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ON LIPSCHITZ CONTINUITY OF VALUE FUNCTIONS FOR INFINITE HORIZON PROBLEM

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Abstract. We investigate conditions of optimality for an infinite-horizon control problem and consider their correspondence with the value function. Assuming Lipschitz continuity of the value function, we prove that sensitivity relations plus the normal form version of the Pontryagin Maximum Principle is a necessary and sufficient condition for the optimality criteria that correspond to this value function. Different criteria of optimality under different asymptotic constraints may be used, including almost strong and classical optimality proposed by D.Bogusz. Special attention is devoted to weakly agreeable criteria. We also obtain the conditions on control system (like controllability) that guarantee the Lipschitz continuity of the value function without any other asymptotic conditions besides finiteness of the value function.

Some examples are discussed. In particular, it was shown that the same control, regarded as agreeable optimal and overtaking optimal control, can correspond to different (everywhere) value functions.

1. Introduction

Optimality conditions on control problems are usually constructed with the Pontryagin Maximum Principle (PMP) [29]. For infinite-horizon control problems, even in the free end-time case, the PMP can be degenerate [1, 16].

For infinite-horizon problems, one can often prove the nondegeneracy (the normality) of the PMP under additional assumptions. One may use uniform coercivity of running cost [30, Corollary 4.1], strong convexity of the Hamiltonian [33], or concavity of the dynamics function [30, Hypothesis 5.1]; or, one may impose asymptotic estimates on motions [3, 8, 34] and costate arcs [2, Theorem 4],[19, Remark 8],[20, Proposition 4], e.g. their total variation [1, 4, 5, 6, 30, 31]. The nondegeneracy of the PMP follows from the Lipschitz continuity of the value function (see [1, Theorem 5.1], [21, 37]) under the Michel condition (see [23]). Besides, the Lipschitz continuity of the value function is also guaranteed by estimates on motions and costate arcs e.g. their total variation, see [1, 2, 3, 5, 6, 30, 37].

Through Dynamic Programming Principle (DPP), we prove that, for a Lipschitz continuous value function, the normal form of PMP with sensitivity relations is a necessary and sufficient condition of optimality in view of this value function. To

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make it a straightforward consequence of the results [11] for finite horizon, we restate the optimality in terms of the value function. Choosing a corresponding value function, we obtain necessary conditions for various optimality criteria under various asymptotic constraints, including the optimality in weighted Lebesgue spaces [4, 27, 34], agreeable [12, 18], almost strong, and classical [9, 28] optimalities. In these results, we only assume the corresponding value function to exist and to be Lipschitz continuous. Generally, the same optimal control can correspond to several different value functions (see Example 7.3). For weakly agreeable optimality criteria, we found a family of value functions parameterized by unboundedly increasing sequences. We also show the conditions on a control system (see Theorems 6.1,6.3) that guarantee the Lipschitz continuity for a finite value function without an assumption on the asymptotic behavior of trajectories or adjoint variables.

We start with definitions, including the value function as a function satisfying DPP and optimal control corresponding to a given value function. In the next section, we obtain a necessary (and sufficient) condition for such optimality. In Section 4, we extend these results to various asymptotic constraints. Next, we study value functions for agreeable optimality. Section 6 is devoted to the conditions on a control system that guarantee the Lipschitz continuity for a finite value function. The last section is devoted to examples.

2. Problem statement and definitions

The infinite horizon control problem. Consider the following optimal control problem for infinite horizon:

$$\text{Minimize} \int_0^{+\infty} f_0(t,x,u)\,dt \tag{2.1}$$

$$\text{subject to } \dot{x} = f(t,x,u), \quad u \in P, \quad x \in \mathbb{R}^m; \tag{2.2}$$

$$x(0) = b_*. \tag{2.3}$$

Let P be a complete separable metric space. As for the class of admissible controls, we consider the set of all Lebesgue measurable functions $u : \mathbb{R}_{\geq 0} \to P$ that are bounded on every time compact and denote it by $\mathbb{U}$.

Assume that $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times P \to \mathbb{R}^m$, $f_0 : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times P \to \mathbb{R}$ are continuous and locally Lipschitz continuous in x; also, let f satisfy the sublinear growth condition, i.e., there exists a continuous function $\varkappa : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $|f(t,x,u)| \leq \varkappa(t)(1 + ||x||)$ for all $(t,x,u) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m \times P$. Then, to each $(b,t) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$ and each $u \in \mathbb{U}$, we can assign a solution of (2.2) with the initial condition $x(t) = b$. This solution is unique and it can be extended to the whole $\mathbb{R}_{\geq 0}$; denote it by $x_{b,t,u}$.

On the dynamic programming principle.

For each $\theta \geq 0, y \in \mathbb{R}^m, T \geq \theta$, and $u \in \mathbb{U}$, set

$$J(\theta, y; u, T) := \int_\theta^T f_0\big(s, x_{y,\theta,u}(s), u(s)\big)\,ds.$$

Let $V^T(\theta, y)$ denote the infimum of the following problem in the interval $[\theta, T]$:

$$\mathbb{P}(T)\begin{cases} \text{Minimize } \int_\theta^T f_0(t,x,u)\,dt \\ \text{subject to } \dot{x} = f(t,x,u), \quad u \in P, \quad x \in \mathbb{R}^m; \\ x(\theta) = y. \end{cases}$$

Definition 2.1. For an interval $I \subset \mathbb{R}_{\geq 0}$ and a function $V : I \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$, we say that V enjoys *DPP (Dynamic Programming Principle)* iff

$$V(t,b) = \inf_{u \in \mathbb{U}} \Big[\int_t^\tau f_0(s, x_{b,t,u}(s), u(s))ds + V(\tau, x_{b,t,u}(\tau)) \Big] \quad \forall b \in \mathbb{R}^m, [t,\tau] \subset I.$$

Remark 2.2. For every $T > 0$, $V^T : [0,T] \times \mathbb{R}^m \to \mathbb{R}$ satisfies DPP.

On conditions of optimality.

Note that the improper integral in (2.1) may not exist; as a consequence, for control problems on infinite horizon, there are several optimality criteria [9, 12, 13, 14, 17, 18, 28, 38]. Hereinafter denote by u^* the optimal control; however, we will always specify which criterion do we use for u^*. Set also $x^* \equiv x_{b_*,0,u^*}$.

Definition 2.3. Let $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$ be a function enjoying DPP. We will say that $u^* \in \mathbb{U}$ *is optimal in view of* $\mathbb{V}$ iff, for all $T \geq 0$,

$$\mathbb{V}(T, x^*(T)) + J(0, x^*(0); u^*, T) = \mathbb{V}(0, x^*(0)). \tag{2.4}$$

3. Necessary and sufficient conditions in terms of value functions.

Consider a Lipschitz continuous function $h : \mathbb{R}^r \to \mathbb{R}$ and a point $x \in \mathbb{R}^r$. The Fréchet superdifferential $\hat{\partial}^+ h(x)$ is the set of vectors $\zeta \in \mathbb{R}^r$ that satisfy

$$\limsup_{y \to x} \frac{h(y) - h(x) - \zeta(y-x)}{||y-x||} \leq 0.$$

The limiting superdifferential $\partial^+ h(x)$ of h at x consists of all ζ in $\mathbb{R}^r$ such that there exist sequences of $x_n \in \mathbb{R}^r, \zeta_n \in \hat{\partial}^+ h(x_n)$ satisfying $x_n \to x$, $h(x_n) \to h(x)$, $\zeta_n \to \zeta$. For different (equivalent) definitions of the limiting superdifferential, see [10].

The Pontryagin Maximum Principle.

Let the Hamilton–Pontryagin function H be given as follows:

$$H(x, u, \psi, \lambda, t) := \psi f\big(t, x, u\big) - \lambda f_0\big(t, x, u\big).$$

Let us introduce the relations

$$-\dot{\psi}(s) = \frac{\partial H}{\partial x}\big(x^*(s), u^*(s), \psi(s), \lambda, s\big), \tag{3.1}$$

$$\sup_{v \in P} H\big(x^*(s), v, \psi(s), \lambda, s\big) = H\big(x^*(s), u^*(s), \psi(s), \lambda, s\big) \text{ a.e}, \tag{3.2}$$

$$-\psi(0) \in \partial_x^+ \mathbb{V}(0, x^*(0)), \tag{3.3}$$

$$\big(H\big(x^*(s), u^*(s), \psi(s), 1, s\big), -\psi(s)\big) \in \partial^+ \mathbb{V}(s, x^*(s)) \text{ a.e.} \tag{3.4}$$

Here, $\partial_x^+ \mathbb{V}$ is the limiting superdifferential of a map $\mathbb{R}^m \ni x \mapsto \mathbb{V}(s,x) \in \mathbb{R}$.

Halkin [16] proved that the Pontryagin Maximum Principle is a necessary condition of optimality for infinite-horizon problems: if an admissible $u^* \in \mathbb{U}$ is finitely optimal [16] for problem (2.1)–(2.3), then there exists a nontrivial solution (ψ^*, λ^*) of system (3.1)-(3.2) for a.a. $s > 0$; here, $\lambda^* \in \{0, 1\}$.

Remember that, for the corresponding control problem with finite horizon T, under rather general conditions on the system, see, for example [11, 32, 36], the Pontryagin Maximum Principle with $\lambda = 1$ plus sensitivity relations (3.3)–(3.4) forms a necessary and sufficient condition of optimality. Similar conditions arise in Hamilton-Jacobi-Isaacs PDEs for infinite horizon, see [5, 6].

Thereinafter in this section we also assume that f and f_0 are differentiable with respect to x.

Necessary conditions in terms of value function.

Theorem 3.1. *Assume a locally Lipschitz continuous function* $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ *satisfies DPP. Let* u^* *be optimal in view of* $\mathbb{V}$*, i.e., satisfy* (2.4).

Then, there exists a co-state arc $\psi^* \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *satisfying PMP relations* (3.1)–(3.2) *with* $\lambda = 1$ *and sensitivity relations* (3.3)–(3.4).

Proof. Fix $T > 0$. Consider the following Bolza problem:

$$\begin{cases} \text{Minimize } \int_0^T f_0(t, x, u)\, dt + \mathbb{V}(T; x(T)) \\ \qquad \text{subject to } \dot{x} = f(t, x, u), \quad u \in P, \\ \qquad\qquad\qquad\qquad x(0) = b. \end{cases} \tag{3.5}$$

Thanks to DPP, this problem has an optimal value, $\mathbb{V}(0, b)$. Now, by (2.4), the control u^* achieves the minimum of this problem, i.e., u^* is optimal for it.

Thanks to [11, Theorem 6.1], for some co-state arc $\psi^T \in C([0, T], \mathbb{R}^m)$, PMP relations (3.1)–(3.2) with $\lambda = 1$ hold; moreover, ψ^T satisfies (3.4) (for a.a. $s \in [0, T]$) and (3.3). So, for every unbounded sequence of positive T_n, co-state arcs ψ^{T_n} with $\lambda = 1$ satisfy (3.1)–(3.2) and relations (3.3),(3.4) for a.a. $s \in [0, T_n]$.

Remember that $\mathbb{V}$ is Lipschitz continuous, therefore $\partial_x^+ \mathbb{V}(0, x^*(0))$ is a compact (see [10]). Consider the sequence of $\psi^{T_n}(0)$. By (3.3), this sequence has a limiting point $\zeta^* \in \partial_x^+ \mathbb{V}(0, x^*(0))$. Consider a solution $\psi^* \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ of (3.1) with $\psi^*(0) = \zeta^*$; it satisfies (3.3). By the theorem on continuous dependence of a differential equation on its initial conditions, this solution is a partial limit of ψ^{T_n} in the compact-open topology (on each time compact). Passing to the partial limit for a.a. positive s, one can provide that the co-state arc ψ^* satisfies (3.1)–(3.2) and sensitivity relation (3.4) for a.a. positive s. □

Sufficient conditions in terms of value function.

Theorem 3.2. *Assume that a locally Lipschitz continuous function* $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ *satisfies DPP. Let some co-state arc* $\psi^* \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ *satisfy PMP relations* (3.1)–(3.2) *with* $\lambda = 1$ *and sensitivity relation* (3.4).

Then, u^* *is optimal in view of* $\mathbb{V}$*, i.e., satisfies* (2.4).

Proof. Fix $T > 0$. Consider Bolza problem (3.5). By DPP, this problem has an optimal value: $\mathbb{V}(0, b)$. Now, (2.4) holds for T iff u^* achieves the minimum of this problem, i.e., if u^* is optimal for this Bolza problem with $b = b_*$.

By [11, Theorem 6.2], u^* is optimal in this Bolza problem with $b = b_*$ iff there exists a co-state arc $\psi^T \in C([0, T], \mathbb{R}^m)$ with $\lambda = 1$ that satisfies (3.1)–(3.2) and sensitivity relation (3.4) for a.a. $s \in [0,T]$. Set $\psi^T \equiv \psi^*|_{[0,T]}$. Since the co-state ψ^* with $\lambda = 1$ satisfies (3.1)–(3.2),(3.4), we obtain (2.4) for all $T > 0$. □

Note that, in these theorems, the local Lipschitz continuity of $\mathbb{V}$ may hold not in $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$ but in a neighborhood of the graph of x^*. In particular, one can require $\mathbb{V}$ to be locally Lipschitz continuous in a strongly invariant (for (2.2)) neighborhood of the graph of x^*.

Of course, one would like to relax the condition of Lipschitz continuity of the value function. On the other hand, Theorem 6.1 provides this based on the finiteness of the value function if the optimal-time function [35] for this control system is Lipschitz continuous.

4. Value functions under asymptotic constraints

Asymptotic Constraints.

For controls $u', u'' \in \mathbb{U}$, for each $T \in \mathbb{R}_{\geq 0}$, the concatenation $u' \diamond_T u'' \in \mathbb{U}$ is as follows: $(u' \diamond_T u'')(t) := u'(t)$ if $t < T$, and $(u' \diamond_T u'')(t) := u''(t)$, if $t \geq T$.

Definition 4.1. We say that a multi-valued map $\Omega_\diamond : \mathbb{R}^m \times \mathbb{R}_{\geq 0} \rightrightarrows \mathbb{U}$ *induces asymptotic constraints* iff, for all $(b,t) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$,

$$\Omega_\diamond(b,t) = \{u \diamond_T u_1 \,:\, u \in \mathbb{U}, u_1 \in \Omega_\diamond(x_{b,t,u}(T), T)\}, \quad \forall T > t. \tag{4.1}$$

It is easy to see that each of the constant multi-valued maps $\mathbb{U}$, $B(\mathbb{R}_{>0}, P)$, $L_p(\mathbb{R}_{\geq 0}, P) \cap \mathbb{U}$ (if $1 \leq p < +\infty$) induces asymptotic constraints.

Let us offer another example of asymptotic constraints. Fix a nonempty set $M \subset \mathbb{R}^m$. For all $(b,t) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$, denote by $\Omega_M(b,t)$ the set of all $u \in \mathbb{U}$ such that

$$\varrho(x_{b,t,u}(s), M) := \inf\{||y - x_{b,t,u}(s)|| \,:\, y \in M\} \to 0$$

as $s \to +\infty$. Necessary conditions of optimality for problems under varying choice of M were considered, for example, in [29, 26]; for exit-time control problems, see [25]. Let us check that Ω_M does also induce asymptotic constraints. Indeed, for all $(t,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m$, $T > t$, $u, u_1 \in \mathbb{U}$, both the statement $u \diamond_T u_1 \in \Omega_M(x_{b,t,u}(t), t)$ and the statement $u_1 \in \Omega_M(x_{b,t,u}(T), T)$ are equivalent to the fact that $\varrho(x_{b_1,T,u_1}(s), M) \to 0$ as $s \to +\infty$; here, $b_1 := x_{b,t,u}(T)$. Thus, Ω_M induces asymptotic constraints.

On conditions of optimality under asymptotic constraints

Define $V^\diamond : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R} \cup \{-\infty, +\infty\}$ as follows: for all $(t,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m$, set $V^\diamond(t,b) = +\infty$ if $\Omega_\diamond(t,b) = \varnothing$; otherwise,

$$V^\diamond(t,b) := \inf_{u \in \Omega_\diamond(b,t)} \liminf_{T \to +\infty} J(t,b;u,T).$$

Theorem 4.2. *Assume that $\Omega_\diamond$ induces asymptotic constraints. Suppose that the function $V^\diamond$ is finite. Then,*

(1) *$V^\diamond$ enjoys the Dynamic Programming Principle (DPP);*

(2) *u^* is optimal in view of $V^\diamond$ if u^* lies in $\Omega_\diamond(b_*, 0)$ and satisfies*

$$\liminf_{T \to +\infty} J(0, b_*; u^*, T) = V^\diamond(0, b_*). \tag{4.2}$$

Proof. Note that, since $V^\diamond$ is everywhere finite, $\Omega_\diamond$ is everywhere nonempty. By the definition of $V^\diamond$, for all $\tau > 0, t \in [0,\tau[, b \in \mathbb{R}^m$, we have

$$\begin{aligned} V^\diamond(t,b) &= \inf_{u\in\Omega_\diamond(b,t)} \liminf_{T\to+\infty} \Big[J(t,b;u,\tau) + J(\tau, x_{b,t,u}(\tau); u, T)\Big] \\ &\overset{(4.1)}{=} \inf_{u'\in\mathbb{U}, u''\in\Omega_\diamond(x_{b,t,u'}(\tau),\tau)} \Big[J(t,b;u'\diamond_\tau u'',\tau) \\ &\qquad + \liminf_{T\to+\infty} J(\tau, x_{b,t,u'\diamond_\tau u''}(\tau); u'\diamond_\tau u'', T)\Big] \\ &= \inf_{u'\in\mathbb{U}} \Big[J(t,b;u',\tau) \\ &\qquad + \inf_{u''\in\Omega_\diamond(x_{b,t,u'}(\tau),\tau)} \liminf_{T\to+\infty} J(\tau, x_{b,t,u'}(\tau); u'', T)\Big] \\ &= \inf_{u'\in\mathbb{U}} \Big[J(t,b;u',\tau) + V^\diamond(\tau, x_{b,t,u'}(\tau))\Big]. \end{aligned}$$

Assume (4.2) holds for $u^* \in \Omega_\diamond(b_*, 0)$. Then, for some unbounded sequence of positive τ_n, we have $\lim_{n\to\infty} J(0,b_*;u^*,\tau_n) = V^\diamond(0,b_*)$. Now,

$$\begin{aligned} V^\diamond(0,b_*) &\overset{(4.2)}{=} \lim_{n\to\infty} \big[J(0,b_*;u^*,T) + J(T,x^*(T);u^*,\tau_n)\big] \\ &= J(0,b_*;u^*,T) + \lim_{n\to\infty} J(T,x^*(T);u^*,\tau_n) \\ &\overset{(4.1)}{\geq} J(0,b_*;u^*,T) + \inf_{u\in\Omega_\diamond(x^*(T),T)} \liminf_{n\to\infty} J(T, x_{b,t,u}(T); u, \tau_n) \\ &\geq J(0,b_*;u^*,T) + V^\diamond(T, x^*(T)) \qquad \forall T \geq 0. \end{aligned}$$

Thanks to the Dynamic Programming Principle, it implies (2.4) for $\mathbb{V} = V^\diamond$. □

On conditions without asymptotic constraints.

Define the function V^{inf} from $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$V^{inf}(t,b) = \inf_{u\in\mathbb{U}} \liminf_{T\to\infty} J(t,b;u,T) \qquad \forall (t,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m.$$

One could easily prove that $V^{inf} \equiv V^\diamond$ if $\Omega_\diamond := \mathbb{U}$. Now, Theorems 3.1 and 4.2 imply the corresponding necessary conditions for u^*.

Corollary 4.3. *Assume that f and f_0 are differentiable with respect to x.*
Let V^{inf} be finite and locally Lipschitz continuous. Let $u^* \in \mathbb{U}$ satisfy

$$\liminf_{T\to+\infty} J(0,b_*;u^*,T) = V^{inf}(0,b_*).$$

Then, there exists a co-state arc $\psi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ satisfying PMP relations (3.1)–(3.2) *with $\lambda = 1$ and sensitivity relations* (3.3)–(3.4) *for $\mathbb{V} = V^{inf}$.*

Note that u^* is optimal in view of V^{inf} if u^* is an overtaking optimal control [18, 12]. Therefore, we obtain necessary conditions of overtaking optimality. These necessary conditions, including the effective transversality condition at infinity, can be found in [3, 7]. However, the necessary conditions in [3, 7] exploit the assumptions that are not required for Corollary 4.3; however, Corollary 4.3 assumes the value function to be known.

Analogously, set $\Omega_\diamond \equiv L_p(\mathbb{R}_{\geq 0}, P) \cap \mathbb{U}$. If u^* achieves $V^\diamond(0, b_*)$, (3.3) holds. For the linear case, this sensitivity relation was proved in [4, (7.13)].

On almost strong and classical optimalities.

Recall the optimality criteria from [9] and [28]. For each $(b,t) \in \mathbb{R}^m \times \mathbb{R}$, denote by $\Omega_{\mathcal{R}}(b,t)$ ($\Omega_{\mathcal{L}}(b,t)$) the set of all $u \in \mathbb{U}$ such that a map $[t, +\infty[\ni s \mapsto f_0(s, x_{b,t,u}(s), u(s))$ has an improper Riemann (Lebesgue) integral. We claim that $\Omega_{\mathcal{R}}$ induces asymptotic constraints. Fix $(b,t) \in \mathbb{R}^m \times \mathbb{R}_{\geq 0}$, $u, u_1 \in \mathbb{U}$, and $T > t$. Set $b_1 := x_{b,t,u}(T)$. By definition, $u_1 \in \Omega_{\mathcal{R}}(b_1, T)$ iff, for all $T > 0$, there exists a limit of $\int_T^\tau f_0(s, x_{b_1,T,u_1}(s), u_1(s))\, ds$, i.e., if, specifically, $\int_t^\tau f_0(s, x_{b_1,T,u\diamond_T u_1}(s), (u\diamond_T u_1)(s))\, ds$ has a limit as $\tau \to +\infty$. By $x_{b_1,T,u\diamond_T u_1} = x_{b,t,u\diamond_T u_1}$, we obtain "$\supseteq$" in (4.1) for $\Omega_\diamond \equiv \Omega_{\mathcal{R}}$. On the other hand, $u \in \Omega_{\mathcal{R}}(b_1,T)$ if $u \in \Omega_{\mathcal{R}}(b,t)$; thanks to $u = u \diamond_T u$, we also obtain the converse inclusion. The proof for $\Omega_{\mathcal{L}}$ is similar.

Let the mappings $V^{\mathcal{L}}$, $V^{\mathcal{R}}$ from $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ be $V^\diamond$ with $\Omega_\diamond = \Omega_{\mathcal{L}}$, $\Omega_\diamond = \Omega_{\mathcal{R}}$, respectively. Assuming $V^{\mathcal{L}}(0, x^*(0)) \in \mathbb{R}$, it is easy to verify that u^* is *classical optimal* [9, Definition 7.5], [28, (L1)] iff $u^* \in \Omega_{\mathcal{L}}(x^*(0), 0)$ holds (4.2) for $V^\diamond = V^{\mathcal{L}}$. Similarly, assuming $V^{\mathcal{R}}(0, b_*) \in \mathbb{R}$, u^* *is almost strongly optimal* [9, Definition 7.8],[28, (R1)] iff $u^* \in \Omega_{\mathcal{R}}(x^*(0), 0)$ holds (4.2) for $V^\diamond = V^{\mathcal{R}}$. Theorems 3.1 and 4.2 imply necessary conditions of optimality for these criteria. In [9, Sect. 7], see rather general conditions of existence for such optimal controls.

5. On agreeable control

Define the function V^∞ from $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$ to $\mathbb{R} \cup \{-\infty, +\infty\}$ as follows:

$$V^\infty(t,b) := \liminf_{n\to\infty} V^{\tau_n}(t,b), \quad \forall (t,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m. \tag{5.1}$$

Lemma 5.1. *Assume that the function V^∞ is finite. Then, V^∞ enjoys the Dynamic Programming Principle (DPP)*

Proof. Fix $T > 0, (t,b) \in [0,T] \times \mathbb{R}^m$. By Remark 2.2, V^{τ_n} satisfies DPP. Passing to the lower limit as $n \to \infty$, we have

$$V^\infty(t,b) = \inf_{u\in\mathbb{U}} \Big[\int_t^T f_0(s, x_{b,t,u}(s), u(s))ds + V^\infty(T, x_{b,t,u}(T)) \Big]$$

for all $T > 0, (t,b) \in [0,T] \times \mathbb{R}^m$. Thus, V^∞ satisfies DPP. □

Definition 5.2. Call $u^* \in \mathbb{U}$ *a weakly agreeable* control [12, Definition 3.2(iii)] iff this control (with its motion x^*, $x^*(0) = b_*$) satisfies

$$\liminf_{T\to+\infty} \Big(J\big(0, b_*; u^*, t\big) + V^T\big(t, x^*(t)\big) - \inf_{u\in\mathbb{U}} J\big(0, b_*; u, T\big) \Big) \leq 0, \quad \forall t \geq 0.$$

Definition 5.3. Call $u^* \in \mathbb{U}$ *an agreeable* control [12, Definition 3.2(ii)] iff this control (with its motion x^*, $x^*(0) = b_*$) satisfies

$$\lim_{T\to+\infty} \Big(J(0, b_*; u^*, t) + V^T(t, x^*(t)) - \inf_{u\in\mathbb{U}} J(0, b_*; u, T) \Big) \leq 0, \quad \forall t \geq 0.$$

Lemma 5.4. *For a control $u^* \in \mathbb{U}$ with its motion x^*, u^* is weakly agreeable iff there exists an unbounded sequence of positive τ_n such that, for all $T > 0$,*

$$J(0, x^*(0); u^*, T) = \lim_{n\to\infty} \big[V^{\tau_n}(0, x^*(0)) - V^{\tau_n}(T, x^*(T)) \big] \tag{5.2}$$

Proof. Let $u^* \in \mathbb{U}$ be weakly agreeable. Then, for a natural n, we can choose $\tau_n > n$ such that $J(0, x^*(0); u^*, n) + V^{\tau_n}(n, x^*(n)) - V^{\tau_n}(0, x^*(0)) < 1/n$. For all $T \in [0, n]$, the relation $u^* \diamond_n \mathbb{U} \subset u^* \diamond_T \mathbb{U}$ holds. Hence,

$$J(0, x^*(0); u^*, T) + V^{\tau_n}(T, x^*(T)) \leq J(0, x^*(0); u^*, n) + V^{\tau_n}(n, x^*(n)),$$

also, by DPP, $0 \leq J(0, x^*(0); u^*, T) + V^{\tau_n}(T, x^*(T)) - V^{\tau_n}(0, x^*(0)) < 1/n$. Passing to the limit as $n \to \infty$, we obtain (5.2) for this sequence of τ_n.

The converse implication is clear. □

Necessary conditions of weakly agreeable optimality

Corollary 5.5. *Let $u^* \in \mathbb{U}$ be a weakly agreeable control in* (2.1)–(2.3), *i.e., for some unbounded sequence $\tau_n \uparrow \infty$, let u^* satisfy* (5.2) *for all $T > 0$.*

Let a function V^∞ be finite and locally Lipschitz continuous.

Then, there exists a co-state arc $\psi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ such that PMP relations (3.1)–(3.2) *for $\lambda = 1$ and sensitivity relations* (3.3)–(3.4) *for $\mathbb{V} = V^\infty$ hold.*

Proof. Fix $T > 0$. By (5.2), there exists a converging to 0 sequence of ε_n such that

$$J(0, x^*(0); u^*, T) + V^{\tau_n}(T, x^*(T)) = V^{\tau_n}(0, x^*(0)) + \varepsilon_n$$

holds. Passing to the lower limit as $\tau_n \to \infty$, we obtain

$$J(0, x^*(0); u^*, T) + V^\infty(T, x^*(T)) = V^\infty(0, x^*(0)), \qquad \forall T > 0.$$

Now, Theorem 3.1 with $\mathbb{V} = V^\infty$ completes the proof. □

Sufficient conditions of weakly agreeable optimality

Lemma 5.6. *Let τ be an unbounded sequence of positive numbers. Let the value $V^\infty(0, b_*)$, defined in* (5.1)*, be finite. Let u^* satisfy* (2.4) *for $\mathbb{V} = V^\infty$.*

Then, u^* is weakly agreeable, and (5.2) *holds for some subsequence of τ.*

Proof. Note that, by (2.4), the finiteness of $V^\infty(0, x^*(0))$ implies the finiteness of $V^\infty(t, x^*(t))$ for all $t > 0$. Now, there exists a subsequence of $\tau'_k := \tau_{n(k)}$ such that $V^{\tau'_k}(k, x^*(k)) \leq V^\infty(k, x^*(k)) + 1/k$ for a natural k. Fix these τ'_k.

For each $k \in \mathbb{N}, T \in [0, k]$, we have $u^* \diamond_k \mathbb{U} \subset u^* \diamond_T \mathbb{U}$. Hence,

$$\begin{aligned} J(0, b_*; u^*, T) + V^{\tau'_k}(T, x^*(T)) &\leq J(0, b_*; u^*, k) + V^{\tau'_k}(k, x^*(k)) \\ &\leq J(0, b_*; u^*, k) + V^\infty(k, x^*(k)) + 1/k \\ &\overset{(2.4)}{=} V^\infty(0, x^*(0)) + 1/k \\ &= V^\infty(T, x^*(T)) + J(0, b_*; u^*, T) + 1/k. \end{aligned}$$

Thus, $V^{\tau'_k}(T, x^*(T))$ converges to $V^\infty(T, x^*(T))$ for all $T \geq 0$ as $k \to \infty$.

Now, for each $T > 0$, passing to the limit, we obtain

$$\begin{aligned} 0 &\overset{(2.4)}{=} J(0, x^*(0); u^*, T) + V^\infty(T, x^*(T)) - V^\infty(0, x^*(0)) \\ &= \lim_{k \to \infty} \big[J(0, x^*(0); u^*, T) + V^{\tau'_k}(T, x^*(T)) - V^{\tau'_k}(0, x^*(0)) \big]. \end{aligned}$$

Thus, (5.2) holds for τ', and, by Lemma 5.4, u^* is weakly agreeable. □

Applying Lemma 5.6 and Theorem 3.2, we obtain a sufficient condition for a weakly agreeable control.

Corollary 5.7. *For some unbounded sequence $\tau \uparrow \infty$, let V^∞ be finite and locally Lipschitz continuous. Let a co-state arc $\psi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ satisfy PMP relations* (3.1)–(3.2) *with $\lambda = 1$ and sensitivity relation* (3.4) *for $\mathbb{V} = V^\infty$.*

Then, (5.2) *holds for some subsequence of τ, and u^* is weakly agreeable.*

See the sufficient conditions of agreeable optimality in terms of the asymptotic behavior of $\psi(s)(x(s) - x^*(s))$ for large s in [13, Theorems 2.5 and 2.6]. Such conditions are often used in proofs of the turnpike property [38].

Conditions for agreeable optimality.

Theorem 5.8. *Let a function V^{all}, defined as follows,*

$$V^{all}(t,b) := \lim_{T\to+\infty} V^T(t,b), \qquad \forall (t,b) \in \mathbb{R}_{\geq 0} \times \mathbb{R}^m, \tag{5.3}$$

be well-defined and locally Lipschitz continuous.

Then, the following conditions are equivalent:

(1) *u^* is weakly agreeable;*
(2) *u^* is agreeable;*
(3) *there exists a co-state arc $\psi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$ satisfying PMP relations* (3.1)–(3.2) *with $\lambda = 1$ and sensitivity relations* (3.3),(3.4) *for $\mathbb{V} = V^{all}$.*

Proof. 2) $\Rightarrow$ 1) was proved in [12, Proposition 3.2].

1) $\Rightarrow$ 3). By Lemma 5.4, u^* satisfies (5.2) for some unbounded sequence of positive τ_n. Note that $V^\infty \equiv V^{all}$, and all conditions of Corollary 5.5 hold. Therefore, for some co-state arc $\psi \in C(\mathbb{R}_{\geq 0}, \mathbb{R}^m)$, relations (3.1)–(3.2) for $\lambda = 1$ and relations (3.3)–(3.4) for $\mathbb{V} = V^\infty$ hold. By $V^\infty \equiv V^{all}$, 1) $\Rightarrow$ 3) is proved.

3) $\Rightarrow$ 2). Assume the contrary. Then, by the definition of agreeable controls, there exist $t \geq 0$ and an unbounded sequence of positive τ_n such that

$$\liminf_{n\to\infty} \Big(J(0, b_*; u^*, t) + V^{\tau_n}(t, x^*(t)) - V^{\tau_n}(0, x^*(0)) \Big) > 0.$$

On the other hand, $V^\infty \equiv V^{all}$ and all conditions of Corollary 5.7 hold, and (5.2) holds for u^* with every unbounded sequence of positive numbers. However, it contradicts the choice of τ_n. □

Some conditions of validity of V^{all} were shown in [15, Theorem 3.3]. In this case, $V^{\inf} \equiv V^{all}$ holds [15, (3.13)]. Then, a weakly agreeable control is also weakly overtaking optimal [12]. Conditions [30, Hypothesis 3.1(i)-(iv), Hypothesis A.1] guarantee the Lipschitz continuity for well-defined V^∞ and V^{all} (it is sufficient to repeat the proof of [30, Theorem A.1] verbatim). On the other hand, in Example 7.3, the value functions $V^{\inf}$ and V^{all} are Lipschitz continuous and there exists a control that is optimal for both value functions, however, $V^{\inf} > V^{all}$ holds everywhere.

6. On conditions of Lipschitz continuity of value functions

Usually, the Lipschitz continuity of a value function is guaranteed by asymptotic conditions on f, f_0, J, or on solutions of (2.2),(3.1), see [1, 2, 3, 5, 6, 30, 37]. We will use the Lipschitz continuity [24, 35] of the optimal-time function of the control system. Let us obtain the conditions to guarantee the Lipschitz continuity of the value function in absence of any asymptotic conditions besides the finiteness of this function.

Fix a non-empty compact $P' \subset P$. By $\mathbb{U}' \subset \mathbb{U}$ denote the class of all admissible controls satisfying $u(t) \in P'$ a.e.

Let G be a non-empty open subset of $\mathbb{R}_{\geq 0} \times \mathbb{R}^m$. Consider nonnegative integers r, s $(r+s = m)$ and a set $\mathbb{W} \subset \mathbb{R}^r$. Define optimal-time function $Q'_{\mathbb{W}}$ under additional condition $u \in \mathbb{U}'$ and maximal-time function $Q^{\mathbb{W}}$ as follows: for all $y' = (w', z') \in G, z \in \mathbb{R}^s, t' \geq 0$,

$$Q'_{\mathbb{W}}(t', y', z) := \inf \big\{\tau \geq 0 : \exists\ u \in \mathbb{U}', x_{y',t',u}(t' + \tau) \in \mathbb{W} \times \{z\}\big\},$$

$$Q^{\mathbb{W}}(t', y', z) := \inf \big\{T \geq 0 : \forall\ u \in \mathbb{U}, \exists\ s \in [0, T], x_{y',t',u}(t' + s) \in \mathbb{W} \times \{z\}\big\},$$

$Q'_{\mathbb{W}}(y', z, t') := +\infty$ if no control $u \in \mathbb{U}'$ transfers (2.2) from (t', y') in $[t', \infty[\times \mathbb{W} \times \{z\}$, and $Q^{\mathbb{W}}(t', y', z) := +\infty$ if there exists a control $u \in \mathbb{U}$ that does not transfer (2.2) from (t', y') in $[t', \infty[\times \mathbb{W} \times \{z\}$.

Let I, W, and Z be non-empty open subsets of $\mathbb{R}_{\geq 0}, int\,\mathbb{W}, \mathbb{R}^s$ respectively.

We will prove the following theorems:

Theorem 6.1. *Let $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ enjoy DPP.*

Assume that a function $S : Z \to \mathbb{R}$ and a positive locally Lipschitz continuous function $R : I \times \mathbb{W} \to \mathbb{R}_{>0}$ satisfy

$$\mathbb{V}(t, (w, z)) = R(t, w)S(z) \qquad \forall (t, w, z) \in I \times \mathbb{W} \times Z.$$

For a point of $I \times W \times Z$, let there exist a neighborhood $I' \times W' \times Z'$ of this point and a positive L such that

$$Q'_{\mathbb{W}}(t', (w', z'), z) \leq L||z' - z|| \quad \forall t' \in I', w' \in W', z, z' \in Z'.$$

Then, $\mathbb{V}$ is locally Lipschitz continuous in $I \times W \times Z$.

Corollary 6.2. *Let $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ satisfy DPP. Assume that*

$$0 \in int\,conv\,\{f(t, x, u)\ :\ u \in P'\} \quad \forall (t, x) \in G$$

holds for some open set $G \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^m$.

Assume that a function $S : \mathbb{R}^m \to \mathbb{R}$ and a locally Lipschitz continuous function $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfy

$$\mathbb{V}(t, x) = R(t)S(x) \qquad \forall (t, x) \in G.$$

Then, $\mathbb{V}$ is locally Lipschitz continuous in G.

Theorem 6.3. *Let $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ enjoy DPP and P be compact. Assume that a function $S : Z \to \mathbb{R}$ and a positive locally Lipschitz continuous function $R : I \times \mathbb{W} \to \mathbb{R}_{>0}$ satisfy*

$$\mathbb{V}(t, (w, z)) = R(t, w)S(z) \qquad \forall (t, w, z) \in I \times \mathbb{W} \times Z.$$

For a point of $I \times W \times Z$, let there exist a neighborhood $I' \times W' \times Z'$ of this point and a positive L such that

$$\min\Big\{Q^{\mathbb{W}}(t', (w', z'), z), Q^{\mathbb{W}}(t', (w', z), z')\Big\} \leq L||z' - z||$$

holds for all $t' \in I', w' \in W', z, z' \in Z'$.

Then, $\mathbb{V}$ is locally Lipschitz continuous in $I \times W \times Z$.

Corollary 6.4. *Let $\mathbb{V} : \mathbb{R}_{\geq 0} \times \mathbb{R}^m \to \mathbb{R}$ satisfy DPP and P be compact. Assume that*

$$0 \notin cl\, conv\, \{f(t,x,u) \,:\, (t,x) \in G, u \in P\}$$

holds for some open set $G \subset \mathbb{R}_{\geq 0} \times \mathbb{R}^m$.

Assume that a function $S : \mathbb{R}^m \to \mathbb{R}$ and a locally Lipschitz continuous function $R : \mathbb{R}_{\geq 0} \to \mathbb{R}_{>0}$ satisfy

$$\mathbb{V}(t,x) = R(t)S(x) \qquad \forall (t,x) \in G.$$

Then, $\mathbb{V}$ is locally Lipschitz continuous in G.

Proof of Theorem 6.1. Fix $(t_*, b_W, b_Z) \in I \times W \times Z$. Take a fitting neighborhood $I' \times W' \times Z'$ of this point with some $L > 0$. It is safe to assume $cl\,(I' \times W' \times Z') \subset I \times W \times Z$ to be compact and I' to be an interval. It will suffice to prove that $\mathbb{V}$ is Lipschitz continuous in another, possibly smaller neighborhood of (t_*, b_W, b_Z). Set $b := (b_W, b_Z)$, $G' := W' \times Z'$.

Thanks to the sublinear growth of f, there exist a compact K ($cl\, G' \subset int\, K \subset \mathbb{W} \times Z$) and $\gamma > 0$ such that $y \in G'$ implies

$$x_{y,t,u}(t+\tau) \in K, t+\tau \in I \quad \forall u \in \mathbb{U}', t \in I', \tau \in [0,\gamma].$$

Now, we can choose $M > \max\{L+1, |\mathbb{V}(t_*, b)|\}$ such that

$$\begin{aligned} &f_0(t+\tau, y, u) < M, \quad 1/M \leq |R(t+\tau, w)| \leq M, \\ &||f(t+\tau, y, u)|| < M, \quad |R(t, w') - R(t+\tau, w)| \leq M||w' - w|| + M\tau \end{aligned}$$

hold for all $y \in K, (w,z), (w',z') \in K \cap (\mathbb{W} \times Z), u \in P', t \in I', \tau \in [0,\gamma]$. Now, we obtain $||x_{y,t,u}(t+\tau) - y|| \leq M\tau$ for all $y \in G', t \in I', \tau \in [0,\gamma]$. Also, we have $|S(b_Z)| = |\mathbb{V}(t_*, b)|/R(t_*, b_W) \leq M^2$. Decreasing the neighborhood $G' := W' \times Z'$ of b, we can assume $diam\, Z' < \gamma/M$.

Fix $t' \in I'$. Consider $w' \in W', z', z'' \in Z'$. Set $y' := (w', z') \in G'$. By the choice of L and M, we have

$$Q'_{\mathbb{W}}((w', z'), z'', \tau) \leq L||z' - z''|| < M\, diam\, Z' < \gamma.$$

By the definition of $Q'_{\mathbb{W}}$, we can choose $u \in \mathbb{U}'$, $\bar{w} \in \mathbb{W}$, and $\tau \geq 0$ enjoying

$$x_{y',t',u}(t'+\tau) = (\bar{w}, z''), \quad \tau \leq (L+1)||z' - z''|| \leq M\, diam\, Z' < \gamma.$$

By $\tau < \gamma$ and $y' \in G'$, we obtain $(\bar{w}, z'') \in K$ and

$$||\bar{w} - w'|| \leq ||x_{y',t',u}(t'+\tau) - y'|| \leq M\tau < M^2||z' - z''||. \tag{6.1}$$

Set $t'' := t' + \tau$. By DPP, we have

$$\mathbb{V}(t', y') \leq J(t', y'; u, t'') + \mathbb{V}(t'', x_{y',t',u}(t'')).$$

In view of $\mathbb{V}(t, (w,z)) = R(t,w)S(z)$ and $f_0(t,y,u) \leq M$, we obtain

$$\begin{aligned} R(t', w')S(z') &= \mathbb{V}(t', y') \\ &\leq M\tau + R(t'', \bar{w})S(z'') = M\tau + \mathbb{V}(t'', (\bar{w}, z'')). \end{aligned} \tag{6.2}$$

In particular, in the case $y' := y, z'' := b_Z$, we can provide $(\bar{w}, z'') \in K$, $\tau < \gamma$, and

$$R(t', w')S(z') \overset{(6.2)}{\leq} R(t'', \bar{w})S(b_Z) + M\tau \leq M|S(b_Z)| + M \leq 2M^3$$

by the choice of M. In the case $y' := b, z'' := z$, we have $(\bar{w}, z) \in K$, $\tau < \gamma$ and, at last,

$$R(t'', \bar{w})S(z) \overset{(6.2)}{\geq} R(t', b_W)S(b_Z) - M\tau \geq -M^3 - M\tau \geq -2M^3.$$

According to $|R| > 1/M$, we also have $|S(z)| \leq 2M^4$ for all $(w, z) \in G'$.

Return to the general case. Consider every $w' \in W', z', z'' \in Z'$ and, additionally, every $w'' \in W', t' \in I'$. Then, we again obtain $(\bar{w}, z'') \in K$ and $\tau < \gamma$ satisfying (6.1) and (6.2). Now, by (6.2), for all $t, t' \in I', w', w'' \in W'$, and $z', z'' \in Z'$, $\mathbb{V}(t', w', z') - \mathbb{V}(t, w'', z'')$ does not exceed

$$\begin{aligned} M\tau + (R(t'+\tau, \bar{w}) - R(t', w') + R(t', w') - R(t, w''))S(z'') \\ \leq M\tau + 2M^5(\tau + ||\bar{w} - w'|| + ||w'' - w'|| + |t' - t|) \\ \overset{(6.1)}{\leq} (M + 2M^5 + 2M^6)\tau + 2M^5(||w'' - w'|| + |t' - t|) \\ \overset{(6.1)}{\leq} 7M^7(||z' - z''|| + ||w'' - w'|| + |t' - t|). \end{aligned}$$

Thus,

$$|\mathbb{V}(t', w', z') - \mathbb{V}(t, w'', z'')| \leq 7M^7(||z' - z''|| + ||w'' - w'|| + |t' - t|)$$

holds for all $(t', w', z'), (t, w'', z'')$ from some neighborhood $I' \times W' \times Z'$ of each $(t_*, b_W, b_Z) \in I \times W \times Z$. □

Proof of Theorem 6.3. Fix $(t_*, b_W, b_Z) \in I \times W \times Z$. Take a fitting neighborhood $I' \times W' \times Z'$ of this point with some $L > 0$. It is safe to assume $cl\,(I' \times W' \times Z') \subset I \times W \times Z$ to be compact and I' to be an interval. It will suffice to prove that $\mathbb{V}$ is Lipschitz continuous in another, possibly smaller neighborhood of (t_*, b_W, b_Z). Set $b := (b_W, b_Z)$, $G' := W' \times Z'$.

Thanks to the sublinear growth of f, there exist a compact K ($cl\,G' \subset int\,K \subset \mathbb{W} \times Z$) and $\gamma > 0$ such that

$$x_{y,t,u}(t + \tau) \in K, t + \tau \in I \qquad \forall y \in G', \tau \in [0, \gamma], u \in \mathbb{U}, t \in I'.$$

Now, we can choose $M > \max\{L + 1, |\mathbb{V}(t_*, b)|\}$ such that

$$\begin{aligned} f_0(t + \tau, y, u) < M, \quad 1/M \leq |R(t + \tau, w)| \leq M, \\ ||f(t + \tau, y, u)|| < M, \quad |R(t, w') - R(t + \tau, w)| \leq M||w' - w|| + M\tau \end{aligned}$$

hold for all $y \in K, (w, z), (w', z') \in K \cap (\mathbb{W} \times Z), u \in P, t \in I, \tau \in [0, \gamma]$. Now, we obtain $||x_{y,t,u}(t + \tau) - y|| \leq M\tau$ for all $y \in G', t \in I', \tau \in [0, \gamma]$. Also, we have $|S(b_Z)| \leq |\mathbb{V}(t_*, b)|/R(t_*, b_W) \leq M^2$. Decreasing the neighborhood $G' := W' \times Z'$ of b, we can assume $diam\, Z' < \gamma/M$.

Fix $t' \in I'$, $w' \in W', z', z'' \in Z'$ such that $Q^{\mathbb{W}}(t', (w', z'), z'') \leq L||z' - z''||$. Set $y' := (w', z') \in G'$. Consider the following exit-time problem:

$$\begin{cases} \text{Minimize } \int_{t'}^{T} f_0(t, x, u)\, dt + \mathbb{V}(T; x(T)) \\ \text{subject to } \dot{x} = f(t, x, u), \quad u \in P, \\ x(t') = y', \quad T = \inf\left\{t \geq t' \,:\, x(t) \in \mathbb{W} \times \{z''\}\right\} \cup \{+\infty\}. \end{cases}$$

By the definition of $Q^{\mathbb{W}}$, for all $u \in \mathbb{U}$,

$$T_u := \inf\{t \geq t' \,:\, x_{y',t',u}(t) \in \mathbb{W} \times \{z''\}\} \leq t' + Q^{\mathbb{W}}(t', (w', z'), z''),$$

i.e.,

$$0 \le T_u - t' \le L||z' - z''||, \qquad x_{y',t',u}(T_u) \in \mathbb{W} \times \{z''\}. \tag{6.3}$$

Since all T_u are uniformly bounded (by $t' + L||z' - z''||$), the optimal value of this exit-time problem is $V(t', y')$; in particular,

$$V(t', y') \le J(t', y'; u, T_u) + V(T_u, x_{y',t',u}(T_u)) \quad \forall u \in \mathbb{U}.$$

In addition, we can choose a control $u' \in \mathbb{U}$ with its motion $x' := x_{y',t',u'}$ such that

$$|V(t', y') - J(t', y'; u', T_{u'}) - V(T_{u'}, x'(T_{u'}))| \le ||z' - z''||.$$

By (6.3) we have $x'(T_{u'}) \in \mathbb{W} \times \{z''\}$. Now, we can choose $\bar{w} \in \mathbb{W}$ such that $x'(T_{u'}) = (\bar{w}, z'')$. So,

$$|R(t', w')S(y') - J(t', y'; u', T_{u'}) - R(T_{u'}, \bar{w})S(z'')| \le ||z' - z''||. \tag{6.4}$$

In addition, $T_{u'} - t' \le L||z' - z''|| \le M\, diam\, Z' < \gamma$. Then, by the choice of the compact K, we have $x'(t) \in K$ for all $t \in [t', T_{u'}]$. By the choice of M, we obtain

$$\begin{aligned} ||\bar{w} - w'|| &\le ||x_{y',t',u'}(T_{u'}) - y'|| \\ &\le M(T_{u'} - t') \overset{(6.3)}{\le} M^2||z' - z''|| \end{aligned} \tag{6.5}$$

and $|J(t', y'; u', T_{u'})| \le M(T_{u'} - t')$. Then,

$$\begin{aligned} |R(t', w')S(z') - R(T_{u'}, \bar{w})S(z'')| &\overset{(6.4)}{\le} ||z' - z''|| + M(T_{u'} - t') \\ &\overset{(6.3)}{\le} 2M^2||z' - z''|| \\ &\le 2M\gamma \le 2M^2. \end{aligned} \tag{6.6}$$

So, we proved that, for all $t' \in I'$, $w' \in W', z', z'' \in Z'$, the inequality $Q^{\mathbb{W}}(t', (w', z'), z'') \le L||z' - z''||$ implies (6.5),(6.6).

We claim that

$$|S(z)| \le 3M^4 \quad \forall z \in Z'. \tag{6.7}$$

Indeed, fix $z \in Z'$ such that $Q^{\mathbb{W}}(t_*, (b_W, z), b_Z) \le L||z - b_Z||$. Now, we can define $t' := t_*, w' := b_W, z' := z, z'' := b_Z$, provide $(\bar{w}, b_Z) \in K$, $t'' \in [t', t' + \gamma]$, and

$$|R(t', w')S(z)| \overset{(6.6)}{\le} |R(t'', \bar{w})S(b_Z)| + 2M^2 \le M|S(b_Z)| + 2M^2 \le 3M^3.$$

Thanks to $|R| > 1/M$, we obtain (6.7) if $Q^{\mathbb{W}}(t_*, (b_W, z), b_Z) \le L||z - b_Z||$.

In the case $Q^{\mathbb{W}}(t_*, (b_W, b_Z), z) \le L||z - b_Z||$ we can define $t' := t_*, w' := b_W, z' := b_Z, z'' := z$, provide $(\bar{w}, z) \in K$, $t'' \in [t', t' + \gamma]$, and

$$|R(t'', \bar{w})S(z)| \overset{(6.6)}{\le} |R(t', w')S(b_Z)| + 2M^2 \le M|S(b_Z)| + 2M^2 \le 3M^3.$$

Thanks to $|R| > 1/M$, we obtain (6.7) if $Q^{\mathbb{W}}(t_*, (b_W, b_Z), z) \le L||z' - b_Z||$.

So, by $\min\left\{Q^{\mathbb{W}}(t_*, (b_W, z), b_Z), Q^{\mathbb{W}}(t_*, (b_W, b_Z), z)\right\} \le L||z - b_Z||$ for all $z \in Z'$, (6.7) proved.

Fix every $t, t' \in I', w', w'' \in W', z', z'' \in Z'$. We shall prove that $|\mathbb{V}(t', w', z') - \mathbb{V}(t, w'', z'')| \le 7M^7(||z' - z''|| + ||w'' - w'|| + |t' - t|)$. Swapping z', z'' if necessary we can assume that $Q^{\mathbb{W}}(t', (w', z'), z'') \le L||z' - z''||$. Then, we again obtain $u' \in$

$\mathbb{U}$, $T_{u'} \in [t', t'+\gamma]$ and $x_{y',t',u'}(T_{u'}) = (\bar{w}, z'') \in K$ satisfying (6.5),(6.6). Now, $|\mathbb{V}(t', w', z') - \mathbb{V}(t, w'', z'')|$ does not exceed

$$\begin{aligned}
&\big|R(t',w')S(z') - R(T_{u'},\bar{w})S(z'')\big| + \big|\big(R(T_{u'},\bar{w}) - R(t,w'')\big)S(z'')\big| \\
\overset{(6.6)}{\leq}\ & 2M^2||z'-z''|| + |(R(T_{u'},\bar{w}) - R(t',w') + R(t',w') - R(t,w'')S(z'')| \\
\overset{(6.7)}{\leq}\ & M^3||z'-z''|| + 3M^5(T_{u'} - t' + ||\bar{w}-w'|| + ||w''-w'|| + |t'-t|) \\
\overset{(6.5)}{\leq}\ & M^3||z'-z''|| + 3M^5(1+M)(T_{u'}-t') + 3M^5(||w''-w'|| + |t'-t|) \\
\overset{(6.3)}{\leq}\ & 7M^7||z'-z''|| + 3M^5(||w''-w'|| + |t'-t|).
\end{aligned}$$

□

7. Examples

Example 7.1. Let x be the capital stock, u the investment, $\nu > 0$ the depreciation rate, and μ the discount rate. Consider the following problem:

$$\text{Minimize} \int_0^{+\infty} e^{\mu t} g(x,u)dt,$$
$$\text{subject to } \dot{x} = -\nu x + u,\ x(0) = x_* \quad u \in [0, U_{max}], \quad k > 0.$$

Set $R(\theta) := e^{\mu\theta}$ for all $\theta \in \mathbb{R}$. It is easy to prove that

$$J(\theta, y; u, T) = R(\theta) J(0, y; u', T-\theta) \qquad \forall \theta \geq 0, y \in \mathbb{R}, T > \theta,$$

where $u'(t) := u(t+\theta)$ for all $t \geq 0$. Since $u \in \Omega$ iff $u' \in \Omega$ for each of $\Omega \in \{\Omega_{\mathcal{L}}, \Omega_{\mathcal{R}}, \mathbb{U}\}$, we have

$$\mathbb{V}(\theta, y) = R(\theta)\mathbb{V}(0, y) \qquad \forall y \in \mathbb{R}^m, \theta \geq 0, \mathbb{V} \in \{V^{\mathcal{R}}, V^{inf}, V^{\mathcal{L}}, V^{all}\}$$

if $\mathbb{V}$ is well-defined.

Set $Z_< :=]0, U_{max}/\nu[$. We obtain $0 \in -\nu x + int\,[0, U_{max}]$ for all $x \in Z_<$. By Corollary 6.3, $\mathbb{V}$ is Lipschitz continuous (if finite) in $\mathbb{R}_{\geq 0} \times Z_<$.

Set $Z_> :=]U_{max}/\nu, \infty[$. We obtain $-\nu x + U_{max} < 0$ for all $x \in Z_>$. By Corollary 6.4, $\mathbb{V}$ is Lipschitz continuous (if finite) in $\mathbb{R}_{\geq 0} \times Z_>$.

So, the value functions are Lipschitz continuous (if finite) in $\{(t,x) \in \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} : x \neq U_{max}/\nu\}$; no assumptions on g besides continuity are needed.

Consider a use case of Theoremae 6.1,6.3 (compare with [39, 22]):

Example 7.2.

$$\text{Minimize} \int_0^{\infty} g(y_1, y_2, u)dt$$
$$\text{subject to } \dot{y_1} = u,\ \dot{y_2} = y_1, \quad (y_1, y_2)(0) = b_*, \tag{7.1}$$
$$y_1, y_2 \in \mathbb{R}, \quad u \in P \subset \mathbb{R}.$$

For a certain $k \in \mathbb{R}$, let us also require

$$g(\nu y_1, \nu^2 y_2, u) = \nu^k g(y_1, y_2, u) \qquad \forall \nu > 0, (y_1, y_2, u) \in \mathbb{R}^3.$$

Clearly, the functions from $\{V^{\mathcal{R}}, V^{inf}, V^{\mathcal{L}}, V^{all}\}$ do not depend on t. More to come. First, note that $u \in \mathbb{U}$ iff all maps $u'_\nu(t) := u(t/\nu)$ are within $\mathbb{U}$ for all $\nu > 0$.

Fix $(y_1, y_2) \in \mathbb{R}^2, u \in \mathbb{U}, \nu > 0$. Set $x := x_{(y_1,y_2),0,u}$ and $x' := x_{(\nu y_1,\nu^2 y_2),0,u'_\nu}$. For all $t \geq 0$, we have

$$x(t) = x'(t/\nu),\ g(x'(t/\nu), u(t/\nu)) = \nu^k g(x(t), u(t)),$$
$$J(0, (\nu y_1, \nu^2 y_2); u'_\nu, T/\nu) = \nu^{k-1} J(0, (y_1, y_2); u, T).$$

Now, for all finite $\mathbb{V} \in \{V^{\mathcal{R}}, V^{inf}, V^{\mathcal{L}}, V^{all}\}$, we have

$$\mathbb{V}(t, \nu y_1, \nu^2 y_2) = \nu^{k-1}\mathbb{V}(0, y_1, y_2) \quad \forall \nu > 0, t \geq 0, (y_1, y_2) \in \mathbb{R}^2.$$

Define $\mathbb{W} := \mathbb{R}_{>0}$, $Z := \mathbb{R}$. Let some $\mathbb{V} \in \{V^{\mathcal{R}}, V^{inf}, V^{\mathcal{L}}, V^{all}\}$ be well-defined and finite.

The case $P = \mathbb{R}$.

Consider $G_+ = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 > 0\}$ and the functions $R : \mathbb{R}_{\geq 0} \times \mathbb{W} \to \mathbb{R}_{>0}$ and $S : Z \to \mathbb{R}$ defined as follows:

$$R(t, w) := w^{k-1}, S(z) := \mathbb{V}(0, z, 1) \qquad \forall z \in \mathbb{R}, w > 0, t \geq 0; \tag{7.2}$$

then,

$$\mathbb{V}(t, y_1, y_2) = R(t, \sqrt{y_2})S(y_1/\sqrt{y_2}) \quad \forall y_1 \in \mathbb{R}, y_2, t \geq 0. \tag{7.3}$$

In the coordinates ($w := \sqrt{y_2}, z := y_1/\sqrt{y_2}$), system (7.1) has the form

$$\dot{w} = z/2, \quad \dot{z} = \frac{u - z^2/2}{w}, \quad u \in \mathbb{R},\ (z, w) \in Z \times \mathbb{W}. \tag{7.4}$$

Note that, for every point $(y_1, y_2) \in G_+$, in its sufficiently small neighborhood, the controls $u := \pm z^2(y_1, y_2) \in P' := [-2y_1^2/y_2, 2y_1^2/y_2]$ provide for z to increase/decrease with the speed at least $1/4\sqrt{y_2}$. Set $L := 4\sqrt{y_2}$; in view of Theorem 6.1, we find out that, in $\mathbb{R}_{\geq 0} \times G_+$ (for $y_2 > 0$), the function $\mathbb{V}$ is Lipschitz continuous. The proof of the case $y_2 < 0$ is similar.

Thus, in the case $P := \mathbb{R}$, the function $\mathbb{V}$ is Lipschitz continuous under

$$\{(y_1, y_2) \in \mathbb{R}^2 \,|\, y_2 \neq 0\}.$$

The subcase $P = [-a^2, a^2]$, $G^a_+ := \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 > 0, y_1^2 < 2y_2a^2\}$.

Consider the functions $R : \mathbb{R}_{\geq 0} \times \mathbb{W} \to \mathbb{R}_{>0}$, $S : Z \to \mathbb{R}$, defined by (7.2). Then, V satisfies (7.3), in the coordinates ($w := \sqrt{y_2}, z := y_1/\sqrt{y_2}$), system (7.1) has the form (7.4). Note that, for every point $(y_1, y_2) \in G^a_+$, we have $|y_1| < a\sqrt{y_2}$, i.e. $z(y_1, y_2) < a$. Hence, in its sufficiently small neighborhood, the controls $u := \pm z^2(y_1, y_2) \in P$ provide for z to increase/decrease with the speed at least $1/4\sqrt{y_2}$. Set $L := 4\sqrt{y_2}$; in view of Theorem 6.1, we find out that, in $\mathbb{R}_{\geq 0} \times G^a_+$ (for $y_1^2 < a^2 y_2$), the function $\mathbb{V}$ is Lipschitz continuous. The proof of the case $-y_1^2 > a^2 y_2$ is similar.

The subcase $P = [-a^2, a^2]$, $\widetilde{G}^a_+ = \{(y_1, y_2) \in \mathbb{R}^2 \,:\, y_2 > 0, y_1^2 > 2a^2 y_2\}$.

Consider the functions $R : \mathbb{R}_{\geq 0} \times \mathbb{W} \to \mathbb{R}_{>0}$ and $S : Z \to \mathbb{R}$ defined as follows:

$$R(t, w) := w^{1-k}, S(z) := \mathbb{V}(0, 1, z) \qquad \forall z \in \mathbb{R}, w > 0, t \geq 0,$$

then,

$$\mathbb{V}(t, y_1, y_2) = R(t, y_1)S(y_2/y_1^2) \qquad \forall y_1 \in \mathbb{R}, y_2, t \geq 0.$$

In the coordinates ($w := y_1, z := y_2/y_1^2$), system (7.1) has the form

$$\dot{w} = u, \quad \dot{z} = \frac{1 - 2zu}{w},\ (z, w) \in Z \times \mathbb{W}.$$

Note that, for every point $(y_1, y_2) \in \widetilde{G}^a_+$, we have $y_1^2 > 2a^2 y_2$, i.e., $2a^2 z(y_1, y_2) > 1$. Hence, in its sufficiently small neighborhood, for every admissible control, the coordinate z strictly increases. Then, in view of Corollary 6.2, we find out that, in $\mathbb{R}_{\geq 0} \times \widetilde{G}^a_+$ (for $y_1^2 > 2a^2 y_2$), the function $\mathbb{V}$ is Lipschitz continuous. The proof of the case $y_1^2 < 2a^2 y_2$ is similar.

Thus, in the case $P := [-a^2, a^2]$ the function $\mathbb{V}$ is Lipschitz continuous under

$$\{(y_1, y_2) \in \mathbb{R}^2 \mid y_2 \neq 0, y_1^2 \neq 2a^2\}.$$

Example 7.3. Let us refine Example 7.1; the new example does not satisfy assumptions [5, Ch.3(A4)], [6, $(\widetilde{A}6)$], [30, Hypothesis 3.1(iv)], [31, (C3)].

$$\text{Minimize} \int_0^{+\infty} \big[2u + |u| - x\big]\, dt$$

$$\text{subject to } \dot{x} = 2u - x,\ x(0) = b_*, \quad u \in P = [-1/2, 1/2], \quad x \in \mathbb{R}.$$

Clearly, $J(0, b; u, T) = x_{b,0,u}(T) - b + ||u||_{L_1([0,T],P)}$ for all $b \in \mathbb{R}$, $u \in \mathbb{U}$, $T > 0$. Since $x_{b,0,u}(T) > -2$ holds for large T, the finiteness of the limit of $J(0, b; u, T)$ as $T \to \infty$ implies $u \in L_1(\mathbb{R}_{\geq 0}, P)$, moreover, we have

$$x_{b,0,u}(T) \to 0,\ b + J(0, b; u, T) \to ||u||_{L_1(\mathbb{R}_{\geq 0},P)} \geq 0 \text{ as } T \to \infty.$$

Then, we obtain $V^{inf}(0, b) = -b$, i.e., $u^* \equiv 0$ is optimal in view of V^{inf}. Moreover, $V^{\mathcal{L}} \equiv V^{\mathcal{R}} \equiv V^{\inf}$.

But, for $T > 1$, we define a control $u_T \in \mathbb{U}$ as follows: $u_T(t) = -1/2$ for all $t \in [T - \ln 2, T]$, and $u_T(t) = 0$ otherwise. Then, we have $x_{b,0,u_T}(t) = be^{-t}$ if $t \in [0, T - \ln 2]$, $x_{b,0,u_T}(t) = e^{-t}(b + e^T/2) - 1$ if $t \in [T - \ln 2, T]$. Now, we obtain

$$x_{b,0,u_T}(T) = be^{-T} - 1/2, V^T(0, b) \leq J(0, b; u_T, T) = be^{-T} - 1/2 - b + \frac{\ln 2}{2}.$$

One easily proves that $V^T(0, b) = J(0, b; u_T, T)$. Passing to the limit, we obtain $V^{all}(0, b) = -b + \frac{-1+\ln 2}{2}$ for all $b \in \mathbb{R}$. Thus,

$$V^{all}(b) < V^{\inf}(b) \quad \forall b \in \mathbb{R}.$$

Since $V^{all} - V^{\inf} \equiv const$, (2.4) guarantees that $u^* \equiv 0$ is optimal in view of V^{inf} and V^{all} at once. Moreover, $u^* \equiv 0$ is DH-optimal (in particular, agreeable) and overtaking optimal but is not strongly optimal [18].

Note that, in this example, the functions $V^{all}, V^{\inf}$ are well-defined and smooth. In [25, Proposition 3.2], for exit-time control problems, it was proved that $V^{all} = V^{\inf}$ if V^{all} is well-defined and continuous. Moreover, in this case $V^{all} = V^{\inf}$ is the unique nonnegative solution of the associated HJB equation. Conditions guaranteeing the continuity of V^{all} were showed in [25, Section 4].

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