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RISK-SENSITIVE CONTROL OF REFLECTED DIFFUSION PROCESSES ON ORTHRANT

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Abstract. In this article, we prove the existence of optimal risk-sensitive control with state constraints. We use near monotone assumption on the running cost to prove the existence of optimal risk-sensitive control.

1. Introduction and problem description

In this paper we study the risk-sensitive control problem when the state dynamics is governed by a controlled reflecting stochastic differential equation in d-dimensional orthrant. We prove that the risk-sensitive value is an eigenvalue of the nonlinear eigenvalue problem with oblique boundary conditions (see, the equation (3.2)) which is the Hamilton Jacobi Bellman (HJB) equation of the risk-sensitive control problem with state constraints. We also show that any minimizing selector in (3.2) corresponding to the eigen function of the risk-sensitive value is a risk-sensitive optimal control. We use near monotone structural condition on the running cost and a blanket recurrence condition for the state dynamics for proving this result. Similar risk-sensitive control problem but with state dynamics given by non degenerate diffusions without state constraints are studied under various set of assumptions. For example [16] initiated risk-sensitive control problems with non degenerate diffusion state dynamics where diffusions are assumed to asymptotically flat. For general non degenerate diffusions, see [5] and the references there in. Note that in [5], author exploits the eigenvalue problem nature of the risk-sensitive control problem. Very recently, in an interesting paper [2], authors consider risk-sensitive control problem with near monotone cost structure for diffusions which may be transient.

The paper is organized as follows. The remaining part of Section 1 contains detailed description of the problem and some results on controlled reflected stochastic differential equations which are used in subsequent sections. Among other results,

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we prove Itô-Krylov formula for reflected stochastic differential equation in a d-orthrant (i.e. an orthrant in $\mathbb{R}^d$) which may be of independent interest. Also results from partial differential equations (pdes) theory we use are sketched in this section. In Section 2, we discuss an auxiliary risk-sensitive control problem with discounted cost structure. We prove the existence of optimal value and control without the structural condition near monotonicity on the running cost. In the final section, we prove our main theorem, i.e. Theorem 3.2. The proof is based on the vanishing discounting method. Also in this section, we discuss multiplicative Poisson equation corresponding to uncontrolled reflected stochastic differential equations. In particular, through an example of a transient reflected Brownian motion, we conjecture that removing the blanket recurrent assumption may lead to a situation where there exists no cost function which is near monotone with respect to β, the risk-sensitive value.

1.1. **Notations.** In this subsection, we introduce frequently used notations. In $\mathbb{R}^d$, the standard norm and inner product are denoted respectively by $|\cdot|$ and $\langle\, ,\, \rangle$. We denote the positive d-orthrant $\{x \in \mathbb{R}^d | x_i > 0 \ \forall \ i\}$ by D. Also U denote a compact metric space. For $A \subseteq \mathbb{R}^d$, the interior, closure, boundary and compliment are denoted respectively by A^o, $\overline{A}, \partial A$ and A^c. Sometimes we use A^c for the compliment of $A \subseteq \overline{D}$ in $\overline{D}$. By domain in $\mathbb{R}^d$, we mean non empty open connected set. By $B(x, R)$ we mean open ball with radius R and center $x \in \mathbb{R}^d$ and when x becomes the origin we use B_R for $B(0, R)$.

For a bounded continuous function $f : \overline{D} \times U \to \mathbb{R}$, denote $\sup\limits_{x,v} |f(x, v)|$ by $\|f\|_\infty$. For $\varphi \in C_b(\overline{D})$, the space of all real-valued bounded continuous functions, for each B, a Borel subset of $\overline{D}$, we denote

$$\|\varphi\|_{\infty,B} \ = \ \sup_{x\in B} |\varphi(x)|, \ \ \|\varphi\|_\infty = \sup_{x\in\overline{D}} |\varphi(x)|.$$

For a Banach space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$, $1 \le p < \infty$, define for $\kappa \ge 0$

$$L^p(\kappa, T; \mathcal{X}) \ = \ \{\varphi : (\kappa,\ T) \to \mathcal{X} | \varphi \text{ is Borel measurable and } \int_\kappa^T \|\varphi(t)\|^p_{\mathcal{X}}\, dt < \infty\}$$

with the norm

$$\|\varphi\|_{p;\mathcal{X}} \ = \ \Big[\int_\kappa^T \|\varphi(t)\|^p_{\mathcal{X}}\, dt\Big]^{\frac{1}{p}}.$$

The norm of the Banach space $L^\infty((\kappa, 1) \times D)$, the space measurable functions on $(\kappa, 1) \times D$ with finite essential supremum norm, is denoted by $\|\cdot\|_{\infty;(\kappa,1)\times D}$.

For $l, k = 0, 1, \cdots, \infty$, $C^{l,k}((\kappa, 1)\times D)$ denote the space of all functions $\varphi : (\kappa, 1)\times D \to \mathbb{R}$ which has continuous derivatives of order up to l in first argument t and up to k in the second argument $x \in \mathbb{R}^d$. The spaces $C^{l,k}([\kappa, 1] \times \overline{D}), C^k(D), C^k(\overline{D})$ are defined similarly. $C^{l,k}_c((\kappa, 1) \times D)$, $l, k = 1, 2, \cdots, \infty$ denotes the space of all functions in $C^{l,k}((\kappa, 1)\times D)$ which are compactly supported. The spaces $C^{l,k}_c((\kappa, 1]\times \overline{D})$, $C^{l,k}_c([\kappa, 1] \times \overline{D})$ and $C^k_c(\overline{D})$ are similarly defined. For any suitably smooth function $\varphi : I \times B \to \mathbb{R}$ where I, B are domains in $[0, \infty)$ and $\mathbb{R}^d$ respectively, $\nabla\varphi$

denote the gradient, $\nabla^2\varphi$ denote the Hessian in $x \in B$. We also use $\frac{\partial \varphi}{\partial t}, \frac{\partial \varphi}{\partial x_i}, \frac{\partial^2 \varphi}{\partial x_i \partial x_j}$ to denote various partial derivatives.

For $\kappa < T < \infty$ and domain B in $\mathbb{R}^d$, $C^{1+\beta/2,2+\beta}((\kappa,T)\times B), \kappa \geq 0$, denotes the set of all continuous functions $\varphi(t,x)$ in $(\kappa,T)\times B$ together with all the derivatives upto order 1 in t and 2 in x with finite Holder norms. The spaces $C^{1+\beta/2,2+\beta}([\kappa,T]\times \overline{B})$ is defined by extending the functions continuously up to the boundary.

For any domain B in $\overline{D}$, the space $\mathcal{W}^{1,2,p}((\kappa,T)\times B)), \kappa \geq 0$, denotes the set of all $\varphi \in L^p(\kappa,T; W^{2,p}(B))$ such that $\frac{\partial \varphi}{\partial t} \in L^p((\kappa,T;L^p(B))$ with the norm given by

$$\|\varphi\|^p_{1,2,p;W^{2,p}(B)} \;=\; \|\varphi\|^p_{p;W^{2,p}(B)} + \|\frac{\partial \varphi}{\partial t}\|^p_{p;L^p(B)}, \;1 \leq p < \infty.$$

Also the local Sobolev spaces $\mathcal{W}^{1,2,p}_{\text{loc}}((\kappa,T)\times B)$ are defined by

$$\begin{aligned} &\mathcal{W}^{1,2,p}_{\text{loc}}(\kappa,T)\times B) \\ = \;& \Big\{\varphi : (\kappa,T)\times B \to \mathbb{R} |\, \varphi \text{ is measurable and } \varphi \in W^{1,2,p}((\kappa,T)\times K), \\ & \text{for each compact subset } K \text{ of } B\Big\}. \end{aligned}$$

Also, define

$$W^{1,2,p}((\kappa,T)\times B) \;=\; \Big\{\varphi : (\kappa,T)\times B \to \mathbb{R}\Big| \|\varphi\|_{1,2,p;(\kappa,T)\times B} < \infty\Big\},$$

where the norm $\|\cdot\|_{1,2,p;(\kappa,T)\times B}$ is defined as

$$\begin{aligned} \|\varphi\|^p_{1,2,p;(\kappa,T)\times B} \;=\;& \int_\kappa^T \int_B |\varphi(t,x)|^p dxdt + \int_\kappa^T \int_B \Big|\frac{\partial \varphi(t,x)}{\partial t}\Big|^p dxdt \\ & + \sum_i \int_\kappa^T \int_B \Big|\frac{\partial \varphi(t,x)}{\partial x_i}\Big|^p dxdt + \sum_{ij} \int_\kappa^T \int_B \Big|\frac{\partial^2 \varphi(t,x)}{\partial x_i x_j}\Big|^p |dxdt. \end{aligned}$$

1.2. **State dynamics.** Now we introduce the state dynamics of the risk-sensitive control problem. For the given functions $b : \overline{D}\times U \longrightarrow \mathbb{R}^d$, $\sigma : \overline{D} \longrightarrow \mathbb{R}^{d\times d}$ and $\gamma : \mathbb{R}^d \longrightarrow \mathbb{R}^d$, consider the controlled reflected diffusion in $\overline{D}$, given by the solution of the reflected stochastic differential equation (in short RSDE)

$$\begin{aligned} dX_t \;&=\; b(X_t, v_t)dt + \sigma(X_t)dW_t - \gamma(X_t)d\xi_t, \\ d\xi_t \;&=\; I_{\{X_t\in\partial D\}}d\xi_t, \\ \xi_0 \;&=\; 0, \quad X_0 = x \in \overline{D}, \end{aligned} \tag{1.1}$$

where $W = (W_1,\cdots,W_d)$ is an $\mathbb{R}^d$-valued standard Wiener process, $v(\cdot)$ is a U-valued measurable process non anticipative with respect to $W(\cdot)$, called an admissible control. In fact the pair $(v(\cdot), W(\cdot))$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ satisfying the usual hypothesis is an admissible control if and only if $v(\cdot)$ is measurable and $\{\mathcal{F}_t\}$-adapted, see Remark 2.1, p.31 of [3]. Henceforth, all filtered probability spaces are assumed to satisfy usual hypothesis. The set of all admissible controls is denoted by $\mathcal{A}$.

By a solution to (1.1) we mean a pair of continuous time processes $(X(\cdot), \xi(\cdot))$ satisfying (1.1) such that the process $X(\cdot)$ is $\overline{D}$-valued and $\xi(\cdot)$ is a $[0,\ \infty)$- non-decreasing process which increases only when $X(\cdot)$ hits the boundary ∂D. The concept of solution in weak and strong sense are analogous to that of the solutions of stochastic differential equation (in short SDE). The above is a special case of the more general definition of solutions of SDEs with reflection, see [15]. In fact we consider the case when the direction of reflection is single valued and continuous.

We use the relaxed control frame work given as follows. The compact metric space $U = \mathcal{P}(S)$ for some compact metric space S, where $\mathcal{P}(S)$ denote the space of probability measures on S endowed with the Prohorov topology, i.e. the topology induced by weak convergence. The drift coefficient b takes the form

$$b(x, v) \ = \ \int_S \bar{b}(x, s) v(ds), \ v \in U, x \in \overline{D}.$$

For $l = 1, 2, \cdots$, set

$$D'_l \ = \ D \cap B_l.$$

From the proof of Theorem A2 (ii) and the remark in p. 28 of [13] there exists open domains $D_{lm} \subseteq \mathbb{R}^d$ with C^∞ boundary such that

- The distance between $\partial D'_l$ and D_{lm} satisfies,
$$d(\partial D_{lm}, \partial D'_l) < \frac{1}{m}, \ l \geq 1,$$
- $D_{ln} \subseteq D_{lm}, \ n \geq m, \ l \geq 1.$

Set

$$D_m \ = \ \cup_{l=1}^{\infty} D_{lm}, \ m \geq 1.$$

Then we have

(i) For each $m \geq 1$, D_m is with C^∞ smooth boundary and $D_m \downarrow \bar{D}$.
(ii) For any compact set $C \subset \bar{D}$, we have $C \subset \overline{D}_{lm}$ for $m \geq 1$ and l sufficiently large.

We make the following assumption which is sufficient to ensure the existence of unique solution to the equation (1.1).

(A1) (i) The function $\bar{b}$ is bounded, jointly continuous, Lipschitz continuous in its first argument uniformly with respect to the second argument.

(ii) The functions $\sigma_{ij} \in C^2(\bar{D}), i, j = 1, \cdots, d$ and bounded.

(iii) The function $a \stackrel{def}{=} \sigma\sigma^\perp$ is uniformly elliptic with ellipticity constant $\delta > 0$, i.e.,

$$x a(x) x^\perp \ \geq \ \delta \, |x|^2, \ x \in \overline{D}\,,$$

where $x^\perp$ denote the transpose of the vector x.

(A2) The function $\gamma = (\gamma_1, \ldots, \gamma_d)$ is such that $\gamma_i \in C_b(\mathbb{R}^d)$, and there exists $\eta > 0$ such that

$$\gamma(x) \cdot n^i(x) \quad \geq \quad \eta \ \text{ for all } x \in F_i, \ i = 1, 2, \ldots, n,$$

and $\gamma(x) \cdot n_m(x) \quad \geq \quad \eta$ for all $x \in G_j \cap \partial D_m$, for all m sufficiently large, $j = 1, 2, \ldots, d$, where $F_i = \{x \in \mathbb{R}^d | x_i = 0\}, G_j$ is a fixed neighbourhood of F_j, and $n^i(\cdot), n_m(\cdot)$ denote the outward normal to F_i and ∂D_m respectively.

We also assume that

(A3) For each $x \in D$,

$$P_x^v(X(t) \in \Gamma' \text{ for some } t > 0) \;=\; 0,$$

where Γ' denote the non smooth boundary part of ∂D.

Example 1.1. Consider the controlled RSDE in the non-negative quadrant $\overline{\mathbb{R}_+^2}$ given by

$$\begin{aligned} dX_t &= b(X_t, v_t)dt + dW_t + (1,1)d\xi_t, \\ d\xi_t &= I_{\{X_t \in \partial \mathbb{R}_+^2\}} d\xi_t, \\ \xi_0 &= 0, \quad X_0 = x \in \overline{\mathbb{R}_+^2}, \end{aligned}$$

where b satisfies (A1). Using Girsanov's theorem, there exists a probability measure Q equivalent to P such that on $(\Omega, \mathcal{F}, Q)$, the process $X(\cdot)$ can be written as

$$\begin{aligned} dX_t &= dB_t + (1,1)d\xi_t, \\ d\xi_t &= I_{\{X_t \in \partial \mathbb{R}_+^2\}} d\xi_t, \\ \xi_0 &= 0, \quad X_0 = x \in \overline{\mathbb{R}_+^2}, \end{aligned}$$

where $B(\cdot)$ is a d-dimensional Wiener process under Q. From [[31], Theorem 2.2], we have $Q(X(t) \in \Gamma' \text{ for some } t > 0) \;=\; 0$. Hence (A3) holds, since P is equivalent to Q.

Assumptions (A1), (A2) and (A3) will be in use for the rest of the paper.

The existence of a unique weak solution of (1.1) for an admissible control has been proved in [4] .

Now we prove a tightness result for solutions of RSDEs on $\overline{D}_m$. Note that we extend the functions $\bar{b}, \sigma$ to D_m satisfying (A1) and (A2). Let $(X^m(\cdot), \xi^m(\cdot))$ denote a unique strong solution to the RSDE (1.1) with zero drift and replacing D with D_m, $m \geq 1$. Existence of unique strong solution for RSDE in $\overline{D}_m$ follows from [4].

Theorem 1.2. *The process $(X^m(\cdot), \xi^m(\cdot))$ converges in law to a unique solution $(X(\cdot), \xi(\cdot))$ of (1.1) with zero drift.*

Proof. Set

$$\begin{aligned} X^m(t) &= Y^m(t) + Z^m(t), \\ Y^m(t) &= x + \int_0^t \sigma(X_s^m) dW_s, \\ Z^m(t) &= -\int_0^t I_{\{X_s^m \in \partial D_m\}} \gamma(X_s^m) d\xi_s^m. \end{aligned}$$

Consider the Skorohod problem for (D_m, γ), i.e. for each $y^m \in C([0,\ \infty);\ \mathbb{R}^d)$, find $(x^m, z^m) \in C([0,\ \infty); \overline{D}_m) \times C([0,\ \infty); \mathbb{R}^d) \cap \mathrm{BV}([0,\ \infty); \mathbb{R}^d)$, where $\mathrm{BV}([0,\ \infty); \mathbb{R}^d)$ denote the set of all functions of bounded variation, such that

$$\begin{aligned} x^m(t) &= y^m(t) + z^m(t), t \geq 0, \\ z^m(t) &= \int_{[0,t)} \gamma(x^m(s)) d|z^m|(s), \\ d|z^m|(t) &= I_{\{x^m(s) \in \partial D_m\}} d|z^m|(t), z^m(0) = 0, \end{aligned}$$

where $|z^m|_t$ denote the total variation of z^m in $[0,t]$.

Set

$$\begin{aligned} a_m(x,\rho) &= \max_{|u|=1, u\in\mathbb{R}^d} \min_{y\in\partial D_m\cap\overline{B(x,\rho)}} \gamma(y)\cdot u, \\ c_m(x,\rho) &= \max_{y\in\partial D_m\cap\overline{B(x,\rho)}} \max\Big\{\frac{\gamma(x)\cdot(x-z)}{|x-z|}, 0\Big\}, \\ e_m(x,\rho) &= \frac{c(x,\rho)}{\max\{a_m^2(x,\rho), \frac{1}{2}a_m(x,\rho)\}}, \end{aligned}$$

From (A2), we have for $x\in\partial D_m$, $\rho>0$,

$$\alpha_m(x,\rho) := \max_{u\in\mathbb{R}^d; |u|=1} \min_{y\in\partial D_m\cap\overline{B(x,\rho)}} n_m(y)\cdot u \ \geq\ \eta.$$

Hence

$$\lim_{\rho\to 0}\inf_{x\in\partial D_m} \alpha_m(x,\rho) \geq \eta > 0.$$

Now from [[14], Proposition 2.3], it follows that

$$\begin{aligned} \lim_{\rho\to 0}\inf_{x\in\partial D_m} a_m(x,\rho) &\geq \eta, \\ \lim_{\rho\to 0}\sup_{x\in\partial D_m} e_m(x,\rho) &= 0, \text{ for all } m\geq 1. \end{aligned} \tag{1.2}$$

Using (1.2), one can easily mimick the proof of Theorem 2.2 of [14] to show that for each $y^m\in\mathcal{Y}$, any compact subset of $C([0,\ T];\ \mathbb{R}^d)$

$$|z^m|_t - |z^m|_s \ \leq\ K\|y^m\|_{s,t},\ 0\leq s\leq t\leq T, \tag{1.3}$$

where $\|y^m\|_{s,t} := \sup_{s\leq t_1\leq t_2\leq t}|y^m(t_2)-y^m(t_1)|$, $K>0$ depends on T and $\mathcal{Y}$ but not on m.

Using the boundedness of b and σ, we can show that

$$E|Y^m(t)-Y^m(s)|^4 \leq K|t-s|^2, 0\leq s<t\leq T, m\geq 1$$

for some $K>0$ which is independent of m. Hence tightness of the laws of $\{Y^m(\cdot)\}$ follows. Now using (1.3), tightness of the laws of $\xi^m(\cdot), Z^m(\cdot)$ follows as in the proof of [[22], Theorem 3.2]. Using the tightness of the laws of $Y^m(\cdot)$ and $Z^m(\cdot)$, the tightness of the laws of $X^m(\cdot)$ follows.

Also, the pathwise uniqueness of solutions to (1.1) follows from Lemma 3.3 of [4]. Now, if $(\tilde{X}(\cdot),\tilde{\xi}(\cdot))$ is a limit point in law of $(X^m(\cdot),\xi^m(\cdot))$, then as in the proof of [[14], Theorem 5.4], $(\tilde{X}(\cdot),\tilde{\xi}(\cdot))$ is a solution to (1.1).

This implies that the law of $(X^m(\cdot),\xi^m(\cdot))$ converges to a unique solution $(X(\cdot),\xi(\cdot))$ of (1.1). □

Remark 1.3. In fact, one can use Theorem 1.2 to give an alternate proof for existence of unique solution to (1.1) as follows. Use Theorem 1.2 to establish a unique solution to (1.1) with zero drift. Now with non zero drift, using Girsanov transformation method to establish existence of unique weak solution under admissible controls, see [[3], pp.42-44]. For a Markov control, one can prove the existence of unique strong solution by adapting the approach by Zovokin and Veretenikov, see [[3], pp.45-46] for the analogous proof for the unconstrained diffusions. See Theorem 3.2 of [4] for details.

The running cost function $r : \overline{D} \times U \longrightarrow [0, \infty)$ is given in the relaxed frame work as

$$r(x, v) = \int_S \bar{r}(x, s) v(ds), x \in \overline{D}, v \in U.$$

Throughout this paper we assume that the cost function $\bar{r}$ is bounded continuous in (x, s) and Lipschitz continuous in the first argument uniformly with respect to the second. We consider two risk-sensitive cost criteria, discounted cost and ergodic cost criteria which are described below.

1.3. **Discounted cost criterion.** Let $\theta \in (0, \ \Theta)$ be the risk-aversion parameter. In the α-discounted cost criterion, controller chooses his control $v(\cdot)$ from the set of all admissible controls $\mathcal{A}$ to minimize his α-discounted risk-sensitive cost given by

$$J_\alpha^v(\theta, x) := \frac{1}{\theta} \ln E_x^v \Big[e^{\theta \int_0^\infty e^{-\alpha t} r(X_t, v_t) dt} \Big], x \in \overline{D}, \tag{1.4}$$

where $\alpha > 0$ is the discount parameter, $X(\cdot)$ is the solution of the s.d.e. (1.1) corresponding to $v(\cdot) \in \mathcal{A}$ and $E_{t,x}^v$ denote the expectation with respect to the law of the process (1.1) corresponding to the admissible control v and initial condition $X_t = x$. If $t = 0$, then we denote $E_{t,x}^v$ by E_x^v. An admissible control $v^*(\cdot) \in \mathcal{A}$ is called an optimal control if

$$J_\alpha^{v^*}(\theta, x) \leq J_\alpha^v(\theta, x), \quad \text{for all } v(\cdot) \in \mathcal{A} \text{ and } \ x \in \overline{D}.$$

1.4. **Ergodic cost criterion.** In this criterion controller chooses his control $v(\cdot) \in \mathcal{A}$ so as to minimize his risk-sensitive accumulated cost given by

$$\rho^v(\theta, x) = \limsup_{T \to \infty} \frac{1}{\theta T} \ln E_x^v \Big[e^{\theta \int_0^T r(X_t, v_t) dt} \Big], x \in \overline{D}. \tag{1.5}$$

The definition of optimal control is analogous. From now onwards, we take $\Theta = 1$ without any loss of generality.

1.5. **Various subclasses of controls.** An admissible control $v(\cdot)$ is said to be a Markov control if there exists a measurable map $\bar{v} : [0, \infty) \times \overline{D} \longrightarrow U$ such that $v(t) = \bar{v}(t, X(t))$. By an abuse of notation, the measurable map $\bar{v} : [0, \infty) \times \overline{D} \longrightarrow U$, itself is refereed as Markov control. If $\bar{v}$ has no explicit time dependence then it is said to be a stationary Markov control. We denote the set of all Markov controls and stationary Markov controls by $\mathcal{M}$ and $\mathcal{S}$ respectively. An admissible control $v(\cdot)$ is said to be a feedback control if it is progressively measurable with respect to $\{\mathcal{F}_t^{X,\xi}\}$, where $(X(\cdot), \xi(\cdot))$ denote the solution of (1.1) and $\mathcal{F}_t^{X,\xi}$ denote sigma field generated by $\{X_s, \xi_s | s \leq t\}, t \geq 0$. This is equivalent to saying that there exists a progressively measurable map $\bar{v} : [0, \ \infty) \times C[[0, \infty); \bar{D}) \times C[[0, \infty); [0, \ \infty)) \to U$ such that $v(t) = \bar{v}(t, X[0, t], \xi[0, t]), t \geq 0$, where $X[0, t], \xi[0, t]$ denote respectively $\{X_s, 0 \leq s \leq t\}, \{\xi_s, 0 \leq s \leq t\}$. Hence by an abuse of notation, we denote the set of feedback controls by all progressively measurable maps. The following lemma tells that we can restrict ourselves to feedback controls. Its proof is a straightforward adaptation of Theorem 2.3.4 (a), p.52 of [3].

Lemma 1.4. *Let $(v(\cdot), W(\cdot))$ be an admissible control and $(X(\cdot), \xi(\cdot))$ be a solution pair to (1.1) on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$. Then on an augmentation $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$ with a $\{\tilde{\mathcal{F}}_t\}$-Wiener process $\tilde{W}(\cdot)$ and a feedback control $\tilde{v}(\cdot)$ such that $(X(\cdot), \xi(\cdot))$ solves (1.1) for the pair $(\tilde{v}(\cdot), \tilde{W}(\cdot))$ on $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde{P})$.*

1.6. **Review of pde results.** Though our basic pdes are quasilinear parabolic, we only need the 'time' variable (which is risk-sensitive parameter for us) as a parameter and hence by suitably freezing 'terms' in the pdes we mostly deal with parametric families of linear elliptic pdes. Hence, we will be using standard estimates from linear elliptic pde literature.

Let $\tilde{D} \subseteq D$ be a bounded domain with C^2 boundary and $u^{\tilde{D}} \in W^{2,2}(\tilde{D})$ be a solution to

$$\begin{aligned} \tilde{\mathcal{L}} u^{\tilde{D}} := \frac{1}{2}\text{trace}(a(x)\nabla^2 u^{\tilde{D}}(x)) + \tilde{b}(x)\cdot \nabla u^{\tilde{D}} &= g(x), x \in \tilde{D}, \\ \nabla u^{\tilde{D}} \cdot \gamma &= 0 \text{ on } \partial\tilde{D}. \end{aligned} \tag{1.6}$$

For $W^{2,2}$ to $W^{2,p}$ regularities, we use the following results, which follows from [[26], Lemma 6.31, p.260] and [[26], Theorem 6.27, p.256] respectively.

Lemma 1.5. *If $u \in W^{2,2}(\tilde{D})$ be such that $\tilde{\mathcal{L}}u = g, g \in L^2(\tilde{D})$ and $H \subseteq \tilde{D}$ such that $\partial D \cap \partial H$ is a $C^{1,\alpha}$ portion of ∂D for some $\alpha > \frac{1}{2}$. If $\tilde{\mathcal{L}}u = g \in L^p(H)$ for $p > 2$, then $u \in W^{2,p}(H)$.*

Theorem 1.6. *Let u and H as in Lemma 1.5. Then there exists a constant C which depends only on $d, p, \tilde{D}, H$, the bounds of $\sigma, \tilde{b}$ and $\eta > 0$ such that*

$$\|\nabla^2 u\|_{p,H} \le C\Big(\frac{1}{\delta}\|g\|_{p;\tilde{D}} + \|u\|_{p,\tilde{D}}\Big).$$

Lemma 1.7. *Let u, g and H be as in Lemma 1.5. Then there exists a constant C which depends only on $\delta, d, p, \tilde{D}, H$, the bounds of $\sigma, \tilde{b}$ and $\eta > 0$ such that*

$$\|u\|_{2,p,H} \le C\Big(\|g\|_{p;\tilde{D}} + \|u\|_{p,\tilde{D}}\Big).$$

Proof. Using Erling-Nirenberg-Gagliardo interpolation inequality, see Theorem 4.14, p.75, [1], for a fixed $\varepsilon > 0$,

$$\|u\|_{1,p;H} \le \varepsilon\|\nabla^2 u\|_{p;H} + C(\varepsilon)\|u\|_{p;H},$$

Now using Theorem 1.6, we get

$$\|u\|_{2,p;H} \le (1+\varepsilon)C(\frac{1}{\delta}\|g\|_{p;\tilde{D}} + \|u\|_{p;\tilde{D}}) + C(\varepsilon)\|u\|_{p;H}.$$

This completes the proof, since $H \subseteq \tilde{D}$. ☐

We use the following Harnack's inequality which follows from [[26], Theorem 1.20, p.28, Theorem 1.27, p.34]. Note that above mentioned theorems in [26] are for $u \in C^2(D \cap B_R) \cap C(\overline{D \cap B_R})$. But the corresponding result for the $W^{2,p}$ class follows by routine approximation argument, so we omit the details.

Theorem 1.8. *Assume (A1)-(A2). Let* $u \in W^{2,d}_{loc}(D)) \cap C^0(\bar{D})$ *be a non negative solution to*

$$\begin{aligned} \tilde{\mathcal{L}}u + c(x)u &= 0, x \in D, \\ \nabla u \cdot \gamma &= 0 \text{ on } \partial D, \end{aligned} \tag{1.7}$$

where c is a bounded measurable function. Then there exists a constant K which depends only on $R > 0$, bounds of $\sigma, \tilde{b}, c$, the constants $\delta > 0, \eta > 0$ from (A1)-(A2) such that

$$\sup_{D\cap B_R} u \leq \; K \inf_{D\cap B_R} u, \;\; R > 0$$

We use the following basic existence uniqueness theorem which can be deduced from [[7], p.80].

Theorem 1.9. *Assume (A1)-(A2), let $c : \mathbb{R}^d \times U \to [0, \; \infty)$ be bounded continuous, $f \in C^2_b(\mathbb{R}^d)$ and $\kappa > 0$. Also $\tilde{D}$ is a C^2 bounded domain in D. Then the pde*

$$\begin{aligned} \frac{\partial \varphi}{\partial t} &= \inf_{v\in U} \Big[\mathcal{L}\varphi(x, v) + c(x, v)\varphi \Big] \\ \varphi(0, x) &= f(x) \text{ in } \tilde{D}, \; \nabla\varphi(t, x) \cdot \gamma(x) \; = \; 0 \text{ on } \partial\tilde{D}. \end{aligned}$$

has a unique solution in $C^{1,2}([\kappa, \; T] \times \overline{\tilde{D}})$.

We use the following approximation procedure. Consider a sequence of elliptic equations

$$\begin{aligned} \frac{1}{2}\text{trace}(a(x)\nabla^2 u_n) + b_n(x) \cdot \nabla u_n &= g_n(x), x \in \tilde{D}_n, \\ \nabla u_n \cdot \gamma &= 0 \text{ on } \partial\tilde{D}_n, \end{aligned} \tag{1.8}$$

where $\tilde{D}_n$ are C^2-open bounded domains in D. If it is known that $\sup_n \|b_n\|_{\infty;D}$ and $\sup_n \|g_n\|_{\infty;D}$ are finite and $u_n \in W^{2,p}(\tilde{D}_n), \; p \geq 2$ satisfies $\sup_n \|u_n\|_{\infty;D} < \infty$, then for each domain H in D with C^2 boundary portion in ∂D (if ∂H intersects ∂D), from Lemma 1.7, it follows that there exists a constant C which is independent of n but depending on the volume and the uniform cone property characteristics of H such that

$$\|u_n\|_{2,p;H} \;\leq\; C, n \geq 1. \tag{1.9}$$

Now by a suitable diagonalization procedure and standard trace results, there exists $u \in W^{2,p}_{loc}(D \cup \Gamma), \; p \geq 2$ such that along a subsequence $u_n \to u$ in $W^{2,p}(H)$ and $\nabla u \cdot \gamma = 0$ on ∂D a.e.

1.7. **Properties of Controlled RSDEs.** We prove some results about the controlled RSDE (1.1) which are used in the subsequent sections. To the best of our knowledge these results are not available for the controlled RSDE (1.1).

First result is about the equivalence of weak solution and martingale problem for reflected diffusions.

For a feedback control $v(\cdot)$, we say that the RSDE (1.1) admits a weak solution if there exists a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$, a $\{\mathcal{F}_t\}$-Wiener process $W(\cdot)$

and a pair of $\{\mathcal{F}_t\}$-adapted processes $(X(\cdot), \xi(\cdot))$ with a.s. continuous paths such that $X(\cdot)$ is $\overline{D}$-valued, $\xi(\cdot)$ is non decreasing and satisfy

$$\begin{aligned} dX(t) &= b(X(t), v(t, X[0,t], \xi[0,t])dt + \sigma(X(t))dW(t) - \gamma(X(t))d\xi(t) \\ d\xi(t) &= I_{\{X(t)\in\partial D\}} d\xi(t), X(0) = x, \xi(0) = 0 \ P \text{ a.s.}. \end{aligned}$$

Set

$$\mathcal{H} = \{f \in C_c^2(\overline{D}) | \nabla f \cdot \gamma \geq 0 \text{ on } \partial D\} \tag{1.10}$$

and

$$\mathcal{L}f(x,v) = b(x,v)\cdot\nabla f(x) + \frac{1}{2}\text{trace}(a(x)\nabla^2 f(x)), f \in \mathcal{D}(\mathcal{L}), \tag{1.11}$$

where the domain $\mathcal{D}(\mathcal{L})$ of the oblique elliptic operator $\mathcal{L}$ contains $C^2_{b,\gamma}(\overline{D})$, the set of all bounded twice continuously differentiable functions satisfying $\nabla f\cdot\gamma \geq 0$ on ∂D.

Constrained controlled martingale problem: A pair of $\{\mathcal{F}_t\}$-adapted processes $(X(\cdot), \xi(\cdot))$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ is said to solve the constrained controlled martingale problem for RSDE (1.1) corresponding to the admissible control $v(\cdot)$ and initial condition $x \in \overline{D}$ if the following holds.

(i) $X(\cdot)$ is $\overline{D}$-valued and $\xi(\cdot)$ is non decreasing and $X(0) = x, \xi(0) = 0$ a.s.

(ii)
$$\int_0^t I_{\{X(s)\in\partial D\}} d\xi(s) = \xi(t), \ P \text{ a.s. for all } t \geq 0,$$

(iii) For all $f \in \mathcal{H}$,
$$M_f(t) := f(X(t)) - \int_0^t \mathcal{L}f(X(s), v(s))ds + \int_0^t \nabla f\cdot\gamma(X(s))d\xi(s), \ t \geq 0$$
is an $\{\mathcal{F}_t\}$-martingale in $(\Omega, \mathcal{F}, P)$.

Theorem 1.10. *For a feedback control $v(\cdot)$, the pair of processes $(X(\cdot), \xi(\cdot))$ defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ solves the constrained controlled martingale problem iff there exists a filtered probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde P)$ and a pair of processes $(\tilde X(\cdot), \tilde\xi(\cdot))$ which is a weak solution to (1.1) such that $(X(\cdot), \xi(\cdot))$ and $(\tilde X(\cdot), \tilde\xi(\cdot))$ agree in law.*

Proof. Suppose $(X(\cdot), \xi(\cdot))$ solves the constrained controlled martingale problem. Hence the law of $X(\cdot)$ solves the corresponding submartingale problem. Now using Theorem 1 of [20], there exists a filtered probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}_t\}, \tilde P)$ and $\{\tilde{\mathcal{F}}_t\}$-adapted processes with continuous paths $(\tilde X(\cdot), \tilde\xi(\cdot))$ and a Wiener process $\tilde W(\cdot)$ such that $(\tilde X(\cdot), \tilde\xi(\cdot))$ is a weak solution to (1.1) and law of $X(\cdot)$ is same as law of $\tilde X(\cdot)$. Now since (1.1) has a unique weak solution, law of $(X(\cdot), \xi(\cdot))$ equals the law of $(\tilde X(\cdot), \tilde\xi(\cdot))$. Converse follows from Itô's formula. ☐

Remark 1.11. Under suitable C^2 smoothness assumption on the domain and bounded, continuity assumption on direction of reflection γ, the equivalence is shown in [29]. The case of domains with piecewise smooth boundaries and with constant direction of reflections is treated in [12].

For an admissible control $v(\cdot)$, if $(X(\cdot), \xi(\cdot))$ denotes a unique weak solution pair to the RSDE (1.1) on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and τ a $\{\mathcal{F}_t\}$-stopping time, then $\mathcal{F}_\tau$ is finitely generated and hence using Theorem 1.3.4, p.34 of [30], it follows that regular conditional probability distribution (rcpd) P_ω of P given $\mathcal{F}_\tau$ exists. Now we prove a result analogous to Lemma 2.3.7 of [3].

Lemma 1.12. *Let $(X(\cdot), \xi(\cdot))$ denote a weak solution pair corresponding to an admissible feedback control $v(\cdot)$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, P)$ and τ be a finite $\{\mathcal{F}_t\}$-stopping time. Then the conditional law μ_ω of the process $X(\tau + \cdot)$ given $\mathcal{F}_\tau$ is a.s. the law of the process $X_\omega(\cdot)$, where $X_\omega(\cdot)$ is a unique weak solution to the RSDE (1.1) on a probability space $(\Omega_\omega, \mathcal{F}_\omega, \{\mathcal{F}_{\omega,t}\}, P_\omega)$ for an admissible control given by $v_\omega(t) = v(t + \tau(\omega), X[0, \tau(\omega) + t], \xi[0, \tau(\omega) + t]), t \geq 0$.*

Proof. For $f \in \mathcal{H}$, since

$$M_t \;=\; f(X_t) - f(X_0) - \int_0^t \mathcal{L}(X_s, v_s)ds + \int_0^t \nabla f \cdot \gamma(X_s)d\xi_s, \;\; t \geq 0,$$

where $\mathcal{L}$ is given by (1.11) is an $\{\mathcal{F}_t\}$-martingale on $(\Omega, \mathcal{F}, P)$, it follows from Theorem 1.2.10, p.28 of [30] that there exist a P-null set N such that for $\omega \notin N$, $M_f^{\tau(\omega)}(t) = M_t - M_{t\wedge\tau(\omega)}, t \geq 0$ is a Martingale with respect to $\{\mathcal{F}_t\}$ on $(\Omega, \mathcal{F}, P_\omega)$. Hence under P_ω,

$$M_f^{\tau(\omega)}(t) \;=\; f(X_t) - f(X_{\tau(\omega)}) - \int_{\tau(\omega)}^t \mathcal{L}f(X_s, v_s)ds + \int_{\tau(\omega)}^t \nabla f \cdot \gamma(X_s)d\xi_s, t \geq \tau(\omega)$$

is a Martingale under $P_\omega, \omega \notin N$. i.e.,

$$\begin{aligned} M_f^{\tau(\omega)}(t) \;&=\; f(X_t) - f(X_{\tau(\omega)}) - \int_0^t \mathcal{L}f(X(\tau(\omega) + s, v_{s+\tau(\omega)})ds \\ &\quad + \int_0^t \nabla f \cdot \gamma(X_{s+\tau(\omega)})d\xi_{s+\tau(\omega)}, t \geq 0 \end{aligned}$$

is a Martingale under $P_\omega, \omega \notin N$. i.e. $(X_\omega(\cdot), \xi_\omega(\cdot)) := (X(\cdot + \omega), \xi(\cdot) + \tau(\omega) - \xi(\tau(\omega))$ solves the constrained controlled martingale problem for the admissible control v_ω and initial distribution $X(\tau(\omega))$. This completes the proof. □

Next we prove the Itô-Krylov formula for controlled RSDE. Note that generalized Itô's formula, i.e. Itô-Krylov formula doesn't seem to be available in a precise form even for uncontrolled RSDE on non smooth domains. But for smooth bounded domains, Itô-Krylov formula can be deduced from the arguments given in [[7], p.89]. So we give details of the Itô-Krylov formula for controlled RSDE for functions from $W^{2,p}(D \cup \Gamma), p \geq d$, where

$$\Gamma \;=\; \Big\{x \in \cup_i F_i | x \notin F_{i_1} \cap \cdots \cap F_{i_k}, k \geq 2, i_l \in \{1, \cdots, d\}\Big\}$$

denotes the smooth part of the boundary ∂D and Γ' denote the remaining part of ∂D. Before proceeding to generalized Itô's formula, we prove the following estimate which is crucial in the proof of generalized Itô's formula.

Lemma 1.13. *Let $(X(\cdot), \xi(\cdot))$ be a unique solution to (1.1) and $f \in L^p((0,\ T) \times D), p \geq d$. Then for $T > 0, R > 0$, there exists a $K > 0$ independent of f such that*

$$E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} |f(s, X(s))| ds\Big] \leq K\|f\|_{p;(0,T)\times D)},$$

where $\tau_R = \tau(B_R)$, $\tau(B) = \inf\{s \geq 0 | X_s \notin B\}$ denote the exit time of a process $X(\cdot)$ from a domain B.

Proof. By an application of Girsanov's theorem as in [[7], Lemma 4.2], we can assume w.l.o.g that $b \equiv 0$. Let $(X^m(\cdot), \xi^m(\cdot))$ be a unique strong solution to the RSDE given by

$$\begin{aligned} dX^m_t &= \sigma(X^m_t)dW_t - \gamma(X^m_t)d\xi^m_t, \\ d\xi^m_t &= I_{\{X^m_t \in \partial D_m\}} d\xi^m_t, \\ \xi^m_0 &= 0, \quad X^m_0 = x \in \overline{D}_m, \end{aligned} \tag{1.12}$$

Then using Theorem 1.2, $(X_m(\cdot), \xi^m(\cdot))$ converges in law to a unique solution $(X(\cdot), \xi(\cdot))$ of (1.1) with $b \equiv 0$. For $f \in C^{1+\frac{\beta}{2}, 2+\beta}((0,\ T) \times D) \cap L^p((0,\ T) \times D)$, $f \geq 0$, consider the pde

$$\begin{aligned} \frac{\partial \varphi^m}{\partial t} + \frac{1}{2}(\text{trace}(a(x)\nabla^2\varphi^m)) + f &= 0 \\ \nabla\varphi^m \cdot \gamma = 0 \text{ on } \partial D_m,\ \varphi &= 0 \text{ on } \partial B_R \cap D_m, \\ \varphi^m(T, x) &= 0. \end{aligned} \tag{1.13}$$

Then (1.13) has a unique solution φ^m in $C^{1,2}([0,\ T] \times \overline{D_m \cap B_R})$, see [[24], Theorem 3]. Now extend φ^m to a bounded domain $(0, T) \times \tilde{D}$ containing $(0, T) \times (D_m \cap B_R)$, where $\tilde{D}$ is C^2, such that ϕ^m satisfies

$$\begin{aligned} \frac{\partial \varphi^m}{\partial t} + \frac{1}{2}(\text{trace}(a(x)\nabla^2\varphi^m)) + f &= 0 \\ \nabla\varphi^m \cdot \gamma = 0 \text{ on } \partial\tilde{D}&, \\ \varphi^m(T, x) &= 0. \end{aligned}$$

Now from [[25], Theorem 7.35, p.185], one can deduce that, for each bounded domain $H \subseteq \overline{D \cap B_R}$ with C^2 boundary,

$$\|\varphi^m\|_{2,p;H} \leq K_1\|f\|_{p;(0,T)\times D)}, \tag{1.14}$$

where $K_1 > 0$ is independent of f and m. Now using the arguments in subsection 1.6, it follows that $\varphi^m \to \varphi \in W^{1,2,p}(0,\ T) \times (D \cap B_R) \cup \Gamma) \cap C^{0,0}([0,\ T] \times \overline{D \cap B_R})$ and satisfies the pde

$$\begin{aligned} \frac{\partial \varphi}{\partial t} + \frac{1}{2}(\text{trace}(a(x)\nabla^2\varphi)) + f &= 0 \\ \nabla\varphi \cdot \gamma = 0 \text{ on } \partial D,\ \varphi &= 0 \text{ on } \partial B_R \cap D \\ \varphi(T, x) &= 0. \end{aligned} \tag{1.15}$$

Using Ito's formula to the process $(X^m(\cdot), \xi^m(\cdot))$ and φ^m, we get

$$\varphi^m(t, x) = E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} f(s, X^m_s))ds\Big].$$

Now using (1.14), it follows that

$$E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} f(s,X^m(s))ds\Big] \le K_1\|f\|_{p;(0,\ T)\times D}. \tag{1.16}$$

Now since $X^m(\cdot)$ converges in law to $X(\cdot)$, $\|f\|_{\infty;(0,T)\times D} < \infty$ and hence by invoking Skohorod's representation theorem, it follows from dominated convergence theorem that

$$E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} f(s,X(s))ds\Big] \le K_1\|f\|_{p;(0,\ T)\times D}. \tag{1.17}$$

For $f \in L^p((0,T)\times D), f \ge 0$, choose $f \in C^{1+\frac{\beta}{2},2+\beta}((0,\ T)\times D)\cap L^p((0,\ T)\times D), f_n \ge 0$ such that $f_n \to f$ in $L^p((0,T)\times D)$. We can assume w.l.o.g. that $f_n \to f$ a.e. by restricting to a subsequence. Now from (1.17) we have

$$E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} f_n(s,X(s))ds\Big] \le K_1\|f_n\|_{p;(0,\ T)\times D},\ n\ge 1.$$

Now by letting $n\to\infty$, with the help of Fatou's lemma for the left hand side, we get

$$E^v_{t,x}\Big[\int_t^{T\wedge\tau_R} f(s,X(s))ds\Big] \le K_1\|f\|_{p;(0,\ T)\times D}.$$

Now for $f \in L^p((0,T)\times D)$, use $f = f^+ - f^-$ to completes the proof.

□

Now we prove generalized Itô's formula for RSDEs.

Theorem 1.14. *For $\varphi \in W^{2,p}_{loc}(D\cup\Gamma), p \ge d$ and the process $(X(\cdot),\xi(\cdot))$ given by (1.1), we have for each bounded domain $H \subseteq D\cup\Gamma,\ t\ge 0$,*

$$\begin{aligned}\varphi(X(t\wedge\tau(H)) &= \varphi(x) + \int_0^{t\wedge\tau(H)} \mathcal{L}\varphi(X_s,v_s)ds + \int_0^{t\wedge\tau(H)} \nabla\varphi(X_s)^{\perp}\sigma(X_s)dW_s \\ &\quad - \int_0^{t\wedge\tau(H)} \nabla\varphi(X_s)\cdot\gamma(X_s)I_{\{X_s\in\partial D\}}d\xi_s.\end{aligned}$$

Proof. Choose $\varphi_n \in C^2(\overline{D})$ such that $\varphi_n \to \varphi$ in $W^{2,p}(D\cup\Gamma)$. Using Itô's formula to φ_n we get

$$\begin{aligned}\varphi_n(X(t\wedge\tau(H)) &= \varphi_n(x) + \int_0^{t\wedge\tau(H)} \mathcal{L}\varphi_n(X_s)ds + \int_0^{t\wedge\tau(H)} \nabla\varphi_n(X_s)^{\perp}\sigma(X_s)dW_s \\ &\quad - \int_0^{t\wedge\tau(H)} \nabla\varphi_n\cdot\gamma(X(s))I_{\{X_s\in\partial D\}}d\xi_s.\end{aligned} \tag{1.18}$$

Using Lemma 1.13, it follows that

$$\lim_{n\to\infty} E^v_x\Big[\int_0^{t\wedge\tau(H)} |\mathcal{L}(\varphi_n-\varphi)(X_s)|ds\Big] = 0. \tag{1.19}$$

Since $\nabla\varphi_n \to \nabla\varphi$ uniformly on $\overline{H}$, it follows that

$$\lim_{n\to\infty}\int_0^{t\wedge\tau(H)} \nabla\varphi_n\cdot\gamma(X(s))I_{\{X_s\in\partial D\}}d\xi_s$$

$$(1.20)\qquad = \lim_{n\to\infty}\int_0^{t\wedge\tau(H)} \nabla\varphi_n\cdot\gamma(X(s))I_{\{X_s\in\Gamma\}}d\xi_s$$
$$= \int_0^{t\wedge\tau(H)} \nabla\varphi\cdot\gamma(X(s))I_{\{X_s\in\partial D\}}d\xi_s.$$

Using Lemma 1.13, we get

$$E_x^v\Big|\int_0^{t\wedge\tau(H)}\nabla(\varphi_n-\varphi)(X_s)^{\perp}\sigma(X_s)dW_s\Big|^2 \;\leq\; K'\|\nabla\varphi_n-\nabla\varphi\|^2_{p;D}.$$

From the above we get

$$(1.21)\qquad \lim_{n\to\infty}\int_0^{t\wedge\tau(H)}\nabla\varphi_n(X_s)^{\perp}\sigma(X_s)dW_s \;=\; \int_0^{t\wedge\tau(H)}\nabla\varphi(X_s)^{\perp}\sigma(X_s)dW_s.$$

Now we complete the proof by combining the above limits with (1.18). □

Now we give a characterization for recurrence of the RSDE (1.1) corresponding to a stationary Markov control in the following lemma.

Lemma 1.15. *Let $v(\cdot)\in\mathcal{S}$ and $X(\cdot)$ be a solution to the RSDE (1.1) corresponding to $v(\cdot)$ and B be an open ball in D. Then $X(\cdot)$ is recurrent iff the pde*

$$\begin{aligned}\mathcal{L}\varphi(x,v(x)) &= 0, \text{ in } D\setminus\bar{B},\\ (1.22)\qquad \varphi &\equiv 1 \text{ on } \partial B,\ \nabla\varphi\cdot\gamma\equiv 0 \text{ on } \partial D.\end{aligned}$$

has a unique non negative bounded solution in $W^{2,d+1}_{loc}((D\setminus\bar{B})\cup\Gamma)\cap C^0(\bar{D}\setminus B)$.

Proof. Note that $\varphi\equiv 1$ is always a positive bounded solution of (1.22) in $W^{2,d+1}_{loc}((D\setminus\bar{B})\cup\Gamma)\cap C^0(\bar{D}\setminus B)$. Also an application of Itô-Krylov formula and Fatou's lemma implies that any bounded non negative solution $\varphi\in W^{2,d+1}_{loc}((D\setminus\bar{B})\cup\Gamma)\cap C^0(\bar{D}\setminus B)$ satisfies

$$\varphi(x)\geq P_x(\tau(\bar{D}\setminus B)<\infty), x\in\overline{D}.$$

Hence the result follows, since non degeneracy of the RSDE implies that $X(\cdot)$ recurrent iff it is B-recurrent for some ball B in D.

□

2. Analysis of the discounted cost criterion

In this section, we study the discounted risk-sensitive control problem with the state dynamics (1.1) and cost criterion

$$J^v_\alpha(\theta,x)=\frac{1}{\theta}\ln E^v_x\left[e^{\theta\int_0^\infty e^{-\alpha t}\, r(X_t,v_t)dt}\right].$$

The α-discounted risk-sensitive control problem is to minimize (1.4) over all admissible controls. We define the so-called 'value function' for the cost (1.4) as

$$(2.1)\qquad \phi_\alpha(\theta,x)=\inf_{v\in\mathcal{A}}J^v_\alpha(\theta,x).$$

Set

$$(2.2)\qquad \bar{J}^v_\alpha(\theta,x)=E^v_x\left[e^{\theta\int_0^\infty e^{-\alpha t}\, r(X_t,v_t)dt}\right].$$

Since logarithm is an increasing function for fixed $\theta > 0$, a minimizer of $\bar{J}^v_\alpha(\theta, x)$ if exists will be a minimizer of $J^v_\alpha(\theta, x)$). Corresponding to the cost (2.2), the value function is defined as

$$u_\alpha(\theta, x) = \inf_{v \in \mathcal{A}} \bar{J}^v_\alpha(\theta, x). \tag{2.3}$$

Note that

$$\phi_\alpha(\theta, x) = \frac{1}{\theta} \ln u_\alpha(\theta, x). \tag{2.4}$$

Since we are dealing with exponential cost we need *multiplicative* version of Dynamic Programming Principle (DPP) in place of additive DPP, given in [[11], pp. 53-59]. We mimic the arguments as in [27] to prove DPP for the value function $u_\alpha(\theta, x)$.

Theorem 2.1 (DPP). *Let τ be any bounded stopping time with respect to the natural filtration of process $X(\cdot)$, i.e., $\{\mathcal{F}^X_t\}$. Then*

$$u_\alpha(\theta, x) = \inf_{v(\cdot)} E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} u_\alpha\left(\theta e^{-\alpha\tau}, X(\tau)\right)\right]. \tag{2.5}$$

where infimum is taken over all feedback controls.

Proof. Note that, given two feedback controls $v_1(t)$ and $v_2(t)$, $t \geq 0$ and τ as above, $v(\cdot)$ given by

$$v(t) = v_1(t) I_{\{t<\tau\}} + v_2(t-\tau) I_{\{t\geq\tau\}},\ t \geq 0, \tag{2.6}$$

is also a feedback control. Indeed, we are given pairs of processes $(X_1(\cdot), \xi_1(\cdot), v_1(\cdot))$ and $(X_2(\cdot), \xi_1(\cdot), v_2(\cdot))$ satisfying (1.1) on some, possibly distinct, probability spaces $(\Omega_1, \mathcal{F}_1, P_1)$, $(\Omega_2, \mathcal{F}_2, P_2)$ respectively, with $v_1(\cdot), v_2(\cdot)$ in feedback form. Also, $X_1(0) = x$ and the law of $X_2(0)$ is same as the law of $X_1(\tau)$, where τ is a prescribed stopping time with respect to the natural filtration of process $X_1(\cdot)$. Now using Lemma 1.12, by augmenting $(\Omega_1, \mathcal{F}_1, P_1)$ suitably, one can construct processes $(X(\cdot), \xi(\cdot))$ and $v(\cdot)$ satisfying (1.1) such that they coincide with $(X_1(\cdot), \xi_1(\cdot))$ and $v_1(\cdot)$ on $[0, \tau]$, and $(X(\tau+\cdot), \xi(\tau+\cdot))$ and $v(\tau+\cdot)$ agree in law with $(X_2(\cdot), \xi(\cdot))$ and $v_2(\cdot)$. Also the conditional law of $X(\tau+\cdot)$ of given $\mathcal{F}_\tau$ is the same as its conditional law given $X(\tau)$ and agrees with the conditional law of $X(\tau+\cdot)$ given $X_2(0)$ a.s. with respect to the common law of $X_2(0), X(\tau)$.

For $\epsilon > 0$, let $X(\cdot)$ be a process (1.1) controlled by $v(\cdot)$ as above with $v_1(\cdot)$ an arbitrary feedback control and $v_2(\cdot)$ an ϵ-optimal feedback control for initial data $X(\tau)$. By (2.3) we have

$$\begin{aligned}
u_\alpha(\theta, x) &\leq E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt + \theta \int_\tau^\infty e^{-\alpha t}\, r(X_t, v_t) dt} \right] \\
&= E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} \times e^{\theta e^{-\alpha\tau} \int_0^\infty e^{-\alpha t}\, r(X_{t+\tau}, v_{t+\tau}) dt} \right] \\
&= E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} E\left[e^{\theta e^{-\alpha\tau} \int_0^\infty e^{-\alpha t}\, r(X_t, v_t) dt} \Big| X(\tau) \right] \right] \\
&\leq E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} \left(u_\alpha\left(\theta e^{-\alpha\tau}, X(\tau)\right) + \epsilon \right) \right] \\
&= E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} u_\alpha\left(\theta e^{-\alpha\tau}, X(\tau)\right) \right] + \epsilon E^v_x \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} \right].
\end{aligned}$$

Since τ, r are bounded and $\epsilon > 0$ is arbitrary we get

$$u_\alpha(\theta, x) \leq \inf_{v(\cdot)} E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} u_\alpha \left(\theta e^{-\alpha\tau}, X(\tau) \right) \right].$$

Conversely, let $\epsilon > 0$ and $v(\cdot)$ is an ϵ-optimal feedback control for initial data $X(0) = x$. Then

$$\begin{aligned} u_\alpha(\theta, x) + \epsilon &\geq E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt + \theta \int_\tau^\infty e^{-\alpha t}\, r(X_t, v_t) dt} \right] \\ &= E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} E \left[e^{\theta e^{-\alpha\tau} \int_0^\infty e^{-\alpha t}\, r(X_t, v_t) dt} \Big| X(\tau) \right] \right] \\ &\geq E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} \inf_{v(\cdot)} E \left[e^{\theta e^{-\alpha\tau} \int_0^\infty e^{-\alpha t}\, r(X_t, v_t) dt} \Big| X(\tau) \right] \right] \\ &= E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} u_\alpha \left(\theta e^{-\alpha\tau}, X(\tau) \right) \right]. \end{aligned}$$

Thus

$$u_\alpha(\theta, x) + \epsilon \geq \inf_{v(\cdot)} E_x^v \left[e^{\theta \int_0^\tau e^{-\alpha t}\, r(X_t, v_t) dt} u_\alpha \left(\theta e^{-\alpha\tau}, X(\tau) \right) \right].$$

Letting $\epsilon \longrightarrow 0$, we complete the proof. □

Using dynamic programming heuristics, the HJB equation for α-discounted cost criterion is given by

$$\begin{aligned} \alpha\theta \frac{\partial u_\alpha}{\partial \theta} &= \inf_{v \in U} [b(x, v) \cdot \nabla u_\alpha + \theta r(x, v) u_\alpha] + \frac{1}{2} trace(a(x) \nabla^2 u_\alpha) \\ u_\alpha(0, x) &= 1 \text{ on } \overline{D}, \quad \nabla u_\alpha(\theta, x).\ \gamma(x) = 0 \text{ on } (0, 1) \times \partial D. \end{aligned} \tag{2.7}$$

First we show that (2.7) has unique a solution. There are two technical difficulties in solving the p.d.e. (2.7). First is the singularity in θ at 0 and the second is the unbounded non smooth nature of the orthrant. We circumvent these difficulties by suitable approximation arguments which involves approximating (2.7) by a family of pdes in the smooth bounded domains D_{ml} given below. For each $m, l \geq 1$ and $0 < \kappa < 1$, consider the p.d.e.

(2.8)

$$\begin{aligned} \alpha\theta \frac{\partial u^\kappa_{\alpha,lm}}{\partial \theta} &= \inf_{v \in U} \left[b(x, v) \cdot \nabla u^\kappa_{\alpha,lm} + \theta r(x, v) u^\kappa_{\alpha,lm} \right] + \frac{1}{2} trace(a(x) \nabla^2 u^\kappa_{\alpha,lm}) \\ u^\kappa_{\alpha,lm}(\kappa, x) &= e^{\frac{\kappa \|r\|_\infty}{\alpha}} \text{ on } \overline{D}_{lm}, \quad \nabla u^\kappa_{\alpha,lm}. \gamma = 0 \text{ on } (\kappa, 1) \times \partial D_{lm}. \end{aligned}$$

Lemma 2.2. *The the p.d.e. (2.8) has a unique solution* $u^\kappa_{\alpha,lm} \in C^{1,2}([\kappa, 1] \times \overline{D}_{lm})$, *and*

$$\|u^\kappa_{\alpha,lm}\|_{\infty;[\kappa,1]\times\overline{D}_{lm}} \leq e^{\frac{\theta\|r\|_\infty}{\alpha}}, \text{ for all } \kappa > 0,\ m, l \geq 1, \tag{2.9}$$

$$\left\| \frac{\partial u^\kappa_{\alpha,lm}}{\partial \theta} \right\|_{\infty;[\kappa,1]\times\overline{D}_{lm}} \leq 3e^{\frac{(\theta+3)\|r\|_\infty}{\alpha}} \frac{\|r\|_\infty}{\alpha}, \text{ for all } \kappa > 0,\ m, l \geq 1. \tag{2.10}$$

Proof. For the existence and uniqueness result we use Theorem 1.9. Set

$$\theta = 1 - t,\ u^\kappa_{\alpha,lm}(\theta, x) = u(t, x).$$

Then (2.8) reduces to

$$\begin{aligned} -\alpha(1-t)\frac{\partial u}{\partial t} &= \inf_{v\in U}\left[b(x,v)\cdot\nabla u + (1-t)r(x,v)u\right] + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u) \\ u(1-\kappa,x) &= e^{\frac{\kappa\|r\|_\infty}{\alpha}}, \text{ for } x\in\overline{D}_m, \\ \nabla u(t,x)\cdot\gamma(x) &= 0 \text{ on } (0,1-\kappa)\times\partial D_m. \end{aligned}$$

Rewrite the above equation as

$$\begin{aligned} \frac{\partial u}{\partial t} + \inf_{v\in U}\left[\frac{b(x,v)}{\alpha(1-t)}\cdot\nabla u + \frac{1}{\alpha}r(x,v)u\right] & \\ +\frac{1}{2}\operatorname{trace}\left(\frac{a(x)}{\alpha(1-t)}\nabla^2 u\right) &= 0 \\ u(1-\kappa,x) &= e^{\frac{\kappa\|r\|_\infty}{\alpha}}, \text{ for } x\in\overline{D}_{lm}, \\ \nabla u(t,x)\cdot\gamma(x) &= 0 \text{ on } (0,1-\kappa)\times\partial D_{lm}. \end{aligned} \tag{2.11}$$

Now, set

$$\begin{aligned} b(t,x,u,p) &= \inf_{v\in U}\left[\frac{b(x,v)}{\alpha(1-t)}\cdot p + \frac{1}{\alpha}r(x,v)u\right] \\ a_{ij}(t,x) &= \frac{a_{ij}(x)}{2\alpha(1-t)} \\ T &= 1-\kappa \\ Q_T &= D_{lm}\times[0,T] \\ \psi_0(x) &= e^{\frac{\kappa\|r\|_\infty}{\alpha}}. \end{aligned} \tag{2.12}$$

Note that $b(t,x,u,p)$ and $a_{ij}(t,x)$ are Lipschitz continuous in x, since $b(x,v)$, $r(x,v), a_{ij}(x)$ are Lipschitz continuous in the first argument uniformly with respect to the second.

Therefore from Theorem 1.9, it follows that (2.11) has a unique solution in $C^{1,2}([\kappa,1]\times\overline{D}_{lm})$. Hence existence of a unique solution to (2.8) in $C^{1,2}([\kappa,1]\times\overline{D}_{lm})$ follows.

Let $v(\cdot)$ be an admissible control and $X(\cdot)$ be the process given by

$$\begin{aligned} dX_t &= b(X_t,v_t)dt + \sigma(X_t)dW_t - \gamma(X_t)d\xi_t \\ d\xi_t &= I_{\{X_t\in\partial D_{lm}\}}d\xi_t \\ \xi_0 &= 0, \quad X_0 = x\in\overline{D}_{lm}\,. \end{aligned}$$

Applying Itô's formula to $e^{\int_0^t \theta_s r(X_s,v_s)ds}\,u^\kappa_{\alpha,lm}(\theta_t,X_t), \theta_t = \theta e^{-\alpha t}$, we get

$$\begin{aligned} d\left(e^{\int_0^t \theta_s r(X_s,v_s)ds}u^\kappa_{\alpha,lm}(\theta_t,X_t)\right) &= e^{\int_0^t \theta_s r(X_s,v_s)ds}du^\kappa_{\alpha,lm}(\theta_t,X_t) \\ &\quad + \theta_t u^\kappa_{\alpha,lm}(\theta_t,X_t)e^{\int_0^t \theta_s r(X_s,v_s)ds}r(X_t,v_t)dt, \end{aligned}$$

where

$$\begin{aligned} du^\kappa_{\alpha,lm}(\theta_t,X_t) &= (\nabla u^\kappa_{\alpha,lm}(\theta_t,X_t))^\perp\sigma(X_t)dW_t - \left[\gamma(X_t)\cdot\nabla u^\kappa_{\alpha,lm}(\theta_t,X_t)\right]I_{\{X_t\in\partial D_{lm}\}}d\xi_t \\ &\quad + \left[\mathcal{L}\,u^\kappa_{\alpha,lm}(\theta_t,X_t,v_t) - \alpha\theta_t\frac{\partial}{\partial\theta}u^\kappa_{\alpha,lm}(\theta_t,X_t)\right]dt, \end{aligned}$$

and $\mathcal{L}$ is defined in (1.11). Using the fact that $u^{\kappa}_{\alpha,lm}$ satisfy the equation (2.8), we get

$$u^{\kappa}_{\alpha,lm}(\theta, x) \ \leq \ E^{v}_{x}\Big[e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} e^{\int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s,v_s)ds}\Big],$$

where $T_\kappa = \dfrac{ln(\frac{\theta}{\kappa})}{\alpha}$. Repeating the above argument with a minimizing selector in (2.8), it follows that

$$u^{\kappa}_{\alpha,lm}(\theta, x) \ = \ \inf_{v(\cdot)} E^{v}_{x}\Big[e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} e^{\int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s,v_s)ds}\Big], \tag{2.13}$$

where infimum is over all admissible controls. Now from (2.13), we get

$$|u^{\kappa}_{\alpha,lm}(\theta, x)| \leq E^{v}_{x}\Big[e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} e^{\int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s,v_s)ds}\Big] \leq e^{\frac{\kappa\|r\|_{\infty}}{\alpha}} e^{\|r\|_{\infty}\frac{(\theta-\kappa)}{\alpha}},$$

which proves the estimate (2.9).

We mimic the arguments of [[8], Theorem 3.1], to prove the estimate (2.10). For ϵ with $|\epsilon|$ sufficiently small, set

$$T^{\epsilon}_{\kappa} = \frac{1}{\alpha}\log\left(\frac{\theta+\epsilon}{\kappa}\right).$$

Now consider for each $v(\cdot)$ admissible

$$\left.\begin{array}{l}
\left|E^{v}_{x}\left[e^{(\theta+\epsilon)\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right] - E^{v}_{x}\left[e^{\theta\int_0^{T_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right]\right| \\
\leq \left|E^{v}_{x}\left[e^{(\theta+\epsilon)\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right] - E^{v}_{x}\left[e^{\theta\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right]\right| \\
\quad + \left|E^{v}_{x}\left[e^{\theta\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right] - E^{v}_{x}\left[e^{\theta\int_0^{T_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right]\right|
\end{array}\right\}. \tag{2.14}$$

Now

$$\left.\begin{array}{l}
\left|E^{v}_{x}\left[e^{(\theta+\epsilon)\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right] - E^{v}_{x}\left[e^{\theta\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right]\right| \\
\leq E^{v}_{x}\left[e^{\theta\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt} \times \left|e^{\epsilon\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt} - 1\right|\right] \\
\leq e^{\frac{\theta\|r\|_{\infty}}{\alpha}\left(1-\frac{\kappa}{\epsilon+\theta}\right)} \times E^{v}_{x}\left|e^{\epsilon\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt} - 1\right| \\
\leq e^{\frac{(\theta+\epsilon)\|r\|_{\infty}}{\alpha}} \frac{\|r\|_{\infty}}{\alpha}|\epsilon|
\end{array}\right\}, \tag{2.15}$$

and

$$\begin{array}{l}
\left|E^{v}_{x}\left[e^{\theta\int_0^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right] - E^{v}_{x}\left[e^{\theta\int_0^{T_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt}\right]\right| \\
\leq E^{v}_{x}\left[e^{\theta\int_0^{T_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt} \times \left|e^{\theta\left|\int_{T_\kappa}^{T^{\epsilon}_{\kappa}} e^{-\alpha t} r(X_t,v_t)dt\right|} - 1\right|\right] \\
\leq e^{\frac{\theta\|r\|_{\infty}}{\alpha}}\left[e^{\frac{\theta\|r\|_{\infty}}{\alpha}|e^{-\alpha T_\kappa} - e^{-\alpha T^{\epsilon}_{\kappa}}|} - 1\right] \\
= e^{\frac{\theta\|r\|_{\infty}}{\alpha}}\left[e^{\frac{\|r\|_{\infty}}{\alpha}\left|\frac{\kappa\epsilon}{\theta+\epsilon}\right|} - 1\right].
\end{array}$$

Note that for each $\theta > 0$, when ϵ is positive, then $\dfrac{\kappa\epsilon}{\theta+\epsilon} \leq 1$ and for $\epsilon < 0$ we can choose a $0 < \epsilon_\theta < 1$ such that $\dfrac{\kappa\epsilon}{\theta+\epsilon} \leq 2$ whenever $|\epsilon| \leq \epsilon_\theta$. Hence we have

$$
(2.16)\quad \left.\begin{array}{l}
\left| E_x^v \left[e^{\theta \int_0^{T_\kappa^\epsilon} e^{-\alpha t} r(X_t,v_t)dt} \right] - E_x^v \left[e^{\theta \int_0^{T_\kappa} e^{-\alpha t} r(X_t,v_t)dt} \right] \right| \\
\leq e^{\frac{\theta\|r\|_\infty}{\alpha}} \left[e^{\frac{2\|r\|_\infty}{\alpha}|\epsilon|} - 1 \right] \text{ whenever } |\epsilon| \leq \epsilon_\theta \\
\leq e^{\frac{\theta\|r\|_\infty}{\alpha}} \dfrac{2\|r\|_\infty}{\alpha} |\epsilon| e^{\frac{2\|r\|_\infty}{\alpha}|\epsilon|} \text{ whenever } |\epsilon| \leq \epsilon_\theta \\
= 2e^{\frac{(\theta+2)\|r\|_\infty}{\alpha}} \dfrac{\|r\|_\infty}{\alpha} |\epsilon| \text{ whenever } |\epsilon| \leq \epsilon_\theta
\end{array}\right\}.
$$

From (2.13), (2.14), (2.15) and (2.16) we have

$$
\begin{aligned}
|u^\kappa_{\alpha,lm}(\theta+\epsilon,x) - u^\kappa_{\alpha,lm}(\theta,x)| &\leq e^{\frac{\kappa\|r\|_\infty}{\alpha}} \sup_{v(\cdot)} \Bigg| E_x^v \left[e^{(\theta+\epsilon) \int_0^{T_\kappa^\epsilon} e^{-\alpha t} r(X_t,v_t)dt} \right] \\
&\qquad - E_x^v \left[e^{\theta \int_0^{T_\kappa} e^{-\alpha t} r(X_t,v_t)dt} \right] \Bigg| \\
&\leq 3e^{\frac{(\theta+3)\|r\|_\infty}{\alpha}} \frac{\|r\|_\infty}{\alpha} |\epsilon| \text{ whenever } |\epsilon| \leq \epsilon_\theta.
\end{aligned}
$$

This completes the proof of the lemma. □

Theorem 2.3. *The equation (2.7) has a solution* $u_\alpha \in W^{1,2,p}_{loc}((0,\,1) \times D \cup \Gamma) \cap C^{0,0}([0,1] \times \overline{D})$, $p \geq d$.

Proof. Let H be an open bounded domain with C^2 boundary in D. Let N be a positive integer such that

$$H \subseteq \overline{D}_{lm}, \text{ for all } m \geq 1,\ l \geq N.$$

From Lemma 2.2, p.d.e. (2.8) has a unique solution $u^\kappa_{\alpha,lm} \in C^{1,2}([\kappa,1] \times \overline{D}_{lm})$ and

$$\|u^\kappa_{\alpha,lm}\|_{\infty;(\kappa,\ 1)\times D_{lm}} \leq e^{\frac{\theta\|r\|_\infty}{\alpha}},\ \forall\, \kappa > 0 \ \&\ m,l \geq 1.$$

Let $v^{lm}(\cdot,\cdot)$ be a minimizing selector in (2.8). Then the p.d.e. (2.8) can be casted as a parametric family of linear elliptic p.d.es given by

$$
\begin{aligned}
\frac{1}{2}\text{trace}(a(x)\nabla^2 u^\kappa_{\alpha,lm}) + b_{lm}(\theta,x)\cdot\nabla u^\kappa_{\alpha,lm} &= g_{lm}(\theta,x) \\
\nabla u^\kappa_{lm}\cdot\gamma &= 0 \text{ on } \partial D_{lm},
\end{aligned}
$$

where

$$
\begin{aligned}
b_{lm}(\theta,x) &= b(x, v^{lm}(\theta,x)), \\
g_{lm}(\theta,x) &= \alpha\theta\frac{\partial u^\kappa_{\alpha,lm}}{\partial\theta} - \theta r(x, v^{lm}(\theta,x)).
\end{aligned}
$$

Now from Lemma 2.2, it is easy to see that for each $\theta \in (0,\ 1]$,

$$\sup_{l,m}\{\|b_{lm}(\theta,\cdot)\|_{\infty,D_{lm}}, \|g_{lm}(\theta,\cdot)\|_{\infty,D_{lm}}\} < \infty.$$

Hence using Lemma 1.7, using the approximation procedure given in subsection 1.6, and using Lemma 2.2, we get

$$\|u^{\kappa}_{\alpha,lm}\|_{1,2,p;(\kappa,1)\times H} < K, \text{ for all } m \geq 1, l \geq N, p \geq 2, \tag{2.17}$$

where K does not depend on l and m. Now choose a sequence of bounded domains $\{H_n\}$ from D such that $\cup_n \overline{H}_n = D \cup \Gamma$ and $\partial D \cap \partial H_n$ is a C^2 portion of ∂D. Now by a standard diagonalization procedure there exists $u^{\kappa}_{\alpha,m} \in W^{1,2,p}_{loc}((\kappa,1)\times D\cup\Gamma)$ such that along a subsequence in $l \to \infty$,

$$u^{\kappa}_{\alpha,lm} \longrightarrow u^{\kappa}_{\alpha,m} \quad \text{weakly in } \; W^{1,2,p}((\kappa,1)\times H). \tag{2.18}$$

Now from (2.17), we have

$$\|u^{\kappa}_{\alpha,m}\|_{1,2,p;(\kappa,1)\times H} < K, \text{ for all } m \geq 1. \tag{2.19}$$

Now by repeating the diagonalization argument there exists $u^{\kappa}_{\alpha} \in W^{1,2,p}_{loc}((\kappa,1)\times D\cup\Gamma)$ such that along a subsequence in $m \to \infty$

$$u^{\kappa}_{\alpha,m} \longrightarrow u^{\kappa}_{\alpha} \quad \text{weakly in } \; W^{1,2,p}((\kappa,1)\times H). \tag{2.20}$$

Using parabolic version of the Morrey's lemma, see [[33], pp.26-27], $W^{1,2,p}((\kappa,1)\times H)$ is compactly embedded in $C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}([\kappa,1]\times\overline{H}), 0 < \hat{\alpha} < 2 - \frac{d+2}{p}$. Hence along a subsequence of $l \to \infty, m \to \infty$, we get

$$\lim_{m\to\infty}\lim_{l\to\infty} u^{\kappa}_{\alpha,lm} \;=\; u^{\kappa}_{\alpha} \;\text{ where the convergence is in } C^{\frac{\hat{\alpha}}{2},\hat{\alpha}}([\kappa,1]\times\overline{H}). \tag{2.21}$$

Now (2.21) implies (along a subsequence in $l, m \to \infty$)

$$\begin{aligned} &\lim_{m\to\infty}\lim_{l\to\infty}\inf_{v}\left[b(x,v)\cdot\nabla u^{\kappa}_{\alpha,m} + \theta r(x,v)u^{\kappa}_{\alpha,m}\right] \\ &= \inf_{v}\left[b(x,v)\cdot\nabla u^{\kappa}_{\alpha} + \theta r(x,v)u^{\kappa}_{\alpha}\right] \text{ in } [\kappa,1]\times\overline{H}. \end{aligned} \tag{2.22}$$

By letting (along a subsequence) $l \to \infty$ and then $m \longrightarrow \infty$ in (2.8), with the help of (2.18) and (2.22), we get

$$\begin{aligned} \alpha\theta\frac{\partial u^{\kappa}_{\alpha}}{\partial\theta} &= \inf_{v}\left[b(x,v)\cdot\nabla u^{\kappa}_{\alpha} + \theta r(x,v)u^{\kappa}_{\alpha}\right] \\ &\quad + \frac{1}{2}\operatorname{trace}(a(x)\nabla^2 u^{\kappa}_{\alpha}) \text{ in } (\kappa,1)\times D \end{aligned}$$

in the sense of distribution and $u^{\kappa}_{\alpha} \in W^{1,2,p}((\kappa,1)\times H)$ for any compact subset $\overline{H}$ of $\overline{D}$ with C^2 smooth boundary. Also from (2.21) it follows that $u^{\kappa}_{\alpha}(\kappa,x) = e^{\frac{\kappa\|r\|_{\infty}}{\alpha}}$.

Now we show that $\nabla u^{\kappa}_{\alpha}\cdot\gamma \equiv 0$ a.e. on ∂D. For $x \in \Gamma$, using the construction of D_{lm} it follows that one can choose $z_{lm} \in \partial D_{lm}$ such that $z_{lm} \to x$ as $l, m \to \infty$. Hence using the fact that $u^{\kappa}_{\alpha}(\theta,\cdot) \in C^1(D\cup\Gamma)$, (2.21) and continuity of γ, we get

$$0 \;=\; \lim_{l,m\to\infty} \nabla u^{\kappa}_{\alpha,lm}(\theta, z_{lm})\cdot\gamma(z_{lm}) \;=\; \nabla u^{\kappa}_{\alpha}(\theta,x)\cdot\gamma(x).$$

Since the surface measure of Γ' is zero, we have $\nabla u^{\kappa}_{\alpha}\cdot\gamma = 0$ a.e. on ∂D.

This proves that (2.7) has a solution $u^{\kappa}_{\alpha} \in W^{1,2,p}((\kappa,1)\times H) \cap C^{\hat{\alpha}/2,\hat{\alpha}}([\kappa,1]\times\overline{H})$, $p \geq 2$ for each bounded C^2 domain H in D. Hence $u^{\kappa}_{\alpha} \in W^{1,2,p}_{loc}((\kappa,1)\times D\cup\Gamma)\cap C^{0,0}([\kappa,1]\times\overline{D})$.

Following the arguments in [[27], Proposition 3.2], extend the function u_α^κ to whole of $[0,\ 1]$ as follows:

$$\bar{u}_\alpha^\kappa(\theta, x) = \begin{cases} u_\alpha^\kappa(\theta, x) & \text{if } \theta > \kappa \\ e^{\frac{\kappa \|r\|_\infty}{\alpha}} & \text{if } 0 \leq \theta \leq \kappa\ . \end{cases}$$

Then it follows that, $\bar{u}_\alpha^\kappa$ is nonnegative, bounded, continuous,

$$\sup_{0<\kappa<1} \left\| \frac{\partial \bar{u}_\alpha^\kappa}{\partial \theta} \right\|_{\infty;(0,1)\times D\cup\Gamma} < \infty.$$

and for each compact $\overline{H} \subset \overline{D}$ with C^2 boundary,

$$\sup_{0<\kappa<1} \|\bar{u}_\alpha^\kappa\|_{2,p;\overline{H}} < \infty,$$

for each $0 < \theta < 1$. The function $\bar{u}_\alpha^\kappa$ is a solution in almost everywhere sense to the following p.d.e

$$\left.\begin{aligned} \alpha\theta \frac{\partial \bar{u}_\alpha^\kappa}{\partial \theta} &= \inf_{v\in U} [b(x,v) \cdot \nabla \bar{u}_\alpha^\kappa + \theta r(x,v) \bar{u}_\alpha^\kappa] \\ &\quad + \frac{1}{2}\text{trace}(a(x) \nabla^2 \bar{u}_\alpha^\kappa) \\ &\quad - \theta e^{\frac{\kappa\|r\|_\infty}{\alpha}} \inf_{v\in U} \{r(x,v)\} I_{\{\theta\leq\kappa\}} \\ \bar{u}_\alpha^\kappa(0,x) &= 1, \quad \nabla \bar{u}_\alpha^\kappa . \gamma = 0 \text{ on } \partial D. \end{aligned}\right\} \tag{2.23}$$

Hence $\bar{u}_\alpha^\kappa \in W_{loc}^{1,2,p}((0,1)\times D\cup\Gamma) \cap C^{\hat{\alpha}/2,\hat{\alpha}}([\kappa,\ 1]\times\overline{H})$ for each bounded C^2 domain H in D, is a weak solution to (2.23). So multiply equation (2.23) with a test function $\hat{\phi} \in C_c^\infty((0,1)\times D)$ and integrate over $(0,1)\times D$, we get

$$\begin{aligned} -\alpha \int_0^1 \theta \left\langle \frac{\partial \bar{u}_\alpha^\kappa}{\partial \theta}, \hat{\phi} \right\rangle d\theta + \int_0^1 \left\langle \inf_{v\in U} \{b(x,v)\cdot \nabla \bar{u}_\alpha^\kappa + \theta r(x,v) \bar{u}_\alpha^\kappa\}, \hat{\phi} \right\rangle d\theta \\ + \frac{1}{2} \int_0^1 \langle \text{trace}(a(x)\nabla^2 \bar{u}_\alpha^\kappa), \hat{\phi} \rangle d\theta = \int_0^\kappa \left\langle \inf_{v\in U} \{\theta r(x,v) e^{\frac{\kappa\|r\|_\infty}{\alpha}}\}, \hat{\phi} \right\rangle d\theta, \end{aligned} \tag{2.24}$$

where $\langle \cdot, \cdot \rangle$ is inner product on $L^2(D)$. By letting $\kappa \longrightarrow 0$ in above, we obtain

$$\begin{aligned} -\alpha \int_0^1 \theta \left\langle \frac{\partial u_\alpha}{\partial \theta}, \hat{\phi} \right\rangle d\theta \quad &+ \quad \int_0^1 \left\langle \inf_{v\in U} \{b(x,v)\cdot \nabla u_\alpha + \theta r(x,v) u_\alpha\}, \hat{\phi} \right\rangle d\theta \\ &+ \quad \frac{1}{2} \int_0^1 \langle \text{trace}(a(x)\nabla^2 u_\alpha), \hat{\phi} \rangle d\theta = 0, \end{aligned}$$

where $u_\alpha \in W_{loc}^{1,2,p}((0,1)\times D\cup\Gamma) \cap C^{\hat{\alpha}/2,\hat{\alpha}}([\kappa,\ 1]\times\overline{H})$ for each bounded C^2 domain H in $D, p \geq 2$. Therefore we have

$$\begin{aligned} \alpha\theta \frac{\partial u_\alpha}{\partial \theta} &= \inf_{v\in U} [b(x,v)\cdot \nabla u_\alpha + \theta r(x,v) u_\alpha] + \frac{1}{2} trace(a(x)\nabla^2 u_\alpha) \\ u_\alpha(0,x) &= 1 \text{ on } \overline{D}. \end{aligned}$$

Let $\theta \in (0,1)$ and H be a domain in D with Lipschitz boundary such that its closure in $\overline{D}$ intersects only with Γ, smooth portion of ∂D. Clearly $\bar{u}_\alpha^\kappa(\theta, \cdot)$ and

$u_\alpha(\theta,\cdot) \in W^{2,p}(H)$. By Morrey Lemma, see [[21], pp. 335-339], $W^{2,p}(H)$ is compactly contained in $C^{1,\hat{\alpha}}(\overline{H})$. Hence for each fixed $\theta > 0$, we have

$$\bar{u}^{\kappa}_{\alpha}(\theta,\cdot) \longrightarrow u_\alpha(\theta,\cdot) \quad \text{in } C^{1,\hat{\alpha}}(\overline{H}).$$

This implies that $\nabla u_\alpha \cdot \gamma = 0$ on $\partial H \cap \partial D$ because $\nabla \bar{u}^{\kappa}_{\alpha} \cdot \gamma = 0$ on ∂D. Since choice of H is arbitrary, it follows that $\nabla u_\alpha \cdot \gamma = 0$ a.e. on ∂D.

Hence we have the existence of a weak solution $u_\alpha \in W^{1,2,p}_{loc}((0,1) \times D \cup \Gamma) \cap C^{0,0}([0,1] \times \overline{D}), p \geq 2$ for the equation (2.7). This completes the proof. □

Now we prove the existence of optimal control for the discounted risk-sensitive control problem. From [6], existence of a measurable minimizing selector in (2.7) follows.

Theorem 2.4. *The equation (2.7) has a unique solution* $u_\alpha \in W^{1,2,p}_{loc}((0,1) \times D \cup \Gamma) \cap C^{0,0}([0,1] \times \overline{D})$, $p \geq 2$, *given by*

$$u_\alpha(\theta, x) \;=\; \inf_{v(\cdot)\in\mathcal{A}} E^v_x\Big[e^{\theta \int_0^\infty e^{-\alpha s} r(X_s, v_s) ds}\Big].$$

Moreover if $v_\alpha(\cdot)$ *is a minimizing selector in (2.7), then* $v_\alpha(\cdot)$ *is optimal for the* α*-discounted risk-sensitive control problem.*

Proof. From the proof of Theorem 2.3 it is clear that for fixed $\theta > 0$, $\bar{u}^{\kappa}_{\alpha}(\theta,x) = u^{\kappa}_{\alpha}(\theta,x)$ for sufficiently small κ. Mimicking the arguments used to prove (2.13), we have the following stochastic representation

$$u^{\kappa}_{\alpha}(\theta,x) = \inf_{v(\cdot)} E^v_x\Big[e^{\frac{\kappa\|r\|_\infty}{\alpha}} e^{\int_0^{T_\kappa} \theta e^{-\alpha s} r(X_s,v_s)ds}\Big]\ ,$$

where $X(\cdot)$ is the process (1.1) corresponding to an admissible control $v(\cdot)$. Since $u^{\kappa}_{\alpha}(\theta,x) \longrightarrow u_\alpha(\theta,x)$ pointwise and $T_\kappa \to \infty$ as $\kappa \longrightarrow 0$ along a subsequence, using dominated convergence theorem, we get

$$u_\alpha(\theta,x) \leq E^v_x\Big[e^{\int_0^\infty \theta e^{-\alpha s} r(X_s,v_s)ds}\Big].$$

Since $v(\cdot)$ is an arbitrary admissible control, we have

$$u_\alpha(\theta,x) \leq \inf_{v(\cdot)} E^v_x\Big[e^{\int_0^\infty \theta e^{-\alpha s} r(X_s,v_s)ds}\Big].$$

In particular we get

$$u_\alpha(\theta,x) \leq E^v_x\Big[e^{\int_0^\infty \theta e^{-\alpha s} r(X_s,v_\alpha(\theta_s,X_s))ds}\Big],$$

where $v_\alpha(\cdot,\cdot)$ is a minimizing selector in (2.7), $\theta_s = \theta e^{-\alpha s}, s \geq 0, \theta > 0$. To prove the reverse inequality we argue as follows. The non-negativity of the function r implies $u^{\kappa}_{\alpha}(\theta,x) \geq 1$ and hence $u_\alpha(\theta,x) \geq 1$. Consider the following RSDE

$$\left.\begin{array}{rcl} dX(t) &=& b(X_t, v_\alpha(\theta_t,X_t))dt + \sigma(X_t)dW(t) - \gamma(X_t)d\xi(t) \\ d\xi(t) &=& I_{\{X_t\in\partial D\}}d\xi(t) \\ \xi(0) &=& 0, \quad X(0) = x \in \overline{D}. \end{array}\right\} \tag{2.25}$$

Choose a sequence of bounded domains $H_k \subseteq D \cup \Gamma, k \geq 1$ such that $H_k \subseteq H_{k+1}$ for all k and

$$\cup_k \overline{H}_k \;=\; D \cup \Gamma.$$

Consider the sequence of stopping times $\tau_k = \tau(H_k), k \geq 1$. Then using (A3), it follows that $\tau_k \to \infty$ a.s. and is also non decreasing.

Apply Itô-Krylov formula to $e^{\int_0^t \theta_s r(X_s, v_\alpha(\theta_s, X_s)ds}u_\alpha(\theta_t, X_t)$, we get

$$\begin{aligned}
& e^{\int_0^{T\wedge\tau_k} \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds}u_\alpha(\theta_{T\wedge\tau_k}, X_{T\wedge\tau_k}) \\
= \quad & u_\alpha(\theta, x) + \int_0^{T\wedge\tau_k} e^{\int_0^t \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} du_\alpha(\theta_t, X_t) \\
& + \int_0^{T\wedge\tau_k} u_\alpha(\theta_t, X_t) e^{\int_0^t \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} \theta_t r(X_t, v_\alpha(\theta_t, X_t))dt,
\end{aligned}$$

where

$$\begin{aligned}
du_\alpha(\theta_t, X_t) \quad = \quad & (\nabla u_\alpha(\theta_t, X_t))^\perp \sigma(X_t) I\{X_t \in \partial D\} dW(t) - \alpha\theta_t \frac{\partial}{\partial\theta} u_\alpha(\theta_t, X_t)dt \\
& + \Big[\nabla u_\alpha(\theta_t, X_t) \cdot b(X_t, v_\alpha(\theta_t, X_t)) + \frac{1}{2}\text{trace}(a(X_t)\nabla^2 u_\alpha(\theta_t, X_t))\Big] dt \\
& - [\gamma(X_t) \cdot \nabla u_\alpha(\theta_t, X_t)] \, I_{\{X_t \in \partial D\}} d\xi_t.
\end{aligned}$$

Using the fact that u_α satisfy the equation (2.7), we get

$$\begin{aligned}
& e^{\int_0^{T\wedge\tau_k} \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds}u_\alpha(\theta_{T\wedge\tau_k}, X_{T\wedge\tau_k}) \\
= \quad & u_\alpha(\theta, x) + \int_0^{T\wedge\tau_k} e^{\int_0^t \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} (\nabla u_\alpha(\theta_t, X_t))^\perp \sigma(X_t) dW(t).
\end{aligned}$$

Since ∇u_α is continuous on $\overline{H}_k$ by the Sobolev embedding Theorem, therefore ∇u_α is bounded on $\overline{H}_k$, which implies that the stochastic integral

$$\int_0^{T\wedge\tau_k} e^{\int_0^t \theta_s r(X_s, v_\alpha(X_s))ds} (\nabla u_\alpha(\theta_t, X_t))^\perp \sigma(X_t) dW(t)$$

is a zero mean martingale for each k. Hence we get

$$u_\alpha(\theta, x) = E_x^v \left[e^{\int_0^{T\wedge\tau_k} \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} u_\alpha(\theta_{T\wedge\tau_k}, X_{T\wedge\tau_k}) \right].$$

Letting $k \to \infty$, we get

$$u_\alpha(\theta, x) = E_x^v \left[e^{\int_0^{T} \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} u_\alpha(\theta_T, X_T) \right] \geq E_x^v \left[e^{\int_0^{T} \theta_s r(X_s, v_\alpha(\theta_s, X_s))ds} \right].$$

Now taking $T \to \infty$, we obtain

$$u_\alpha(\theta, x) \geq E_x^v \left[e^{\int_0^{\infty} \theta e^{-\alpha s} r(X_s, v_\alpha(\theta_s, X_s))ds} \right].$$

Thus,

$$u_\alpha(\theta, x) \quad = \quad \inf_{v(\cdot)\in\mathcal{A}} E_x^v \left[e^{\theta \int_0^\infty e^{-\alpha s} r(X_s, v_s)ds} \right] \quad = \quad E_x^v \left[e^{\theta \int_0^\infty e^{-\alpha s} r(X_s, v_\alpha(\theta_s, X_s))ds} \right].$$

This proves $v_\alpha(\cdot, \cdot)$ is optimal and u_α is the unique solution to the equation (2.7), which completes the proof. □

3. Risk-sensitive control with near monotone cost

In this section we prove existence of optimal control for the risk-sensitive control problem described in Section 1, under a structural condition on the cost function $r(x, v)$, called "near monotonicity". We also use an additional assumption that the process given by (1.1) is recurrent for each admissible control. Let $X(\cdot)$ be the process (1.1) corresponding to the admissible control $v(\cdot)$. For any domain $O \subset \overline{D}$, recall that $\tau(O)$ denotes the first exit time of the process $X(\cdot)$ from O, i.e.,

$$\tau(O) = \inf\{t > 0 : X(t) \notin O\}.$$

Definition 3.1. Let $X(\cdot)$ be the process given by (1.1) corresponding to an admissible control $v(\cdot)$ with initial condition x. We say controlled process $X(\cdot)$ is *recurrent*, if for any domain $O \subset \overline{D}$ the first hitting time of the set O, i.e., $\tau(O^c)$, satisfies $P(\tau(O^c) < \infty) = 1$, for all $x \in \overline{D}$. If $E[\tau(O^c)] < \infty$ for all $x \in \overline{D}$, then $X(\cdot)$ is said to be positive recurrent. Correspondingly, the control $v(\cdot)$ is called a stable control. We denote the set of stable, stationary Markov controls by $\mathcal{M}_s$.

Define the optimal risk-sensitive values as follows

$$\beta \;=\; \inf_{v(\cdot)\in\mathcal{A}} \limsup_{T\longrightarrow\infty} \frac{1}{T\theta} \ln E_x^v \left[e^{\theta\int_0^T r(X_t,v_t)dt} \right].$$

Though in principle, β can depend on x, we are suppressing the dependence, since under our assumption β is independent of x which we will see later.

Now we state the near-monotonicity assumption.

(A4) The cost function r satisfy the following

$$\liminf_{|x|\longrightarrow\infty} \inf_{v\in U} r(x, v) > \beta, \;\; x \in \overline{D}, \tag{3.1}$$

i.e., r is near monotone with respect to β.

Also we use the following recurrent condition.

(A5) For each stationary Markov control $v(\cdot)$, the corresponding the process $X(\cdot)$ given by (1.1) is recurrent.

See Lemma 1.15 for a characterization of (A5).

In fact, we see at the end of this section that (A4) is non existent without (A5) for most interesting situations.

We adapt the vanishing discount approach to prove the existence of optimal risk-sensitive ergodic control under the near-monotonicity assumption. To prove existence of solution for risk-sensitive ergodic HJB, we study the limiting behaviour of the equation (2.7) as $\alpha \longrightarrow 0$.

Theorem 3.2. *There exist a solution* $(\rho, \hat{u}) \in \mathbb{R} \times W^{2,p}_{loc}(D \cup \Gamma) \cap C^0(\overline{D})$ *to the equation*

$$\left.\begin{aligned} \theta\rho\hat{u} &= \inf_{v\in U} [b(x, v)\cdot\nabla\hat{u} + \theta r(x, v)\hat{u}] + \frac{1}{2} trace(a(x)\nabla^2\hat{u}) \\ \nabla\hat{u}.\,\gamma &= 0 \text{ on } \partial D, \;\; \hat{u}(x_0) = 1\,. \end{aligned}\right\} \tag{3.2}$$

Moreover

$$\rho \le \beta.$$

Proof. For $k \geq 1$, let χ_k denote a nonnegative smooth function such that $\chi_k \equiv 1$ in B_k, $\chi_k \equiv 0$ in B^c_{k+1} and $0 \leq \chi_k \leq 1$. Let $r_k = r\chi_k$. Then

$$\|r_k\|_\infty \leq \|r\|_{\infty, B_{k+1}}.$$

Define for $\alpha > 0$

$$u^k_\alpha(\theta, x) := \inf_{v(\cdot)\in\mathcal{A}} E^v_x \left[e^{\theta \int_0^\infty e^{-\alpha t} r_k(X_t, v_t) dt} \right]. \tag{3.3}$$

Consider the p.d.e.

$$\left.\begin{array}{rcl} \alpha\theta \frac{\partial u^k_\alpha}{\partial\theta} &=& \inf_{v\in U} \left[b(x,v)\cdot\nabla u^k_\alpha + \theta r_k(x,v) u^k_\alpha \right] \\ && + \frac{1}{2}\, trace(a(x)\nabla^2 u^k_\alpha) \\ \nabla u^k_\alpha \cdot \gamma(x) &=& 0 \text{ on } \partial D,\ u^k_\alpha(0,x) = 1\,. \end{array}\right\} \tag{3.4}$$

Mimicking the arguments as in Theorem 2.3 and 2.4, one can see that p.d.e. (3.4) has a unique solution u^k_α in $W^{1,2,p}_{loc}((0,1)\times D\cup\Gamma)\cap C^{0,0}([0,\ 1]\times\overline{D}),\ p\geq 2$, and u^k_α has the representation (3.3).

Set

$$\phi^k_\alpha(\theta,x) = \frac{1}{\theta}\ln u^k_\alpha(\theta,x),\ g^k_\alpha(\theta,x) = \alpha\phi^k_\alpha + \alpha\theta\frac{\partial\phi^k_\alpha}{\partial\theta} \tag{3.5}$$

Mimicking the arguments as in [[8], Lemma 2.1] we have

$$\|\alpha\phi^k_\alpha\|_\infty + \left\|\alpha\theta\frac{\partial\phi^k_\alpha}{\partial\theta}\right\|_\infty \leq 3\|r_k\|_\infty,\ \forall\, 0<\alpha<1,\ 0<\theta\leq 1. \tag{3.6}$$

Let $\tau = \tau(B^c_{k+1})$, i.e. the hitting time of the process (1.1) to the set B_{k+1} under the admissible control $v(\cdot)\in\mathcal{A}$. For $x\in B^c_{k+1}$, dynamic programming principle (2.5) implies

$$\begin{aligned} u^k_\alpha(\theta,x) &= \inf_{v(\cdot)} E^v_x\left[e^{\theta\int_0^\tau e^{-\alpha t}\, r_k(X_t,v_t)dt} u^k_\alpha\left(\theta e^{-\alpha\tau}, X_\tau\right)\right] \\ &= \inf_{v(\cdot)} E^v_x\left[u^k_\alpha(\theta e^{-\alpha\tau}, X_\tau)\right] \quad (\because r_k\equiv 0 \text{ on } B^c_{k+1}) \\ &\leq \inf_{v(\cdot)} E^v_x\left[u^k_\alpha(\theta, X_\tau)\right] \quad (\because e^{-\alpha\tau}<1 \text{ a.s. and } u^k_\alpha \text{ is increasing in } \theta) \\ &\leq \sup_{y\in\partial B_{k+1}\cap\overline{D}} u^k_\alpha(\theta,y). \end{aligned}$$

Rewrite the equation (3.4) as the following parametric family of elliptic pdes

$$\left.\begin{array}{rcl} 0 &=& \mathcal{L}u^k_\alpha(\theta,x,v^{\alpha,k}(\theta,x)) + \theta(r_k(x,v^{\alpha,k}(\theta,x)) - g^k_\alpha)u^k_\alpha \\ \nabla u^k_\alpha\cdot\gamma(x) &=& 0 \text{ on } \partial D,\ u^k_\alpha(0,x)=1\,, \end{array}\right\} \tag{3.7}$$

where $v^{\alpha,k}(\cdot,\cdot)$ denote a minimizing selector of (3.4). Using the weak Harnack's inequality, Theorem 1.8, we have

$$\sup_{x\in B_{k+1}\cap\overline{D}} u^k_\alpha(\theta,x) \leq K_{4\cdot 1}(k), \tag{3.8}$$

where $K_{4\cdot 1}(k)$ is independent of α.

Set

$$\bar{u}_\alpha^k(\theta, x) := \frac{u_\alpha^k(\theta, x)}{u_\alpha^k(\theta, x_0)} \quad \text{for some } x_0 \in D.$$

Then $\bar{u}_\alpha^k$ is a solution to

$$\begin{aligned} 0 &= \inf_{v \in U} \left[\mathcal{L}\bar{u}_\alpha^k(\theta, x, v) + \theta(r_k(x, v) - g_\alpha^k)\bar{u}_\alpha^k \right] \\ \nabla \bar{u}_\alpha^k \cdot \gamma(x) &= 0 \text{ on } \partial D, \ \bar{u}_\alpha^k(\theta, x_0) = 1\,. \end{aligned} \tag{3.9}$$

From (3.8) it follows that

$$\sup_{x \in B_{k+1} \cap \overline{D}} \bar{u}_\alpha^k(\theta, x) \leq K_{4\cdot 1}(k).$$

But the foregoing arguments show that for $x \in B_{k+1}^c$,

$$\bar{u}_\alpha^k(\theta, x) \leq \sup_{y \in \partial B_{k+1} \cap \overline{D}} \left[\frac{u_\alpha^k(\theta, y)}{u_\alpha^k(\theta, x_0)} \right] \leq K_{4\cdot 1}(k),$$

where $K_{4\cdot 1}(k)$ can be chosen independent of x, α. Now using the approximation arguments given in subsection 1.6, we have for each $R < k+1$

$$\|\bar{u}_\alpha^k(\theta, \cdot)\|_{2,p;B_R \cap (D \cup \Gamma)} \leq K_{4\cdot 2}, \tag{3.10}$$

where $K_{4\cdot 2} > 0$ is independent of $\alpha > 0$. Now using compact and continuous Sobolev embedding theorems, for each fixed $\theta > 0$, without loss of generality $\theta = 1$, there exists $\hat{u}^k \in W_{loc}^{2,p}(D \cup \Gamma)$ such that

$$\begin{aligned} \bar{u}_\alpha^k(1, \cdot) &\longrightarrow \hat{u}^k \text{ strongly in } W_{loc}^{1,p}(D \cup \Gamma), \\ \bar{u}_\alpha^k(1, \cdot) &\longrightarrow \hat{u}^k \text{ weakly in } W_{loc}^{2,p}(D \cup \Gamma), \end{aligned}$$

along a subsequence as $\alpha \downarrow 0$. By Sobolev embedding theorem, the convergence is uniform on compact subsets of $D \cup \Gamma$, hence we have $\hat{u}^k$ is bounded above by $K_{4\cdot 1}(k)$. Now we show that

$$g_\alpha^k(1, x) \longrightarrow \rho_k \in \mathbb{R}.$$

From (3.6), along a further subsequence,

$$\alpha \phi_\alpha^k(\theta, x) \longrightarrow \rho_1^k(\theta, x), \quad \text{in weak* topology of } L^\infty((0,\,1) \times D). \tag{3.11}$$

We show that ρ_1^k is a function of θ alone. From (3.10) and

$$\frac{1}{\theta} \nabla \ln \bar{u}_\alpha^k(\theta, x) = \nabla \phi_\alpha^k(\theta, x),$$

we have for any $R > 0$

$$\|\nabla \phi_\alpha^k(\theta, \cdot)\|_{1,p;(D \cup \Gamma) \cap B_R} \leq K_{4\cdot 3}, \tag{3.12}$$

where $K_{4\cdot 3} > 0$ is independent of $\alpha > 0$. By (3.12),

$$\lim_{\alpha \downarrow 0} \int_D \alpha \nabla \phi_\alpha^k(\theta, x) f(x) = 0,$$

for each $f \in C_c^\infty(D)$. Thus the distributional derivative of ρ_1^k in x is identically zero, proving the claim. Also by (3.6), for each fixed $\theta = \theta_0 > 0$, $\left\{ \alpha \frac{\partial \phi_\alpha^k}{\partial \theta} \Big| \alpha > 0 \right\}$ is bounded in $L^\infty([\theta_0, 1] \times D)$. Hence along a further subsequence

$$\alpha \frac{\partial \phi_\alpha^k}{\partial \theta} \longrightarrow \rho_2^k(\theta, x), \quad \text{weakly in } L^2_{loc}([\theta_0, 1] \times D). \tag{3.13}$$

It follows from (3.11) and (3.13) that $\rho_2^k(\cdot,\cdot) = (\rho_1^k)'$ in the sense of distribution, where $(\rho_1^k)'$ is the distributional derivative (in θ) of ρ_1^k. Hence $\rho_2^k(\cdot,\cdot)$ is also a function of θ alone. Thus we have: for each $\theta > 0$ there exists a constant ρ_k such that along a subsequence

$$\alpha \phi_\alpha^k + \alpha \theta \frac{\partial \phi_\alpha^k}{\partial \theta} \longrightarrow \rho_k.$$

Now letting $\alpha \longrightarrow 0$ in (3.9) along the subsequence, we have $(\rho_k, \hat{u}^k) \in \mathbb{R} \times W^{2,p}_{loc}(D \cup \Gamma)$ satisfying the following equation ($\theta = 1$)

$$\left.\begin{aligned} \rho_k \hat{u}_k &= \inf_{v \in U} [b(x,v) \cdot \nabla \hat{u}_k + r_k(x,v) \hat{u}_k] + \frac{1}{2} trace(a(x) \nabla^2 \hat{u}_k) \\ \nabla \hat{u}_k \cdot \gamma(x) &= 0 \text{ on } \partial D, \ \hat{u}_k(x_0) = 1\,. \end{aligned}\right\} \tag{3.14}$$

For $n \geq 1$, let $\tau_n = \tau(H_n)$, where $H_n \subseteq D \cup \Gamma, n \geq 1$ is an increasing sequence of bounded domains such that

$$\cup_n \overline{H}_n = D \cup \Gamma.$$

Applying Itô-Krylov formula to the process (1.1) corresponding to $v(\cdot) \in \mathcal{A}$,

$$\begin{aligned} d\left(e^{\int_0^t (r_k(X_s,v_s) - \rho_k) ds} \ \hat{u}^k(X_t)\right) &= e^{\int_0^t (r_k(X_s,v_s) - \rho_k) ds} \ d\Big(\hat{u}^k(X_t)\Big) \\ &+ \ (r_k(X_t, v_t) - \rho_k) e^{\int_0^t (r_k(X_s,v_s) - \rho_k) ds} \ \hat{u}^k(X_t) \ dt, \end{aligned}$$

where

$$\begin{aligned} d\left(\hat{u}^k(X_t)\right) = &\left[b(X_t, v_t)) \cdot \nabla \hat{u}^k(X_t) + \frac{1}{2} \text{trace}(a(X_t) \nabla^2 \hat{u}^k(X_t))\right] I\{X_t \in \partial D\} dt \\ &- \Big(\gamma(X_t) \cdot \nabla \hat{u}^k(X_t)\Big) I_{\{X_t \in \partial D\}} d\xi_t + (\nabla \hat{u}^k(X_t))^\perp \sigma(X_t) dW_t, \\ &t \leq \tau_n, n \geq 1. \end{aligned}$$

Hence it follows that

$$\begin{aligned} e^{\int_0^{T \wedge \tau_n} (r_k(X_s,v_s) - \rho_k) ds} \hat{u}^k(X_{T \wedge \tau_n}) &- \ \hat{u}^k(x) \\ &\geq \int_0^{T \wedge \tau_n} e^{\int_0^t (r_k(X_s,v_s) - \rho_k) ds} \nabla \hat{u}^k(X_t)^\perp \sigma(X_t) dW_t. \end{aligned} \tag{3.15}$$

Since $\hat{u}^k \in W^{2,p}_{loc}(D \cup \Gamma), p \geq d$, we have $\nabla \hat{u}^k$ is bounded on $\overline{H}_n$, and hence

$$\int_0^{T \wedge \tau_n} e^{\int_0^t (r_k(X_s,v_s) - \rho_k) ds} (\nabla \hat{u}^k(X_t)) \sigma(X_t) dW_t,$$

is a zero mean martingale. Taking expectation in (3.15) we obtain

$$E_x^v \left[e^{\int_0^{T \wedge \tau_n} (r_k(X_s,v_s) - \rho_k) ds} \hat{u}^k(X_{T \wedge \tau_n})\right] - \hat{u}^k(x) \geq 0.$$

Since $\hat{u}^k$ is bounded above, we have

$$\hat{u}^k(x) \leq K_{4\cdot 1}(k) E_x^v \left[e^{\int_0^{T\wedge\tau_n} (r_k(X_s,v_s)-\rho_k)ds} \right] \leq K_{4\cdot 1}(k) E_x^v \left[e^{\int_0^{T} (r_k(X_s,v_s)-\rho_k)ds} \right].$$

Taking ln and divide by T we get

$$\frac{1}{T}\ln \hat{u}^k(x) \leq \frac{\ln K(k)}{T} + \frac{1}{T}\ln E_x^v \left[e^{\int_0^{T} (r_k(X_s,v_s)-\rho_k)ds} \right].$$

Since $u_\alpha^k \geq 1$, by definition $\bar{u}_\alpha^k$ is bounded below, hence uniform convergence on compact sets gives that $\hat{u}^k$ is bounded below say by $K_{4\cdot 3}(k) > 0$. Therefore

$$\frac{1}{T}\ln K_{4\cdot 3}(k) \leq \frac{\ln K(k)}{T} + \frac{1}{T}\ln E_x^v \left[e^{\int_0^{T} r_k(X_s,v_s ds} \right] - \rho_k.$$

Now taking $T \longrightarrow \infty$ we get

$$\rho_k \leq \limsup_{T\longrightarrow\infty} \frac{1}{T}\ln E_x^v \left[e^{\int_0^{T} r_k(X_s,v_s)ds} \right].$$

Since $|r_k| \leq |r|$,

$$\rho_k \leq \limsup_{T\longrightarrow\infty} \frac{1}{T}\log E_x^v \left[e^{\int_0^{T} r(X_s,v_s)ds} \right].$$

Taking infimum over all admissible controls in the right hand side of above, we get

$$\rho_k \leq \beta, \ \forall\, k. \tag{3.16}$$

Since the coefficients of (3.14) are bounded, by repeating the arguments given after eq. (3.6), we can use the Harnack's inequality Theorem 1.8 to show that $|\hat{u}^k|$ is bounded uniformly in k. Thus we have $\hat{u}^k \longrightarrow \hat{u}$ in $W^{1,p}_{loc}(D \cup \Gamma)$ and $\rho_k \longrightarrow \rho$ along a subsequence. Furthermore, it follows from Harnack's inequality that $\hat{u} > 0$ on compacts, in fact one has uniform positive lower bounds for $\hat{u}^k$ on compacts. Letting $k \longrightarrow \infty$ in (3.14), by repeating the argument as in Theorem 2.3, it follows that $(\rho, \hat{u})$ satisfy

$$\left.\begin{aligned} \rho\hat{u} &= \inf_{v\in U}\left[b(x,v)\cdot\nabla\hat{u} + r(x,v)\hat{u}\right] + \frac{1}{2}trace(a(x)\nabla^2\hat{u}) \\ \nabla\hat{u}\cdot\gamma(x) &= 0 \text{ on } \partial D,\ \hat{u}(x_0) = 1\,, \end{aligned}\right\} \tag{3.17}$$

In view of (3.16) it follows that $\rho \leq \beta$, which completes the proof. □

Theorem 3.3. *The ergodic risk-sensitive HJB equation (3.2) has a solution* $(\rho, \hat{\phi})$ *such that* $\rho = \beta$. *Also, minimizing selector in (3.2) is an optimal control.*

Proof. In view of Theorem 3.2 it remains to show $\beta \leq \rho$. By assumption (3.1) we have

$$\inf_v r(\cdot, v) > \beta > \rho \quad \text{outside } O \subset \overline{D},$$

for some bounded open O. We know that for some $\nu > 0$, $\hat{u} \geq \nu > 0$ in O. Let $x \in O^c \cap \overline{D}$. Set $T_n = n \wedge \tau_n$, n is chosen sufficiently large. Let $v^*(\cdot)$ be minimizing selector in (3.2), applying Itô-Krylov's formula

$$\begin{aligned} & e^{\int_0^{\tau(O^c)\wedge T_n}(r(X_s,v^*(X_s))-\rho)ds}\hat{u}(X_{\tau(O^c)\wedge T_n}) - \hat{u}(x) \\ = & \int_0^{\tau(O^c)\wedge T_n} e^{\int_0^t (r(X_s,v^*(X_s))-\rho)ds}(\nabla\hat{u}(X_t))^{\perp}\sigma(X_t)dW_t. \end{aligned}$$

Since $\hat{u} \in W^{2,p}_{loc}(D \cup \Gamma)$, it follows that $\nabla \hat{u}$ is locally bounded and using the boundedness of r, σ,

$$\int_0^{t \wedge \tau(O^c) \wedge T_n} e^{\int_0^{t'} (r(X_s, v^*(X_s)) - \rho) ds} (\nabla \hat{u}(X_{t'})) \sigma(X_{t'}) dW_{t'},$$

is zero mean martingale. Hence we have

$$E^v_x \left[e^{\int_0^{\tau(O^c) \wedge T_n} (r(X_s, v^*(X_s)) - \rho) ds} \hat{u}(X_{\tau(O^c) \wedge T_n}) \right] - \hat{u}(x) = 0$$

Using the Fatou's lemma, letting $n \longrightarrow \infty$ we get

$$\hat{u}(x) \geq E^v_x \left[e^{\int_0^{\tau(O^c)} (r(X_t, v^*(X_t)) - \rho) dt} \hat{u}(X_{\tau(O^c)}) \right].$$

Using (A5), it follows that $\tau(O^c) < \infty$ a.s. Hence

$$E^v_x \left[e^{\int_0^{\tau(O^c)} (r(X_t, v^*(X_t)) - \rho) dt} \hat{u}(X_{\tau(O^c)}) \right] \geq \nu.$$

This proves that $\hat{u}$ is bounded below by ν. Repeating the previous argument, we also have for any $T > 0$,

$$\hat{u}(x) \geq E^v_x \left[e^{\int_0^T (r(X_t, v^*(X_t)) - \rho) dt} \hat{u}(X_T) \right] \geq \nu E^v_x \left[e^{\int_0^T (r(X_t, v^*(X_t)) - \rho) dt} \right].$$

Taking logarithm and dividing by T

$$\frac{1}{T} \ln E^v_x \left[e^{\int_0^T (r(X_t, v^*(X_t)) - \rho) dt} \right] + \frac{1}{T} \ln \nu \leq \frac{1}{T} \ln \hat{u}(x).$$

Letting $T \longrightarrow \infty$ on both sides, we have

$$\limsup_{T \longrightarrow \infty} \frac{1}{T} \ln E^v_x \left[e^{\int_0^T (r(X_t, v^*(X_t)) - \rho) dt} \right] \leq 0.$$

i.e.,

$$\limsup_{T \longrightarrow \infty} \frac{1}{T} \ln E^v_x \left[e^{\int_0^T r(X_t, v^*(X_t)) dt} \right] \leq \rho.$$

Thus $\beta \leq \rho$. This completes the proof of the theorem. □

3.1. **Multiplicative Poisson equation.** The main aim of this subsection is to indicate that it may be useless to remove (A4). i.e., we give an example of a transient uncontrolled RSDE in which the near monotone condition becomes meaningless. This example in fact point to the fact that it could be the case for general uncontrolled transient diffusions though we don't have a proof.

To this end, along the lines of [2], we can prove the following lemma. Frame work is uncontrolled, so we consider the uncontrolled RSDE given by

$$\begin{aligned} dX_t &= b(X_t)dt + \sigma(X_t)dW_t - \gamma(X_t)d\xi_t, \\ d\xi_t &= I_{\{X_t \in \partial D\}} d\xi_t, \\ \xi_0 &= 0, \quad X_0 = x \in \overline{D} \end{aligned} \tag{3.18}$$

and the corresponding multiplicative Poisson equation is

$$\begin{aligned} \rho \hat{u} &= \mathcal{L}\hat{u} + r(x)\hat{u} \text{ in } D \\ \nabla \hat{u} \cdot \gamma(x) &= 0 \text{ on } \partial D. \end{aligned} \tag{3.19}$$

Lemma 3.4. *Let r be near monotone with respect to ρ and $(\rho, \hat{u}) \in W^{2,d}_{loc}(D \cup \Gamma) \cap C^0(\overline{D})$ be a nonnegative solution to (3.19) satisfying $\hat{u}(x_0) > 0$ for some x_0. Then the following are equivalent.*

1. *The RSDE (3.18) is recurrent.*
2. $\inf_{\overline{D}} \hat{u} > 0$.
3. *The function $\hat{u}$ is inf-compact.*
4. *RSDE (3.18) is geometrically ergodic.*

Moreover any one of the above implies that $\beta \leq \rho$.

Now we will provide an example which indicates that removal of the recurrent condition makes the near monotone condition impossible to hold.

Example 3.5. Let D be the positive quadrant and $X(\cdot)$ denote a Brownian motion on D with direction of reflection given by $\gamma(x) = -(1, 1), x \in \partial D$. Then $(X(\cdot), Z(\cdot))$ can be seen as a unique strong solution to the reflecting Brownian motion given by

$$X(t) \; = x + W(t) - RZ(t), t \geq 0,$$

where

$$R \;=\; \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

and $W(\cdot)$, 2-dimensional standard Wiener process. Then it follows from Theorem 3.1 of [32] that $X(\cdot)$ is transient.

Choose r as follows.

- r increases in $\|x\|$ and $r(x) < \|r\|_\infty$ for all x.
- $\lim_{\|x\| \to \infty} r(x) = \|r\|_\infty$.

In particular r is near monotone with respect to $\lambda < \|r\|_\infty$. Set

$$\beta = \limsup_{t \to \infty} \frac{1}{T} \ln E_x \left[e^{\int_0^T r(X_t) dt} \right]$$

and

$$\mathcal{L}_0 \;=\; \frac{1}{2}\Delta + r.$$

Consider the eigenvalue problem associated with $\mathcal{L}_0$ with the oblique boundary condition γ defined above.

$$\begin{aligned} \mathcal{L}_0 \hat{u} \;&=\; \lambda \hat{u} \text{ in } D, \\ \nabla \hat{u} \cdot \gamma(x) \;&=\; 0 \text{ on } \partial D, \;\; \hat{u}(x_0) = 1, \end{aligned} \tag{3.20}$$

where $x_0 \in D$ is fixed. Along the lines of [10], we define generalized principal eigenvalues as follows.

$$\begin{aligned} \lambda_1(-\mathcal{L}_0, D) \;=\; \inf \Big\{ & \lambda \in \mathbb{R} | \exists\, \varphi \in W^{2,2}_{loc}(D), \varphi > 0, \mathcal{L}_0 \varphi \leq \lambda \varphi \text{ a.e. } D, \\ & \nabla \varphi \cdot \gamma \geq 0 \text{ a.e. on } \partial D \Big\}. \end{aligned} \tag{3.21}$$

It is easy to see from the definitions that

$$\lambda_1(-\mathcal{L}_0, D) \; \leq \|r\|_\infty. \tag{3.22}$$

First we show the existence of principal eigenvalue. Let H_n be an increasing sequence of bounded C^2 domains in $D \cup \Gamma$ such that

$$\cup_n \overline{H}_n = D \cup \Gamma.$$

First we state the following theorem on eigenvalue problem for mixed boundary conditions which is a restatement of Proposition 2.3 of [19].

Theorem 3.6. *For $n \geq 1$, there exists a unique pair $(\lambda_n, \varphi_n) \in \mathbb{R} \times W^{2,p}(H_n) \cap C^0(\bar{H}_n), p \geq d$ such that*

$$\begin{aligned} \mathcal{L}_0 \varphi_n &= \lambda_n \varphi_n \text{ in } H_n \\ \nabla \varphi_n \cdot \gamma &= 0 \text{ on } \partial D \cap \partial H_n,\ \varphi_n \ = \ 0 \text{ on } \partial H_n \setminus (\partial D \cap \partial H_n). \end{aligned} \tag{3.23}$$

A simple application of Itô-Krylov's formula implies that

$$\lambda_n \ = \ \limsup_{T \to \infty} \frac{1}{T} \ln E\Big[e^{\int_0^{T \wedge \tau_n} r(X_t) dt}\Big], \tag{3.24}$$

where $\tau_n \ = \ \tau(H_n)$ This implies that λ_n is monotonically increasing and

$$\lambda_n \leq \beta.$$

Let $\tilde{\lambda}$ be the limit of λ_n's. Now repeating the arguments as in the proof of Theorem 3.2 with the help of Harnack's inequality given in Theorem 1.8, it follows that ϕ_n has a limit point $\varphi \in W^{2,p}_{loc}(D \cup \Gamma) \cap C^0(\bar{D}), p \geq d, \varphi > 0$ and $(\tilde{\lambda}, \varphi)$ satisfies (3.20). This in particular implies that

$$\tilde{\lambda} \geq \lambda_1(-\mathcal{L}_0, D).$$

Now straightford mimicking of the arguments in the proof of Proposition 1 of [10] it follows that

$$\lambda_n \leq \lambda_1(-\mathcal{L}_0, D), \ \forall \ n.$$

Thus we have

$$\tilde{\lambda} = \lambda_1(-\mathcal{L}_0, D).$$

i.e.

$$\lambda_1(-\mathcal{L}_0, D) \leq \beta \leq \|r\|_\infty. \tag{3.25}$$

Now suppose

$$\lambda_1(-\mathcal{L}_0, D) < \|r\|_\infty.$$

Then r is near monotone with respect to $\lambda_1(-\mathcal{L}_0, D)$. Therefore using Lemma 3.4, it follows that φ is not inf compact and $\inf_D \varphi = 0$. In fact by closely mimicking of the arguments in the proof of [[23], Theorem 3.1] , one can show that φ is bounded.

Using [[28], Theorem 6.1], it follows that

$$|X_t| \leq |x| + K \max_{0 \leq s \leq t} |W(s)|, t \geq 0, \tag{3.26}$$

for some $K > 0$. Hence using Doob's maximal inequality, it follows that

$$\lim_{t \to \infty} \frac{1}{t} E|X_t| \ = \ 0. \tag{3.27}$$

Using the arguments given in subsection 1.7 to prove (1.9), we can show that

$$\|\varphi\|_{2,p;B(x,1) \cap D} \ \leq K \|\varphi\|_{\infty; B(x,1) \cap D}, \tag{3.28}$$

where $K > 0$ is independent of $x \in D$. Now using Sobolev imbedding theorem and Harnack's inequality,Thereon 1.8 , we get

$$\sup_{y\in B(x,1)} |\nabla\varphi(y)| \leq K_1\varphi(x)$$

Hence we get

$$\varphi(x) \geq e^{-K_1(1+|x|)},\ x \in D. \tag{3.29}$$

Now by an application of Itô's formula, we get

$$\varphi(x) \ = \ E_x\Big[e^{\int_0^T r(X_t)-\lambda_1(-\mathcal{L}_0,D))dt}\varphi(X_T)\Big],\ T > 0.$$

Using Jensen's inequality, we get

$$\frac{1}{T}\int_0^T r(X_t)dt - \lambda_1(-\mathcal{L}_0, D) + \frac{1}{T}E_x[\ln\varphi(X_T)] \leq \frac{1}{T}\ln\varphi(x).$$

Now by taking $T \to \infty$ in view of (3.27) and (3.29) we get

$$\lim_{T\to\infty}\frac{1}{T}E_x\Big[\int_0^T r(X_t)dt\Big] \ \leq \ \lambda_1(-\mathcal{L}_0, D).$$

Since $|X_t| \to \infty$ a.s as $t \to \infty$, using dominated convergence theorem, we get

$$\|r\|_\infty \ = \ \lim_{T\to\infty}\frac{1}{T}E_x\Big[\int_0^T r(X_t)dt\Big].$$

Hence $\lambda_1(-\mathcal{L}_0, D) = \beta \ = \ \|r\|_\infty$. Thus we have proved the following.

Theorem 3.7. *There exists* $\varphi \in W^{2,p}_{loc}(D \cup \Gamma) \cap C^0(\bar{D}), p \geq d,\ \varphi > 0$ *such that the pair* $(\lambda_1(-\mathcal{L}_0, D), \varphi)$ *solves the eigen value problem (3.20). More over* $\lambda_1(-\mathcal{L}_0, D) = \beta = \|r\|_\infty$.

The above theorem implies that since $X(\cdot)$ is transient, there exists no continuous function which is near monotone with respect to β. We would like to conjecture that this is indeed the case for any non degenerate transient RSDE in D.

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