Pure and Applied Functional Analysis
Volume 2, Number 3, 2017, 511–534

Yokohama Publishers
ISSN 2189-3764 ONLINE JOURNAL

UPPER VALUE OF A SINGULAR INFINITE HORIZON ZERO-SUM LINEAR-QUADRATIC DIFFERENTIAL GAME

VALERY Y. GLIZER AND OLEG KELIS

Abstract. An infinite horizon zero-sum linear-quadratic differential game is considered. For this game, we study the case where a weight matrix of the minimizer's control cost in the cost functional is singular. In such a case, a part of the coordinates of the minimizer's control is singular, meaning that the game itself is singular. We study the upper value of this game. For this purpose, a regularization method is proposed. Namely, the singular game is associated with a new differential game, having the same equation of dynamics. The cost functional in the new game is the sum of the original cost functional and an infinite horizon integral of the squares of the minimizer's singular control coordinates with a small positive weight. The new game is regular. Moreover, it is a partial cheap control game. Using the solvability conditions, this game is associated with a Riccati matrix algebraic equation, perturbed by a small parameter. Based on an asymptotic solution of this equation, the finiteness of the upper value in the original (singular) game is established and estimates of this value are derived. In a reduced set of minimizer's admissible strategies, an expression of the upper value is obtained. Illustrative example is presented.

1. Introduction

A zero-sum differential game is called singular, if it cannot be solved by application of the Isaacs MinMax principle and the Bellman–Isaacs equation method ([6, 16, 17]). This occurs, because the problem of minimization (maximization) of its variational Hamiltonian with respect to the minimizer (maximizer) control either has no solution or has infinitely many solutions. In these cases, one can use higher order optimality conditions to solve the game (see, e.g., [7, 8, 24, 27] and references therein). However, such conditions do not yield a candidate optimal control for the game, having no an optimal control of at least of one player in the class of regular (non generalized) functions, even if the cost functional has a finite infsup (supinf) in this class of functions. Such a case was studied in several works. In [1], the open-loop solution of a singular finite horizon differential game was derived in a class of generalized functions. In [14, 25], singular finite horizon differential games were analyzed by a regularization method, yielding a game value, an optimal state-feedback strategy of the maximizer and a minimizing sequence of state-feedback controls. Based on the minimizing sequence of state-feedback controls, in [14] the notion of optimal trajectory sequence in a singular differential game was introduced,

2010 *Mathematics Subject Classification.* 91A05, 91A10, 91A23, 93C73.
Key words and phrases. Singular differential game, infinite horizon zero-sum differential game, game's upper value.

and such a sequence was constructed. In [29, 30], a singular infinite horizon differential game was considered. The existence of an almost equilibria in this game was established by using the Riccati matrix inequality.

In the present paper, an infinite horizon zero-sum linear-quadratic differential game is considered. The feature of this game is that a weight matrix of the minimizer's control cost in the cost functional is non-invertible (singular). Due to this feature, the game can be solved neither by application of the Isaacs MinMax principle, nor using the Bellman-Isaacs equation approach, i.e. this game is singular. Moreover, the minimizer's optimal control does not belong, in general, to the class of regular functions. We analyze this game from the minimizer's viewpoint. Namely, we look for the minimum guaranteed game outcome (the upper value of the game). To realize this aim, a regularization method is applied, yielding a new game (partial cheap control game). This partial cheap control game is associated with a Riccati matrix algebraic equation, which coefficients depend on a small positive parameter. Using perturbation techniques, an asymptotic solution to this equation is constructed and justified. Based on this solution, the finiteness of the upper value in the original (singular) game is established, and upper and lower estimates of this value are obtained. In a reduced set of admissible minimizer's state-feedback controls, an expression for the upper value is derived.

The paper is organized as follows. In Section 2, we formulate rigorously the problem, to be solved, and the objectives of the paper. General conditions for the finiteness of the game's upper value are derived in Section 3. In Section 4, a regularization of the original singular game is carried out, yielding a partial cheap control game. An asymptotic analysis of the Riccati matrix algebraic equation, associated with the partial cheap control game, is done in Section 5. Upper and lower estimates of the upper value of the original singular game are derived in Section 6. In Section 7, an expression of the upper value is obtained for the reduced set of admissible minimizer's state-feedback controls. Section 8 deals with an illustrative example. Complex proof of one lemma is placed in Section 9. Conclusions are presented in Section 10.

Completing the introduction, let us note that the notation $O_{n_1 \times n_2}$ is used in the paper for the zero matrix of dimension $n_1 \times n_2$, excepting the cases where the dimension of zero matrix is obvious. In such cases, we use the notation 0 for the zero matrix.

2. Problem statement

2.1. **Game formulation and main assumptions .** Consider the following differential equation controlled by two decision makers (players):

$$\frac{dZ(t)}{dt} = \mathcal{A}Z(t) + \mathcal{B}U(t) + \mathcal{C}v(t), \quad Z(0) = Z_0, \quad t \geq 0, \tag{2.1}$$

where $Z(t) \in \mathbb{R}^n$ is the state vector; $U(t) \in \mathbb{R}^r$, $(r \leq n)$, $v(t) \in \mathbb{R}^s$ are the players' controls; $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ are given constant matrices of corresponding dimensions; $Z_0 \in \mathbb{R}^n$ is a given vector.

The cost functional, to be minimized by U (the minimizer) and maximized by v (the maximizer), is

$$\mathcal{J}\big(U,v\big) \triangleq \int\limits_0^{+\infty} \big[Z^T(t)\mathcal{D}Z(t) + U^T(t)\mathcal{G}_U U(t) - v^T(t)\mathcal{G}_v v(t)\big]\, dt, \tag{2.2}$$

where $\mathcal{D}$ and $\mathcal{G}_v$ are given constant symmetric matrices of corresponding dimensions; the given constant $r \times r$-matrix $\mathcal{G}_U$ has the form

$$\mathcal{G}_U = \mathrm{diag}\big(g_{u_1}, ..., g_{u_q}, \underbrace{0, ..., 0}_{r-q}\big), \qquad 0 \le q < r. \tag{2.3}$$

In what follows, we assume:

(**A1**) The matrix $\mathcal{B}$ has full column rank r;
(**A2**) $\mathcal{D} \ge 0$;
(**A3**) $\mathcal{G}_v > 0$;
(**A4**) $g_{u_k} > 0$, $k = 1, ..., q$.

2.2. **Upper value of the game (2.1)-(2.2).** In what follows, we assume that the minimizer knows perfectly the current state value of the game.

Consider the set $\mathcal{F}$ of all functions $f(w,t) : \mathbb{R}^n \times [0,+\infty) \to \mathbb{R}^r$, which are measurable w.r.t. $t \ge 0$ for any fixed $w \in \mathbb{R}^n$ and satisfy the local Lipschitz condition w.r.t. $w \in \mathbb{R}^n$ uniformly in $t \ge 0$.

Definition 2.1. Let $U(Z,t)$, $(Z,t) \in \mathbb{R}^n \times [0,+\infty)$, be a function, belonging to $\mathcal{F}$. The function $U(Z,t)$ is called an admissible state-feedback control (strategy) of the minimizer in the game (2.1)-(2.2) if the following conditions hold: (1) the initial-value problem (2.1) for $U(t) = U(Z,t)$ and any fixed $v(t) \in L^2[0,+\infty;\mathbb{R}^s]$ has the unique locally absolutely continuous solution $Z(t)$ on the entire interval $[0,+\infty)$; (2) $Z(t) \in L^2[0,+\infty;\mathbb{R}^n]$; (3) $U\big(Z(t),t\big) \in L^2[0,+\infty;\mathbb{R}^r]$. The set of all such $U(Z,t)$ is denoted by $\mathcal{N}_U$.

Definition 2.2. For a given $U(Z,t) \in \mathcal{N}_U$, the value

$$\mathcal{J}_U\big(U(Z,t);Z_0\big) = \sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} \mathcal{J}\big(U(Z,t),v(t)\big) \tag{2.4}$$

is called the guaranteed result of $U(Z,t)$ in the game (2.1)-(2.2).

Definition 2.3. The value

$$\mathcal{J}_{\mathrm{up}}\big(Z_0\big) = \inf_{U(Z,t)\in\mathcal{N}_U} \mathcal{J}_U\big(U(Z,t);Z_0\big) \tag{2.5}$$

is called the upper value of the game (2.1)-(2.2).

2.3. **Transformation of the game (2.1)-(2.2).** Let us partition the matrix $\mathcal{B}$ into blocks as follows:

$$\mathcal{B} = \Big(\mathcal{B}_1, \mathcal{B}_2\Big), \tag{2.6}$$

where the blocks $\mathcal{B}_1$ and $\mathcal{B}_2$ are of dimensions $n \times q$ and $n \times (r-q)$, respectively.

We assumed that:

(**A5**) $\det\big(\mathcal{B}_2^T \mathcal{D}\mathcal{B}_2\big) \ne 0$;

Let $\mathcal{B}_c$ be a complement matrix to the matrix $\mathcal{B}$, i.e., the dimension of $\mathcal{B}_c$ is $n \times (n-r)$, and the block matrix $(\mathcal{B}_c, \mathcal{B})$ is nonsingular. Hence, the block matrix

$$\widetilde{\mathcal{B}}_c = (\mathcal{B}_c, \mathcal{B}_1) \tag{2.7}$$

is a complement matrix to $\mathcal{B}_2$.

Consider the following matrices:

$$\mathcal{H} = (\mathcal{B}_2^T \mathcal{D} B_2)^{-1} \mathcal{B}_2^T \mathcal{D} \widetilde{\mathcal{B}}_c, \tag{2.8}$$

$$\mathcal{L} = \widetilde{\mathcal{B}}_c - \mathcal{B}_2 \mathcal{H}. \tag{2.9}$$

Now, we construct the block matrix $(\mathcal{L}, \mathcal{B}_2)$ and, using this matrix, we transform the state in the differential game (2.1)-(2.2) as follows:

$$Z(t) = (\mathcal{L}, \mathcal{B}_2)\, z(t), \tag{2.10}$$

where $z(t) \in \mathbb{R}^n$ is a new state.

Using the work [13], one can conclude that the transformation (2.10) is nonsingular.

Let us partition the matrix $\mathcal{H}$ into blocks as:

$$\mathcal{H} = \big(\mathcal{H}_1, \mathcal{H}_2\big), \tag{2.11}$$

where the blocks $\mathcal{H}_1$ and $\mathcal{H}_2$ are of the dimensions $(r-q) \times (n-r)$ and $(r-q) \times q$, respectively.

Now, similarly to the results of [14] (Lemma 1), we have the following proposition.

Proposition 2.4. *Let the assumptions (A1), (A2), (A5) be valid. Then, transforming the state variable of the game (2.1)-(2.2) in accordance with (2.10), and redenoting the minimizer's control as $u(t)$, we obtain the differential game with the dynamics*

$$\frac{dz(t)}{dt} = Az(t) + Bu(t) + Cv(t), \quad z(0) = z_0, \quad t \geq 0, \tag{2.12}$$

and the cost functional

$$J(u,v) = \int_0^{+\infty} \big[z^T(t) D z(t) + u^T(t) G_u u(t) - v^T(t) G_v v(t)\big]\, dt, \tag{2.13}$$

where

$$A = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{A} \, (\mathcal{L}, \mathcal{B}_2)\,, \tag{2.14}$$

$$B = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{B} = \begin{pmatrix} O_{(n-r)\times q} & O_{(n-r)\times(r-q)} \\ I_q & O_{q\times(r-q)} \\ \mathcal{H}_2 & I_{r-q} \end{pmatrix}, \tag{2.15}$$

$$C = (\mathcal{L}, \mathcal{B}_2)^{-1} \mathcal{C}, \tag{2.16}$$

$$D = (\mathcal{L}, \mathcal{B}_2)^T \mathcal{D} \, (\mathcal{L}, \mathcal{B}_2) = \begin{pmatrix} D_1 & O_{(n-r+q)\times(r-q)} \\ O_{(r-q)\times(n-r+q)} & D_2 \end{pmatrix},$$

$$D_1 = \mathcal{L}^T \mathcal{D} \mathcal{L}, \qquad D_2 = \mathcal{B}_2^T \mathcal{D} \mathcal{B}_2,$$

(2.17)

$$G_u = \mathcal{G}_U, \qquad G_v = \mathcal{G}_v, \tag{2.18}$$

$$z_0 = (\mathcal{L}, \mathcal{B}_2)^{-1} Z_0, \tag{2.19}$$

the matrices D_1 *and* D_2 *are symmetric, and* $D_1 \geq 0$, $D_2 > 0$.

Remark 2.5. In the differential game (2.12)-(2.13), the cost functional $J(u,v)$ is minimized by the control $u(t)$ and maximized by the control $v(t)$. Since the weight matrix of the minimizer's control cost in the cost functional $J(u,v)$ is singular, the solution (if any) of this game can be obtained neither by the Isaacs's Min-Max principle nor by the Bellman–Isaacs equation method, meaning that the game (2.12)-(2.13) is singular. The set N_u of admissible state-feedback minimizer's controls (strategies) $u(z,t)$ in the game (2.12)-(2.13) is defined similarly to the set $\mathcal{N}_U$ in the game (2.1)-(2.2). For any $u(z,t) \in N_u$, its guaranteed result $J_u\big(u(z,t); z_0\big)$ in the game (2.12)-(2.13) and the upper value $J_{\rm up}(z_0)$ of this game are defined similarly to those in the game (2.1)-(2.2), (see (2.4) and (2.5), respectively).

2.4. **Equivalence of the games (2.1)-(2.2) and (2.12)-(2.13) .**

Lemma 2.6. *Let the assumptions (A1), (A2), (A5) be valid. Then, the existence of a strategy* $U(Z,t) \in \mathcal{N}_U$*, having the finite guaranteed result* $\mathcal{J}_U\big(U(Z,t); Z_0\big)$ *in the game (2.1)-(2.2), yields the existence of a strategy* $u(z,t) \in N_u$*, having the finite guaranteed result* $J_u\big(u(z,t); z_0\big)$ *in the game (2.12)-(2.13), and vice versa.*

Proof. Let $U(Z,t) \in \mathcal{N}_U$ be a strategy such that its guaranteed result $\mathcal{J}_U\big(U(Z,t); Z_0\big)$ in the game (2.1)-(2.2) is finite. Let us define

$$u(z,t) \stackrel{\triangle}{=} U\big((\mathcal{L}, \mathcal{B}_2)z, t\big), \qquad (z,t) \in \mathbb{R}^n \times [0, +\infty)\,. \tag{2.20}$$

Let us show that $u(z,t) \in N_u$. Indeed, since $U(Z,t) \in \mathcal{N}_U$, then for any $v(t) \in L^2[0,+\infty; \mathbb{R}^s]$ the initial-value problem

$$\frac{dZ(t)}{dt} = \mathcal{A}Z(t) + \mathcal{B}U\big(Z(t),t\big) + \mathcal{C}v(t), \quad Z(0) = Z_0, \quad t \geq 0 \tag{2.21}$$

has the unique locally absolutely continuous solution $Z(t)$ on the entire interval $[0,+\infty)$. This solution belongs to $L^2\,[0,+\infty; \mathbb{R}^n]$, and $U\big(Z(t),t\big) \in L^2\,[0,+\infty; \mathbb{R}^r]$.

Due to Proposition 2.4 and the equation (2.20), the invertible transformation (2.10) converts the initial-value problem (2.21) to the initial-value problem

$$\frac{dz(t)}{dt} = Az(t) + Bu\big(z(t),t\big) + Cv(t), \quad z(0) = z_0, \quad t \geq 0. \tag{2.22}$$

For any $v(t) \in L^2[0,+\infty; \mathbb{R}^s]$, this initial-value problem has the unique locally absolutely continuous solution $z(t) = (\mathcal{L}, \mathcal{B}_2)^{-1}Z(t)$ on the entire interval $[0,+\infty)$. Moreover, since $Z(t) \in L^2\,[0,+\infty; \mathbb{R}^n]$ and $U\big(Z(t),t\big) \in L^2\,[0,+\infty; \mathbb{R}^r]$, then $z(t) \in L^2\,[0,+\infty; \mathbb{R}^n]$ and, due to (2.20), $u\big(z(t),t\big) \in L^2\,[0,+\infty; \mathbb{R}^r]$. Thus, $u(z,t) \in N_u$.

Now, using Proposition 2.4 and the equation (2.20), we obtain the following equality for any $v(t) \in L^2[0,+\infty; \mathbb{R}^s]$:

$$J\big(u(z,t), v(t)\big) = \mathcal{J}\big(U(Z,t), v(t)\big). \tag{2.23}$$

The latter, along with Definition 2.2 for $\mathcal{J}_U\big(U(Z,t);Z_0\big)$, the definition for $J_u\big(u(z,t);z_0\big)$ (see Remark 2.5), and the finiteness of $\mathcal{J}_U\big(U(Z,t);Z_0\big)$, directly yields the finiteness of $J_u\big(u(z,t);z_0\big)$ and the equality $J_u\big(u(z,t);z_0\big) = \mathcal{J}_U\big(U(Z,t);Z_0\big)$. The vice versa statement of the lemma is proven similarly. Moreover, for a given strategy $u(z,t) \in N_u$ with the finite guaranteed result $J_u\big(u(z,t);z_0\big)$ in the game (2.12)-(2.13), the strategy $U(Z,t) \stackrel{\triangle}{=} u\big((\mathcal{L},\mathcal{B}_2)^{-1}Z,t\big) \in \mathcal{N}_U$ is such that its guaranteed result $\mathcal{J}_U\big(U(Z,t);Z_0\big)$ in the game (2.1)-(2.2) is finite and satisfies the equality $\mathcal{J}_U\big(U(Z,t);Z_0\big) = J_u\big(u(z,t);z_0\big)$. This completes the proof of the lemma. □

Lemma 2.7. *Let the assumptions (A1), (A2), (A5) be valid. Let there exists a strategy* $\tilde{u}(z,t) \in N_u$*, having the finite guaranteed result* $J_u\big(\tilde{u}(z,t);z_0\big)$ *in the game (2.12)-(2.13). Then, the upper values* $J_{\rm up}(z_0)$ *and* $\mathcal{J}_{\rm up}(Z_0)$ *of the games (2.12)-(2.13) and (2.1)-(2.2), respectively, are finite, nonnegative and equal to each other.*

Proof. First of all, let us note that for $v(t) \equiv 0$, and any $U(Z,t) \in \mathcal{N}_U$ and $u(z,t) \in N_u$, the corresponding values of the cost functionals in the games (2.1)-(2.2) and (2.12)-(2.13) are nonnegative.

Due to the conditions of the lemma, and by virtue of Lemma 2.6, there exists a strategy $\widetilde{U}(Z,t) \stackrel{\triangle}{=} \tilde{u}\big((\mathcal{L},\mathcal{B}_2)^{-1}Z,t\big) \in \mathcal{N}_U$, having the finite guaranteed result $\mathcal{J}_U\big(\widetilde{U}(Z,t);Z_0\big)$ in the game (2.1)-(2.2). All the above mentioned, along with the definition for $J_{\rm up}(z_0)$ (see Remark 2.5), Definition 2.3 for $\mathcal{J}_{\rm up}(Z_0)$ and the proof of Lemma 2.6, means that these upper values are finite and satisfy the inequalities

$$0 \le J_{\rm up}(z_0) \le J_u\big(\tilde{u}(z,t);z_0\big), \tag{2.24}$$

$$0 \le \mathcal{J}_{\rm up}(Z_0) \le J_u\big(\tilde{u}(z,t);z_0\big). \tag{2.25}$$

Due to the definition for $J_{\rm up}(z_0)$, there exists a control sequence $\{u_k(z,t)\}$, $u_k(z,t) \in N_u$, $(k = 1,2,...)$, such that

$$\begin{aligned} &\lim_{k\to+\infty} J_u\big(u_k(z,t);z_0\big) = J_{\rm up}(z_0),\\ &J_u\big(u_k(z,t);z_0\big) \ge J_{\rm up}(z_0), \quad k = 1,2,... . \end{aligned} \tag{2.26}$$

Similarly, due to the definition for $\mathcal{J}_{\rm up}(Z_0)$, there exists a control sequence $\{\widehat{U}_k(Z,t)\}$, $\widehat{U}_k(Z,t) \in \mathcal{N}_U$, $(k = 1,2,...)$, such that

$$\begin{aligned} &\lim_{k\to+\infty} \mathcal{J}_U\big(\widehat{U}_k(Z,t);Z_0\big) = \mathcal{J}_{\rm up}(Z_0),\\ &\mathcal{J}_U\big(\widehat{U}_k(Z,t);Z_0\big) \ge \mathcal{J}_{\rm up}(Z_0), \quad k = 1,2,.. . \end{aligned} \tag{2.27}$$

Let us define the strategies

$$U_k(Z,t) \stackrel{\triangle}{=} u_k\big((\mathcal{L},\mathcal{B}_2)^{-1}Z,t\big), \quad \hat{u}_k(z,t) \stackrel{\triangle}{=} \widehat{U}_k\big((\mathcal{L},\mathcal{B}_2)z,t\big), \quad k = 1,2,... . \tag{2.28}$$

Similarly to the proof of Lemma 2.6, one can show that

$$U_k(Z,t) \in \mathcal{N}_U, \quad \hat{u}_k(z,t) \in N_u, \quad k = 1,2,... , \tag{2.29}$$

and, for any $(k = 1, 2, ...)$,

$$\mathcal{J}_U\big(U_k(Z,t); Z_0\big) = J_u\big(u_k(z,t); z_0\big), \quad J_u\big(\hat{u}_k(z,t); z_0\big) = \mathcal{J}_U\big(\widehat{U}_k(Z,t); Z_0\big). \tag{2.30}$$

The equations (2.26)-(2.27) and (2.30) imply the inequalities

$$\mathcal{J}_{\rm up}(Z_0) \le J_{\rm up}(z_0), \quad J_{\rm up}(z_0) \le \mathcal{J}_{\rm up}(Z_0), \tag{2.31}$$

which yield the equality

$$J_{\rm up}(z_0) = \mathcal{J}_{\rm up}(Z_0). \tag{2.32}$$

Thus, the lemma is proven. □

In the sequel of this paper, we deal with the differential game (2.12)-(2.13). We call this game the Original Differential Game (ODG). As it was mentioned above, the ODG is singular. Moreover, this game does not have, in general, an optimal control of the minimizer among regular functions.

2.5. **Objectives of the paper .** The objectives of this paper are:
(I) to establish general sufficient conditions for the finiteness of the ODG upper value;
(II) to derive upper and lower estimates of this value;
(III) for a reduced set of admissible minimizer's strategies, to derive an expression of the ODG upper value.

3. General conditions for the finiteness of the upper value in the ODG

Let the pair $\{A, B\}$ be stabilizable, i.e., there exists a $r \times n$-matrix M such that the trivial solution of the system

$$\frac{dz(t)}{dt} = \Big(A + BM\Big)z(t), \quad t \ge 0 \tag{3.1}$$

is asymptotically stable.

Consider the following strategy of the minimizer:

$$u = u_M(z) = Mz. \tag{3.2}$$

Along with this strategy, let us consider the Riccati matrix algebraic equation

$$K\left(A + BM\right) + \left(A + BM\right)^T K + KCG_v^{-1}C^TK + M^TG_uM + D = 0. \tag{3.3}$$

Lemma 3.1. *Let the assumptions (A1)-(A5) be satisfied. Let the equation (3.3) have a solution $K = K_M$, $(K_M^T = K_M)$, such that the trivial solution of the system*

$$\frac{dz(t)}{dt} = \left(A + BM + CG_v^{-1}C^TK_M\right)z(t), \quad t \ge 0 \tag{3.4}$$

is asymptotically stable. Then:
(i) $K_M \ge 0$*;*
(ii) *the following equality holds*

$$J_u\big(u_M(z); z_0\big) = \sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} J\big(u_M(z), v(t)\big) = z_0^TK_Mz_0; \tag{3.5}$$

(iii) the supremum value in (3.5) is attained for

$$v(t) = v_M\big(z_M(t;z_0)\big) \stackrel{\triangle}{=} G_v^{-1} C^T K_M z_M(t;z_0), \quad t \geq 0, \tag{3.6}$$

where $z_M(t;z_0)$ is the solution of (3.4) subject to the initial condition $z(0) = z_0$;
(iv) the minimizer's strategy $u_M(z)$ is admissible in the ODG.

The proof of the lemma is presented in Section 9.

Corollary 3.2. *Let the conditions of Lemma 3.1 hold. Then, the upper value of the ODG is finite and satisfies the inequality*

$$0 \leq J_{\mathrm{up}}(z_0) \leq z_0^T K_M z_0. \tag{3.7}$$

Proof. The left-hand side inequality in (3.7) follows from (2.24). Proceed to the proof of the right-hand side inequality in (3.7). Using the definition of the ODG upper value (see Remark 2.5) and the equation (3.5), we obtain the following chain of equalities and inequality

$$J_{\mathrm{up}}(z_0) = \inf_{u(z,t)\in N_u} J_u\big(u(z,t);z_0\big)$$

$$= \inf_{u(z,t)\in N_u} \left(\sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} J\big(u(z,t),v(t)\big) \right)$$

$$\leq \sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} J\big(u_M(z),v(t)\big) = z_0^T K_M z_0.$$

This completes the proof of the corollary. □

4. Regularization of the ODG

4.1. **Partial cheap control game.** We start to analyze the ODG with a regularization of this game. Namely, we replace the original game with a regular differential game, which is close in some sense to the ODG. This new game has the same dynamics (2.12) as the ODG. However, the cost functional in the new game differs from the original one. This new cost functional has the "regular" form, i.e., it contains the quadratic control cost of the minimizer with a "regular" (positive definite) weight matrix:

$$J_\varepsilon(u,v) = \int_0^{+\infty} \big(z^T(t)Dz(t) + u^T(t)\left(G_u + \mathcal{E}\right)u(t) - v^T(t)G_v v(t)\big)\,dt, \tag{4.1}$$

where

$$\mathcal{E} = \mathrm{diag}\Big(\underbrace{0,...,0}_{q}, \underbrace{\varepsilon^2,...,\varepsilon^2}_{r-q}\Big), \tag{4.2}$$

and $\varepsilon > 0$ is a small parameter.

Then

$$G_u + \mathcal{E} = \mathrm{diag}\Big(g_{u_1},...,g_{u_q}, \underbrace{\varepsilon^2,...,\varepsilon^2}_{r-q}\Big) \tag{4.3}$$

is a positive definite matrix.

Remark 4.1. The regularization approach was applied widely enough in the literature to analysis of singular optimal control problems (see e.g. [4, 10, 11, 12, 15, 18] and references therein). However, to the best of our knowledge, such an approach to the analysis and solution of singular zero-sum linear-quadratic differential games was applied only in the papers [14] and [25].

Remark 4.2. Since the parameter $\varepsilon > 0$ is small, the problem (2.12), (4.1) is a partial cheap control differential game, i.e., a differential game with a cost of some control coordinates of at least one of the players much smaller than costs of the other control coordinates and a state cost in the cost functional. In what follows, we call this game the Partial Cheap Control Game (PCCG). Differential games with a total cheap control of at least one of the players were studied in the literature (see [9, 21, 28, 31, 32, 33]). However, to the best of our knowledge, a differential game with a partial cheap control of one of the players has been studied only in the work [14].

4.2. **Minimizer's control optimality conditions in the PCCG.** First of all, it should be noted that the set of admissible state-feedback minimizer's controls in the PCCG coincides with such a set in the ODG, i.e., it is N_u. Moreover, the guaranteed result $J_{\varepsilon,u}\big(u(z,t);z_0\big)$ of an admissible minimizer's state-feedback control $u(z,t)$ of the PCCG and the upper value $J_{\varepsilon,\mathrm{up}}(z_0)$ of the PCCG are defined similarly to those of ODG.

Let us consider the following Riccati matrix algebraic equation:

$$PA + A^T P - P\big(S_u(\varepsilon) - S_v\big)P + D = 0, \tag{4.4}$$

where

$$S_u(\varepsilon) = B(G_u + \mathcal{E})^{-1}B^T, \qquad S_v = CG_v^{-1}C^T. \tag{4.5}$$

Let F be a matrix such that

$$D = F^T F. \tag{4.6}$$

In what follows, we assume that the pair $\big\{A, F\big\}$ is observable. Based on this assumption and using the results of [3, 19, 20], one obtains the following proposition.

Proposition 4.3. *Let, for a given $\varepsilon > 0$, the equation (4.4) have a symmetric minimal positive definite solution $P = P^*(\varepsilon)$. Then, the upper value of the PCCG is finite and has the form*

$$J_{\varepsilon,\mathrm{up}}(z_0) = z_0^T P^*(\varepsilon) z_0. \tag{4.7}$$

This value is achieved for the minimizer's strategy (optimal one)

$$u_\varepsilon^*(z) = -(G_u + \mathcal{E})^{-1}B^T P^*(\varepsilon) z. \tag{4.8}$$

The supremum

$$\sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} J_\varepsilon\big(u_\varepsilon^*(z), v(t)\big) = J_{\varepsilon,\mathrm{up}}(z_0) \tag{4.9}$$

is achieved for the maximizer's control

$$v(t) = v_\varepsilon^*(t) \overset{\triangle}{=} G_v^{-1}C^T P^*(\varepsilon) z^*(t,\varepsilon), \tag{4.10}$$

where $z^*(t,\varepsilon)$ is the solution of the system

$$\frac{dz(t)}{dt} = \Big(A - S_u(\varepsilon)P^*(\varepsilon) + S_v P^*(\varepsilon)\Big)z(t), \quad t \geq 0 \tag{4.11}$$

subject to the initial condition $z(0) = z_0$.

Moreover, the trivial solution of (4.11) and of the system

$$\frac{dz(t)}{dt} = \Big(A - S_u(\varepsilon)P^*(\varepsilon)\Big)z(t), \quad t \geq 0 \tag{4.12}$$

is asymptotically stable.

5. Asymptotic analysis of the Riccati equation (4.4)

5.1. **Transformation of the equation (4.4).** First of all, let as note that by substitution of the block representations of the matrices B and $G_u + \mathcal{E}$ (see the equations (2.15) and (4.3)) into the expression for $S_u(\varepsilon)$ (see (4.5)), we obtain after a routine algebra the following block representation of this matrix:

$$S_u(\varepsilon) = \begin{pmatrix} S_{u_1} & S_{u_2} \\ S_{u_2}^T & (1/\varepsilon^2)S_{u_3}(\varepsilon) \end{pmatrix}, \tag{5.1}$$

where

$$S_{u_1} = \begin{pmatrix} O_{(n-r)\times(n-r)} & O_{(n-r)\times q} \\ O_{q\times(n-r)} & \widetilde{G}_u^{-1} \end{pmatrix}, \qquad S_{u_2} = \begin{pmatrix} O_{(n-r)\times(r-q)} \\ \widetilde{G}_u^{-1}\mathcal{H}_2^T \end{pmatrix}, \tag{5.2}$$

$$S_{u_3}(\varepsilon) = \varepsilon^2\mathcal{H}_2\widetilde{G}_u^{-1}\mathcal{H}_2^T + I_{r-q}, \qquad \widetilde{G}_u = \mathrm{diag}\big(g_{u_1}, ..., g_{u_q}\big). \tag{5.3}$$

Due to (5.1) and (5.3), the left-hand side of the equation (4.4) has a singularity at $\varepsilon = 0$. To remove this singularity, we seek the symmetric solution $P(\varepsilon)$ of the equation (4.4) in the block form

$$P(\varepsilon) = \begin{pmatrix} P_1(\varepsilon) & \varepsilon P_2(\varepsilon) \\ \varepsilon P_2^T(\varepsilon) & \varepsilon P_3(\varepsilon) \end{pmatrix}, \tag{5.4}$$

where the blocks $P_1(\varepsilon)$, $P_2(\varepsilon)$ and $P_3(\varepsilon)$ have the dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$ and $(r-q)\times(r-q)$, respectively, and

$$P_1^T(\varepsilon) = P_1(\varepsilon), \quad P_3^T(\varepsilon) = P_3(\varepsilon). \tag{5.5}$$

We also partition the matrices A and S_v into blocks as follows:

$$A = \begin{pmatrix} A_1 & A_2 \\ A_3 & A_4 \end{pmatrix}, \qquad S_v = \begin{pmatrix} S_{v_1} & S_{v_2} \\ S_{v_2}^T & S_{v_3} \end{pmatrix}, \tag{5.6}$$

where the blocks A_1, A_2, A_3 and A_4 have the dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$, $(r-q)\times(n-r+q)$ and $(r-q)\times(r-q)$, respectively; the blocks S_{v_1}, S_{v_2} and S_{v_3} have the form

$$S_{v_1} = C_1G_v^{-1}C_1^T, \quad S_{v_2} = C_1G_v^{-1}C_2^T, \quad S_{v_3} = C_2G_v^{-1}C_2^T, \tag{5.7}$$

C_1 and C_2 are the upper and lower blocks of the matrix C of the dimensions $(n-r+q)\times s$ and $(r-q)\times s$, respectively, i.e.,

$$C=\begin{pmatrix} C_1 \\ C_2 \end{pmatrix}. \tag{5.8}$$

Substitution of (2.17), (5.1), (5.4) and (5.6) into the equation (4.4) yields after a routine rearrangement the following equivalent set of Riccati-type matrix algebraic equations with respect to $P_1(\varepsilon)$, $P_2(\varepsilon)$ and $P_3(\varepsilon)$:

$$\begin{aligned} &P_1(\varepsilon)A_1+\varepsilon P_2(\varepsilon)A_3+A_1^TP_1(\varepsilon)+\varepsilon A_3^TP_2^T(\varepsilon)\\ &\quad-P_1(\varepsilon)\left(S_{u_1}-S_{v_1}\right)P_1(\varepsilon)-\varepsilon P_2(\varepsilon)\left(S_{u_2}^T-S_{v_2}^T\right)P_1(\varepsilon)\\ &\quad-\varepsilon P_1(\varepsilon)\left(S_{u_2}-S_{v_2}\right)P_2^T(\varepsilon)-P_2(\varepsilon)\left(S_{u_3}(\varepsilon)-\varepsilon^2S_{v_3}\right)P_2^T(\varepsilon)+D_1=0, \end{aligned} \tag{5.9}$$

$$\begin{aligned} &P_1(\varepsilon)A_2+\varepsilon P_2(\varepsilon)A_4+\varepsilon A_1^TP_2(\varepsilon)+\varepsilon A_3^TP_3(\varepsilon)\\ &\quad-\varepsilon P_1(\varepsilon)\left(S_{u_1}-S_{v_1}\right)P_2(\varepsilon)-\varepsilon^2P_2(\varepsilon)\left(S_{u_2}^T-S_{v_2}^T\right)P_2(\varepsilon)\\ &\quad-\varepsilon P_1(\varepsilon)\left(S_{u_2}-S_{v_2}\right)P_3(\varepsilon)-P_2(\varepsilon)\left(S_{u_3}(\varepsilon)-\varepsilon^2S_{v_3}\right)P_3(\varepsilon)=0, \end{aligned} \tag{5.10}$$

$$\begin{aligned} &\varepsilon P_2^T(\varepsilon)A_2+\varepsilon P_3(\varepsilon)A_4+\varepsilon A_2^TP_2(\varepsilon)+\varepsilon A_4^TP_3(\varepsilon)\\ &\quad-\varepsilon^2P_2^T(\varepsilon)\left(S_{u_1}-S_{v_1}\right)P_2(\varepsilon)-\varepsilon^2P_3(\varepsilon)\left(S_{u_2}^T-S_{v_2}^T\right)P_2(\varepsilon)\\ &\quad-\varepsilon^2P_2^T(\varepsilon)\left(S_{u_2}-S_{v_2}\right)P_3(\varepsilon)-P_3(\varepsilon)\left(S_{u_3}(\varepsilon)-\varepsilon^2S_{v_3}\right)P_3(\varepsilon)+D_2=0. \end{aligned} \tag{5.11}$$

5.2. **Zero-order asymptotic solution of (5.9)-(5.11).** We seek the zero-order asymptotic solution $\{P_{10},P_{20},P_{30}\}$ of the system (5.9)-(5.11). Equations for this asymptotic solution terms are obtained by setting formally $\varepsilon=0$ in (5.9)-(5.11). Thus, we have the system

$$P_{10}A_1+A_1^TP_{10}-P_{10}\left(S_{u_1}-S_{v_1}\right)P_{10}-P_{20}P_{20}^T+D_1=0, \tag{5.12}$$

$$P_{10}A_2-P_{20}P_{30}=0, \tag{5.13}$$

$$(P_{30})^2-D_2=0. \tag{5.14}$$

The equation (5.14) has the solution

$$P_{30}=P_{30}^*\overset{\triangle}{=}\left(D_2\right)^{1/2}, \tag{5.15}$$

where the superscript "1/2" denotes the unique symmetric positive definite square root of the corresponding symmetric positive definite matrix.

Due to (5.15), the equation (5.13) yields the unique expression for P_{20}

$$P_{20}=P_{10}A_2\left(D_2\right)^{-1/2}, \tag{5.16}$$

where the superscript "$-1/2$" denotes the inverse matrix for the unique symmetric positive definite square root of the corresponding symmetric positive definite matrix.

Substituting (5.16) into (5.12), one obtains after some rearrangement the equation with respect to P_{10}

$$P_{10}A_1+A_1^TP_{10}-P_{10}S_1P_{10}+D_1=0, \tag{5.17}$$

where

$$S_1 = A_2 D_2^{-1} A_2^T + S_{u_1} - S_{v_1}. \tag{5.18}$$

Due to the results of [14] (Lemma 5), the matrix S_1 can be represented in the form

$$S_1 = \bar{B}\Theta^{-1}\bar{B}^T - S_{v_1}, \tag{5.19}$$

where

$$\bar{B} = \begin{pmatrix} \widetilde{B}\ , & A_2 \end{pmatrix}, \qquad \widetilde{B} = \begin{pmatrix} O_{(n-r)\times q} \\ I_q \end{pmatrix}, \tag{5.20}$$

$$\Theta = \begin{pmatrix} \widetilde{G}_u & O_{q\times(r-q)} \\ O_{(r-q)\times q} & D_2 \end{pmatrix}, \tag{5.21}$$

$\widetilde{G}_u$ is defined in (5.3).

Let F_1 be a matrix such that

$$D_1 = F_1^T F_1. \tag{5.22}$$

In what follows, we assume:

(A6) The pair (A_1, F_1) is observable;

(A7) The Riccati matrix algebraic equation (5.17) has a symmetric minimal positive definite solution $P_{10} = P_{10}^*$.

Using the above mentioned solution of (5.17) and the equation (5.16), we obtain the second component of the solution to the system (5.12)-(5.14) as

$$P_{20} = P_{20}^* \stackrel{\triangle}{=} P_{10}^* A_2 \big(D_2\big)^{-1/2}. \tag{5.23}$$

5.2.1. *Game-theoretic interpretation of the equation (5.17).* Consider the zero-sum linear-quadratic differential game with the dynamics

$$\frac{d\bar{x}(t)}{dt} = A_1\bar{x}(t) + \bar{B}\bar{u}(t) + C_1\bar{v}(t), \quad t \ge 0, \quad \bar{x}(0) = x_0, \tag{5.24}$$

where $\bar{x}(t) \in \mathbb{R}^{n-r+q}$ is the state vector; $\bar{u}(t) \in \mathbb{R}^r$, $\bar{v}(t) \in \mathbb{R}^s$ are the players' controls; $x_0 \in \mathbb{R}^{n-r+q}$ is the upper block of the vector z_0 given by (2.19).

The cost functional, to be minimized by $\bar{u}(t)$ and maximized by $\bar{v}(t)$, has the form

$$\bar{J}(\bar{u}, \bar{v}) = \int_0^{+\infty} \Big[\big(\bar{x}(t)\big)^T D_1 \bar{x}(t) + \big(\bar{u}(t)\big)^T \Theta \bar{u}(t) - \big(\bar{v}(t)\big)^T G_v \bar{v}(t)\Big]\, dt. \tag{5.25}$$

We call the game (5.24)-(5.25) the Reduced Differential Game (RDG).

Lemma 5.1. *Let the assumptions (A1)-(A7) be satisfied. Then, the upper value of the RDG is finite and has the form*

$$\bar{J}_{\rm up}(x_0) = x_0^T P_{10}^* x_0. \tag{5.26}$$

This value is achieved for the minimizer's strategy (optimal one)

$$\bar{u}^*(\bar{x}) = -\Theta^{-1}\bar{B}^T P_{10}^* \bar{x}, \qquad \bar{x} \in \mathbb{R}^{n-r+q}. \tag{5.27}$$

The supremum

$$\sup_{\bar{v}(t)\in L^2[0,+\infty;\mathbb{R}^s]} \bar{J}\big(\bar{u}^*(\bar{x}), \bar{v}(t)\big) = \bar{J}_{\mathrm{up}}(x_0) \tag{5.28}$$

is achieved for the maximizer's control

$$\bar{v}^*(t) = G_v^{-1} C_1^T P_{10}^* \bar{x}^*(t), \tag{5.29}$$

where $\bar{x}^*(t)$ is the solution of the system

$$\frac{d\bar{x}(t)}{dt} = \Big(A_1 - S_1 P_{10}^*\Big)\bar{x}(t), \quad t \geq 0 \tag{5.30}$$

subject to the initial condition $\bar{x}(0) = x_0$. Moreover, the trivial solution of (5.30) and of the system

$$\frac{d\bar{x}(t)}{dt} = \big(A_1 - \bar{B}\Theta^{-1}\bar{B}^T P_{10}^*\big)\bar{x}(t), \quad t \geq 0 \tag{5.31}$$

is asymptotically stable.

Proof. The statements of the lemma directly follow from Proposition 2.4, the first equation in (5.7), the equation (5.19), and the results of [3, 19, 20]. □

Remark 5.2. Using (5.20) and (5.21), the minimizer's optimal strategy (5.27) in the RDG can be represented in the block form as:

$$\bar{u}^*(\bar{x}) = \begin{pmatrix} \bar{u}_1^*(\bar{x}) \\ \bar{u}_2^*(\bar{x}) \end{pmatrix}, \tag{5.32}$$

where

$$\bar{u}_1^*(\bar{x}) = -\widetilde{G}_u^{-1}\widetilde{B}^T P_{10}^* \bar{x}, \quad \bar{u}_2^*(\bar{x}) = -\big(D_2\big)^{-1} A_2^T P_{10}^* \bar{x}. \tag{5.33}$$

5.2.2. *Justification of the asymptotic solution to the set (5.9)-(5.11) and the equation (4.4).*

Lemma 5.3. *Let the assumptions (A1)-(A7) be satisfied. Then, there exists a positive number ε_0, such that for all $\varepsilon \in (0, \varepsilon_0]$ the equation (4.4) has the symmetric minimal positive definite solution $P^*(\varepsilon)$. This solution has the block form*

$$P^*(\varepsilon) = \begin{pmatrix} P_1^*(\varepsilon) & \varepsilon P_2^*(\varepsilon) \\ \varepsilon\big(P_2^*(\varepsilon)\big)^T & \varepsilon P_3^*(\varepsilon) \end{pmatrix}, \tag{5.34}$$

where the blocks $P_1^*(\varepsilon)$, $P_2^*(\varepsilon)$, $P_3^*(\varepsilon)$ are of dimensions $(n-r+q)\times(n-r+q)$, $(n-r+q)\times(r-q)$, $(r-q)\times(r-q)$, respectively. These blocks satisfy the inequalities

$$\|P_i^*(\varepsilon) - P_{i0}^*\| \leq a\varepsilon, \quad i = 1,2,3, \quad \varepsilon \in (0, \varepsilon_0], \tag{5.35}$$

where $a > 0$ is some constant independent of ε.

Proof. The lemma is proven similarly to item (2) of Theorem 1 in [20]. □

6. Estimates of the upper value of the ODG

In this section, we obtain more accurate estimates of the ODG upper value and subject to simpler assumptions than the estimates and the assumptions of Corollary 3.2.

6.1. **Upper estimate.**

Theorem 6.1. *Let the assumptions (A1)-(A7) be valid. Then, the upper value of the ODG* $J_{\rm up}(z_0)$ *satisfies the inequality*

$$J_{\rm up}(z_0) \le \bar{J}_{\rm up}(x_0), \tag{6.1}$$

where $\bar{J}_{\rm up}(x_0)$ *is the upper value of the RDG given by (5.26).*

Proof. First of all let us note that, from Proposition 4.3 (see (4.7)) and Lemma 5.3, we directly obtain the following inequality for all sufficiently small $\varepsilon > 0$:

$$\bar{J}_{\rm up}(x_0) - a\varepsilon \le J_{\varepsilon,{\rm up}}(z_0) \le \bar{J}_{\rm up}(x_0) + a\varepsilon, \tag{6.2}$$

where $a > 0$ is some constant independent of ε.

From the expressions for the cost functionals of the ODG and PCCG (see (2.13) and (4.1)), as well as from the definition of the guaranteed result of an admissible strategy and the definition of the upper value in these games, we have for all sufficiently small $\varepsilon > 0$

$$J_{\rm up}(z_0) \le J_{\varepsilon,{\rm up}}(z_0). \tag{6.3}$$

The inequalities (6.2) and (6.3) imply the inequality

$$J_{\rm up}(z_0) \le \bar{J}_{\rm up}(x_0) + a\varepsilon, \tag{6.4}$$

valid for all sufficiently small $\varepsilon > 0$. The latter inequality directly yields the statement of the theorem. □

6.2. **Lower estimate.** Consider the optimal control problem, consisting of the equation of dynamics

$$\frac{dz(t)}{dt} = Az(t) + Bu(t), \quad z(0) = z_0, \quad t \ge 0, \tag{6.5}$$

and the performance index

$$\widetilde{J}(u) \triangleq \int_0^{+\infty} \big(z^T(t)Dz(t) + u^T(t)G_u u(t)\big)dt \to \inf_u . \tag{6.6}$$

Since the matrix G_u is singular, the problem (6.5)-(6.6) is singular, i.e., the Pontryagin's Maximum Principle [22] and the Hamilton-Jacobi-Bellman equation [5] are not applicable to solution of this problem.

Definition 6.2. Let $u(z,t)$, $(z,t) \in \mathbb{R}^n \times [0,+\infty)$, be a function, belonging to $\mathcal{F}$. The function $u(z,t)$ is called an admissible state-feedback control in the problem (6.5)-(6.6) if the following conditions hold: (1) the initial-value problem (6.5) for $u(t) = u(z,t)$ has the unique locally absolutely continuous solution $z(t)$ on the entire interval $[0,+\infty)$; (2) $z(t) \in L^2\,[0,+\infty;\mathbb{R}^n]$; (3) $u\,(z(t),t) \in L^2\,[0,+\infty;\mathbb{R}^r]$. The set of all such $u(z,t)$ is denoted by $\widetilde{N}_u$.

Comparison of the Definitions 2.1 and 6.2 directly yields the inclusion

$$N_u \subseteq \widetilde{N}_u. \tag{6.7}$$

Consider the Riccati matrix algebraic equation with respect to $\widetilde{P}$

$$\widetilde{P}A_1 + A_1^T\widetilde{P} - \widetilde{P}\bar{B}\Theta^{-1}\bar{B}^T\widetilde{P} + D_1 = 0, \tag{6.8}$$

where the $(n-r+q)\times r$-matrix $\bar{B}$ and the $r\times r$-matrix Θ are given by (5.20) and (5.21), respectively.

We assume

(A8) The pair $\big(A_1,\bar{B}\big)$ is stabilizable.

Due to the results of [2], subject to the assumptions (A6) and (A8), the equation (6.8) has the unique symmetric solution $\widetilde{P}=\widetilde{P}^*\geq 0$. Moreover, the matrix $\big(A_1-\bar{B}\Theta^{-1}\bar{B}^T\widetilde{P}^*\big)$ is a Hurwitz one.

The following lemma is a particular case of the results of [15] (Theorem 20) where a singular infinite horizon linear-quadratic control problem for systems with known disturbances is analyzed. In the problem (6.5)-(6.6), such a disturbance equals zero.

Lemma 6.3. *Let the assumptions (A1), (A2), (A4)-(A6), (A8) be valid. Then, the infimum* $\widetilde{J}^*\stackrel{\triangle}{=}\inf_{u(z,t)\in\widetilde{N}_u}\widetilde{J}\big(u(z,t)\big)$ *of the cost functional in the optimal control problem (6.5)-(6.6) is finite and has the form*

$$\widetilde{J}^*=\widetilde{J}^*(x_0)\stackrel{\triangle}{=}x_0^T\widetilde{P}^*x_0, \tag{6.9}$$

where $x_0\in\mathbb{R}^{n-r+q}$ *is (like in (5.24)) the upper block of the vector* z_0 *given by (2.19) and used in (6.5).*

Based on this lemma, the lower estimate for the ODG upper value $J_{\rm up}(z_0)$ is obtained in the following theorem.

Theorem 6.4. *Let the assumptions (A1)-(A8) be valid. Then, the upper value* $J_{\rm up}(z_0)$ *of the ODG satisfies the inequality*

$$J_{\rm up}(z_0)\geq\widetilde{J}^*(x_0). \tag{6.10}$$

Proof. The statement of the theorem directly follows from the definition of the ODG upper value (see Remark 2.5) and the inclusion (6.7). □

Remark 6.5. If the upper block C_1 of the dimension $(n-r+q)\times s$ of the matrix C in the equation (2.12) is zero matrix, then the inequalities (6.1) and (6.10) become the equalities $J_{\rm up}(z_0)=\bar{J}_{\rm up}(x_0)=\widetilde{J}^*(x_0)$. Also, for any block C_1 and $x_0=0$, $J_{\rm up}(z_0)=\bar{J}_{\rm up}(x_0)=\widetilde{J}^*(x_0)=0$.

7. Upper value of the ODG in a reduced set of minimizer's admissible strategies

In this section, we consider the ODG for the set of all minimizer's strategies $u_M(z)$ of the form (3.2), satisfying the conditions of Lemma 3.1, i.e., such that: (1) the matrix $\big(A+BM\big)$ is a Hurwitz one; (2) there exists a symmetric solution $K=K_M$ of the Riccati matrix algebraic equation (3.3), for which the matrix $\big(A+BM+CG_v^{-1}C^TK_M\big)$ is a Hurwitz one. The set of all such minimizer's strategies is called the reduced set of the admissible strategies for the ODG, and it is denoted as $N_{u,\rm r}$. Due to Lemma 3.1, $N_{u,\rm r}\subset N_u$. The guaranteed result of any strategy $u_M(z)\in N_{u,\rm r}$ in the ODG and the upper value $J_{\rm up,r}(z_0)$ of the ODG in $N_{u,\rm r}$ are defined similarly to Definitions 2.2 and 2.3.

Remark 7.1. It is important to note that for any $\varepsilon \in (0, \varepsilon_0]$ (ε_0 is defined in Lemma 5.3), the minimizer's optimal strategy in PCCG $u_\varepsilon^*(z)$ belongs to the set $N_{u,\mathrm{r}}$. Therefore, the upper value $J_{\varepsilon,\mathrm{up,r}}(z_0)$ of the PCCG in $N_{u,\mathrm{r}}$ coincides with the upper value of this game in the set N_u, i.e., $J_{\varepsilon,\mathrm{up,r}}(z_0) = J_{\varepsilon,\mathrm{up}}(z_0)$.

Theorem 7.2. *Let the assumptions (A1)-(A7) be valid. Then,*

$$J_{\mathrm{up,r}}(z_0) = \bar{J}_{\mathrm{up}}(x_0), \tag{7.1}$$

where $\bar{J}_{\mathrm{up}}(x_0)$ is the upper value of the RDG given by (5.26).

Proof. Using Remark 7.1, we obtain similarly to Theorem 6.1 the following inequality:

$$J_{\mathrm{up,r}}(z_0) \le \bar{J}_{\mathrm{up}}(x_0), \tag{7.2}$$

Now, let us assume that the statement of the theorem is wrong, i.e., $J_{\mathrm{up,r}}(z_0) \neq \bar{J}_{\mathrm{up}}(x_0)$. This inequality, along with (7.2), yields

$$J_{\mathrm{up,r}}(z_0) < \bar{J}_{\mathrm{up}}(x_0). \tag{7.3}$$

Due to (7.3), there exists an admissible strategy of the ODG $\tilde{u}(z) = \widetilde{M}z \in N_{u,\mathrm{r}}$ such that

$$J_{\mathrm{up,r}}(z_0) < J_u\big(\tilde{u}(z); z_0\big) < \bar{J}_{\mathrm{up}}(x_0). \tag{7.4}$$

Using Proposition 4.3, the definition of the upper value in the PCCG and Remark 7.1, we obtain for all sufficiently small $\varepsilon > 0$

$$\begin{aligned} J_{\varepsilon,\mathrm{up,r}}(z_0) &= \sup_{v(t)\in L^2[0,+\infty;\mathbb{R}^s]} J_\varepsilon\big(u_\varepsilon^*(z), v(t)\big) \\ &\le \sup_{v(t)\in L^2[0,+\infty;E^s]} J_\varepsilon\big(\tilde{u}(z), v(t)\big) = J_{\varepsilon,u}\big(\tilde{u}(z); z_0\big). \end{aligned} \tag{7.5}$$

Consider the differential system

$$\frac{dz(t)}{dt} = \Big(A + B\widetilde{M}\Big)z(t), \quad t \ge 0, \tag{7.6}$$

and the Riccati matrix algebraic equation with respect to $\widetilde{K}$

$$\widetilde{K}\left(A + B\widetilde{M}\right) + \left(A + B\widetilde{M}\right)^T \widetilde{K} + \widetilde{K}CG_v^{-1}C^T\widetilde{K} + \widetilde{M}^T G_u \widetilde{M} + D = 0. \tag{7.7}$$

Since $\tilde{u}(z) \in N_{u,\mathrm{r}}$, the trivial solution of (7.6) is asymptotically stable. Moreover, the equation (7.7) has a symmetric solution $\widetilde{K} = \widetilde{K}_{\widetilde{M}}$ such that the trivial solution of the system

$$\frac{dz(t)}{dt} = \left(A + B\widetilde{M} + CG_v^{-1}C^T\widetilde{K}_{\widetilde{M}}\right)z(t), \quad t \ge 0 \tag{7.8}$$

is asymptotically stable.

By virtue of Lemma 3.1, $\widetilde{K}_{\widetilde{M}} \ge 0$, and

$$J_u\big(\tilde{u}(z); z_0\big) = z_0^T \widetilde{K}_{\widetilde{M}} z_0. \tag{7.9}$$

Now, consider the following Riccati matrix algebraic equation with respect to $\widehat{K}$:

$$\widehat{K}\left(A + B\widetilde{M}\right) + \left(A + B\widetilde{M}\right)^T \widehat{K} + \widehat{K}CG_v^{-1}C^T\widehat{K}$$

$$+\widetilde{M}^T\big(G_u+\mathcal{E}\big)\widetilde{M}+D=0. \tag{7.10}$$

Due to (4.2), the equation (7.10) is perturbed by the small parameter ε, and this equation becomes (7.7) for $\varepsilon = 0$.

Similarly to Lemma 3.1, one can prove the following assertion. If for a given $\varepsilon > 0$ the equation (7.10) has a symmetric solution $\widehat{K} = \widehat{K}(\varepsilon)$ such that the trivial solution of the system

$$\frac{dz(t)}{dt} = \Big(A + B\widetilde{M} + CG_v^{-1}C^T\widehat{K}(\varepsilon)\Big)\, z(t), \quad t \geq 0 \tag{7.11}$$

is asymptotically stable, then

$$J_{\varepsilon,u}\big(\tilde{u}(z); z_0\big) = z_0^T\widehat{K}(\varepsilon)z_0. \tag{7.12}$$

Similarly to the proof of Lemma 3.1 (see (9.11)-(9.15)), we obtain the existence of the following solution to (7.10) for all sufficiently small $\varepsilon > 0$:

$$\widehat{K}(\varepsilon) = \widetilde{K}_{\widetilde{M}} + \widehat{R}(\varepsilon), \tag{7.13}$$

where $\widehat{R}(\varepsilon)$ is some symmetric matrix, satisfying the inequality

$$\big\|\widehat{R}(\varepsilon)\big\| \leq a\varepsilon^2, \tag{7.14}$$

$a > 0$ is some constant independent of ε.

Now, using the equation (7.13), the inequality (7.14), the asymptotic stability of the trivial solution to the system (7.8) and the results of [26] on the continuity of eigenvalues of quadratic matrices with respect to an independent variable, we immediately obtain the asymptotic stability of the trivial solution to the system (7.11) for all sufficiently small $\varepsilon > 0$. Hence, using the equations (7.9), (7.12), (7.13) and the inequality (7.14), we have for these ε,

$$\Big|J_{\varepsilon,u}\big(\tilde{u}(z); z_0\big) - J_u\big(\tilde{u}(z); z_0\big)\Big| \leq a\varepsilon^2, \tag{7.15}$$

where $a > 0$ is some constant independent of ε.

Due to Remark 7.1, we obtain similarly to (6.2) the inequality

$$\big|J_{\varepsilon,\mathrm{up,r}}(z_0) - \bar{J}_{\mathrm{up}}(x_0)\big| \leq a\varepsilon, \tag{7.16}$$

where $a > 0$ is some constant independent of ε.

The inequalities (7.5), (7.15) and (7.16) directly yield the inequality

$$\bar{J}_{\mathrm{up}}(x_0) \leq J_u\big(\tilde{u}(z); z_0\big), \tag{7.17}$$

which contradicts the right-hand inequality in (7.4). This contradiction means that the assumption $J_{\mathrm{up,r}}(z_0) \neq \bar{J}_{\mathrm{up}}(x_0)$ is wrong. Therefore, the equality (7.1) is correct. □

8. Example

Consider a particular case of the initially formulated differential game (2.1)-(2.2). Namely, $n = r = s = 2$, $q = 1$, and

$$\mathcal{A} = \begin{pmatrix} -3 & 2 \\ 2 & 4 \end{pmatrix}, \quad \mathcal{B} = \begin{pmatrix} 2 & -3 \\ 4 & 2 \end{pmatrix}, \quad \mathcal{C} = \begin{pmatrix} 4 & 7 \\ -8 & 6 \end{pmatrix},$$
$$\mathcal{D} = \begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}, \quad \mathcal{G}_U = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{G}_v = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \quad Z_0 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}. \tag{8.1}$$

For these data, $\mathcal{B}_2 = \begin{pmatrix} -3 \\ 2 \end{pmatrix}$ and $\widetilde{B}_c = \mathcal{B}_1 = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$. Therefore, due to (2.8)-(2.9), $\mathcal{H} = 2$ and $\mathcal{L} = \begin{pmatrix} 8 \\ 0 \end{pmatrix}$. Hence, the state transformation (2.10) in the game (2.1)-(2.2),(8.1) becomes as $Z(t) = \begin{pmatrix} 8 & -3 \\ 0 & 2 \end{pmatrix} z(t)$. This transformation converts the game (2.1)-(2.2),(8.1) to the equivalent game (2.12)-(2.13) (the ODG), where, due to (2.14)-(2.19),

$$A = \begin{pmatrix} 0 & 2 \\ 8 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} -1 & 2 \\ -4 & 3 \end{pmatrix},$$
$$D = \begin{pmatrix} 128 & 0 \\ 0 & 2 \end{pmatrix}, \quad G_u = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad G_v = \begin{pmatrix} 5 & 0 \\ 0 & 5 \end{pmatrix}, \quad z_0 = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}. \tag{8.2}$$

Due to the data (8.2), the matrix equation (5.17) becomes the scalar one

$$-2P_{10}^2 + 128 = 0, \tag{8.3}$$

yielding the unique positive solution $P_{10}^* = 8$. Thus, by Lemma 5.1, the upper value of the RDG in this example is

$$\bar{J}(x_0) = 2. \tag{8.4}$$

Similarly, the equation (6.8) becomes

$$-3\widetilde{P}^2 + 128 = 0, \tag{8.5}$$

yielding the unique positive solution $\widetilde{P}^* = \frac{8\sqrt{6}}{3}$. For this solution, the scalar $\big(A_1 - \bar{B}\Theta^{-1}\bar{B}^T\widetilde{P}^*\big) = -8\sqrt{6} < 0$. Hence, due to Lemma 6.3, the infimum of the cost functional of the optimal control problem (6.5)-(6.6) in this example is

$$\widetilde{J}^*(x_0) = \frac{2\sqrt{6}}{3} \approx 1.633. \tag{8.6}$$

Thus, by virtue of Theorems 6.1 and 6.4, the upper value of the ODG in the set N_u of the admissible minimizer's state-feedback controls satisfies the inequality

$$\frac{2\sqrt{6}}{3} \le J_{\mathrm{up}}(z_0) \le 2. \tag{8.7}$$

Furthermore, by virtue of Theorem 7.2, the upper value of the ODG in the set $N_{u,\mathrm{r}}$ of the admissible minimizer's state-feedback controls is

$$J_{\mathrm{up,r}}(z_0) = 2. \tag{8.8}$$

9. Proof of Lemma 3.1

We start with the item (i). Consider the Lyapunov-like function

$$V(z) = z^T K_M z. \tag{9.1}$$

Let, for any given $v(t) \in L^2[0, +\infty; \mathbb{R}^s]$ and any given $z_0 \in \mathbb{R}^n$, $z_{Mv}(t; z_0)$ be the solution of the problem (2.12) with $u(t) = u_M(z)$. Then, differentiating $V\big(z_{Mv}(t; z_0)\big)$ with respect to t, we obtain after a simple algebra:

$$\begin{aligned}\frac{dV\big(z_{Mv}(t;z_0)\big)}{dt} &= 2\frac{dz_{Mv}^T(t;z_0)}{dt}K_M z_{Mv}(t;z_0)\\ &= 2\big[(A+BM)z_{Mv}(t;z_0) + Cv(t)\big]^T K_M z_{Mv}(t;z_0)\\ &= z_{Mv}^T(t;z_0)\big[K_M(A+BM) + (A+BM)^T K_M\big] z_{Mv}(t;z_0)\\ &\qquad + 2v^T(t)C^T K_M z_{Mv}(t;z_0), \qquad t \ge 0.\end{aligned} \tag{9.2}$$

Due to (3.3), we can rewrite the equation (9.2) as

$$\begin{aligned}\frac{dV\big(z_{Mv}(t;z_0)\big)}{dt} &= -z_{Mv}^T(t;z_0)K_M C G_v^{-1} C^T K_M z_{Mv}(t;z_0)\\ &\quad - z_{Mv}^T(t;z_0) D z_{Mv}(t;z_0) - z_{Mv}^T(t;z_0) M^T G_u M z_{Mv}(t;z_0)\\ &\quad + 2v^T(t)C^T K_M z_{Mv}(t;z_0), \qquad t \ge 0.\end{aligned} \tag{9.3}$$

Using (3.2) and (3.6), the equation (9.3) can be represented in the form

$$\begin{aligned}\frac{dV\big(z_{Mv}(t;z_0)\big)}{dt} &= -z_{Mv}^T(t;z_0) D z_{Mv}(t;z_0)\\ &\quad - u_M^T\big(z_{Mv}(t;z_0)\big) G_u u_M\big(z_{Mv}(t;z_0)\big)\\ &\quad - \big[v(t) - v_M\big(z_{Mv}(t;z_0)\big)\big]^T G_v \big[v(t) - v_M\big(z_{Mv}(t;z_0)\big)\big]\\ &\quad + v^T(t) G_v v(t), \qquad t \ge 0.\end{aligned} \tag{9.4}$$

Let $z_{M0}(t;z_0) \triangleq z_{Mv}(t;z_0)|_{v(t)\equiv 0}$. Since the trivial solution of the equation (3.1) is asymptotically stable, $\lim_{t\to+\infty} z_{M0}(t;z_0) = 0$.

Since $D \ge 0$, $G_u \ge 0$ and $G_v > 0$, the equation (9.4) yields the inequality

$$\frac{dV\big(z_{M0}(t;z_0)\big)}{dt} \le 0, \qquad t \ge 0. \tag{9.5}$$

Integrating this inequality with respect to t from 0 to $+\infty$, we obtain that $V(z_0) \ge 0$, i.e., $z_0^T K_M z_0 \ge 0$ for all $z_0 \in \mathbb{R}^n$. Hence, $K_M \ge 0$ which completes the proof of the item (i).

Now, let us proceed to the proof of the items (ii) and (iii). Equation (9.4) directly yields the inequality

$$\begin{aligned}&z_{Mv}^T(t;z_0) D z_{Mv}(t;z_0) + u_M^T\big(z_{Mv}(t;z_0)\big) G_u u_M\big(z_{Mv}(t;z_0)\big)\\ &\qquad -v^T(t) G_v v(t) \le -\frac{dV\big(z_{Mv}(t;z_0)\big)}{dt}, \qquad t \ge 0.\end{aligned} \tag{9.6}$$

Integrating this inequality with respect to the time from 0 to any fixed $t \geq 0$ and taking into account that $K_M \geq 0$, we obtain

$$\int_0^t \Big[z_{Mv}^T(\xi;z_0)Dz_{Mv}(\xi;z_0) + u_M^T\big(z_{Mv}(\xi;z_0)\big)G_u u_M\big(z_{Mv}(\xi;z_0)\big) - v^T(\xi)G_v v(\xi)\Big]d\xi \leq z_0^T K_M z_0 - z_{Mv}^T(t;z_0)K_M z_{Mv}(t;z_0) \leq z_0^T K_M z_0. \tag{9.7}$$

Since $v(t) \in L^2[0,+\infty;\mathbb{R}^s]$, the integral $\int_0^{+\infty} v^T(\xi)G_v v(\xi)d\xi$ converges. Moreover, since $G_v > 0$, then $\int_0^{+\infty} v^T(\xi)G_v v(\xi)d\xi \geq 0$. Therefore, due to the inequality (9.7) and the positive semi-definiteness of the matrices D, G_u, the integral $\int_0^{+\infty}\big[z_{Mv}^T(\xi;z_0)Dz_{Mv}(\xi;z_0) + u_M^T\big(z_{Mv}(\xi;z_0)\big)G_u u_M\big(z_{Mv}(\xi;z_0)\big)\big]d\xi$ also converges. Thus, the equation (2.13) and the inequality (9.7) yield

$$J\big(u_M(z), v(t)\big) \leq z_0^T K_M z_0. \tag{9.8}$$

Now, setting $v(t) = v_M\big(z_M(t;z_0)\big)$ and, therefore, $z_{Mv}(t;z_0) = z_M(t;z_0)$ in the equation (9.4), we have

$$\frac{dV\big(z_M(t;z_0)\big)}{dt} + z_M^T(t;z_0)Dz_M(t;z_0) + u_M^T\big(z_M(t;z_0)\big)G_u u_M\big(z_M(t;z_0)\big) - v_M^T\big(z_M(t;z_0)\big)G_v v_M\big(z_M(t;z_0)\big) = 0, \quad t \geq 0. \tag{9.9}$$

Integration of this equality with respect to t from 0 to $+\infty$, and use of the equations (2.13), (9.1) and the limit equality $\lim_{t\to+\infty} z_M(t;z_0) = 0$ yield after a simple rearrangement

$$J\Big(u_M(z), v_M\big(z_M(t;z_0)\big)\Big) = z_0^T K_M z_0. \tag{9.10}$$

The comparison of the inequality (9.8) and the equality (9.10) immediately implies the validity of the items (ii) and (iii).

Finally, let us prove the item (iv). The existence of the unique locally absolutely continuous solution $z(t)$ of the problem (2.12) for $u(t) = u_M(z)$ and any $v(t) \in L^2[0,+\infty;\mathbb{R}^s]$ on the entire interval $[0,+\infty)$ is clear.

We are going to show that $z(t) \in L^2[0,+\infty;\mathbb{R}^n]$. This inclusion is proven by combining Lyapunov-like function and small perturbation approaches. Consider the Riccati matrix algebraic equation

$$K(A+BM) + (A+BM)^T K + KCG_v^{-1}C^T K + \big(M^T G_u M + D + \nu I_n\big) = 0, \tag{9.11}$$

where $\nu > 0$ is a small parameter.

Since $G_u \geq 0$ and $D \geq 0$, then $M^T G_u M + D + \nu I_n > 0$ for all $\nu > 0$.

Consider the matrix

$$\mathcal{A}_{MK} \stackrel{\triangle}{=} A + BM + CG_v^{-1}C^T K_M. \tag{9.12}$$

Using the asymptotic expansion (with respect to ν) of solution to (9.11) and the Implicit Function Theorem [23] (Chapter III, paragraph 8), one can show that this equation has the solution

$$K = K_\nu \stackrel{\triangle}{=} K_M + \nu K_{M1} + R_\nu, \tag{9.13}$$

where

$$K_{M1} = \int_0^{+\infty} \exp\big(\mathcal{A}_{MK}^T\xi\big)\exp\big(\mathcal{A}_{MK}\xi\big)d\xi, \tag{9.14}$$

R_ν is some symmetric matrix, satisfying the following inequality for all sufficiently small $\nu > 0$:

$$\|R_\nu\| \le c\nu^2, \tag{9.15}$$

$c > 0$ is some constant independent of ν.

Since the matrix $\mathcal{A}_{MK}$ is a Hurwitz one, the integral in (9.14) converges and $K_{M1} > 0$. Hence, due to the positive semi-definiteness of the matrix K_M, the equation (9.13) and the inequality (9.15) imply $K_\nu > 0$ for all sufficiently small $\nu > 0$.

Now, consider the Lyapunov-like function

$$V_\nu(z) \triangleq z^T K_\nu z, \qquad z \in \mathbb{R}^n. \tag{9.16}$$

Using the Riccati equation (9.11), we obtain

$$\begin{aligned}\frac{dV_\nu\big(z(t)\big)}{dt} &= -z^T(t)\big(M^TG_uM + D + \nu I_n\big)z(t)\\ -\big(v(t) - v_\nu(t)\big)^T G_v\big(v(t) - v_\nu(t)\big) &+ v^T(t)G_v v(t), \qquad t \ge 0,\end{aligned} \tag{9.17}$$

where $z(t)$ is the solution of the initial-value problem (2.12) for $u(t) = u_M(z)$ and any $v(t) \in L^2[0, +\infty; \mathbb{R}^s]$;

$$v_\nu(t) = G^{-1}C^TK_\nu z\,(t)\,. \tag{9.18}$$

Since $G_v > 0$, the equation (9.17) leads to the following inequality for all $t \ge 0$:

$$0 \le z^T\,(t)\,\big(M^TG_uM + D + \nu I_n\big)z\,(t) \le -\frac{dV_\nu\big(z(t)\big)}{dt} + v^T(t)G_v v(t), \tag{9.19}$$

which yields by the integration

$$\begin{aligned}0 &\le \int_0^t z^T\,(\xi)\,\big(M^TG_uM + D + \nu I_n\big)z\,(\xi)\,d\xi\\ &\le V_\nu(z_0) - V_\nu\big(z(t)\big) + \int_0^t v^T(\xi)G_v v(\xi)d\xi, \qquad t \ge 0,\end{aligned} \tag{9.20}$$

or, due to $K_\nu > 0$,

$$\begin{aligned}0 &\le \int_0^t z^T\,(\xi)\,\big(M^TG_uM + D + \nu I_n\big)z\,(\xi)\,d\rho\\ &\le V_\nu(z_0) + \int_0^t v^T(\xi)G_v v(\xi)d\xi, \qquad t \ge 0.\end{aligned} \tag{9.21}$$

Since $v(t) \in L^2[0, +\infty; \mathbb{R}^s]$, then the integral in the right-hand side of (9.21) converges for $t \to +\infty$. Therefore, due to the inequality (9.21), the integral $\int_0^{+\infty} z^T(\xi)\big(M^TG_uM + D + \nu I_n\big)z\,(\xi)\,d\xi$ exists and is finite. The latter, along with the positive definiteness of the matrix $\big(M^TG_uM + D + \nu I_n\big)$, implies the inclusions $z(t) \in L^2[0, +\infty; \mathbb{R}^n]$. Moreover, due to (3.2), $u_M\big(z(t)\big) \in L^2[0, +\infty; \mathbb{R}^r]$ meaning that the strategy $u_M(z)$ is admissible in the ODG, which completes the proof of the item (iv). Thus, the lemma is proven.

10. Conclusions

In this paper, an infinite horizon zero-sum differential game with linear dynamics and quadratic cost functional was considered. A weight matrix of the control cost of a minimizing player (the minimizer) in the cost functional is singular but, in general, non-zero. This means that the game is singular. However, if the weight matrix is non-zero, only a part of the coordinates of the minimizer's control is singular, while the others are regular. Using proper assumptions, the linear system of the game dynamics was equivalently converted to a new system consisting of three modes. The first mode is controlled directly only by a maximizing player (the maximizer), the second mode is controlled directly by the maximizer and the regular coordinates of the minimizer's control, while the third mode is controlled directly by the maximizer and the entire control of the minimizer. Due to this transformation, a new singular game was obtained. Its equivalence to the initially formulated game was shown. In the sequel of the paper, this new singular game was considered as an original one. The original game was analyzed using a regularization approach, i.e., it was approximately replaced with an auxiliary regular game. This regular game has the same equation of dynamics and a similar cost functional augmented by an infinite horizon integral of the squares of the minimizer's singular control coordinates with a small positive weight. Hence, the auxiliary game is an infinite horizon zero-sum linear-quadratic differential game with partial cheap control of the minimizer. For this game, the minimizer's optimal state-feedback control was written down. This control depends on the minimal positive definite solution of a Riccati matrix algebraic equation perturbed by the small parameter. An asymptotic expansion of this solution was constructed and justified. Using this asymptotic expansion, upper and lower estimates of the upper value of the original (singular) game were established in a wide set of the minimizer's admissible state-feedback controls. In a reduced set of the minimizer's admissible state-feedback controls (linear stabilizing controls), the upper value of the original (singular) game was explicitly derived. It was shown that this upper value coincides with the upper value of a reduced dimension regular differential game (reduced game). The reduced game was obtained from the asymptotic expansion of the minimal positive definite solution to the Riccati matrix algebraic equation, associated with the auxiliary partial cheap control game.

References

[1] F. Amato and A. Pironti, *A note on singular zero-sum linear quadratic differential games*, in: M. Peshkin (ed) Proceedings of the 33rd Conference on Decision and Control, Lake Buena Vista, FL, 1994, pp. 1533–1535.

[2] B. D. O. Anderson and J. B. Moore, *Linear Optimal Control*, Prentice-Hall, Englewood Cliffs, NJ, 1971.

[3] T. Basar and P. Bernhard, *H^∞-Optimal Control and Related Minimax Design Problems*, Birkhauser, Boston, MA, 1995.

[4] D. J. Bell and D. H. Jacobson, *Singular Optimal Control Problems*, Academic Prees, New York, NY, 1975.

[5] R. Bellman, *Dynamic Programming*, Princeton University Press, Princeton, NJ, 1957.
[6] A. E. Bryson and Y. C. Ho, *Applied Optimal Control*, Hemisphere, New York, NY, 1975.
[7] K. Forouhar, S. J. Gibson and C. T. Leondes, *Singular linear quadratic differential games with bounds on control*, J. Optim. Theory Appl. **41** (1983), 341–348.
[8] K. Forouhar and C. T. Leondes, *Singular differential game numerical techniques*, J. Optim. Theory Appl. **37** (1982), 69–87.
[9] V. Y. Glizer, *Asymptotic solution of zero-sum linear-quadratic differential game with cheap control for the minimizer*, NoDEA Nonlinear Differential Equations Appl. **7** (2000), 231–258.
[10] V. Y. Glizer, *Solution of a singular optimal control problem with state delays: a cheap control approach*, in: Optimization Theory and Related Topics, Contemporary Mathematics Series, S. Reich and A. J. Zaslavski (eds) , vol. 568, American Mathematical Society, Providence, RI, 2012, pp. 77–107.
[11] V. Y. Glizer, *Stochastic singular optimal control problem with state delays: regularization, singular perturbation, and minimizing sequence*, SIAM J. Control Optim. **50** (2012), 2862–2888.
[12] V. Y. Glizer, *Singular solution of an infinite horizon linear-quadratic optimal control problem with state delays*, in: Variational and Optimal Control Problems on Unbounded Domains, Contemporary Mathematics Series, G. Wolansky and A. J. Zaslavski (eds), vol. 619, American Mathematical Society, Providence, RI, 2014, pp. 59–98.
[13] V. Y. Glizer, L. M. Fridman and V. Turetsky, *Cheap suboptimal control of an integral sliding mode for uncertain systems with state delays*, IEEE Trans. Automat. Control **52** (2007), 1892–1898.
[14] V. Y. Glizer and O. Kelis, *Solution of a zero-sum linear quadratic differential game with singular control cost of minimiser*, J. Control Decis. **2** (2015), 155–184.
[15] V. Y. Glizer and O. Kelis, *Singular infinite horizon quadratic control of linear systems with known disturbances: a regularization approach*, arXiv: 1603.01839v1 [math.OC], (2016), 36 p.
[16] R. Isaacs, *Differential Games*, John Wiley and Sons, New York, NY, 1967.
[17] N. N. Krasovskii and A. I. Subbotin, *Game-Theoretical Control Problems*, Springer Verlag, New York, NY, 1988.
[18] G. A. Kurina, *On a degenerate optimal control problem and singular perturbations*, Soviet Math. Dokl. **18** (1977). 1452–1456.
[19] E. F. Mageirou, *Values and strategies for infinite time linear quadratic games*, IEEE Trans. Automat. Control **21** (1976), 547–550.
[20] Z. Pan and T. Basar, *H^∞-optimal control for singularly perturbed systems. part I: perfect state measurements*, Automatica J. IFAC **29** (1993), 401–423.
[21] I. R. Petersen, *Linear-quadratic differential games with cheap control*, Systems Control Lett. **8** (1986), 181–188.
[22] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Gordon & Breach, New York, NY, 1986.
[23] L. Schwartz, *Analyse Mathematique*, vol. I, Hermann, Paris, France, 1967.
[24] J. Shinar and V. Y. Glizer, *Application of receding horizon control strategy to pursuit-evasion problems*, Optim. Control Appl. Methods **16** (1995), 127–141.
[25] J. Shinar, V. Y. Glizer and V. Turetsky *Solution of a singular zero-sum linear-quadratic differential game by regularization*, Int. Game Theory Rev. **16** (2014), 1440007-1 – 1440007-32.
[26] Y. Sibuya, *Some global properties of matrices of functions of one variable*, Math. Annalen **161** (1965), 67–77.
[27] E. N. Simakova, *Differential pursuit game*, Autom. Remote Control **28** (1967), 173–181.
[28] A. W. Starr and Y. C. Ho, *Nonzero-sum differential games*, J. Optim. Theory Appl. **3** (1969), 184–206.
[29] A. A. Stoorvogel, *The singular zero-sum differential game with stability using H_∞ control theory*, Math. Control Signals Systems **4** (1991), 121–138.
[30] A.A. Stoorvogel, *The H_∞ Control Problem: A State Space Approach*, University of Michigan Press, Ann Arbor, MI, 2000.

[31] V. Turetsky and V. Y. Glizer, *Robust state-feedback controllability of linear systems to a hyperplane in a class of bounded controls*, J. Optim. Theory Appl. **123**, (2004), 639–667.
[32] V. Turetsky and V. Y. Glizer, *Robust solution of a time-variable interception problem: a cheap control approach*, Int. Game Theory Rev. **9** (2007), 637–655.
[33] V. Turetsky, V. Y. Glizer and J. Shinar, *Robust trajectory tracking: differential game/cheap control approach*, Internat. J. Systems Sci. **45** (2014), 2260–2274.

Manuscript received May 3 2016
revised September 14 2016

V.Y. Glizer
Department of Mathematics, Ort Braude College of Engineering, P. O. Box 78, Karmiel 2161002, Israel
E-mail address: valery48@braude.ac.il

O. Kelis
Department of Mathematics, Ort Braude College of Engineering, P. O. Box 78, Karmiel 2161002, Israel and Department of Mathematics, University of Haifa, Haifa 31905, Israel
E-mail address: olegkelis@braude.ac.il