, \end{cases} \tag{4.1}$$

where $N_1 = N_1(t)$ is the number of prey (for example, rabbits), $N_2 = N_2(t)$ is the number of some predator (for example, foxes) at time t, $\dot{N}_1 = \dot{N}_1(t)$ and $\dot{N}_2 = \dot{N}_2(t)$ represent the growth rates of the two populations over time, and the dot indicates, as usual, the time derivative. The positive real numbers $\varepsilon_1, \gamma_1, \varepsilon_2, \gamma_2$ are parameters describing the interaction of the two species; for technical reasons we assume that

$$\varepsilon_2 \gamma_1 \geq \varepsilon_1 \gamma_2 \tag{4.2}$$

in the sequel. The underlying eco-system is supposed to be *closed* which means that no migration is allowed into or out of the system.

Let us spend some time on the meaning of these four parameters. If only animals of prey type live in the region (i.e., $N_2(t) \equiv 0$), their growth rate is a fixed positive number, say $\dot{N}_1(t) \equiv \varepsilon_1$. Conversely, if the predator-type animals which live only (or mostly) on animals of the prey species stay in an isolated part of the region, their growth rate is a fixed negative number, say $\dot{N}_2(t) \equiv -\varepsilon_2$. So in absence of the prey population (i.e., $N_1(t) \equiv 0$), the predator population would decay to zero due to starvation. Now, if these two species live together in a common region, the number of preys will increase more slowly if the number of predators is large, while the number of predators will increase faster if the number of preys is large.

The first problem connected with the system (4.1) is to find its *equilibrium states* and to analyze its *stability*. Clearly, the system

$$\begin{cases} N_1(\varepsilon_1 - \gamma_1 N_2) = 0, \\ N_2(\varepsilon_2 - \gamma_2 N_1) = 0 \end{cases} \tag{4.3}$$

has the two solutions $(N_1, N_2) = (0, 0)$ and $(N_1, N_2) = (\varepsilon_2/\gamma_2, \varepsilon_1/\gamma_1)$; they represent equilibria of (4.1). For the second equilibrium in (4.3) we introduce the new coordinates

$$\frac{\varepsilon_2}{\gamma_2} =: \tilde{X}_1, \quad \frac{\varepsilon_1}{\gamma_1} =: \tilde{X}_2, \quad \tilde{X} := \begin{pmatrix} \tilde{X}_1 \\ \tilde{X}_2 \end{pmatrix}.$$

By (4.2), we have then $\tilde{X}_1 \geq \tilde{X}_2$, which means that the non-zero equilibrium $\tilde{X}$ lies below the diagonal in the plane. For the stability analysis of (4.1) one uses Lyapunov's stability theorem for the first approximation. The Jacobian matrix containing the partial derivatives of the right-hand side of (4.1) with respect to N_1 and N_2 is

$$A(N_1, N_2) = \begin{pmatrix} \varepsilon_1 - \gamma_1 N_2 & -\gamma_1 N_1 \\ \gamma_2 N_2 & -\varepsilon_2 + \gamma_2 N_1 \end{pmatrix}.$$

In particular,

$$A(0,0) = \begin{pmatrix} \varepsilon_1 & 0 \\ 0 & -\varepsilon_2 \end{pmatrix}, \qquad A(\tilde{X}_1, \tilde{X}_2) = \begin{pmatrix} 0 & -\gamma_1\varepsilon_2/\gamma_2 \\ \gamma_2\varepsilon_1/\gamma_1 & 0 \end{pmatrix}.$$

An easy calculation shows that $A(0,0)$ has the eigenvalues ε_1 and $-\varepsilon_2$, and $A(\tilde{X}_1, \tilde{X}_2)$ has the eigenvalues $\pm\omega i$, where $\omega := \sqrt{\varepsilon_1\varepsilon_2}$. This shows that $(0,0)$ is an unstable saddle point, while $(\tilde{X}_1, \tilde{X}_2)$ is a stable center.

The linearization of (4.1) near the equilibrium $\tilde{X} := (\tilde{X}_1, \tilde{X}_2)$ leads to the system

$$\begin{cases} \dot{x} = -\dfrac{\gamma_1\varepsilon_2}{\gamma_2} y, \\ \dot{y} = \dfrac{\gamma_2\varepsilon_1}{\gamma_1} x, \end{cases} \tag{4.4}$$

where $x = x(t)$ and $y = y(t)$ approximate $N_1 - \tilde{X}_1$ and $N_2 - \tilde{X}_2$, respectively. The solutions of the linearized system (4.4) are called *small fluctuations*; their period is given in explicit form by

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{\varepsilon_1\varepsilon_2}}.$$

The behaviour of the trajectories of the nonlinear system (4.1) close to the equilibrium $\tilde{X}$ is rather similar to that of (4.4).

Let us now study predator-prey equations from the viewpoint of the theory of DN-operators which we discussed in the preceding two sections. On a closed convex set $\tilde{Q} \subseteq \mathbb{R}^2_+$ we consider the DN-system

$$\dot{X} = \tau_X[F(X) + \delta(X - \tilde{X})]. \tag{4.5}$$

Here $X = (X_1, X_2)$ is the pair of prey population $X_1(t)$ and predator population $X_2(t)$ at time t as before, and the nonlinearity $F(X)$ denotes the right-hand side of the Volterra-Lotka system (4.1), i.e.,

$$F(X) := \begin{pmatrix} (\varepsilon_1 - \gamma_1 X_2)X_1 \\ -(\varepsilon_2 - \gamma_2 X_1)X_2 \end{pmatrix}.$$

The term $\delta(X - \tilde{X})$ in (4.5) takes into account exterior effects on the population of the two species. If $\delta = 0$ then (4.5) is a classical predator-prey system; in this case we have $F(X) \in T_Q(X)$, hence $\tau_X F(X) = F(X)$, for every $X \in \tilde{Q}$. On the other hand, if $\delta > 0$ and $\varepsilon_1 = \gamma_1 = \varepsilon_2 = \gamma_2 = 0$ (hence $F(X) \equiv 0$), then (4.5) splits into the two simple separated equations

$$\begin{cases} \dot{X}_1 = \delta X_1, \\ \dot{X}_2 = \delta X_2. \end{cases}$$

Let us now establish a criterion for auto-oscillations of (4.5), i.e., for the existence of a unique limit cycle [3,4]. To this end, it is useful to introduce the new coordinates

$$x_1 := X_1 - \tilde{X}_1, \quad x_2 := X_2 - \tilde{X}_2. \tag{4.6}$$

So if $X = (X_1, X_2)$ runs over the domain $\tilde{Q}$, $x = (x_1, x_2)$ runs over the domain $\tilde{Q} - \tilde{X} =: Q$ (which is still closed and convex). For the corresponding normal cone and tangent cone we get then

$$N_{\tilde{Q}}(X) = N_Q(x), \quad T_{\tilde{Q}}(X) = T_Q(x). \tag{4.7}$$

In fact, to see that the first equality in (4.7) is true observe that $y \in N_{\tilde{Q}}(X)$ means that

$$\langle y, Z - X \rangle \leq 0 \qquad (Z \in \tilde{Q}),$$

by definition of the normal cone (2.2). Since $Z \in \tilde{Q}$ if and only if $z := Z - \tilde{X} \in Q$, this is equivalent to

$$\langle y, z - x \rangle = \langle y, (z - \tilde{X}) - (x - \tilde{X}) \rangle = \langle y, Z - X \rangle \leq 0 \qquad (z \in Q),$$

and this in turn means nothing else but $y \in N_Q(x)$. The second relation in (4.7) follows from the first, by duality.

Let us now see how the DN-system (4.5) transforms into the new coordinates (4.6). The definition of $\tilde{X}_1$ and $\tilde{X}_2$ shows that $\gamma_2 \tilde{X}_1 = \varepsilon_2$ and $\gamma_1 \tilde{X}_2 = \varepsilon_1$. Consequently,

$$\begin{aligned} F(X) + \delta(X - \tilde{X}) &= \begin{pmatrix} (\varepsilon_1 - \gamma_1 x_2 - \gamma_1 \tilde{X}_2)(x_1 + \tilde{X}_1) + \delta x_1 \\ (-\varepsilon_2 + \gamma_2 x_1 - \gamma_2 \tilde{X}_1)(x_2 + \tilde{X}_2) + \delta x_2 \end{pmatrix} \\ &= \begin{pmatrix} -\gamma_1 x_1 x_2 - \dfrac{\gamma_1 \varepsilon_2}{\gamma_2} x_2 + \delta x_1 \\ \gamma_2 x_1 x_2 + \dfrac{\gamma_2 \varepsilon_1}{\gamma_1} x_1 + \delta x_2 \end{pmatrix} =: f(x). \end{aligned} \tag{4.8}$$

So (4.5) takes now in the new coordinates the form (1.1), where x is given by (4.6) and $f(x)$ by (4.8). To apply the Lobanova-Sadovskij theorem we have to verify the Hypotheses (A) – (E).

Clearly, the Hypotheses (A) and (B) are satisfied, since f is a polynomial in x. For verifying (C) we distinguish two cases.

Suppose first that x is an interior point of Q. Then $N_Q(x) = \{0\}$, and so the relation $f(x) \in N_Q(x)$ means that

$$\left\{\begin{array}{l} -\gamma_1 x_1 x_2 - \dfrac{\gamma_1\varepsilon_2}{\gamma_2} x_2 + \delta x_1 = 0, \\ \gamma_2 x_1 x_2 - \dfrac{\gamma_2\varepsilon_1}{\gamma_1} x_1 + \delta x_2 = 0. \end{array}\right. \tag{4.9}$$

Multiplying the first equality in (4.9) by x_2, the second by x_1, and subtracting the first from the second gives

$$\gamma_2\left(x_2 + \frac{\varepsilon_1}{\gamma_1}\right)x_1^2 + \gamma_1\left(x_1 + \frac{\varepsilon_2}{\gamma_2}\right)x_2^2 = 0.$$

Since $\gamma_1 \geq 0$, $\gamma_2 \geq 0$, $x_1 > -\tilde{X}_1$ and $x_2 > -\tilde{X}_2$, this implies that $x_1 = x_2 = 0$, and so Hypothesis (C) is fulfilled in this case.

On the other hand, suppose that x is a boundary point of Q, and recall that $f(x) \in N_Q(x)$ if and only if $\langle z - x, f(x)\rangle \leq 0$ for each $z \in Q$. In the original coordinates this conditions means that

$$\langle Z - X, F(X) + \delta(X - \tilde{X})\rangle \leq 0 \qquad (Z \in \tilde{Q}). \tag{4.10}$$

In view of (4.10) we add yet another hypothesis to the Hypotheses (A) – (E) in the Lobanova-Sadovskij theorem:

(F) For each $X \in \partial\tilde{Q}$ there exists $\xi \in T_{\tilde{Q}}(X)$ such that $\langle \xi, F(X)+\delta(X-\tilde{X})\rangle > 0$.

Hypothesis (F) guarantees that (C) is fulfilled also for boundary points of Q.

Now we verify Hypothesis (D) for the positive definite diagonal matrix

$$B := \begin{pmatrix} \dfrac{\gamma_2\varepsilon_1}{\gamma_1} & 0 \\ 0 & \dfrac{\gamma_1\varepsilon_2}{\gamma_2} \end{pmatrix}.$$

By definition of f we obtain for $x = (x_1, x_2) \in Q$

$$\langle f(x), Bx\rangle = \begin{pmatrix} \left(-\gamma_1 x_1 x_2 - \dfrac{\gamma_1\varepsilon_2}{\gamma_2}x_2 + \delta x_1\right)\dfrac{\gamma_2\varepsilon_1}{\gamma_1}x_1 \\ \left(\gamma_2 x_1 x_2 + \dfrac{\gamma_2\varepsilon_1}{\gamma_1}x_1 + \delta x_2\right)\dfrac{\gamma_1\varepsilon_2}{\gamma_2}x_2 \end{pmatrix}$$

$$= \gamma_2\varepsilon_1\left(\frac{\delta}{\gamma_1} - x_2\right)x_1^2 + \gamma_1\varepsilon_2\left(\frac{\delta}{\gamma_2} + x_1\right)x_2^2 + \frac{\gamma_1^2\varepsilon_2^2 - \gamma_2^2\varepsilon_1^2}{\gamma_1\gamma_2}x_1 x_2.$$

The last fraction is nonnegative, by our assumption (4.2). So if we choose $\eta > 0$ in such a way that

$$x_1 \geq \eta - \frac{\delta}{\gamma_2}, \quad x_2 \leq \eta + \frac{\delta}{\gamma_1} \qquad ((x_1, x_2) \in Q)$$

or, equivalently,

$$X_1 \geq \eta - \frac{\delta - \varepsilon_2}{\gamma_2}, \quad X_2 \leq \eta + \frac{\delta + \varepsilon_1}{\gamma_1} \qquad ((X_1, X_2) \in \tilde{Q}) \tag{4.11}$$

Hypothesis (D) is satisfied for the function $\mu(t) := \eta t^2$.

It remains to analyze Hypothesis (E), which is easy. In rather the same way as for the matrix B, we get here

$$\langle f(x), Cx\rangle = \gamma_1 x_1 x_2^2 + \frac{\gamma_1\varepsilon_2}{\gamma_2}x_2^2 - \delta x_1 x_2 + \gamma_2 x_1^2 x_2 + \frac{\gamma_2\varepsilon_1}{\gamma_1}x_1^2 + \delta x_1 x_2$$

$$= \gamma_1\left(x_1 + \frac{\varepsilon_2}{\gamma_2}\right)x_2^2 + \gamma_2\left(x_2 + \frac{\varepsilon_1}{\gamma_1}\right)x_1^2.$$

The terms in the last brackets are $x_1 + \tilde{X}_1 = X_1$ and $x_2 + \tilde{X}_2 = X_2$, respectively. So if we suppose that

$$\omega_i := \inf\{X_i : (X_1, X_2) \in \tilde{Q}\} > 0 \qquad (i = 1, 2)$$

we may choose

$$\nu := \min\{\gamma_1\omega_1, \gamma_2\omega_2\} > 0, \tag{4.12}$$

to fulfill Hypothesis (E). Of course, condition (4.12) is only sufficient, but not necessary for Hypothesis (E) to hold.

In this way we have proved the following theorem on the existence of a stable closed trajectory for a generalized predator-prey system building on the Lobanova-Sadovskij theorem.

Theorem 4.1. *Let $\tilde{Q} \subseteq \mathbb{R}^2$ be nonempty, convex, and closed, and suppose that the parameters ε_1, γ_1, ε_2, and γ_2 occuring in the system* (4.1) are positive and satisfy the estimates (4.11) for every $X = (X_1, X_2) \in \tilde{Q}$ and some $\eta > 0$. Moreover, assume that the stable center $\tilde{X}$ of (4.1) is an interior point of Q, that (4.12) is satisfied, and that Hypothesis (F) holds for every $X \in \partial\tilde{Q}$. Then the system (4.5) admits a unique closed trajectory which attracts all nonconstant solutions.

5. An example

To illustrate our theorem we close with the following simple

Example. Consider the predator-prey system

$$\begin{cases} \dot{N}_1 = 2N_1 - N_1N_2, \\ \dot{N}_2 = N_1N_2 - 3N_2 \end{cases} \tag{5.1}$$

on the rectangle $\tilde{Q} := [1, 4] \times [1, 3]$. So we have

$$\gamma_1 = \gamma_2 = 1,\ \varepsilon_1 = 2,\ \varepsilon_2 = 3,\ \tilde{X} = (3, 2),\ \omega_1 = \omega_2 = 1,\ \nu = 1, \tag{5.2}$$

which shows that (4.2) and (4.12) are fulfilled. The conditions (4.11) connecting the parameters δ and η read here

$$X_1 \geq \eta - \delta + 3, \quad X_2 \leq \eta + \delta + 2 \qquad ((X_1, X_2) \in [1, 4] \times [1, 3]),$$

so we may choose, for example, $\delta = 3$ and $\eta = 1$. For this choice of δ and η, the nonlinearity in (4.8) becomes

$$F(X) + \delta(X - \tilde{X}) = \begin{pmatrix} 5X_1 - X_1X_2 - 3 \\ \\ X_1X_2 - 2 \end{pmatrix} = \begin{pmatrix} -x_1x_2 + 3x_1 - 3x_2 \\ \\ x_1x_2 + 2x_1 + 3x_2 \end{pmatrix}.$$

It remains to check Hypothesis (F). Fix $X \in \partial\tilde{Q}$; here we have to distinguish two cases.

<u>1st case</u>: X is a corner point, say $X = (4, 3)$. In this case the tangent cone at X is the angular domain

$$T_{\tilde{Q}}(X) = \{(\xi_1, \xi_2) : \xi_1 \leq 4, \xi_2 \leq 3\} = (-\infty, 4] \times (-\infty, 3].$$

So choosing, for example, $\xi := (1, 0)$, we get

$$\langle \xi, F(X) + \delta(X - \tilde{X})\rangle = 5\xi_1 + 10\xi_2 = 5 > 0.$$

<u>2nd case</u>: X is not a corner point, say $X = (4, X_2)$ with $1 \leq X_2 \leq 3$. In this case the tangent cone at X is the halfplane

$$T_{\tilde{Q}}(X) = \{(\xi_1, \xi_2) : \xi_1 \leq 4\} = (-\infty, 4] \times \mathbb{R}.$$

So choosing again $\xi := (1, 0)$ we get

$$\langle \xi, F(X) + \delta(X - \tilde{X})\rangle = (17 - 4X_2)\xi_1 + (4X_2 - 2)\xi_2 = 5 > 0.$$

The other boundary points of $\tilde{Q}$ may be treated analogously to show that Hypothesis (F) is fulfilled for (5.1).

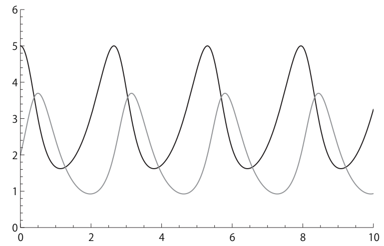

Figure 1

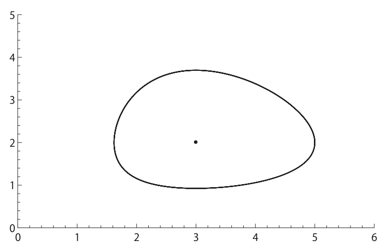

Figure 2

In Figure 1 above we have plotted $N_1(t)$ and $N_2(t)$ separately against the time axis. The graph of the predator function N_2 slightly slags the graph of the prey function N_1, reflecting the fact that the amount of prey has a retarded effect on the number of predators. In Figure 2 we have sketched the phase portraits of the predator-prey type system (5.1), where the rectangle $\tilde{Q} = [1, 4] \times [1, 3]$ is compact, and the parameters of the system are given by (5.2). The closed trajectory is the boundary of the dinosaurian egg in Q, and the equilibrium $\tilde{X}$ is marked in the interior of the egg.

Acknowledgements

The authors are grateful to both referees for several helpful remarks.

References

[1] P. Hartman, *Ordinary Differential Equations*, John Wiley & Sons, New York 1964.
[2] M. A. Krasnosel'skij and A. V. Pokrovskj, *System with Hysteresis* (Russian), Nauka, Moscow 1982; Engl. transl.: Springer, Berlin 1989.
[3] O. A. Lobanova and B. N. Sadovskj, *On two-dimensional dynamical systems with constraint* (Russian), Diff. Uravn. **43** (2007), 449–456; Engl. transl.: Diff. Uravn. **43** (2007), 460–469.
[4] L. P. Petrov and B. N. Sadovskj, *On the mathematical theory of electric circuits with a diode-type voltage converter* (Russian), Izd. Voron. Gos. Univ., Voronezh 1982.

Manuscript received May 22 2016
revised July 1 2016

J. Appell
Universität Würzburg, Mathematisches Institut, Emil-Fischer-Str. 30, D-97074 Würzburg, Germany
E-mail address: jurgen@dmuw.de

N. Merentes
Universidad Central de Venezuela, Facultad de Ciencias, Escuela de Mátematicas, Paseo Los Ilustres, Caracas 1020 A, Venezuela
E-mail address: nelson.merentes@ciens.ucv.ve

J. L. Sánchez
Universidad Central de Venezuela, Facultad de Ciencias, Escuela de Mátematicas, Paseo Los Ilustres, Caracas 1020 A, Venezuela
E-mail address: jose.sanchez@ciens.ucv.ve