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ON STABILITY UNDER PERTURBATIONS OF LONG-RUN AVERAGE OPTIMAL CONTROL PROBLEMS

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Abstract. We state conditions that ensure a continuity of the optimal value of a long run average optimal control (LRAOC) problem with respect to a perturbation parameter. A distinctive feature of our approach is that the perturbation analysis of the LRAOC problem is carried out on the basis of the perturbation analysis of a certain infinite-dimensional linear programming (LP) problem and that of the corresponding approximating semi-infinite LP problems.

1. Introduction

Consider a control system

$$y'(t) = f_0(u(t), y(t)) + \varepsilon f_1(u(t), y(t)), \tag{1.1}$$

depending on a small parameter $\varepsilon \geq 0$, where $f_0(u,y) : U \times \mathbf{R}^m \to \mathbf{R}^m$, $f_1(u,y) : U \times \mathbf{R}^m \to \mathbf{R}^m$ are continuous in (u,y) and satisfy Lipschitz conditions in y. Controls $u(\cdot)$ are Lebesgue measurable functions of time that take values in a given compact metric space U. A pair $(u(\cdot), y(\cdot))$, where $u(\cdot)$ is a control and $y(\cdot)$ is a solution of (1.1), will be called *admissible* if

$$y(t) \in Y, \tag{1.2}$$

where Y is a given compact subset of $\mathbf{R}^m$ with the nonempty interior ((1.2) being interpreted as a state constraint).

In this paper, we discuss conditions, under which the optimal value $G^*(\varepsilon)$ of the long-run average optimal control (LRAOC) problem

$$G^*(\varepsilon) \stackrel{def}{=} \inf_{(u(\cdot),y(\cdot))} \liminf_{\mathcal{T}\to\infty} \frac{1}{\mathcal{T}} \int_0^{\mathcal{T}} q(u(t), y(t))dt, \tag{1.3}$$

where $q(u,y) : U \times \mathbf{R}^m \to \mathbf{R}^1$ is a given continuous function and *inf* is sought over all admissible pairs of the system (1.1) has the following continuity property

$$|G^*(\varepsilon) - G^*(0)| \leq L\varepsilon, \qquad L = \text{const}. \tag{1.4}$$

For convenience, we will refer to this property as to "*stability under perturbations*".

The perturbation analysis of optimal control problems was in the focus of attention of many researchers (see, e.g., [2,5,6,8,13–16,18,20,21] and references therein).

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However, to the best of our knowledge, no results on stability under perturbations for LRAOC problems have been discussed in the literature.

A distinctive feature of our approach is that it is based on the fact that, under nonrestrictive conditions, the optimal value of the problem (1.3) is equal to the optimal value of the infinite-dimensional linear programming (IDLP) problem (see [7,9,10] and [11])

$$\min_{\gamma \in W(\varepsilon)} \int_{U\times Y} q(u,y)\gamma(du,dy) = G^*(\varepsilon), \tag{1.5}$$

where

$$W(\varepsilon) = \Big\{\gamma | \gamma \in \mathcal{P}(U\times Y), \int_{U\times Y} \nabla\phi_i(y)^T \Big[f_0(u,y) + \varepsilon f_1(u,y)\Big]\gamma(du,dy) = 0, \quad \forall\ i = 1,2,...\Big\}. \tag{1.6}$$

Here $\mathcal{P}(U\times Y)$ stands for the space of probability measures defined on Borel subsets of $U\times Y$ and $\{\phi_i(\cdot),\ i = 1,2,...,\}$ is a sequence of functions such that any $\phi(\cdot) \in C^1$ and its gradient can be simultaneously approximated on Y by linear combinations of functions from this sequence and their respective gradients. An example of such a sequence are monomials $y_1^{i_1}...y_m^{i_m}$, $i_1,...,i_m = 0,1,...$, where y_l $(l = 1,...,m)$ stand for the components of y (see, e.g., [17]).

Assuming that the conditions ensuring the equality of the optimal values of the LRAOC problem (1.3) and the IDLP problem (1.5) are satisfied for any $\varepsilon \geq 0$ (this is the case, for example, if Y is forward invariant with respect to the solutions of (1.1); see [9] and [11]) we replace the stability under perturbation analysis of the former by that of the latter.

Our consideration is also based on the fact that the optimal values of the IDLP problem (1.5) is approximated by the optimal values of the following semi-infinite linear programming (SILP) problems

$$\min_{\gamma \in W_N(\varepsilon)} \int_{U\times Y} q(u,y)\gamma(du,dy) = G^N(\varepsilon), \qquad N = 1,2,..., \tag{1.7}$$

where

$$W_N(\varepsilon) = \Big\{\gamma | \gamma \in \mathcal{P}(U\times Y), \int_{U\times Y} (\nabla\phi_i(y))^T \Big[f_0(u,y) + \varepsilon f_1(u,y)\Big]\gamma(du,dy) = 0, \quad i = 1,2,...,N\Big\}. \tag{1.8}$$

It is straightforward to verify that the optimal value of the problem (1.5) is approximated by the optimal values of (1.7). Namely (see, e.g., Proposition 7 in [11]),

$$\lim_{N\to\infty} G^N(\varepsilon) = G^*(\varepsilon), \tag{1.9}$$

(the convergence being valid for both $\varepsilon > 0$ and $\varepsilon = 0$).

Our line of argument will be as follows. We will, first, investigate conditions, under which

$$|G^N(\varepsilon) - G^N(0)| \leq L\varepsilon \tag{1.10}$$

(the constant in the right-hand-side of (1.10) may depend on N), and we will then give a condition, under which the estimate in (1.10) is uniform in N. The latter and (1.9) will lead to the validity of (1.4).

The paper is organized as follows. This introductory section is followed by Section 2, in which we establish the stability under perturbations result for a class of SILP problems that satisfies a certain regularity condition (see Theorem 2.2). Note that this result is established without any references to LRAOC problems. These references are made in Section 3. In this section, we show: (a) that the regularity condition of Section 2 is satisfied if the reduced system (that is, the system (1.1) considered with $\varepsilon = 0$) satisfies a readily verifiable controllability condition implying that (1.10) is true (see Theorem 3.2) and (b) that, under a stronger controllability condition, the estimate in (1.10) is uniform in N (see Theorem 3.3 and its Corollary). The last Section 4 contains proofs of some results from Section 2.

Let us conclude this section with some preliminaries. Given a compact metric space X, $\mathcal{P}(X)$ will stand for the set of probability measures defined on Borel subsets of X. This set will be treated as a compact metric space with a metric ρ which is consistent with its weak* topology (see, e.g., [3] or [19]). That is, a sequence $\gamma^k \in \mathcal{P}(X)$, $k = 1, 2, \ldots$, converges to $\gamma \in \mathcal{P}(X)$ in this metric if and only if

$$\lim_{k\to\infty} \int_X c(x)\gamma^k(dx) = \int_X c(x)\gamma(dx)$$

for any continuous function $c(\cdot)$.

2. Perturbation analysis of SILP problems

Consider a family of semi-infinite dimensional linear programming (SILP) problems depending on a small parameter ε and N equality constraints; namely

$$G^N(\varepsilon) \stackrel{def}{=} \min_{\gamma\in W_N(\varepsilon)} \int_X q(x)\gamma(dx), \tag{2.1}$$

where

$$W_N(\varepsilon) \stackrel{def}{=} \Big\{\gamma : \gamma \in \mathcal{P}(X), \int_X [h_i^0(x) + \varepsilon h_i^1(x)]\gamma(dx) = 0, \quad i = 1, ..., N\Big\}, \tag{2.2}$$

where X is a compact metric space and $q(x), h_i^0(x), h_i^1(x)$ are continuous functions on X. Consider also, a SILP problem obtained from problem (2.1) by taking $\varepsilon = 0$. That is,

$$G^N(0) \stackrel{def}{=} \min_{\gamma\in W_N(0)} \int_X q(x)\gamma(dx), \tag{2.3}$$

where

$$W_N(0) \stackrel{def}{=} \Big\{\gamma : \gamma \in \mathcal{P}(X), \quad \int_X h_i^0(x)\gamma(dx) = 0, \quad i = 1, 2, ..., N\Big\}. \tag{2.4}$$

The family (2.1) is referred to as the *perturbed problem* and (2.3) is referred to as the *reduced problem*. Note that the sets $W_N(\epsilon)$ and $W_N(0)$ are compact subsets of $\mathcal{P}(X)$. Hence, an optimal solution of (2.1) (respectively, (2.3)) exist if $W_N(\epsilon)$(respectively, $W_N(0)$) are not empty.

Definition 2.1. The reduced problem will be said to satisfy the *regularity condition* if the inequality

$$\sum_{i=1}^{N} v_i h_i^0(x) \geq 0, \qquad \forall x \in X,$$

implies $v_i = 0, \quad i = 1, ..., N$.

The main result of this section is the following theorem.

Theorem 2.2. *If the reduced problem satisfies regularity condition, then*

$$|G^N(\varepsilon) - G^N(0)| \leq L\varepsilon, \qquad \forall \varepsilon \in [0, \bar{\varepsilon}), \tag{2.5}$$

where L and $\bar{\varepsilon}$ are positive constants.

The proof of the theorem is given at the end of the section and is based on several propositions that are stated below.

Along with the problems (2.1) and (2.3), let us consider their respective duals that can be written as follows (see, e.g., [7] and [12]):

$$D^N(\varepsilon) \stackrel{def}{=} \sup_{(v,d)} \Big\{ d : \ d \leq q(x) + \sum_{i=1}^{N} v_i \Big(h_i^0(x) + \varepsilon h_i^1(x) \Big), \\ v = (v_1, \ldots, v_N) \in \mathbf{R}^N, \forall x \in X \Big\}, \tag{2.6}$$

(2.7)

$$D^N(0) \stackrel{def}{=} \sup_{(v,d)} \Big\{ d : \ d \leq q(x) + \sum_{i=1}^{N} v_i h_i^0(x), \quad v = (v_1, \ldots, v_N) \in \mathbf{R}^N, \forall x \in X \Big\}.$$

The problems (2.6) and (2.7) will be referred to as *perturbed* and *reduced* dual problems, respectively. Duality relationships between the perturbed and reduced problems and their duals are described by the following proposition.

Proposition 2.3. *The following relations hold:*

(i): $D^N(0) < \infty$ *if and only if* $W_N(0) \neq \emptyset$;
(ii): *If* $D^N(0) < \infty$ *then* $D^N(0) = G^N(0)$;
(iii): $D^N(\varepsilon) < \infty$ *if and only if* $W_N(\varepsilon) \neq \emptyset$;
(iv): *If* $D^N(\varepsilon) < \infty$ *then* $D^N(\varepsilon) = G^N(\varepsilon)$.

The proof of Proposition 2.3 is based on the following lemma

Lemma 2.4. *The following relations hold:*

(i): $D^N(0) \geq \underline{Q}, \ D^N(\varepsilon) \geq \underline{Q}, \forall \varepsilon$, *where* $\underline{Q} \stackrel{def}{=} \min_{x \in X} q(x) > -\infty$;
(ii): $D^N(0) = \infty$ *if and only if there exists* $v = (v_1, \ldots, v_N)$ *such that*

$$\min_{x \in X} \sum_{i=1}^{N} v_i h_i^0(x) > 0.$$

(iii): $D^N(\varepsilon) = \infty$ *if and only if there exists* $v = (v_1, \ldots, v_N)$ *such that*

$$\min_{x \in X} \sum_{i=1}^{N} v_i (h_i^0(x) + \varepsilon h_i^1(x)) > 0.$$

Proof. The proofs of Proposition 2.3 and of Lemma 2.4 are similar to the proofs of Theorem 4.1 in [7] and Theorem 3.1 in [12]. These proofs are provided in Section 4 (for the sake of completeness). □

A vector $v^*(\varepsilon) = (v_1^*(\varepsilon), \ldots, v_N^*(\varepsilon))$ will be called an optimal solution of the problem (2.6) if

$$D^N(\varepsilon) = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^{N} v_i^*(\varepsilon) \Big(h_i^0(x) + \varepsilon h_i^1(x) \Big) \Big\}. \tag{2.8}$$

Similarly, a vector $v^* = (v_1^*, \ldots, v_N^*)$ will be called an optimal solution of the problem (2.7) if

$$D^N(0) = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^{N} v_i^* h_i^0(x) \Big\}. \tag{2.9}$$

Note: Proposition 2.3 implies that, if the dual problems (2.6), (2.7) have solutions $v^*(\varepsilon) = (v_1^*(\varepsilon), \ldots, v_N^*(\varepsilon))$ and $v^* = (v_1^*, \ldots, v_N^*)$, respectively, then the sets $W_N(\varepsilon)$, $W_N(0)$ are not empty and $G^N(\varepsilon)$ and $G^N(0)$ can be represented in the form

$$G^N(\varepsilon) = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^{N} v_i^*(\varepsilon) \Big(h_i^0(x) + \varepsilon h_i^1(x) \Big) \Big\};$$

$$G^N(0) = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^{N} v_i^* h_i^0(x) \Big\}.$$

Definition 2.5. Given $\varepsilon > 0$, the perturbed problem will be said to satisfy the *regularity condition* if the inequality

$$\sum_{i=1}^{N} v_i \Big(h_i^0(x) + \varepsilon h_i^1(x) \Big) \geq 0, \qquad \forall x \in X,$$

implies $\quad v_i = 0, \quad i = 1, ..., N.$

Proposition 2.6. *If the reduced problem satisfies the regularity condition, then there exists $\varepsilon^* > 0$ such that for any $\varepsilon \in (0, \varepsilon^*)$ the perturbed problem also satisfies the regularity condition.*

Proof. Assume that the perturbed problem does not satisfy the regularity condition. Then, there exists a sequence $\varepsilon_l \to 0$ and a sequence of vectors $v(\varepsilon_l) = (v_1(\varepsilon_l), \ldots, v_N(\varepsilon_l))$ such that $||v(\varepsilon_l)|| > 0$ $\forall l$, and the following holds

$$\sum_{i=1}^{N} v_i(\varepsilon_l) \Big(h_i^0(x) + \varepsilon_l h_i^1(x) \Big) \geq 0, \ \forall x \in X, \quad \forall l = 1, 2, \ldots. \tag{2.10}$$

Without loss of generality, one may assume that

$$\lim_{l \to \infty} \frac{v(\varepsilon_l)}{||v(\varepsilon_l)||} \stackrel{def}{=} \tilde{v}, \qquad ||\tilde{v}|| = 1. \tag{2.11}$$

Dividing (2.10) by $||v(\varepsilon_l)||$ and taking the limit as $l \to \infty$, one can obtain

$$\sum_{i=1}^{N} \tilde{v}_i h_i^0(x) \geq 0, \qquad \forall x \in X. \tag{2.12}$$

Hence, by the assumption of the proposition $\tilde{v} = (\tilde{v}_1, \ldots, \tilde{v}_N) = 0$, which is in contradiction with (2.11). Thus, the perturbed problem satisfies the regularity condition for all sufficiently small $\varepsilon > 0$. □

The next proposition shows that the regularity condition guarantees the existence and bounded-ness of optimal solutions to both the perturbed dual and the reduced dual problems

Proposition 2.7. *If the reduced problem satisfies the regularity condition, then optimal solutions of both the perturbed dual problem with $\varepsilon \in (0, \varepsilon^*)$ and of the reduced dual problem exist. Here ε^* is as in Proposition 2.6.*

Proof. The proof is the same for both perturbed and reduced dual problems and, therefore, we will use the following notations $h_i(x) = h_i^0(x) + \varepsilon h_i^1(x)$ and $D = D^N(\varepsilon)$ (valid for both $\varepsilon \in (0, \varepsilon^*)$ and for $\varepsilon = 0$).

Take any sequences $D_k \to D$ and $v^k = (v_1^k, \ldots, v_N^k)$ such that

$$D_k \leq q(x) + \sum_{i=1}^{N} v_i^k \ h_i(x), \quad \forall x \in X, \quad k = 1, 2, \ldots . \tag{2.13}$$

First, let us show that $D < \infty$. Indeed, otherwise, there would exist k such that $D_k - \max\{q(x) : x \in X\} \stackrel{def}{=} d > 0$ and $d \leq \sum_{i=1}^{N} v_i^k \ h_i(x), \quad \forall x \in X$. This contradicts the regularity condition.

Let us show now that the sequence v^k is bounded. Assume that it is not the case and there exists a sequence v^{k_l} such that $\| v^{k_l} \| \to \infty$ and

$$\lim_{l \to \infty} \frac{v^{k_l}}{\| v^{k_l} \|} \stackrel{def}{=} \tilde{v} \quad \text{and} \quad \| \tilde{v} \| = 1.$$

Then from the fact that $D < \infty$ and from (2.13) it follows that

$$0 \leq \sum_{i=1}^{N} \tilde{v} h_i(x), \quad \forall x \in X.$$

Due to the regularity condition, this relation implies $\tilde{v} = 0$, which contradicts $\| \tilde{v} \| = 1$. Thus, the sequence v^k is bounded and, hence, it has a limit point $v^* = (v_1^*, \ldots, v_N^*)$. From (2.13) it follows that

$$D \leq q(x) + \sum_{i=1}^{N} v_i^* \ h_i(x), \quad \forall x \in X;$$

that is, v^* is an optimal solution. □

Note. If the regularity condition holds for the reduced problem, then Proposition 2.3 (parts (i), (iii)) and Proposition 2.7 imply that the sets $W_N(0)$ and $W_N(\varepsilon)$ are not empty.

Proposition 2.8. *If the reduced problem satisfies the regularity condition then the set*

$$\mathcal{V}(\varepsilon) = \{v = (v_1, \ldots, v_N) : D^N(\varepsilon) \leq q(x) + \sum_{i=1}^{N} v_i(h_i^0(x) + \varepsilon h_i^1(x)),\ \forall x \in X\}$$

of optimal solutions of the perturbed dual problem (2.6) is bounded for all sufficiently small ε. *That is, there exist* $K < \infty$ *and* $\varepsilon^{**} > 0$ *such that*

$$\sup\{\| v \| \mid v \in \mathcal{V}(\varepsilon)\} \leq K \tag{2.14}$$

for all $\varepsilon \in [0, \varepsilon^{**})$. *Here* $\mathcal{V}(0)$ *stands for the set of optimal solutions of the reduced problem (2.7).*

Proof. Assume that the statement of the proposition is not true, then there exist sequences $\varepsilon_l \to 0$, and $v(\varepsilon_l)$ such that the following holds

$$D^N(\varepsilon_l) \leq q(x) + \sum_{i=1}^{N} v_i(\varepsilon_l)(h_i^0(x) + \varepsilon_l h_i^1(x)), \quad \forall x \in X \tag{2.15}$$

and

$$\lim_{\varepsilon_l \to 0} \| v(\varepsilon_l) \| = \infty.$$

Let

$$\lim_{\varepsilon_l \to 0} \frac{v(\varepsilon_l)}{\| v(\varepsilon_l) \|} \stackrel{def}{=} \tilde{v} \quad \text{and} \quad \| \tilde{v} \| = 1.$$

Due to Proposition 2.3(iv), $D^N(\varepsilon_l)$ is bounded. Hence, dividing (2.15) by $||v(\varepsilon_l)||$ and passing to the limit as $\varepsilon_l \to 0$, one can obtain

$$0 \leq \sum_{i=1}^{N} \tilde{v}_i h_i^0(x), \quad \forall x \in X.$$

By the regularity condition, this implies $\tilde{v} = (\tilde{v}_i) = 0$. This is a contradiction. Thus, the validity of (2.14) is established. □

Proof of Theorem 2.2. We prove this theorem in two steps. First we prove the relation (2.5) and then we prove the Lipschitz continuity of $G^N(\varepsilon)$

1. Denote $\bar{\varepsilon} = \min\{\varepsilon^*, \varepsilon^{**}\}$, where ε^* and ε^{**} are defined in Propositions 2.6 and 2.8, respectively. First, let us show that there exists $L < \infty$ such that

$$G^N(\varepsilon) \leq G^N(0) + L\,\varepsilon, \qquad \forall \varepsilon \in (0, \bar{\varepsilon}). \tag{2.16}$$

Take any positive number $\varepsilon < \bar{\varepsilon}$ and consider the dual problem (2.6). Proposition 2.7 ensures that an optimal solution of the problem (2.6) exists. Let $v(\varepsilon) = (v_1(\varepsilon), \ldots, v_N(\varepsilon))$ be such optimal solution; that is

$$D^N(\varepsilon) \leq q(x) + \sum_{i=1}^{N} v_i(\varepsilon)(h_i^0(x) + \varepsilon h_i^1(x)), \quad \forall x \in X. \tag{2.17}$$

Let $\gamma \in W_N(0)$ be a solution of (2.3). From Proposition 2.3 (iv) we have $G^N(\varepsilon) = D^N(\varepsilon)$ and therefore by integrating (2.17) we obtain

$$G^N(\varepsilon) \le G^N(0) + \varepsilon \int_X \sum_{i=1}^{N} v_i(\varepsilon) h_i^1(x)\gamma(dx). \tag{2.18}$$

Due to Proposition 2.8, the relation $\|v(\varepsilon)\| \le K$ holds. Moreover, since functions $h_i^1(x)$ are continuous on compact set X, there exists $M < \infty$ such that $\|h^1(x)\| \le M$, for all $x \in X$. Therefore (2.18) yields (2.16) where $L = KM$.

Now we show that the following relation also holds

$$G^N(\varepsilon) \ge G^N(0) - L\,\varepsilon, \qquad \forall \varepsilon \in (0, \bar{\varepsilon}). \tag{2.19}$$

Let $\varepsilon \in (0, \bar{\varepsilon})$ and consider the reduced dual problem (2.7). According to Proposition 2.7, problem (2.7) has an optimal solution that is denoted by $v = (v_1, \ldots, v_N)$. In this case, taking into account the relation $G^N(0) = D^N(0)$ (see Proposition 2.3 (ii)) we have

$$G^N(0) \le q(x) + \sum_{i=1}^{N} v_i h_i^0(x), \quad \forall x \in X. \tag{2.20}$$

Let $\gamma_\varepsilon \in W_N(\varepsilon)$ be a solution of (2.1); that is, $G^N(\varepsilon) = \int_X q(x)\gamma_\varepsilon(dx)$ and

$$\int_X h_i^0(x)\gamma_\varepsilon(dx) = -\varepsilon \int_X h_i^1(x)\gamma_\varepsilon(dx), \quad i = 1, ..., N. \tag{2.21}$$

By integrating (2.20) and taking (2.21) into account, we have

$$G^N(0) \le G^N(\varepsilon) - \varepsilon \int_X \sum_{i=1}^{N} v_i h_i^1(x)\gamma_\varepsilon(dx). \tag{2.22}$$

Now, since $\|h^1(x)\| \le M$ for all $x \in X$ and $\|v\| \le K$ (Proposition 2.8), from (2.22) we obtain

$$G^N(\varepsilon) \ge G^N(0) - KM\varepsilon.$$

Therefore (2.19) is true with $L = KM$. Finally, the relation (2.5) follows from (2.16) and (2.19). This completes the proof. □

Remark 2.9. Due to Proposition 2.6, the perturbed problem satisfies the regularity condition for all sufficiently small $\varepsilon > 0$. Hence, one can use an argument similar to one used above to establish that the optimal value function $G^N(\varepsilon)$ satisfied Lipschitz condition on the interval $[0, \bar{\bar{\varepsilon}}]$, where $\bar{\bar{\varepsilon}}$ is sufficiently small positive constant.

3. Stability under perturbations of the SILP and IDLP problems related to the long run average optimal control problem

As can be readily seen, the SILP problem (1.7) is a special case of the perturbed problem (2.1) obtained with the change of the notations

$$x = (u, y), \quad X = U \times Y, \quad h_i^0(x) = \nabla\phi_i(y) f_0(u, y), \quad h_i^1(x) = \nabla\phi_i(y) f_1(u, y). \tag{3.1}$$

The corresponding reduced problem obtained from (1.7) with $\varepsilon = 0$ is of the form

$$\min_{\gamma \in W_N(0)} \int_{U \times Y} q(u,y)\gamma(du,dy) = G^N(0), \tag{3.2}$$

where

$$W_N(0) = \Big\{\gamma | \gamma \in \mathcal{P}(U \times Y), \int_{U \times Y} (\nabla \phi_i(y))^T f_0(u,y)\gamma(du,dy) = 0, \quad i = 1,2,...,N\Big\}. \tag{3.3}$$

In what follows, it is assumed that the gradients of the functions $\phi_i(\cdot)$ are linearly independent in the sense that, if for some open set $Q \subset \mathbf{R}^m$

$$\sum_{i=1}^{N} v_i \nabla \phi_i(y) = 0 \quad \forall y \in Q,$$

then $v_i = 0, \; i = 1,...,M$. Note that this linear independence condition is satisfied if the monomials are used as $\phi_i(\cdot)$ (see comments after the introduction of $W(\varepsilon)$ in (1.6)).

Let us consider the system obtained by equating ε to zero in (1.1),

$$y'(t) = f_0(u(t), y(t)) \tag{3.4}$$

(this will be referred to as the *reduced system*) and let us introduce the following definition.

Definition 3.1. The reduced system (3.4) is locally approximately controllable in Y if there exists a set $Y^0 \subset Y$ such that the closure of Y^0 has a nonempty interior and such that any two points in Y^0 can be connected by an admissible trajectory of (3.4) (that is, for any $y', y'' \in Y^0$, there exist a control $u(\cdot)$ and the corresponding solution $y(\cdot)$ of (3.4) such that $y(0) = y', \; y(S) = y''$ for some $S > 0$, and $y(t) \in Y \; \forall t \in [0, S]$).

Theorem 3.2. *If the reduced system (3.4) is locally approximately controllable in Y, then the reduced problem (3.2) satisfies the regularity condition of Definition 2.1 and, hence (by Theorem 2.2), the SILP problem (1.7) is stable under perturbations. That is, the estimate (1.10) is valid (the estimate may not be uniform in N).*

Proof. By Definition 2.1 (see also (3.1)), to prove the theorem, one needs to show that from the fact that

$$\sum_{i=1}^{N} v_i (\nabla \phi_i(y))^T f_0(u,y) \geq 0, \; \forall (u,y) \in U \times Y \tag{3.5}$$

it follows that

$$v_i = 0, \; \forall i = 1,...,N. \tag{3.6}$$

For convenience, let introduce the notation

$$\eta(y) = \sum_{i=1}^{N} v_i \phi_i(y)$$

and rewrite (3.5) in the form

$$\nabla\eta(y)^T f_0(u,y) \geq 0, \quad \forall (u,y) \in U \times Y. \tag{3.7}$$

Take arbitrary two points $y', y'' \in Y^0$, where Y^0 is as in the definition of local approximate controllability (see Definition 3.1). By the assumption of the theorem, there exists an admissible pair $(u(\cdot), y(\cdot))$ such that $y(0) = y'$ and $y(S) = y''$. From (3.7) it follows that

$$\eta(y_2) - \eta(y_1) = \textstyle\int_0^S (\nabla\eta(y(t)))^T f_0(u(t), y(t))dt \geq 0 \quad \Rightarrow \quad \eta(y_2) \geq \eta(y_1).$$

Since y_1, y_2 are arbitrary points in Y_0, the above inequality allows one to conclude that

$$\eta(y) = const \ \ \forall y \in Y_0 \qquad \Rightarrow \qquad \eta(y) = const \ \ \forall y \in clY^0,$$

the latter implying that $\nabla\eta(y) = 0 \ \ \forall y \in int(clY^0)$ and, consequently, leading to the fact that (3.6) is true. This proves the theorem. $\square$

Let us now introduce a stronger controllability assumption.

Assumption I. There exists $r > 0$ such that

$$rB \subset co\ f_0(U,y), \quad \forall y \in Y, \tag{3.8}$$

where B is a closed unit ball in $\mathbf{R}^m$, *co* stands for the convex hull and

$$f_0(U,y) \stackrel{def}{=} \{v : v = f_0(u,y), \ u \in U\}.$$

Note that Assumption I is quite common in the optimal control theory (see, e.g., [1]). Note also that, as can be readily seen, Assumption I implies the approximate controllability of the reduced system in Y. In fact Y^0 can be taken to be any open ball in Y in this case (remind that Y is assumed to have a nonempty interior).

Theorem 3.3. *Let Assumption I be satisfied. Then*

$$|G^N(\varepsilon) - G^N(0)| \leq L\varepsilon, \qquad \forall \varepsilon \in (0, \bar{\varepsilon}), \ N = 1, 2, \dots\ , \tag{3.9}$$

where L and $\bar{\varepsilon}$ are positive constants independent of N.

Proof. The problem dual to (1.7) is of the form

$$\sup_{\mu,\lambda}\{\mu : q(u,y) + \sum_{i=1}^N \lambda_i (\nabla\phi_i(y))^T (f_0(u,y) + \varepsilon f_1(u,y)) \geq \mu \ \forall (u,y) \in U \times Y\} = G^N(\varepsilon), \tag{3.10}$$

while the problem dual to (3.2) is

$$\sup_{\mu,\lambda}\{\mu : q(u,y) + \sum_{i=1}^N \lambda_i (\nabla\phi_i(y))^T f_0(u,y) \geq \mu \ \forall (u,y) \in U \times Y\} = G^N(0) \tag{3.11}$$

(see (2.6), (2.7) and (3.1)). Due to Theorem 3.2 and Proposition 2.7, an optimal solution of the problem (3.10) exists for all ε small enough. That is, there exists $\lambda(\varepsilon) = (\lambda_1(\varepsilon), \dots, \lambda_N(\varepsilon))$ such that

$$q(u,y) + \sum_{i=1}^N \lambda_i(\varepsilon)(\nabla\phi_i(y))^T (f_0(u,y) + \varepsilon f_1(u,y)) \geq G^N(\varepsilon) \quad \forall (u,y) \in U \times Y. \tag{3.12}$$

Since $q(u,y)$ is continuous, there exists a constant M such that $|q(u,y)| \le M$ holds for all $(u,y) \in U \times Y$. In this case, we have $|G^N(\varepsilon)| \le M$ (due to the fact that $G^N(\varepsilon)$ is the optimal value of the problem (1.7)). Hence, by (3.12),

$$\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T(f_0(u,y)+\varepsilon f_1(u,y)) \ge -2M \quad \forall (u,y) \in U \times Y. \tag{3.13}$$

For a given $y \in Y$, let u_y be such that

$$\begin{aligned} &\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T f_0(u_y,y) = \min_{u\in U}\{\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T f_0(u,y)\} \\ &= \min_{v\in f_0(U,y)}\{\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T v\} = \min_{v\in co f_0(U,y)}\{\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T v\} \\ &\le \min_{v\in rB}\{\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T v\} = -r||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))||, \end{aligned} \tag{3.14}$$

where the inequality in (3.14) follows from Assumption I. By (3.13) and (3.14),

$$-r||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| + \varepsilon \sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))^T f_1(u_y,y) \ge -2M \quad \forall y \in Y$$

$$\Rightarrow \quad -r||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| + \varepsilon||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| \; ||f_1(u_y,y)|| \ge -2M \quad \forall y \in Y$$

Assuming that $||f_1(u,y)|| \le M_1 \ \forall (u,y) \in U \times Y$ (where M_1 is a constant), one can obtain

$$\begin{aligned} &-r||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| + \varepsilon M_1||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| \ge -2M \quad \forall y \in Y \\ &\Rightarrow \quad (r-\varepsilon M_1)||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| \le 2M \quad \forall y \in Y \\ &\Rightarrow \quad ||\sum_{i=1}^{N} \lambda_i(\varepsilon)(\nabla\phi_i(y))|| \le \frac{4M}{r}, \quad \forall y \in Y, \quad \forall \varepsilon \in [0, \frac{r}{2M_1}] \end{aligned} \tag{3.15}$$

Let now γ_N^* be an optimal solution of (3.2). From (3.12) it follows that

$$\int_{U\times Y} \left[q(u,y) + \sum_{i=1}^{N} \lambda_i^i(\varepsilon)(\nabla\phi_i(y))^T(f_0(u,y)+\varepsilon f_1(u,y)) \right] \gamma_N^*(du,dy) \ge G^N(\varepsilon)$$

$$\Rightarrow \quad G^N(0) + \varepsilon| \int_{U\times Y} \sum_{i=1}^{N} \lambda_i^i(\varepsilon)(\nabla\phi_i(y))^T f_1(u,y))\gamma_N^*(du,dy) \;| \ge G^N(\varepsilon)$$

$$\Rightarrow \quad G^N(0) + \varepsilon \int_{U\times Y} ||\sum_{i=1}^{N} \lambda_i^i(\varepsilon)(\nabla\phi_i(y))|| \; ||f_1(u,y))||\gamma_N^*(du,dy) \ge G^N(\varepsilon).$$

The latter and (3.15) imply that

$$G^N(0) + \varepsilon C_1 \geq G^N(\varepsilon) \quad \forall \varepsilon \in [0, \frac{r}{2M_1}], \quad \text{where} \quad C_1 \stackrel{def}{=} \frac{4MM_1}{r}.$$

To complete the proof of the theorem we need to show that, for all sufficiently small positive ε,

$$G^N(\varepsilon) \geq G^N(0) - \varepsilon C_2, \tag{3.16}$$

where $C_2 > 0$ is a constant. Let $\lambda = (\lambda_1, \ldots, \lambda_N)$ be an optimal solution of the problem (3.11) (it exists due to the fact that the reduced problem (3.2) satisfies the regularity condition; see Theorem 3.2 and Proposition 2.7). That is,

$$q(u,y) + \sum_{i=1}^{N} \lambda_i (\nabla \phi_i(y))^T f_0(u,y) \geq G^N(0) \quad \forall (u,y) \in U \times Y. \tag{3.17}$$

Using (3.17) and making the steps similar to (3.12), (3.13), (3.14) and (3.15), one can establish the validity of the estimate

$$|| \sum_{i=1}^{N} \lambda_i (\nabla \phi_i(y)) || \leq \frac{2M}{r} \quad \forall y \in Y. \tag{3.18}$$

Let $\gamma_N^{*\varepsilon}$ be an optimal solution of (1.7). That is,

$$\int_{U \times Y} q(u,y) \gamma_N^{*,\varepsilon}(du,dy) = G^N(\varepsilon),$$

and for all $i = 1, 2, ..., N$,

$$\int_{U \times Y} (\nabla \phi_i(y))^T f_0(u,y) \gamma_N^{*,\varepsilon}(du,dy) = -\varepsilon \int_{U \times Y} (\nabla \phi_i(y))^T f_1(u,y) \gamma_N^{*,\varepsilon}(du,dy). \tag{3.19}$$

Integrating (3.17) over $\gamma_N^{*\varepsilon}$ and taking into account the relationships above, we obtain

$$G^N(\varepsilon) - \varepsilon \int_{U \times Y} \sum_{i=1}^{N} \lambda_i (\nabla \phi_i(y))^T f_1(u,y) \gamma_N^{*,\varepsilon}(du,dy) \geq G^N(0).$$

From (3.18) and from the fact that $||f_1(u,y)|| \leq M_1 \ \ \forall (u,y) \in U \times Y$, it follows that

$$| \int_{U \times Y} \sum_{i=1}^{N} \lambda_i (\nabla \phi_i(y))^T f_1(u,y) \gamma_N^{*,\varepsilon}(du,dy)| \leq \frac{2MM_1}{r}.$$

Thus,

$$G^N(\varepsilon) \geq G^N(0) + \varepsilon \int_{U \times Y} \sum_{i=1}^{N} \lambda_i (\nabla \phi_i(y))^T f_1(u,y) \gamma_N^{*,\varepsilon}(du,dy)$$
$$\geq G^N(0) - \varepsilon \frac{2MM_1}{r}.$$

That is, (3.16) holds with $C_2 = \frac{2MM_1}{r}$. This proves (3.9) with $L = \max\{C_1, C_2\} = \frac{4MM_1}{r}$ and $\bar{\varepsilon} = \frac{r}{2M_1}$ (see (3.15)). □

Corollary 3.4. *If Assumption I holds, then*

$$|G^*(\varepsilon) - G^*(0)| \le L\varepsilon \quad \forall \varepsilon \in (0, \bar{\varepsilon}), \tag{3.20}$$

where L and $\bar{\varepsilon}$ are the constants from (3.9).

Proof. By (3.9),

$$\begin{aligned} |G^*(\varepsilon) - G^*(0)| &\le |G^*(\varepsilon) - G^N(\varepsilon)| + |G^N(\varepsilon) - G^N(0)| + |G^N(0) - G^*(0)| \\ &\le |G^*(\varepsilon) - G^N(\varepsilon)| + L\varepsilon + |G^N(0) - G^*(0)|. \end{aligned}$$

Due to (1.9), one can obtain (3.20) by passing in the last expression to the limit with $N \to \infty$. □

Thus, under the assumptions made, the LRAOC problem is stable under perturbations.

4. Proofs of Lemma 2.4 and Proposition 2.3

Proof of Lemma 2.4.

1. The statement (i) is obtained from (2.6) and (2.7) by setting $v = 0$.

2. First, let us prove the "*if*" statement in (ii). Let $D(0) = \infty$. Then, there exists a sequence (d^n, v^n) such that

$$d^n = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^N v_i^n h_i^0(x) \Big\} \to \infty \text{ as } n \to \infty.$$

We have

$$d^n \le \max_{x \in X} q(x) + \min_{x \in X} \sum_{i=1}^N v_i^n h_i^0(x),$$

which implies

$$\min_{x \in X} \sum_{i=1}^N v_i^n h_i^0(x) \to \infty \text{ as } n \to \infty.$$

Therefore, for sufficiently large n, the inequality

$$\min_{x \in X} \sum_{i=1}^N v_i^n h_i^0(x) > 0$$

is true. This proves the "*if*" statement.

Let us now show that the "*only if*" statement in (ii) is true too. Let v be such that

$$\min_{x \in X} \sum_{i=1}^N v_i h_i^0(x) \ge \alpha > 0$$

holds. Take any $\lambda > 0$ and let

$$d(\lambda) = \min_{x \in X} \Big\{ q(x) + \sum_{i=1}^N (\lambda\, v_i) h_i^0(x) \Big\}.$$

The we obtain

$$D(0) \geq d(\lambda) \geq \min_{x \in X} q(x) + \lambda \min_{x \in X} \sum_{i=1}^{N} v_i h_i^0(x) \to \infty, \text{ as } \lambda \to \infty.$$

This proves the "*only if*" statement.

3. The proof of (iii) is follows exactly the same way as that of (ii). One just need to replace $h_i^0(x)$ by $h_i^0(x) + \varepsilon h_i^1(x)$. □

Proof of Proposition 2.3.

1. To prove (i), let us first show that if $W(0) \neq \emptyset$, then $D(0) \leq G(0) < \infty$. For any given $d < D(0)$, there exists $v = (v_1, \ldots, v_n)$ such that

$$d \leq q(x) + \sum_{i=1}^{N} v_i h_i^0(x), \ \forall x \in X.$$

Then

$$d \leq \int_X q(x)\gamma(dx) + \sum_{i=1}^{N} v_i \int_X h_i^0(x)\gamma(dx), \ \forall \gamma \in \mathcal{P}(X)$$

and

$$d \leq \int_X q(x)\gamma(dx) \ \forall \gamma \in W(0).$$

Therefore,

$$d \leq \min_{\gamma \in W(0)} \int_X q(x)\gamma(dx) \stackrel{def}{=} G(0).$$

Since this relation holds for any $d < D(0)$, we obtain $D(0) \leq G(0)$.

To prove a converse statement, let us assume that $D(0) < \infty$. Consider the set

$$Q = \{(\xi_1, \ldots, \xi_N) : \ \xi_i = \int_X h_i^0(x)\gamma(dx), i = 1, \ldots, N; \ \gamma \in \mathcal{P}(X)\}.$$

It is not difficult to verify that Q is a convex closed subset of $\mathbf{R}^N$. Now, if the set $W(0) = \emptyset$, then $(0, \ldots, 0) \notin Q$ and, therefore, there exists a non-zero vector $(v_1, \ldots, v_N)$ such that

$$0 < v_1\xi_1 + \cdots + v_N\xi_N, \ \forall (\xi_1, \ldots, \xi_N) \in Q;$$

that is,

$$0 < \int_X \sum_{i=1}^{N} v_i h_i^0(x)\gamma(dx), \ \forall \gamma \in \mathcal{P}(X).$$

Then

$$0 < \min_{\gamma \in \mathcal{P}(X)} \int_X \sum_{i=1}^{N} v_i h_i^0(x)\gamma(dx) = \min_{x \in X} \sum_{i=1}^{N} v_i h_i^0(x).$$

By Proposition 2.4(ii), the latter implies that $D(0) = \infty$, which is a contradiction. Thus (i) is proved.

2. Now we will prove (ii). Assume that $D(0) < \infty$. In this case part (i) of this proposition yields $W(0) \neq \emptyset$ and $D(0) \leq G(0) < \infty$. Thus, to prove (ii), it is sufficient to show that $D(0) \geq G(0)$.

Consider the set

$$Q = \Big\{(d, \xi_1, \dots, \xi_N) : \; d \ge \int_X q(x)\gamma(dx), \; \xi_i = \int_X h_i^0(x)\gamma(dx), \\ i = 1, \dots, N; \; \gamma \in \mathcal{P}(X)\Big\}.$$

It is not difficult to verify that Q is a convex closed subset of $\mathbf{R}^N$. Take an arbitrary $\delta > 0$. Clearly $(G(0) - \delta, 0, \dots, 0) \notin Q$ and, therefore, there exists a non-zero vector $(c, v_1, \dots, v_N)$ and a number $\eta > 0$ such that

$$c(G(0) - \delta) + \eta \le \inf_{(d,\xi)\in Q} \{cd + \sum_{i=1}^N v_i \xi_i\} \\ \le \inf_{d\in\mathbf{R}, \; \gamma\in\mathcal{P}(X)} \Big\{cd + \int_X \sum_{i=1}^N v_i h_i^0(x)\gamma(dx) : \; d \ge \int_X q(x)\gamma(dx)\Big\}.$$

From this inequality it immediately follows that $c \ge 0$. Also, if $c = 0$, then

$$0 < \eta \le \inf_{\gamma\in\mathcal{P}(X)} \int_X \sum_{i=1}^N v_i h_i^0(x)\gamma(dx) = \min_{x\in X} \sum_{i=1}^N v_i h_i^0(x),$$

which, by Proposition 2.4(ii), would contradict to our assumption that $D(0) < \infty$. Thus, $c > 0$. From (**??**) we obtain

$$G(0) - \delta + \frac{\eta}{c} \le \inf_{\gamma\in\mathcal{P}(X)} \int_X \Big\{q(x) + \sum_{i=1}^N \frac{v_i}{c} h_i^0(x)\Big\}\gamma(dx) \\ = \min_{x\in X} \Big\{q(x) + \sum_{i=1}^N \frac{v_i}{c} h_i^0(x)\Big\} \le D(0).$$

Therefore, $G(0) - \delta \le D(0)$. Since $\delta > 0$ can be arbitrary small, we may conclude that $G(0) \le D(0)$. This completes the proof of (ii).

3. The proofs of (iii) and (iv) are the same as to the proofs of (i) and (ii). One just needs to replace $h_i^0(x)$ with $h_i^0(x) + \varepsilon h_i^1(x)$. ☐

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