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NONLINEAR NONCONVEX SECOND ORDER MULTIVALUED SYSTEMS WITH MAXIMAL MONOTONE TERMS

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Abstract. We consider a multivalued system in $\mathbb{R}^N$ driven by the vector $p-$Laplacian, with maximal monotone terms and multivalued perturbations. The boundary condition is nonlinear and general and incorporate as special cases the Dirichlet, Neumann and periodic problems. We first prove the existence of extremal trajectories. Then, for the semilinear systems (that is, $p = 2$) and for particular boundary conditions, we prove a strong relaxation theorem, showing that the extremal trajectories are dense in the solution set of the convexified system.

1. Introduction

In this paper we study the following multivalued boundary value problem

$$\left\{ \begin{array}{l} \varphi\left(u'\left(t\right)\right)' \in A\left(u\left(t\right)\right) + extF\left(t, u\left(t\right)\right) \text{ for a.a. } t \in T := [0, b] \\ \left(\varphi\left(u'\left(0\right)\right), -\varphi\left(u'\left(b\right)\right)\right) \in \xi\left(u\left(0\right), u\left(b\right)\right). \end{array} \right. \tag{1.1}$$

In this problem $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is the homeomorphism defined by

$$\varphi\left(x\right) = \left|x\right|^{p-2} x \text{ for all } x \in \mathbb{R}^N,\ 2 \le p < \infty. \tag{1.2}$$

Also $A : \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map and $F : T \times \mathbb{R}^N \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ is a multivalued vector field, with $extF\left(t, x\right)$ denoting the extreme points of the set $F\left(t, x\right)$. In the boundary condition, $\xi : \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map. The conditions on $\xi\left(.,.\right)$ are such that we recover as special cases the Dirichlet, Neumann and periodic problems. We stress that we do not require that $\mathrm{dom}A = \mathbb{R}^N$. This way we incorporate in our framework variational inequalities.

First, we prove an existence theorem for problem (1.1), and then for $p = 2$ (semilinear system) and for particular boundary conditions, we show that the solutions of problem (1.1) are dense with respect to the topology of the space $C\left(T, \mathbb{R}^N\right)$ in the solution set of the convexified problem (strong relaxation theorem). Such a result is important in control theory in connection with the so-called "bang-bang principle".

This kind of second order systems in $\mathbb{R}^N$ were studied by Erbe-Krawcewicz [3], Frigon [4], Frigon-Montoki [5], Halidias-Papageorgiou [9], Hu-Papageorgiou [13] (section III.1) and Kyritsi-Matzakos-Papageorgiou [15]. All the aforementioned

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works deal with semilinear systems (that is, $p = 2$). Systems driven by the vector $p-$Laplacian were considered by Halidias-Papageorgiou [10] and Papageorgiou-Papageorgiou [16] (periodic problems). None of these works addresses the issue of "extremal" solutions and of strong relaxation.

2. Mathematical background

Our analysis of problem (1.1) uses tools from multivalued analysis and from the theory of nonlinear operators of monotone type, which we recall in this section. Details can be found in the books of Hu-Papageorgiou [12] and Zeidler [17].

Let (Ω, Σ) be a measurable space and $(X, \|.\|)$ a separable Banach space. We introduce the following notation:

$$P_{f(c)}(X) = \{C \subseteq X : C \text{ is nonempty, closed, (convex)}\},$$

$$P_{(w)k(c)}(X) = \{C \subseteq X : C \text{ is nonempty, (weakly-) compact, (convex)}\}.$$

If $F : \Omega \to 2^X \setminus \{\varnothing\}$ is a multifunction (set valued map), then the *graph* of F is the set

$$Gr\ F = \{(\omega, x) \in \Omega \times X : x \in F(\omega)\}.$$

We say that $F(.)$ is *graph measurable* if $Gr\ F \in \Sigma \times \mathcal{B}(X)$ where $\mathcal{B}(X)$ is the Borel $\sigma-$field of X. Suppose that μ is a $\sigma-$finite measure on Σ and $F : \Omega \to 2^X \setminus \{\varnothing\}$ is graph measurable. Then the Yankov-von Neumann-Aumann selection theorem (see Hu-Papageorgiou [12], p.158) implies that $F(.)$ admits a measurable selection, that is, there exists a $\Sigma-$measurable function $f : \Omega \to X$ such that

$$f(\omega) \in F(\omega) \text{ for } \mu - \text{a.a..}\omega \in \Omega.$$

In fact, there is a whole sequence of $\Sigma-$measurable selections $f_n : \Omega \to X$, $n \in N$ such that

$$F(\omega) \subseteq \overline{\{f_n(\omega)\}}_{n\in\mathbb{N}} \text{ for } \mu - \text{a.a. } \omega \in \Omega.$$

(see Hu-Papageorgiou [12], p.159). Moreover, this result remains true if X is only a Souslin space. Recall that a Souslin space is always separable but needs not be metrizable (see Hu-Papageorgiou [12], p.145).

A multifunction $F : \Omega \to P_f(X)$ is said to be *measurable*, if for every $v \in X$, the function

$$\omega \to d(v, F(\omega)) := \inf\{\|v - x\| : x \in F(\omega)\}$$

is $\Sigma-$measurable. A measurable multifunction $F : \Omega \to P_f(X)$ is also graph measurable. The converse is true if (Ω, Σ) admits a complete $\sigma-$finite measure μ.

Suppose that (Ω, Σ, μ) is a $\sigma-$finite measure space, $1 \leq p \leq \infty$ and $F : \Omega \to 2^X \setminus \{\varnothing\}$ is a multifunction. We define the set

$$S_F^p = \{f \in L^p(\Omega, X) : f(\omega) \in F(\omega)\ \ \mu - \text{a.e. in } \Omega\}.$$

Using the Yankov-von Neumann-Aumann selection theorem, we see that for a graph measurable multifunction $F(.)$ we have $S_F^p \neq \varnothing$, if and only if

$$\inf\{\|x\| : x \in F(\omega)\} \in L^p(\Omega).$$

The set S_F^p is *decomposable,* in the sense that if $(C, f_1, f_2) \in \Sigma \times S_F^p \times S_F^p$, then

$$f_1\chi_C + f_2\chi_{\Omega\setminus C} \in S_F^p.$$

Here, for any $E \in \Sigma$, χ_E denotes the characteristic function of E.

For any Banach space Y, we can define the Hausdorff-Pompeiu generalized metric on $P_f(Y)$ by setting

$$\begin{aligned} h(A,B) &= \sup\{|d(v,A) - d(v,B)| : v \in Y\} \\ &= \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\right\} \text{ for all } A,\ B \in P_f(Y). \end{aligned}$$

We know that $(P_f(Y), h)$ is complete.

If Z is a Hausdorff topological space, a multifunction $G : Z \to P_f(Y)$ is said to be Hausdorff continuous (h-continuous, for short) if it is a continuous map from Z into $(P_f(Y), h)$.

Also, if Z, Y are Hausdorff topological spaces and $F : Z \to 2^Y \setminus \{\varnothing\}$ is a multifunction, we say that $F(.)$ is lower semicontinuous (lsc for short), if for every open subset U of Y,

$$F^-(U) := \{z \in Z : F(z) \cap U \neq \varnothing\}$$

is open. Recall that if $F(.)$ is lsc, then so is $z \to \overline{F(z)}$ (see Gasinski-Papageorgiou [8]).

Let X be a real reflexive Banach space and X^* its topological dual. By $(\cdot,\cdot)$ we denote the duality brackets for the pair (X^*, X). We will use the symbol $\xrightarrow{w}$ to designate weak convergence.

A multivalued map $A : D(A) \subseteq X \to 2^{X^*}$ is said to be *monotone* if

$$(x^* - y^*, x - y) \geq 0 \text{ for all } (x,x^*),\ (y,y^*) \in GrA.$$

We say that A is *strictly monotone* if

$$(x^* - y^*, x - y) = 0 \implies x = y.$$

The map A is called *maximal monotone* if

$$[(x^* - y^*, x - y) \geq 0 \text{ for all } (x,x^*) \in GrA] \Longrightarrow (y,y^*) \in GrA.$$

This means that $Gr\ A$ is maximal with respect to inclusion among the graphs of all monotone maps. Then it is easy to see that for a maximal monotone map $A(.)$, Gr A is sequentially closed in $X_w \times X^*$ and in $X \times X^*_{w.}$. Here by X_w (resp. X^*_w we denote the space X (resp. X^*) furnished with the weak topology. Also, if $A(.)$ is maximal monotone, then for every $x \in D(A)$ one has $A(x) \in P_{fc}(X^*)$.

Let $X = H$ be a real Hilbert space with norm $\|\cdot\|_H$ and let A be a maximal monotone map in H with domain

$$D(A) := \left\{x \in \mathbb{R}^N : A(x) \neq \varnothing\right\}.$$

For $\lambda > 0$, we define the following single valued maps approximating the identity operator and A, respectively:

$$\begin{aligned} J_\lambda &:= (I + \lambda A)^{-1} \text{ (the \textit{resolvent} of } A), \\ A_\lambda &:= \frac{1}{\lambda}(I - J_\lambda) \text{ (the \textit{Yosida approximation} of } A). \end{aligned}$$

Recall that $A(.)$ is maximal monotone if and only if for every $\lambda > 0$ (equivalently, for some $\lambda > 0$)

$$R(I + \lambda A) = H$$

(that is, the operator $I+\lambda A$ is surjective). The result is known as Minty's theorem (see Hu-Papageorgiou [12], p. 321) and Zeidler [17], p. 855). The maps J_λ and A_λ exhibit several interesting properties which are collected in the next proposition (see Hu-Papageorgiou [12], p. 329).

Proposition 2.1. *If $A : D(A) \subseteq H \to 2^H$ is a maximal monotone map and $\lambda > 0$, then*

(*a*) *$J_\lambda : H \to H$ is nonexpansive (that is, $\|J_\lambda(x) - J_\lambda(y)\|_H \leq \|x-y\|_H$ for all $x, y \in H$);*
(*b*) *$A_\lambda(x) \in A(J_\lambda(x))$ for all $x \in H$;*
(*c*) *$A_\lambda(.)$ is monotone and Lipschtz continuous with Lipschitz constant $\frac{1}{\lambda}$ (hence $A_\lambda(.)$ is maximal monotone).*

We conclude this section by finalizing our notation. So, suppose that $C \subseteq \mathbb{R}^N$, $C \neq \varnothing$. We set

$$|C| = \sup\{|x| : x \in C\},$$

where $|.|$ denotes the norm of $\mathbb{R}^N$.

Also, by $(\cdot,\cdot)_{\mathbb{R}^N}$ we denote the inner product on $\mathbb{R}^N$, by $\mathcal{L}_T$ the Lebesgue $\sigma-$field on $T := [0,b]$ and by $\mathcal{B}(\mathbb{R}^N)$ the Borel $\sigma-$field of $\mathbb{R}^N$. If $1 \leq p < \infty$ then $1 < p' \leq \infty$ satisfies $\frac{1}{p} + \frac{1}{p'} = 1$. The classical norm in $L^p(T, \mathbb{R}^N)$ is denoted by $\|.\|_p$, while the duality pairing between $L^p(T,\mathbb{R}^N)$ and $L^{p'}(T,\mathbb{R}^N)$, $1 \leq p < \infty$ is designated by $(\cdot,\cdot)_{p,p'}$.

For a Banach space V, the *weak norm* on $L^1(T,V)$, denoted by $\|.\|_w$, is defined by

$$\|f\|_w = \sup\left\{\left\|\int_s^t f(\tau)\,d\tau\right\|_V : 0 \leq s \leq t \leq b\right\}$$

or equivalently by

$$\|f\|_w = \sup\left\{\left\|\int_0^t f(\tau)\,d\tau\right\|_V : 0 \leq t \leq b\right\}.$$

This is equivalent to the Pettis norm (see Egghe [2]). By $L^1_w(T,V)$ we denote the space $L^1(T,V)$ furnished with the weak norm $\|.\|_w$.

By $\Gamma_0(V)$ we denote the cone of functions $\psi : V \to \overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ which are proper (that is, not identically $+\infty$), convex and lower semicontinuous. By $\partial\psi$ we denote the subdifferential in the sense of convex analysis of ψ, defined by

$$\partial\psi(v) = \{v^* \in V^* : \psi(x) - \psi(v) \geq (v^*, y-x) \text{ for all } x \in V\}.$$

If $K \in P_{fc}(V)$, then the indicator function of K is defined by

$$i_K(v) = \begin{cases} 0 & \text{if } v \in K \\ +\infty & \text{if } v \notin K. \end{cases}$$

Then $i_K \in \Gamma_0(V)$ and $\partial i_K = N_K$ is the normal cone to K.

3. Extremal trajectories

In this section we produce solutions for problem (1.1). For this, we impose the following conditions on the data of problem (1.1).

$\mathbf{H}(A):$ $A : D(A) \subseteq \mathbb{R}^N \to 2^{\mathbb{R}^N}$ is a maximal monotone map with $0 \in A(0)$.

$\mathbf{H}(\xi):$ $\xi : D(\xi) \subseteq \mathbb{R}^N \times \mathbb{R}^N \to 2^{\mathbb{R}^N \times \mathbb{R}^N}$ is a maximal monotone map such that $(0,0) \in \xi(0,0)$ and one of the following conditions holds:
(i) for every $(e,e') \in \xi(d,d')$ we have $(e,d)_{\mathbb{R}^N} \geq 0$ and $(e',d')_{\mathbb{R}^N} \geq 0$
or
(ii) $D(\xi) = \left\{(d,d') \in \mathbb{R}^N \times \mathbb{R}^N : d = d'\right\}$.

We will also need a hypothesis relating $A(.)$ and $\xi(.,.)$:

$\mathbf{H}_0:$ For every $\lambda > 0$ and all $((d,d'),(e,e')) \in Gr\ \xi$ we have

$$(A_\lambda(d), e)_{\mathbb{R}^N} + \left(A_\lambda\left(d'\right), e'\right)_{\mathbb{R}^N} \geq 0.$$

Remark. Suppose that $D(A) = \mathbb{R}^N$ and $\xi = \partial\psi$ with $\psi \in \Gamma_0\left(\mathbb{R}^N \times \mathbb{R}^N\right)$. Assume that

$$\psi\left(\left((I+\lambda A)^{-1} x - (I+\lambda A)^{-1} x'\right), y\right) \leq \psi\left(x - x', y\right)$$
$$\psi\left(x, \left((I+\lambda A)^{-1} y - (I+\lambda A)^{-1} y'\right)\right) \leq \psi\left(x, y - y'\right)$$

for all x, x', y, $y' \in \mathbb{R}^N$, $\lambda > 0$. Then $\mathbf{H}_0$ is satisfied (see Barbu [1], p. 187).

The conditions on the multivalued vector field $F(t,x)$ are the following:

$\mathbf{H}(F)_1:$ $F : T \times \mathbb{R}^N \to P_{kc}\left(\mathbb{R}^N\right)$ is a multifunction such that
(i) for every $x \in \mathbb{R}^N$, $t \to F(t,x)$ is graph measurable;
(ii) for a.a. $t \in T$, $x \to F(t,x)$ is $h-$continuous;
(iii) for every $r > 0$, there exists $a_r \in L^2(T)$ such that

$$|F(t,x)| \leq a_r(t) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N \text{ with } |x| \leq r;$$

(iv) there exists $M > 0$ such that for a.a. $t \in T$, all $x \in \mathbb{R}^N$ with $|x| = M$ and all $v \in F(t,x)$, we have

$$(v,x)_{\mathbb{R}^N} \geq 0.$$

Remarks. Hypotheses $\mathbf{H}(F)_1\,(i)$, (ii) imply that $(t,x) \to F(t,x)$ is measurable. In particular it follows that F is superpositionally measurable, that is, for all $u : T \to \mathbb{R}^N$ measurable, $t \to F(t,u(t))$ is measurable.

Hypothesis $\mathbf{H}(F)_1\,(iv)$ is a multivalued version of the so-called *Hartman condition.* It was first used by Hartman [11] in the context of second order Dirichlet systems with a single valued continuous vector field $f(t,x)$. Later it was used by Knobloch [14] for periodic systems.

Let $\mathcal{A} : L^p\left(T, \mathbb{R}^N\right) \to 2^{L^{p'}\left(T,\mathbb{R}^N\right)}$ be the lifting (realization) of A on the dual pair $\left(L^{p'}\left(T, \mathbb{R}^N\right), L^p\left(T, \mathbb{R}^N\right)\right)$, that is, $\mathcal{A}$ is defined by

$$\mathcal{A}(u) = \left\{g \in L^{p'}\left(T, \mathbb{R}^N\right) : g(t) \in A(u(t)) \text{ for a.a. } t \in T\right\},$$

with

$$D(\mathcal{A}) = \left\{u \in L^p\left(T, \mathbb{R}^N\right) : S^{p'}_{A(.,u(.))} \neq \varnothing\right\}.$$

Evidently, $D(\mathcal{A}) \neq \varnothing$ (see hypothesis $\mathbf{H}(A)$).

Lemma 3.1. *If hypothesis* $\mathbf{H}(A)$ *holds, then the map* $\mathcal{A} : L^p\left(T, \mathbb{R}^N\right) \to 2^{L^{p'}\left(T,\mathbb{R}^N\right)}$ *is maximal monotone.*

Proof. Let $\widehat{J} : L^p\left(T, \mathbb{R}^N\right) \to L^{p'}\left(T, \mathbb{R}^N\right)$ be the map defined by

$$\widehat{J}(u)(.) = |u(.)|^{p-2}\, u(.) \text{ for all } u \in L^p\left(T, \mathbb{R}^N\right). \tag{3.1}$$

Evidently $\widehat{J}(.)$ is continuous and strictly monotone, thus maximal monotone, too.

Claim 1. $R\left(\mathcal{A} + \widehat{J}\right) = L^{p'}\left(T, \mathbb{R}^N\right)$.

Let $h \in L^{p'}\left(T, \mathbb{R}^N\right)$ and consider the multifunction $\widehat{E} : T \to 2^{\mathbb{R}^N}$ defined by

$$\widehat{E}(t) = \left\{x \in \mathbb{R}^N : h(t) \in A(x) + \varphi(x)\right\} \text{ a.e. on } T,$$

where $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is defined by (1.2). The map $x \to A(x) + \varphi(x)$ is maximal monotone (see Hu-Papageorgiou [12], p.344) and since $0 \in A(0)$, we have

$$(A(x) + \varphi(x), x)_{\mathbb{R}^N} \geq |x|^p \text{ for all } x \in \mathbb{R}^N.$$

Therefore $x \to A(x)+\varphi(x)$ is coercive. It follows that $x \to A(x)+\varphi(x)$ is surjective (see Hu-Papageorgiou [12], p.322), and so $\widehat{E}(t) \neq \varnothing$ for all $t \in T$.

We have

$$Gr\ \widehat{E} = \left\{(t, x) \in T \times \mathbb{R}^N : (x, h(t) - \varphi(x)) \in GrA\right\}. \tag{3.2}$$

Let $\eta : T \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$ be defined by

$$\eta(t, x) = (x, h(t) - \varphi(x)) \text{ for almost all } t \in T, \text{ all } x \in \mathbb{R}^N.$$

Clearly η is a Carathéodory function (that is, for all $x \in \mathbb{R}^N$, $t \to \eta(t, x)$ is measurable and for a.a. $t \in T$, $x \to \eta(t, x)$ is continuous). Hence η is jointly measurable (see Hu-Papageorgiou [12], p.142).

Also, since A is maximal monotone, $GrA \subseteq \mathbb{R}^N \times \mathbb{R}^N$ is closed. So, from (3.2), it follows that

$$Gr\ \widehat{E} = \eta^{-1}(GrA) \in \mathcal{L}_T \times \mathcal{B}\left(\mathbb{R}^N\right).$$

Therefore, we can use the Yankov-von Neumann-Aumann theorem and find a measurable function $u : T \to \mathbb{R}^N$ such that

$$u(t) \in \widehat{E}(t) \text{ for a.a. } t \in T,$$

hence

$$h(t) \in A(u(t)) + \varphi(u(t)) \text{ for a.a. } t \in T.$$

Taking the inner product with $u(t)$ and recalling that $0 \in A(0)$, we obtain

$$|u(t)|^p \leq |h(t)|\, |u(t)| \text{ for a.a. } t \in T$$

hence

$$|u(t)|^{p-1} \leq |h(t)| \text{ for a.a. } t \in T,$$

therefore $u \in L^p(T, \mathbb{R}^N)$ and we have

$$h \in \mathcal{A}(u) + \widehat{J}(u).$$

Since $h \in L^{p'}(T, \mathbb{R}^N)$ is arbitrary, we conclude that

$$R\left(\mathcal{A} + \widehat{J}\right) = L^{p'}(T, \mathbb{R}^N).$$

This proves Claim 1.

Clearly $\mathcal{A}(.)$ is monotone. Using Claim 1, we can show the maximality of $\mathcal{A}$.

To this end, suppose that $(v, h) \in L^p(T, \mathbb{R}^N) \times L^{p'}(T, \mathbb{R}^N)$ satisfies

(3.3)

$$(g - h, u - v)_{p,p'} = \int_0^b (g(t) - h(t), u(t) - v(t))_{\mathbb{R}^N} \geq 0 \text{ for all } (u, g) \in Gr\mathcal{A}.$$

Invoking the Claim, we can find $(u_1, g_1) \in Gr\mathcal{A}$ such that

$$g_1 + \widehat{J}(u_1) = h + \widehat{J}(v). \tag{3.4}$$

Using (3.4) in (3.3) with $(u, g) = (u_1, g_1)$ we obtain

$$\left(\widehat{J}(v) - \widehat{J}(u_1), u_1 - v\right)_{p,p'} \geq 0,$$

hence

$$u_1 = v \text{ (recall that } \varphi \text{ is strictly monotone)},$$

therefore $(v, h) \in Gr\mathcal{A}$ and so, $\mathcal{A}$ is maximal monotone. □

Given $h \in L^2(T, \mathbb{R}^N)$, we consider the following auxiliary system

$$\begin{cases} \varphi(u'(t))' - \varphi(u(t)) \in Au(t) + h(t) \text{ for a.a. } t \in T := [0, b] \\ (\varphi(u'(0)), -\varphi(u'(b))) \in \xi(u(0), u(b)). \end{cases} \tag{3.5}$$

Recall that $\varphi : \mathbb{R}^N \to \mathbb{R}^N$ is the homeomorphism defined by (1.2).

Proposition 3.2. *If hypotheses* $\mathbf{H}(A)$, $\mathbf{H}(\xi)$, $\mathbf{H}_0$ *hold, then for every* $h \in L^2(T, \mathbb{R}^N)$, *problem* (3.5) *admits a unique solution* $u = \theta(h) \in C(T, \mathbb{R}^N)$ *and the map* $\theta : L^2(T, \mathbb{R}^N) \to C(T, \mathbb{R}^N)$ *is completely continuous (that is, if* $h_n \xrightarrow{w} h$ *in* $L^2(T, \mathbb{R}^N)$, *then* $\theta(h_n) \to \theta(h)$ *in* $C(T, \mathbb{R}^N)$*).*

Proof. First we show the uniqueness of the solution of problem (3.5). So, suppose that $u, v \in C(T, \mathbb{R}^N)$ are two solutions of (3.5). Then exploiting the monotonicity of $A(.)$, we have

$$\begin{aligned} &(\varphi(u(t)) - \varphi(v(t)), u(t) - v(t))_{\mathbb{R}^N} \\ &\leq \left(\varphi(u'(t))' - \varphi(v'(t))', u(t) - v(t)\right)_{\mathbb{R}^N} \text{ for a.a. } t \in T. \end{aligned}$$

Since $2 \leq p$, it follows that

$$2^{2-p} |u(t) - v(t)|^p \leq (\varphi(u(t)) - \varphi(v(t)), u(t) - v(t))_{\mathbb{R}^N} \text{ for a.a. } t \in T,$$

hence

$$\begin{aligned}
2^{2-p}\left\|u-v\right\|_p^p &\le \int_0^b \Big(\varphi\left(u'(t)\right)'-\varphi\left(v'(t)\right)', u(t)-v(t)\Big)_{\mathbb{R}^N} dt \\
&= \Big(\varphi\left(u'(b)\right)'-\varphi\left(v'(b)\right)', u(b)-v(b)\Big)_{\mathbb{R}^N} \\
&\quad - \Big(\varphi\left(u'(0)\right)'-\varphi\left(v'(0)\right)', u(0)-v(0)\Big)_{\mathbb{R}^N} \\
&\quad - \int_0^b \left(\varphi\left(u'\right)-\varphi\left(v'\right), u'-v'\right)_{\mathbb{R}^N} dt \text{ (by integration by parts)} \\
&\le -\int_0^b \left(\varphi\left(u'\right)-\varphi\left(v'\right), u'-v'\right)_{\mathbb{R}^N} dt \text{ (from the monotonicity of } \xi) \\
&\le -2^{2-p}\left\|u'-v'\right\|_p^p,
\end{aligned}$$

therefore

$$\frac{1}{2^{p-2}}\left[\left\|u-v\right\|_p^p+\left\|u'-v'\right\|_p^p\right] \le 0.$$

It follows that

$$\left\|u-v\right\|_{W^{1,p}(\Omega)}^p \le 0$$

hence

$$u=v.$$

This proves the uniqueness of the solution of (3.5). Next we show the existence of such a solution. To this end, let $V: D \subseteq L^p\left(T,\mathbb{R}^N\right) \to L^{p'}\left(T,\mathbb{R}^N\right)$ be the nonlinear map defined by

$$V(u) = -\varphi\left(u'\right)'$$

for all $u \in D$ with

$$\begin{aligned}
D = \Big\{ & v \in W^{1,p}\left((0,b),\mathbb{R}^N\right) : \varphi\left(v'(.)\right) \in W^{1,p'}\left((0,b),\mathbb{R}^N\right), \\
& \left(\varphi\left(v'(0)\right), -\varphi\left(v'(b)\right)\right) \in \xi\left(v(0), v(b)\right)\Big\}.
\end{aligned}$$

Note that if $v \in D$, then

$$v \in C\left(T,\mathbb{R}^N\right) \text{ and } \varphi\left(v'(.)\right) \in C\left(T,\mathbb{R}^N\right).$$

Because φ is a homeomorphism it follows that

$$v' \in C\left(T,\mathbb{R}^N\right)$$

hence

$$v \in C^1\left(T,\mathbb{R}^N\right).$$

So, the evaluations at $t=0$ and $t=b$ of v and v' in the definition of D make sense.

We show that $V(.)$ is maximal monotone. So, as in the proof of Lemma 3.1, let $\widehat{J}: L^p\left(T,\mathbb{R}^N\right) \to L^{p'}\left(T,\mathbb{R}^N\right)$ be the map defined by (3.1). From Proposition 3 of Halidias-Papageorgiou [10] (see also Halidias-Papageorgiou, Theorem 1, Claim 1), we conclude that

$$R\left(V+\widehat{J}\right) = L^{p'}\left(T,\mathbb{R}^N\right). \tag{3.6}$$

Since V is monotone, as in the proof of Lemma 3.1, using (3.6), we show the maximality of V.

For $\lambda > 0$, let $N_{A_\lambda} : L^p\left(T, \mathbb{R}^N\right) \to L^{p'}\left(T, \mathbb{R}^N\right)$ be defined by

$$N_{A_\lambda}(u)(.) := A_\lambda(u(.)) \text{ for all } u \in L^p\left(T, \mathbb{R}^N\right).$$

Evidently, N_{A_λ} is monotone continuous (see Proposition 2.1 and recall that $p \geq 2$) and so, it is maximal monotone.

Let

$$K_\lambda := V + \widehat{J} + N_{A_\lambda}.$$

Then K_λ is maximal monotone (see Hu-Papageorgiou [13], p.334) and for every $u \in D$, we have

$$\begin{aligned}
&(K_\lambda(u), u)_{p,p'} \\
&= (V(u), u)_{p,p'} + \|u\|_p^p + \int_0^b (A_\lambda(u), u)_{\mathbb{R}^N} dt \\
&\geq \int_0^b \left(-\varphi\left(u'(t)\right)', u(t)\right)_{\mathbb{R}^N} dt + \|u\|_p^p \text{ (since } A_\lambda \text{ is monotone and } A_\lambda(0) = 0 \\
&\geq \int_0^b \left(\varphi\left(u'(t)\right), u'(t)\right)_{\mathbb{R}^N} dt + \|u\|_p^p \text{ (by integration by parts, since } u \in D) \\
&= \left\|u'\right\|_p^p + \|u\|_p^p \\
&= \|u\|_{W^{1,p}}^p ,
\end{aligned}$$

hence K_λ is coercive.

Recall that a maximal monotone coercive map is surjective (see Hu-Papageorgiou [12], p.322). Hence we can find $u_\lambda \in D \subseteq C^1\left(T, \mathbb{R}^N\right)$ such that

$$V(u_\lambda) + \widehat{J}(u_\lambda) + N_{A_\lambda}(u_\lambda) = -h. \tag{3.7}$$

Let $\lambda_n \downarrow 0$ and let $u_n = u_{\lambda_n} \in D \subseteq C^1\left(T, \mathbb{R}^N\right)$ be the solution of (3.7) established above (in fact, from the first part of the proof we know that this solution is unique). We have

$$(V(u_n), u_n)_{p,p'} + \|u_n\|_p^p + \int_0^b (A_{\lambda_n}(u_n), u_n)_{\mathbb{R}^N} dt = -\int_0^b (h, u_n)_{\mathbb{R}^N} dt,$$

hence

$$(V(u_n), u_n)_{p,p'} + \|u_n\|_p^p \leq \|h\|_{p'} \|u_n\|_p \text{ (since } A_{\lambda_n}(0) = 0 \text{ for all } n \in \mathbb{N}). \tag{3.8}$$

As before, using integration by parts and the fact that $u_n \in D$ for all $n \in \mathbb{N}$, we obtain

$$\left\|u_n'\right\|_p^p \leq (V(u_n), u_n)_{p,p'} \text{ for all } n \in \mathbb{N}. \tag{3.9}$$

Returning to (3.8) and using (3.9), we obtain

$$\|u_n\|_{W^{1,p}}^{p-1} \leq C_1 \|h\|_{p'} \text{ for some } C_1 > 0, \text{ all } n \in \mathbb{N},$$

hence

$$\{u_n\}_{n \in \mathbb{N}} \subseteq W^{1,p}\left((0,b), \mathbb{R}^N\right) \text{ is bounded.}$$

So, by passing to a subsequence if necessary, we may assume that

$$u_n \xrightarrow{w} u \text{ in } W^{1,p}\left((0,b), \mathbb{R}^N\right) \text{ and } u_n \to u \text{ in } C\left(T, \mathbb{R}^N\right). \tag{3.10}$$

Note that $N_{A_{\lambda_n}}(u_n) \in C(T, \mathbb{R}^N)$. Acting with it on (3.7) we obtain

$$\begin{aligned}(3.11)\qquad &\left(V(u_n), N_{A_{\lambda_n}}(u_n)\right)_{p,p'} + \int_0^b |u_n|^{p-2}(u_n, A_{\lambda_n}(u_n))_{\mathbb{R}^N} dt \\ &+ \left\|N_{A_{\lambda_n}}(u_n)\right\|_2^2 = -\int_0^b (h, A_{\lambda_n}(u_n))_{\mathbb{R}^N} dt.\end{aligned}$$

Since $A_{\lambda_n}(0) = 0$ and $A_{\lambda_n}(.)$ is monotone, we have

$$(3.12)\qquad \int_0^b |u_n|^{p-1}(u_n, A_{\lambda_n}(u_n))_{\mathbb{R}^N} dt \geq 0.$$

Also, we have

$$\begin{aligned}(3.13)\qquad \left(V(u_n), N_{A_{\lambda_n}}(u_n)\right)_{p,p'} &= \int_0^b \left(-\varphi(u_n')', A_{\lambda_n}(u_n)\right)_{\mathbb{R}^N} dt \\ &= -\left(\varphi(u_n'(b))', A_{\lambda_n}(u_n(b))\right)_{\mathbb{R}^N} + \left(\varphi(u_n'(0)), A_{\lambda_n}(u_n(0))\right)_{\mathbb{R}^N} \\ &+ \int_0^b \left(\varphi(u_n'), \frac{d}{dt}A_{\lambda_n}(u_n)\right)_{\mathbb{R}^N} dt \text{ (by integration by parts).}\end{aligned}$$

Since $A_{\lambda_n}: \mathbb{R}^N \to \mathbb{R}^N$ is Lipschitz continuous (see Proposition 2.1), by Rademacher's theorem (see Gasinski-Papageorgiou [8], Theorem 3.120, p. 433) it is differentiable at almost all $x \in \mathbb{R}^N$. Also, $A_{\lambda_n}(.)$ is monotone (see Proposition 2.1). So, if $x \in \mathbb{R}^N$ is a point of differentiability of $A_{\lambda_n}(.)$, exploiting the monotonicity of $A_{\lambda_n}(.)$, we have

$$\left(\frac{1}{\varepsilon}\left[A_{\lambda_n}(x + \varepsilon h) - A_{\lambda_n}(x)\right], h\right)_{\mathbb{R}^N} \geq 0,\ \forall \varepsilon > 0,$$

hence

$$(3.14)\qquad \left(A_{\lambda_n}'(x)h, h\right)_{\mathbb{R}^N} \geq 0 \text{ for all } h \in \mathbb{R}^N.$$

In addition, from the chain rule for Sobolev functions (see Gasinski-Papageorgiou [6], p.195), we have

$$(3.15)\qquad \frac{d}{dt}A_{\lambda_n}(u_n(t)) = A_{\lambda_n}(u_n(t))\left(u_n'(t)\right) \text{ for a.a. } t \in T.$$

Finally, hypothesis $\mathbf{H}_0$ implies that

$$(3.16)\qquad \left(-\varphi(u_n'(b)), A_{\lambda_n}(u_n(b))\right)_{\mathbb{R}^N} + \left(\varphi(u_n'(0)), A_{\lambda_n}(u_n(0))\right)_{\mathbb{R}^N} \geq 0.$$

Returning to (3.13) and using (3.14), (3.15), (3.16), we obtain

$$(3.17)\qquad \left(V(u_n), N_{A_{\lambda_n}}(u_n)\right)_{p,p'} \geq \int_0^b |u_n'|^{p-2}\left(u_n', A_{\lambda_n}'(u_n)u_n'\right)_{\mathbb{R}^N} dt \geq 0.$$

We use (3.12) and (3.17) in (3.11). Then

$$\left\|N_{A_{\lambda_n}}(u_n)\right\|_2^2 \leq \|h\|_2 \left\|N_{A_{\lambda_n}}(u_n)\right\|_2 \text{ for all } n \in \mathbb{N},$$

hence

$$(3.18)\qquad \left\|N_{A_{\lambda_n}}(u_n)\right\|_2 \leq \|h\|_2 \text{ for all } n \in \mathbb{N}.$$

Therefore $\left\{N_{A_{\lambda_n}}(u_n)\right\}_{n\geq 1}\subseteq L^2\left(T,\mathbb{R}^N\right)$ is bounded and so, by passing to a subsequence if necessary, we may assume that

$$N_{A_{\lambda_n}}(u_n)\xrightarrow{w}\gamma \text{ in } L^2\left(T,\mathbb{R}^N\right). \tag{3.19}$$

On (3.7) we act with $u_n-u\in W^{1,p}\left((0,b),\mathbb{R}^N\right)$ (recall that $N_{A_{\lambda_n}}(u_n)\in C\left(T,\mathbb{R}^N\right)$ for all $n\in\mathbb{N}$). We obtain

$$\begin{aligned}&(V(u_n),u_n-u)_{p,p'}+\left(\widehat{J}(u_n),u_n-u\right)_{p,p'}+\left(N_{A_{\lambda_n}}(u_n),u_n-u\right)_{p,p'}\\&=-(h,u_n-u)_{p,p'},\ \forall n\in\mathbb{N}.\end{aligned} \tag{3.20}$$

Evidently

$$\left(\widehat{J}(u_n),u_n-u\right)_{p,p'},\ \ \left(N_{A_{\lambda_n}}(u_n),u_n-u\right)_{p,p'},\ \ (h,u_n-u)_{p,p'}\to 0,$$

hence

$$\lim_{n\to\infty}(V(u_n),u_n-u)_{p,p'}=0 \tag{3.21}$$

(see (3.20)). On account of the maximal monotonicity of $V(.)$, from (3.21) and Proposition 3.2.47 of Gasinski-Papageorgiou [6], p.330) it follows that $u\in D$ (in particular $u\in C^1\left(T,\mathbb{R}^N\right)$) and

$$V(u_n)\xrightarrow{w}V(u)\ \text{ in } L^{p'}\left(T,\mathbb{R}^N\right). \tag{3.22}$$

So, if in (3.20) we pass to the limit as $n\to\infty$ and use (3.10), (3.19) and (3.22), we obtain

$$V(u)+\widehat{J}(u)+\gamma=-h. \tag{3.23}$$

Note that

$$\left(J_{\lambda_n}(u_n),N_{A_{\lambda_n}}(u_n)\right)\in Gr\mathcal{A} \text{ for all } n\in\mathbb{N}. \tag{3.24}$$

Since $\mathcal{A}$ is maximal monotone (see Lemma 3.1), from (3.10), (3.19) and (3.24) we obtain

$$(u,\gamma)\in Gr\mathcal{A}$$

hence

$$\gamma(t)\in A(u(t))\ \text{ for a.a. } t\in T.$$

So, $u\in C^1\left(T,\mathbb{R}^N\right)$ is the unique solution of the auxiliary problem (3.5).

Now we show the complete continuity of the solution map θ. To this end let $h_n\xrightarrow{w}h$ in $L^2\left(T,\mathbb{R}^N\right)$ and for all $n\in\mathbb{N}$, let $u_n=\theta(h_n)$ be the unique solution of (3.5) for the input h_n. From the previous part of the proof, we know that

$$u_{nm}\to u_n \text{ in } C\left(T,\mathbb{R}^N\right)\ \text{ as } m\to\infty \tag{3.25}$$

where $u_{nm}\in D\subseteq C^1\left(T,\mathbb{R}^N\right)$ is the unique solution of the nonlinear operator equation

$$V(u)+\widehat{J}(u)+N_{A_{\lambda_m}}(u)=-h_n \text{ for all } m,n\in\mathbb{N}. \tag{3.26}$$

Also, we have

$$\left(\varphi\left(u_n'(t)\right)\right)'-\varphi(u_n(t))\in A(u_n(t))+h_n(t)\ \text{ for a.a. } t\in T,\ \text{ all } \in\mathbb{N}. \tag{3.27}$$

We take the inner product of (3.27) with $u_n(t)$, integrate over $T=[0,b]$ and use integration by parts and the boundary condition. We obtain

$$\left\|u_n'\right\|_p^p+\|u_n\|_p^p\le C_2\|h_n\|_{p'}\|u_n\|_p \text{ for some } C_2>0, \text{all } n\in\mathbb{N},$$

hence $\{u_n\}_{n\in\mathbb{N}}$ is bounded in $W^{1,p}\left((0,b),\mathbb{R}^N\right)$. So, by passing to a subsequence if necessary, we may assume that

$$u_n\to u \text{ in } C\left(T,\mathbb{R}^N\right). \tag{3.28}$$

From (3.25), (3.28) and Problem 1.175 of Gasinski-Papageorgiou ([7], p.61), it follows that there exists a sequence $n\to m(n)$ increasing (not necessarily strictly) to $+\infty$ such that

$$u_{nm(n)}\to u \text{ in } C\left(T,\mathbb{R}^N\right) \text{ as } n\to\infty. \tag{3.29}$$

As before (see (3.18)) we conclude that

$$\left\|N_{A_{\lambda_{m(n)}}}\right\|_2\le\|h_n\|_2.$$

Since $h_n\xrightarrow{w}h$ in $L^2\left(T,\mathbb{R}^N\right)$, it follows that

$$N_{A_{\lambda_{m(n)}}} \text{is bounded in } L^2\left(T,\mathbb{R}^N\right). \tag{3.30}$$

Recall that

$$V\left(u_{nm(n)}\right)+\widehat{J}\left(u_{nm(n)}\right)+N_{A_{\lambda_{m(n)}}}\left(u_{nm(n)}\right)=-h_n \text{ for all } n\in\mathbb{N} \tag{3.31}$$

(see (3.26)). From (3.29), (3.30) and (3.31) it follows that

$$\left\{V\left(u_{nm(n)}\right)=-\varphi\left(u_{nm(n)}'\right)'\right\}_{n\ge1}\subseteq L^{p'}\left(T,\mathbb{R}^N\right) \text{ is bounded.} \tag{3.32}$$

Also, as we did for $(u_n)_{n\ge1}$, we show that $\{u_{nm(n)}\}_{n\ge1}\subseteq W^{1,p}\left((0,b),\mathbb{R}^N\right)$ is bounded, from which it follows that $\{\varphi(u_{nm(n)}')\}_{n\ge1}\subseteq W^{1,p'}\left((0,b),\mathbb{R}^N\right)$ is bounded (see (3.32)). Therefore we may assume that

$$\varphi\left(u_{nm(n)}'\right)\xrightarrow{w}\beta \text{ in } W^{1,p'}\left((0,b),\mathbb{R}^N\right) \text{ and } \varphi\left(u_{nm(n)}'\right)\to\beta \text{ in } C\left(T,\mathbb{R}^N\right).$$

Then

$$\varphi\left(u_{nm(n)}'(t)\right)\longrightarrow\beta(t) \text{ for all } t\in T,$$

hence

$$u_{nm(n)}'(t)\longrightarrow\varphi^{-1}(\beta(t)) \text{ for all } t\in T,$$

(recall that φ is a homeomorphism), therefore

$$u_{nm(n)}'\xrightarrow{w}\varphi^{-1}(\beta) \text{ in } L^{p'}\left(T,\mathbb{R}^N\right). \tag{3.33}$$

The boundedness of $\left\{u_{nm(n)}\right\}_{n\ge1}\subseteq W^{1,p}\left((0,b),\mathbb{R}^N\right)$ and (3.29) imply that

$$u_{nm(n)}\xrightarrow{w}u \text{ in } W^{1,p}\left((0,b),\mathbb{R}^N\right),$$

hence

$$u'=\varphi^{-1}(\beta)$$

(see (3.33)), therefore

$$\varphi\left(u'(t)\right) = \beta(t) \text{ for all } t \in T. \tag{3.34}$$

We have (cf (3.31) and (3.34))

$$A_{\lambda_{m(n)}}\left(u_{nm(n)}\right) \xrightarrow{w} -h + \varphi\left(u'\right) - \varphi(u) \text{ in } L^2\left(T, \mathbb{R}^N\right)$$

whence

$$\varphi\left(u'(t)\right)' - \varphi(u(t)) \in A(u(t)) + h(t) \text{ for a.a. } t \in T.$$

The closedness of $Gr\ \xi$ (recall that ξ is maximal monotone, see hypotheses $\mathbf{H}(\xi)$), implies that

$$\left(\varphi\left(u'(0)\right), -\varphi\left(u'(b)\right)\right) \in \xi(u(0), u(b)).$$

So, we conclude that $u = \theta(h)$, and this proves the complete continuity of the solution map θ. ☐

Now we are ready to establish the existence of a solution of problem (1.1).

Theorem 3.3. *If hypotheses* $\mathbf{H}(A)$, $\mathbf{H}(\xi)$, $\mathbf{H}_0$ *and* $\mathbf{H}(F)_1$ *hold, then problem* (1.1) *admits a solution* $u \in C^1\left(T, \mathbb{R}^N\right)$.

Proof. Let $M > 0$ be as postulated by hypothesis $\mathbf{H}(F)_1(iv)$ and let $a_M \in L^2(T)$ be the bound from $\mathbf{H}(F)_1(iii)$. We introduce the set $E \subseteq L^{p'}\left(T, \mathbb{R}^N\right)$ by

$$E = \left\{h \in L^2\left(T, \mathbb{R}^N\right) : |h(t)| \leq a_1(t) \text{ for a.a. } t \in T\right\}, \tag{3.35}$$

where $a_1(t) := a_M(t) + M^{p-1}$.

Let $C := \theta(E)$ where $\theta : L^2\left(T, \mathbb{R}^N\right) \to C\left(T, \mathbb{R}^N\right)$ is the solution map from Proposition 3.2.

The complete continuity of θ (see Proposition 3.2), implies that $C \subseteq C\left(T, \mathbb{R}^N\right)$ is compact. Then, Theorem 5.86 of Gasinski-Papageorgiou [7], implies that

$$\widehat{C} := \overline{conv}\ C \in P_{kc}\left(C\left(T, \mathbb{R}^N\right)\right). \tag{3.36}$$

Let $p_M : \mathbb{R}^N \to \mathbb{R}^N$ be the $M-$ radial retraction, that is,

$$p_M(x) = \begin{cases} x & \text{if } |x| \leq M \\ M\frac{x}{|x|} & \text{if } |x| > M \end{cases} \text{ for all } x \in \mathbb{R}^N.$$

We know that $p_M(.)$ is nonexpansive, that is

$$|p_M(x) - p_M(y)| \leq |x - y| \text{ for all } x, y \in \mathbb{R}^N.$$

We introduce the following multifunction

$$F_1(t, x) = F(t, p_M(x)) - \varphi(p_M(x)) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N. \tag{3.37}$$

Evidently, for all $x \in \mathbb{R}^N$, $t \to F_1(t, x)$ is graph measurable, for a.a. $t \in T$, $x \to F_1(t, x)$ is h-continuous and

$$|F_1(t, x)| \leq a_1(t) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N,$$

with $a_1 \in L^2(T)$. We consider the multivalued map $N_1 : \widehat{C} \to P_{wkc}\left(L^2\left(T, \mathbb{R}^N\right)\right)$ defined by

$$N_1(u) = S^2_{F_1(.,u(.))} \text{ for all } u \in \widehat{C}.$$

Using Theorem II.8.31 of Hu-Papageorgiou ([12], p. 260), we can find a continuous map $\sigma : \widehat{C} \to L^1_w\left(T, \mathbb{R}^N\right)$ such that

$$\sigma(u) \in extN_1(u) = extS^2_{F_1(.,u(.))} = S^2_{extF_1(.,u(.))} \text{ for all } u \in \widehat{C} \tag{3.38}$$

(for the last equality see Theorem II.4.5 of Hu-Papageorgiou ([12],p. 191).

Using Lemma I.2.8 of Hu-Papageorgiou ([13], p. 24), we infer that σ is also continuous from $\widehat{C}$ into $L^2\left(T, \mathbb{R}^N\right)_w$ (the space $L^2\left(T, \mathbb{R}^N\right)$ furnished with the weak topology).

We consider the following system

$$\begin{cases} \varphi(u'(t))' - \varphi(u(t)) \in Au(t) + \sigma(u)(t) \text{ for a.a. } t \in T \\ (\varphi(u'(0)), -\varphi(u'(b))) \in \xi(u(0), u(b)). \end{cases} \tag{3.39}$$

We show that problem (3.39) has a solution. To this end, it is sufficient to establish the existence of a fixed point of the map

$$\alpha = \theta \circ \sigma \tag{3.40}$$

in $C\left(T, \mathbb{R}^N\right)$. By (3.35), (3.36), (3.37), (3.38), (3.40) and the properties of θ and σ, it follows that α is a continuous self-map of the compact and convex set $\widehat{C} \subseteq C\left(T, \mathbb{R}^N\right)$. An application of Schauder's fixed point theorem then yields a fixed point u of α in $\widehat{C}$. This obviously satisfies (3.39). Recalling (3.37) and (3.38), we conclude that there exists $f \in S^2_{extF(.,p_M(u(.)))}$ such that

$$\begin{cases} \varphi(u'(t))' - \varphi(u(t)) \in Au(t) + f(t) - p_M(u(t)) \text{ a.e. on } T \\ (\varphi(u'(0)), -\varphi(u'(b))) \in \xi(u(0), u(b)). \end{cases} \tag{3.41}$$

Claim 2. $\|u\|_{C(T,\mathbb{R}^N)} \leq M$.

We argue by contradiction. So, suppose that Claim 2 is not true. Two things can happen:

(a) we can find $t_1, t_2 \in T = [0, b]$, $t_1 < t_2$ such that

$$|u(t_1)| = M, \ |u(t_2)| = \max_{t \in T} |u(t)| > M, \ |u(t)| > M \text{ for all } t \in (t_1, t_2]$$

(The case $|u(t_2)| = M$, $|u(t_1)| = \max_{t\in T} |u(t)| > M$, $|u(t)| > M$ for all $t \in [t_1, t_2)$ is treated similarly).

(b) $|u(t)| > M$ for all $t \in [0, b]$.

First we examine (a). We know that $t \to (\varphi(u'(t)), u(t))_{\mathbb{R}^N}$ is absolutely continuous on $T = [0, b]$ (recall that $\varphi(u'(.)) \in W^{1,p'}\left((0, b), \mathbb{R}^N\right)$). So, it is differentiable almost everywhere on T. We have

$$\begin{aligned} \frac{d}{dt}\left(\varphi(u'(t)), u(t)\right)_{\mathbb{R}^N} &= \Big(\varphi(u'(t))', u(t)\Big)_{\mathbb{R}^N} + \left(\varphi(u'(t)), u'(t)\right)_{\mathbb{R}^N} \\ &\in (A(u(t)) + f(t) - \varphi(p_M(u(t))) + \varphi(u(t)), u(t))_{\mathbb{R}^N} \\ &\quad + \left|(u'(t))\right|^p \ \text{(see (3.41))} \\ &\geq \frac{|u(t)|}{M}(f(t), p_M(u(t)))_{\mathbb{R}^N} - M^{p-1}|u(t)| + |(u(t))|^p \\ &\qquad \text{for a.a. } t \in [t_1, t_2]. \end{aligned} \tag{3.42}$$

Hypothesis $\mathbf{H}(F)_1(iv)$ (the multivalued Hartman condition) implies that

$$\frac{|u(t)|}{M}(f(t), p_M(u(t)))_{\mathbb{R}^N} \geq 0 \text{ for a.a. } t \in [t_1, t_2].$$

So, from (3.42) we have

$$\frac{d}{dt}\left(\varphi\left(u'(t)\right), u(t)\right)_{\mathbb{R}^N} \geq |u(t)|\left[|u(t)|^{p-1} - M^{p-1}\right] > 0 \text{ for a.a. } t \in [t_1, t_2]. \tag{3.43}$$

First suppose that $t_2 \in (0, b)$ and let $r(t) := |u(t)|^2$. Then

$$r'(t_2) = 0$$

hence

$$\left(u'(t_2), u(t_2)\right)_{\mathbb{R}^N} = 0.$$

From (3.43) we see that $t \to (\varphi(u'(t)), u(t))_{\mathbb{R}^N}$ is strictly increasing on $[t_1, t_2]$. Hence

$$\left|u'(t)\right|^{p-2}\left(u'(t), u(t)\right)_{\mathbb{R}^N} < \left|u'(t_2)\right|^{p-2}\left(u'(t_2), u(t_2)\right)_{\mathbb{R}^N} = 0 \text{ for all } t \in [t_1, t_2),$$

hence

$$\left(u'(t), u(t)\right)_{\mathbb{R}^N} < 0 \text{ for all } t \in [t_1, t_2),$$

therefore

$$r'(t) < 0 \text{ for all } t \in [t_1, t_2).$$

It follows that

$$M^2 < r(t_2) < r(t_1) = M^2, \text{ a contradiction.}$$

If $t_2 = b$, then $r'(b) \geq 0$. Suppose that hypothesis $\mathbf{H}(\xi)(i)$ holds. Then

$$\left|u'(b)\right|^{p-2}\left(u'(b), u(b)\right)_{\mathbb{R}^N} \leq 0$$

hence

$$\left(u'(b), u(b)\right)_{\mathbb{R}^N} \leq 0,$$

therefore

$$r'(b) \leq 0$$

and we conclude that

$$r'(b) = 0.$$

So, the previous argument applies and again we reach a contradiction.

If hypothesis $\mathbf{H}(\xi)(ii)$ holds, then

$$|u(0)|^2 = |u(b)|^2 = \max_{t \in T} |u(t)|^2.$$

This implies

$$r'(0) \leq 0, \ r'(b) \geq 0.$$

Using the boundary condition in (3.41) we arrive at

$$\left|u'(b)\right|^{p-2}\left(u'(b), u(b)\right)_{\mathbb{R}^N} \leq \left|u'(0)\right|^{p-2}\left(u'(0), u(0)\right)_{\mathbb{R}^N}. \tag{3.44}$$

So, we have

$$0 \leq r'(b) \leq r'(0) \leq 0,$$

therefore

$$r'(0) = r'(b) = 0.$$

and the previous argument applies.

Now suppose that (b) holds. Then as above we have

$$\frac{d}{dt}\left(\varphi\left(u'(t)\right), u(t)\right)_{\mathbb{R}^N} > 0 \text{ for a.a. } t \in [0,b],$$

hence

$$\left|u'(0)\right|^{p-2}\left(u'(0), u(0)\right)_{\mathbb{R}^N} < \left|u'(b)\right|^{p-2}\left(u'(b), u(b)\right)_{\mathbb{R}^N}. \tag{3.45}$$

But since $u \in D$, from the boundary condition in (3.41) we again obtain (3.44)

Comparing (3.45) and (3.44), we reach a contradiction. This proves Claim 2.

On account of Claim 2, we have

$$p_M(u(t)) = u(t) \text{ for all } t \in T.$$

So, (3.41) reduces to

$$\begin{cases} \varphi(u'(t))' \in Au(t) + f(t) \text{ for a.a. } t \in T \\ (\varphi(u'(0)), -\varphi(u'(b))) \in \xi(u(0), u(b)) \end{cases} \tag{3.46}$$

with $f \in S^2_{extF(.,u(.))}$. Consequently $u \in W^{1,p}\left((0,b), \mathbb{R}^N\right)$ is a solution of problem (1.1). □

4. Strong relaxation

We prove a strong relaxation theorem for a particular version of our system. So, now $p = 2$ (semilinear system) and we consider the following convexified version of (1.1):

$$\begin{cases} u'' \in A(u(t)) + F(t, u(t)) \text{ for a.a. } t \in T \\ u \text{ satisfies } BC \end{cases} \tag{4.1}$$

where $F(t,x)$ is convex valued and BC denotes the Dirichlet or Sturm-Liouville boundary conditions:

(Dirichlet problem) $\qquad u(0) = u(b) = 0$

(Sturm-Liouville problem) $\qquad u'(0) = L_0 u(0), u'(b) = -L_1 u(b).$

Here L_0, L_1 are $N \times N$ matrices which are positive definite. So, the first eigenvalue $\widehat{\lambda}_1$ of the corresponding Sturm-Liouville eigenvalue problem is positive (that is $\widehat{\lambda}_1 > 0$). Of course this is also the case for the first eigenvalue of the Dirichlet problem. Both boundary conditions fall within the framework of Section 3. Indeed we have:

$$\xi\left(d, d'\right) = \mathbb{R}^N \times \mathbb{R}^N \text{ and } D(\xi) = \{0, 0\} \text{ (Dirichlet problem, see also Section 5)}$$

and

$$\xi\left(d, d'\right) = \left(L_0 d, -L_1 d'\right) \text{ for all } \left(d, d'\right) \in \mathbb{R}^N \times \mathbb{R}^N \text{ (Sturm-Liouville problem).}$$

Note that for the Dirichlet problem, hypothesis $\mathbf{H}_0$ is automatically satisfied.

In what follows, problem (1.1) is understood with $p = 2$, and the boundary condition is BC (that is, Dirichlet or Sturm-Liouville).

By $\mathcal{S}_e$ (resp. $\mathcal{S}_c$) we denote the solution set of (1.1) (resp. of (4.1)). Evidently, $\mathcal{S}_e \subseteq \mathcal{S}_c$ and from Theorem 3.3, we have

$$\varnothing \neq \mathcal{S}_e \subseteq C^1\left(T, \mathbb{R}^N\right).$$

By strengthening the regularity of $F(t,.)$, we will show that

$$\overline{\mathcal{S}_e}^{C(T,\mathbb{R}^N)} = \mathcal{S}_c \qquad \text{(strong relaxation).}$$

In the context of control systems, such a density result means that the states of the system can be approximated by states generated by bang-bang controls. Hence, for such systems we can economize in the use of controls.

The new stronger conditions on the multifunction $F(t,x)$ are the following:

$\mathbf{H}(F)_2$: $F : T \times \mathbb{R}^N \to P_{kc}(\mathbb{R}^N)$ is a multifunction such that

(i) for every $x \in \mathbb{R}^N$, $t \to F(t,x)$ is graph measurable;

(ii) there exists $k \in L^\infty(T)$ with $\|k\|_\infty < \widehat{\lambda}_1$ such that for a.a. $t \in T$, all $x, y \in \mathbb{R}^N$, we have

$$h(F(t,x), F(t,y)) \le k(t)|x-y|;$$

(iii) for every $r > 0$, there exists $a_r \in L^2(T)$ such that

$$|F(t,x)| \le a_r(t) \text{ for a.a. } t \in T, \text{ all } x \in \mathbb{R}^N \text{ with } |x| \le r;$$

(iv) there exists $M > 0$ such that for a.a. $t \in T$, all $x \in \mathbb{R}^N$ with $|x| = M$ and all $v \in F(t,x)$, we have

$$(v,x)_{\mathbb{R}^N} \ge 0.$$

Theorem 4.1. *If hypotheses* $\mathbf{H}(A)$, $\mathbf{H}_0$ *and* $\mathbf{H}(F)_2$ *hold, then*

$$\overline{\mathcal{S}_e}^{C(T,\mathbb{R}^N)} = \mathcal{S}_c.$$

Proof. Let $\widehat{C} \subseteq W^{1,2}((0,b),\mathbb{R}^N) \subseteq C(T,\mathbb{R}^N)$ be defined by (3.36) in the proof of Theorem 3.3, that is

$$\widehat{C} = \overline{conv}\,\theta(E)$$

where E is given by (3.35) and $\theta : L^2(T,\mathbb{R}^N) \to C(T,\mathbb{R}^N)$ is the solution map from Proposition 3.2. We know that

$$\widehat{C} \in P_{wkc}\left(W^{1,2}((0,b),\mathbb{R}^N)\right) \text{ and } \widehat{C} \in P_{kc}\left(C(T,\mathbb{R}^N)\right).$$

Let $u \in \mathcal{S}_c$. We have

$$\begin{cases} u''(t) \in A(u(t)) + F(t,u(t)) \text{ for a.a. } t \in T \\ u \text{ satisfies } BC \end{cases}$$

with $f \in S^2_{extF(.,u(.))}$. Let $v \in \widehat{C}$ and $\varepsilon > 0$ be given. We consider the multifunction $\Gamma_{v,\varepsilon} : T \to 2^{\mathbb{R}^N} \setminus \{\varnothing\}$ defined by

$$\Gamma_{v,\varepsilon}(t) = \{y \in F_1(t,v(t)) : |f(t) - y| < \varepsilon + d(f(t), F_1(t,v(t)))\}$$

where $F_1(t,x)$ is the multifunction introduced in the proof of Theorem 3.3 (see 3.37), that is,

$$F_1(t,x) = F(t,p_M(x)) - p_M(x).$$

Evidently, $Gr\ \Gamma_{v,\varepsilon} \in \mathcal{L}_T \times \mathcal{B}(\mathbb{R}^N)$. So, we can use the Yankov-von Neumann-Aumann selection theorem and produce a measurable map $g : T \to \mathbb{R}^N$ such that

$$g(t) \in \Gamma_{v,\varepsilon}(t) \text{ for a.a. } t \in T.$$

Then we consider the multifunction $L_\varepsilon : \widehat{C} \to 2^{L^2(T,\mathbb{R}^N)}$ defined by

$$L_\varepsilon(v) = \Big\{ g \in S^2_{F_1(.,v(.))} : |f(t) - g(t)| < \varepsilon \\ + d(f(t), F_1(t, v(t))) \text{ for a.a. } t \in T \Big\}.$$

From the first part of the proof, we have

$$L_\varepsilon(v) \neq \varnothing \text{ for all } v \in \widehat{C}.$$

Moreover, from Lemma II.8.3 of Hu-Papageorgiou ([12], p. 239) it follows that $v \to L_\varepsilon(v)$ is lower semicontinuous, hence $v \to \overline{L_\varepsilon(v)}$ is lower semicontinuous and has closed values. So, using Theorem II.8.7 of Hu-Papageorgiou ([12], p. 245), we can find a continuous map $\eta_\varepsilon : \widehat{C} \to L^2(T, \mathbb{R}^N)$ such that

$$\eta_\varepsilon(v) \in \overline{L_\varepsilon(v)} \text{ for all } v \in \widehat{C}.$$

In addition, via Theorem II.8.31 of Hu-Papageorgiou ([12], p. 260), there is a continuous map $\sigma_\varepsilon : \widehat{C} \to L^1_w(T, \mathbb{R}^N)$ such that

$$\begin{array}{l} \sigma_\varepsilon(v) \in ext S^2_{F_1(.,v(.))} = S^2_{ext F_1(.,v(.))} \text{ and} \\ \|\sigma_\varepsilon(v) - \eta_\varepsilon(v)\|_w \leq \varepsilon \text{ for all } u \in \widehat{C}. \end{array} \tag{4.2}$$

Let $\varepsilon_n \downarrow 0$ and set $\eta_n = \eta_{\varepsilon_n}$, $\sigma_n = \sigma_{\varepsilon_n}$ for all $n \in \mathbb{N}$. We consider the following system

$$\left\{ \begin{array}{l} v''(t) - v(t) \in A(v(t)) + \sigma_n(v)(t) \text{ for a.a. } t \in T \\ v \text{ satisfies } BC. \end{array} \right. \tag{4.3}$$

From the proof of Theorem 3.3, we know that problem (4.3) admits a solution u_n such that

$$u_n \in C^1(T, \mathbb{R}^N) \text{ and } \|u_n\|_{C(T,\mathbb{R}^N)} \leq M \text{ for all } n \in \mathbb{N}.$$

Moreover, from (4.3) as before, via integration by parts and use of the boundary condition, we infer that

$$\{u_n\}_{n\geq 1} \subseteq W^{1,2}((0,b), \mathbb{R}^N) \text{ is bounded.}$$

Also we have

$$\{u_n\}_{n\geq 1} \subseteq \widehat{C} \in P_{kc}(C(T, \mathbb{R}^N)).$$

So, by passing to a suitable subsequence if necessary, we have

$$u_n \xrightarrow{w} \widehat{u} \text{ in } W^{1,2}((0,b), \mathbb{R}^N) \text{ and } u_n \longrightarrow \widehat{u} \text{ in } C(T, \mathbb{R}^N), \ \widehat{u} \in D. \tag{4.4}$$

Exploiting the monotonicity of $A(.)$ we have

$$(u_n'' - u, u - u_n)_2 \leq \int_0^b (\sigma_n(u_n) - f, u - u_n)_{\mathbb{R}^N} dt$$

hence

$$\|u_n' - u'\|_2^2 \leq \int_0^b (\sigma_n(u_n) - \eta_n(u_n), u - u_n)_{\mathbb{R}^N} dt \\ + \int_0^b (\eta_n(u_n) - f, u - u_n)_{\mathbb{R}^N} dt,$$

(by integration by parts and use of the boundary condition), therefore

$$\left\|u_n' - u'\right\|_2^2 \leq \varepsilon_n' + \int_0^b k(t)\,|u_n - u|^2\,dt \text{ (see (4.2) and } \mathbf{H}(F)_2(ii))$$
$$\leq \varepsilon_n' + \|k\|_\infty \|u_n - u\|_2^2, \text{ with } \varepsilon_n' \downarrow 0.$$

We conclude that

$$\left\|\widehat{u}' - u'\right\|_2^2 \leq \|k\|_\infty \|\widehat{u} - u\|_2^2 \leq \frac{\|k\|_\infty}{\widehat{\lambda}_1} \left\|\widehat{u}' - u'\right\|_2^2 < \left\|\widehat{u}' - u'\right\|_2^2$$

(from the variational characterization of $\widehat{\lambda}_1$ and $\mathbf{H}(F)_2(ii)$), a contradiction, unless $\widehat{u}' = u'$. Hence

$$\widehat{u} = u + \widetilde{c}, \text{ with } \widetilde{c} \in \mathbb{R}.$$

Using the boundary conditions, we see that $\widetilde{c} = 0$, therefore

$$\widehat{u} = u.$$

Since $|u_n(t)| \leq M$ for all $t \in T$, from (4.3) it follows that $u_n \in \mathcal{S}_e$ and so we conclude that

$$\overline{\mathcal{S}_e}^{C(T,\mathbb{R}^N)} = \mathcal{S}_c.$$

□

We present another situation where strong relaxation holds.

Let $\varphi_0 \in \Gamma_0(\mathbb{R}^N)$, $\varphi_0 \geq 0$ and $\varphi_0(0) = 0$. Then $0 \in \partial\varphi_0(0)$. Assume that

$\mathbf{H}(\xi)'$: $\xi(x,y) = N_{\{0\}}(x) \times \partial\varphi_0(y)$ or $\xi(x,y) = \partial\varphi_0(x) \times N_{\{0\}}(y)$

Note that these multifunctions satisfy hypothesis $\mathbf{H}(\xi)$. Indeed, $\xi(.,.)$ is maximal monotone, $(0,0) \in \xi(0,0)$ and for $u^* \in \partial\varphi_0(y)$ we have $(u^*, y)_{\mathbb{R}^N} \geq \varphi_0(y) - \varphi_0(0) = \varphi_0(y)$, hence

$$(u^*, y)_{\mathbb{R}^N} \geq 0$$

and so condition $\mathbf{H}(\xi)(i)$ holds.

If $\varphi_0 = i_K$ (the indicator function of K) with $K \in P_{fc}(\mathbb{R}^N)$, $0 \in K$, then $\partial\varphi_0(x) = \partial i_K(x) = N_K(x)$ where $N_K(x)$ is the normal cone to the set K at $x \in K$ (see Hu-Papageorgiou [12], p.634). In this case $D(\partial\varphi_0) = K$.

Also hypotheses $\mathbf{H}(F)_2$ are slightly modified as hollows:

$\mathbf{H}(F)_2'$: The same as $\mathbf{H}(F)_2$ but with $k \in L^1(T)_+$, $b\|k\|_1 < 1$.

Remark. If $k \in L^\infty(T)$ then $\|k\|_1 \leq b\|k\|_\infty$. Recall that for the Dirichlet problem $\widehat{\lambda}_1 = \left(\frac{2\pi}{b}\right)^2$ (see Gasinski-Papageorgiou [6]). Then the condition $\|k\|_\infty < \widehat{\lambda}_1$ from hypothesis $\mathbf{H}(F)_2(ii)$ implies that $\|k\|_1 < \frac{(2\pi)^2}{b}$ which is less restrictive than $\|k\|_1 < \frac{1}{b}$ used in $\mathbf{H}(F)_2'$ for the Dirichlet problem.

Theorem 4.2. *If hypotheses* $\mathbf{H}(A)$, $\mathbf{H}(\xi)'$, $\mathbf{H}_0$ *and* $\mathbf{H}(F)_2'$ *hold, then*

$$\overline{\mathcal{S}_e}^{C(T,\mathbb{R}^N)} = \mathcal{S}_c.$$

Proof. The proof of Theorem 4.1 remains unchanged up to the point where we have

$$\left\|u_n' - u'\right\|_2^2 \le \varepsilon_n' + \int_0^b k(t)\left|u_n - u\right|^2 dt \text{ for all } n \in \mathbb{N}$$

hence

$$\left\|\widehat{u}' - u'\right\|_2^2 \le \int_0^b k(t)\left|\widehat{u} - u\right|^2 dt \text{ for all } n \in \mathbb{N} \text{ (see (4.4))}.$$

We assume that the first option in hypothesis $\mathbf{H}(\xi)'$ holds. (The proof is similar if the other option holds). Then we have

$$\begin{aligned}\int_0^b k(t)\left|\widehat{u} - u\right|^2 dt &\le \int_0^b k(t)\left|\int_0^t \left|\widehat{u}' - u'\right| ds\right|^2 dt \\ &\le \int_0^b k(t)\, b \int_0^b \left|\widehat{u}' - u'\right|^2 ds dt \text{ (using Jensen's inequality)} \\ &= \left\|\widehat{u}' - u'\right\|_2^2 b\left\|k\right\|_1.\end{aligned}$$

So, we have

$$\left\|\widehat{u}' - u'\right\|_2^2 \le b\left\|k\right\|_1 \left\|\widehat{u}' - u'\right\|_2^2,$$

a contradiction (since $b\left\|k\right\|_1 < 1$) unless $\widehat{u}' = u'$. Then

$$\widehat{u} = u + \widetilde{c}, \text{ with } \widetilde{c} \in \mathbb{R}.$$

The boundary condition implies $\widetilde{c} = 0$, and so

$$\widehat{u} = u.$$

So, as before (see the proof of Theorem 4.1) we conclude that

$$\overline{\mathcal{S}_e}^{C(T,\mathbb{R}^N)} = \mathcal{S}_c.$$

□

Remark. It is an interesting open problem, if strong relaxation holds under the general boundary condition of Section 3 and even more generally when $p \ne 2$.

5. Special cases

We present some special cases which fit in our framework.

(*a*) Suppose $K_1, K_2 \in P_{fc}\left(\mathbb{R}^N\right)$ with $0 \in K_1 \cap K_2$. Then $i_{K_1 \times K_2} \in \Gamma_0\left(\mathbb{R}^N \times \mathbb{R}^N\right)$ and

$$\partial i_{K_1 \times K_2} = N_{K_1 \times K_2} = N_{K_1} \times N_{K_2}$$

(see Hu-Papageorgiou ([12], p. 636). We set

$$\xi(x, y) = N_{K_1}(x) \times N_{K_2}(y) \text{ for all } (x, y) \in \mathbb{R}^N \times \mathbb{R}^N.$$

Evidently hypothesis $\mathbf{H}(\xi)$ is satisfied (recall that $0 \in K_1 \cap K_2$). Then problem (1.1) becomes

$$\begin{cases} \varphi(u'(t))' \in A(u(t)) + extF(t, u(t)) \text{ for a.a. } t \in T := [0, b] \\ u(0) \in K_1, \ u(b) \in K_2 \\ (u'(0), u(0))_{\mathbb{R}^N} = \overline{\sigma}(u'(0), K_1) \\ (-u'(b), u(b))_{\mathbb{R}^N} = \overline{\sigma}(-u'(b), K_2) \end{cases}$$

where $\overline{\sigma}$ is the support function . According to Theorem 3.3, this problem has a solution $u \in C^1\left(T, \mathbb{R}^N\right)$.

More generally we can choose

$$\xi(x, y) = \partial\varphi_1(x) \times \partial\varphi_2(y)$$

with $\varphi_1,\ \varphi_2 \in \Gamma_0\left(\mathbb{R}^N\right)$, such that $\varphi_1,\ \varphi_2 \geq 0,\ 0 = \varphi_1(0) = \varphi_2(0)$. If $\varphi_1 = i_K$ or $\varphi_2 = i_K$ with $K \in P_{fc}\left(\mathbb{R}^N\right)$, $0 \in K$ and $p = 2$, then strong relaxation holds, provided hypotheses $\mathbf{H}(F)_2'$ hold (see Theorem 4.2)

(b) If in the above example, $K_1 = K_2 = \{0\}$, then we have the Dirichlet problem. For this problem when $p = 2$ strong relaxation holds provided hypotheses $\mathbf{H}(F)_2$ are satisfied (see Theorem 4.1).

(c) If in example (a), $K_1 = K_2 = \left\{\mathbb{R}^N\right\}$, then $\xi(x, y) = \{(0,0)\}$ for all x, $y \in \mathbb{R}^N$ and the resulting problem is the Neumann problem. Theorem 3.3 applies and we have extremal solutions for the system.

(d) Suppose $K = \left\{(x, y) \in \mathbb{R}^N \times \mathbb{R}^N : x = y\right\}$. Then

$$\xi(x, y) = N_K(x, y) = \left\{(u^*, v^*) \in \mathbb{R}^N \times \mathbb{R}^N : u^* = -v^*\right\}$$

satisfies hypothesis $\mathbf{H}(\xi)$ and the resulting problem is the periodic problem. According to Theorem 3.3, we have extremal periodic trajectories.

(e) Let

$$\xi(x, y) = \left(|x|^{p-2} L_0(x), -|y|^{p-2} L_1(y)\right), p \geq 2$$

where L_0, L_1 are nonnegative definite $N \times N-$matrices. Then hypothesis $\mathbf{H}(\xi)$ holds and the system has extremal solutions. If $p = 2$ and L_0, L_1 are positive definite, then strong relaxation holds.

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