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VIABLE NASH EQUILIBRIA IN THE PROBLEM OF COMMON POLLUTION

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Abstract. Two countries produce goods and are penalized by the common pollution they generate. Each country maximizes an inter-temporal utility criterion, taking account of the pollution stock to which both contribute. The dynamic is in continuous time with possible sudden switches to less polluting technologies. The set of Nash equilibria, for which solutions also remain in the set of constraints, is the intersection of two manifolds in a certain state space. At the Nash equilibrium, the choices of the two countries are interdependent: different productivity levels after switching lead the more productive country to hasten and the less productive to delay the switch. In the absence of cooperation, efforts by one country to pollute less motivate the other to pollute more, or encourage the country that will be cleaner or less productive country after switching to delay its transition.

1. Introduction

The economics of the commons [10, 11, 12] with two non-cooperating countries sharing a common pollution may lead to Nash equilibria. The complexity of the dynamics usually makes it hard to find an analytic solution, and this holds even more so when trajectories have to remain within constraints and when the continuous dynamics can be reset by impulses on state variables.

Taking the production function linear in input and the input as a control variable, Boucekkine, Krawczyk, and Vallée [7] proved the existence of a Nash equilibrium, where each player chooses the technology without regard for the other's choice. Here, we consider a Cobb-Douglas production function and the input as a state variable governed by a differential equation. This precludes finding analytic solutions, but we can deal with non-linearities and with the switching times from the more to the less polluting regime. Our solution is numerical and the procedure can be used whenever looking for Nash equilibria between two rivals.

Our solution to the problem of the commons innovates by exploiting the properties of *capture-viability kernels*. The capture-viability kernel of a closed set $\mathcal{K}$ with closed target Ω under a set-valued dynamic F is the set of all initial states from which there exists at least one solution remaining in $\mathcal{K}$ and reaching Ω at a given finite time horizon. The fact that an upper Painlevé-Kuratowski limit of closed viability domains is a viability domain [1] guarantees the convergence of the value functions in finite time to the value function in infinite time. Such initial

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states are called *viable*. Our innovation is to show that a state is a viable Nash equilibrium if and only if it is located at the intersection of certain boundaries of two capture-viability kernels.

In contrast to [7], who compute open-loop Nash equilibria, the solution by capture-viability kernels yields all value functions without solving the first-order Pontryagin necessary conditions for each set of initial conditions ([5, 6]). Moreover, state constraints are inherently taken into account, contrary to Pontryagin's or Hamilton-Jacobi-Bellman's methods, where they must be treated one by one. In the viability framework, control values are selected from successive $R(x(t))$, where R is a set-valued map from the state space to the control space. They are then feedback; they depend on the state $x(t)$.

In dynamic programming, functional relationships often have to be assumed in order to derive the optimal feedback rules. This kind of assumption is not necessary here, which opens the way to searching for possible a posteriori relationships between controls and state variables at Nash equilibria, as well as a posteriori dependence between the two players.

We show that different productivity levels after the switch to a cleaner technology gives the more productive country a lead in the value function: this country hastens, and the less productive country delays, making the switch.

We reveal the non-corner solutions of the game: at Nash equilibrium, the players are not selfish as in [7]. Our consideration of non-linearities, through the Cobb-Douglas production functions, enriches the problem and leads to a more differentiated set of equilibria. The problem has infinite time horizon support. We solve it by finite time horizon approximation, which [4] proved to be convergent. Because we deal with non-linearities, our solution, based on [3]'s algorithm, is numerical.

After presenting the problem of common pollution (section 2) and viability theory (section 3.1), we explain how to infer Nash equilibria from capture-viability kernels (section 3.2). Numerical solving is achieved with the viability algorithm (section 4.1), which we use to consider three exemplary cases (section 4.2): same response from each country in terms of productivity and reducing pollution to the introduction of a cleaner technology; same productivity but different levels of pollution reduction, and different productivity levels with similar effect on pollution. We determine the viable Nash equilibria (section 4.3).

2. The problem

Two non-cooperating countries called 1 and 2 produce goods, generating a pollution equally detrimental to both countries. They do not trade in goods. Each country $j = 1, 2$ has consumption C_j and capital K_j, with common capital share ν in the production function $A_j {K_j}^{\nu}$. Technological progress is exogenous in each technological regime, so that returns to scale decrease as in the standard neoclassical model. It uses a technology with progress level A_j and marginal contribution to pollution α_j. It has the possibility, at a date t_j that it chooses, to switch from a technology characterized by $A_j = A_j^{(1)}$ and $\alpha_j = \alpha_j^{(1)}$ to a less polluting technology characterized by $A_j^{(2)}$ and marginal contribution to pollution $\alpha_j^{(2)}$. Pollution comes from both countries and is proportional to production.

Country $j = 1, 2$ solves the program:

$$\max_{C_j, t_j} \int_0^\infty U(C_j(t), P(t)) e^{-\rho t}\, dt, \tag{2.1}$$

where U is the utility function and ρ the discount rate, under the continuous-time dynamic:

$$\begin{cases} K_j'(t) &= A_j(t) K_j(t)^\nu - C_j(t) - \delta_K K_j(t) \\ P'(t) &= \sum_{i=1}^2 \alpha_i(t) A_i(t) K_i(t)^\nu - \delta_P P(t) \\ A_j'(t) &= 0 \\ \alpha_j'(t) &= 0 \\ A_j(0) &= A_j^{(1)},\ \alpha_j(0) = \alpha_j^{(1)},\ K_j(0) = K_j^0, P(0) = P^0, \quad j = 1, 2, \end{cases} \tag{2.2}$$

and the impulse (or discrete-time part) at t_j, $j = 1, 2$:

$$\begin{cases} {K_j}^+ &= {K_j}^- \\ P^+ &= P^- \\ {A_j}^+ &= {A_j}^- + A_j^{(2)} - A_j^{(1)} \\ {\alpha_j}^+ &= {\alpha_j}^- + \alpha_j^{(2)} - \alpha_j^{(1)},\ j = 1, 2. \end{cases} \tag{2.3}$$

The third equation presents the impulse from ${A_j}^-$ to ${A_j}^+$; it amounts to $A_j = A_j^{(1)}$ for $t < t_j$ and $A_j = A_j^{(2)}$ for $t \geq t_j$. Likewise, the fourth equation amounts to $\alpha_j = \alpha_j^{(1)}$ for $t < t_j$ and $\alpha_j = \alpha_j^{(2)}$ for $t \geq t_j$.

We use the conventional utility function (Boucekkine et al., 2011):

$$U_j(C, P) = \ln(C) - \beta_j P, \tag{2.4}$$

where β_j is a country-specific parameter, $j = 1, 2$. The controls are t_1, t_2, C_1, and C_2, and the state variables K_1, K_2, P, A_1, A_2, α_1, and α_2.

System {2.2, 2.3} constitutes a differential system in continuous-discrete time, also called hybrid dynamic [2] under the constraints

$$\begin{aligned} \mathcal{K} \ := \ & \{K_1 \geq K_1^{(\min)}, K_2 \geq K_2^{(\min)}, P \geq 0, A_1 \geq A_1^{(\min)}, A_2 \geq A_2^{(\min)}, \\ & \alpha_1 \geq \alpha_1^{(\min)}, \alpha_2 \geq \alpha_2^{(\min)}\} \subset \mathbb{R}^{+7}. \end{aligned} \tag{2.5}$$

Nash equilibria are obtained when country 1 maximizes its inter-temporal utility, that is, solves (2.1) with $j = 1$, considering that country 2 does the same, that is, solves (2.1) with $j = 2$.

3. Method: Capture-Viability, Optimization, Algorithm

3.1. **Impulse Dynamical Systems and Capture-Viability.** An impulse differential inclusion (F, R) consists in a continuous-time set-valued map $F : X \to X$:

$$x' \in F(x) = \{f(x, u, v), u \in V_u(x), v \in V_v(x)\}, \tag{3.1}$$

where f is a function such that F satisfies the regularity properties (3.1) stated below and $V_u(x)$ and $V_v(x)$ are closed sets, and a discrete-time reset map $R : X \to X$, giving a successor state

$$x^{i+1} \in R(x^i) \tag{3.2}$$

to the state x^i at control impulse time t_i.

For $\mathcal{K} \subset \mathbb{R}^m$ a closed set, the problem of capture-viability is to delineate all states $x^0 \in \mathcal{K}$ from which there exists at least one trajectory under the dynamic (F, R) remaining in $\mathcal{K}$ until a given time horizon T and hitting a closed subset $\Omega \subset \mathcal{K}$ in finite time. Such states are said to be *viable* in $\mathcal{K}$ under (F, R) with target Ω. Under the Marchaud assumptions:

Hypothesis 3.1.

(3.3)

(i) F is upper semi-continuous with non empty compact, convex values,

(ii) $\exists\, c \in \mathbb{R}$ such that $\sup_{y\in F(x)} \| y \| < c(\| x \| + 1)$,

(iii) R is upper semi-continuous with compact domain and compact values

hold true, then there exists a maximal set of viable states —called *capture-viability kernel*— containing all sets of viable states ([1]). The hybrid capture-viability kernel of $\mathcal{K}$ under the hybrid dynamic (F, R) is denoted $\mathrm{Capt}_{(F,R)}(\mathcal{K}, \Omega)$.

$S_F(x^0, v(\cdot))$ is the set of absolutely continuous solutions to (3.1) starting from x^0, and for any subset X of $\mathbb{R}^m$, $S_F(X, v(\cdot)) := \bigcup_{x^0\in X} S_F(x^0, v(\cdot))$. The scalar product is denoted by $<\ ,\ >$.

Definition 3.2. Consider set-valued maps $F : \mathbb{R}^m \mapsto \mathbb{R}^m$ and $R : \mathbb{R}^m \mapsto \mathbb{R}^m$ satisfying Assumptions 3.1.

- For a continuous-time system described by the differential inclusion $x' \in F(x)$, $\mathcal{K}$ is a viability domain with target Ω under F if and only if

$$\forall x \in \mathcal{K} \setminus \Omega,\ \ \forall p \in NP_{\mathcal{K}}(x), \quad \exists y \in F(x),\ < y, -p > \geq 0\,, \tag{3.4}$$

  where $NP_{\mathcal{K}}(x)$ is the normal cone to $\mathcal{K}$ in x. If $\mathcal{K}$ is not a viability domain with target, there exists a largest closed capture-viability domain contained in $\mathcal{K}$ denoted by $\mathrm{Capt}_F(\mathcal{K}, \Omega)$ and called the *capture-viability kernel of* $\mathcal{K}$ *with target* Ω. It is the largest closed set of initial conditions in $\mathcal{K}$ from which there exists at least one trajectory viable in $\mathcal{K}$ with target Ω.
- For an impulse system (F, R), $\mathcal{K}$ is an impulse capture-viability domain with target Ω under (F, R) if and only if it is a viability domain with target $(\Omega \cup R^{-1}(\mathcal{K} \cup \Omega))$ under F, namely

$$\forall x \in \mathcal{K} \setminus (\Omega \cup R^{-1}(\mathcal{K} \cup \Omega)),\ \ \forall p \in NP_{\mathcal{K}}(x), \quad \exists y \in F(x),\ < y, -p > \geq 0. \tag{3.5}$$

  If $\mathcal{K}$ is not an impulse capture-viability domain with target, there exists a largest closed impulse capture-viability domain contained in $\mathcal{K}$, called *impulse capture-viability kernel of* $\mathcal{K}$ *with target* Ω and denoted $\mathrm{Capt}_{(F,R)}(\mathcal{K}, \Omega)$. It is the set of initial conditions in $\mathcal{K}$ from which there exists at least one trajectory of (F, R) viable in $\mathcal{K}$ with target Ω.

[13] proved the first point, [14] the second point.

3.2. **Viable Nash equilibria by capture-viability kernels.** The problem {2.2, 2.3} is a problem of capture-viability with impulse. It can be written as a differential inclusion with impulse:

$$
\left\{\begin{array}{lcl}
x'(t) & \in & F(x(t)) := \{\big(A_1(t)K_1(t)^\nu - C_1(t) - \delta_K K_1(t), \\
& & A_2(t)K_2(t)^\nu - C_2(t) - \delta_K K_2(t), \\
& & \sum_{i=1}^2 \alpha_i(t)A_i(t)K_i(t)^\nu - \delta_P P(t), 0,0,0,0\big) \\
& & \quad \mid C_1 \in V_1, C_2 \in V_2, t_1 \geq 0, t_2 \geq 0\} \\
x^+ & = & R(x^-) := \{{K_1}^-, {K_2}^-, P^-, {A_1}^- + A_2^{(1)} - A_1^{(1)}, {A_2}^-, \\
& & {\alpha_1}^- + \alpha_2^{(1)} - \alpha_1^{(1)}, {\alpha_2}^-\} \quad \text{at } t_1, \\
x^+ & = & R(x^-) := \{{K_1}^-, {K_2}^-, P^-, {A_1}^-, {A_2}^- + A_2^{(2)} - A_1^{(2)}, \\
& & {\alpha_1}^-, {\alpha_2}^- + \alpha_2^{(2)} - \alpha_1^{(2)}\} \quad \text{at } t_2,
\end{array}\right. \tag{3.6}
$$

where $x := (K_1, K_2, P, A_1, A_2, \alpha_1, \alpha_2)$, and where $V_j \subset \mathbb{R}^+$, $j = 1, 2$, are closed sets, under the state constraints defined by $\mathcal{K}$ (defined in (2.5)).

Moreover, Bonneuil (2012) showed that $x^0 \in \mathbb{R}^n$ is a solution to the optimization problem

$$
\left\{\begin{array}{l}
\max_{u \in V_u} \int_0^\infty U(x(t))\, dt \\
x'(t) \in F(x(t)) := \{f(x(t), u(t)) \ \mid \ u(t) \in V_u\} \\
x^+ = R(x^-) \quad \text{at } t_1, \cdots, t_n \\
\forall t,\ x(t) \in \mathcal{K} \\
x(0) = x^0,
\end{array}\right. \tag{3.7}
$$

where U is a continuous function $\mathcal{L}^1(I\!R^{2m+1}, I\!R^+)$ and $t_1, \cdots, t_n$ are discrete times, if and only if it is located on the upper boundary in the direction of high y of the capture-viability kernel $\text{Capt}_{(F,R,-U)}(\mathcal{K} \times \mathbb{R}, \mathcal{K} \times \{0\})$ for the augmented dynamic

$$
\left\{\begin{array}{rcl}
x'(t) & \in & F(x(t)) \\
x^+ & = & R(x^-) \quad \text{at } t_1, \cdots, t_n \\
y'(t) & = & -U(x(t)) \\
\forall t,\ x(t) & \in & \mathcal{K} \\
x(0) & = & x^0 \\
y(0) & = & y^0.
\end{array}\right. \tag{3.8}
$$

Bonneuil (2012) also showed that the infinite time problem ($T = \infty$) is conveniently approximated by $T < \infty$ large enough, which is made possible by the discount term $\exp(-\rho t)$. The numerical results below correspond to $T = 30$ and $\rho = 5\%$. We present the method for the approximation of (2.1) in finite time horizon T. The criterion becomes

$$
\max_{C_j, t_j} \int_0^T U(C_j(t), P(t)) e^{-\rho t}\, dt. \tag{3.9}
$$

By increasing the time horizon T to infinity, the solution of problem (3.9) converges to problem (2.1).

Proposition 3.3. The value function in the infinitesimal horizon control problem:

$$
\max_{C_j, t_j} \int_0^\infty U(C_j(t), P(t)) e^{-\rho t}\, dt, \quad j = 1, 2, \tag{3.10}
$$

with $(C_1(0), C_2(0), P(0)) = (C_1^0, C_2^0, P^0)$, is related to the capture-viability kernel

$$\text{Capt}^{(\infty)}_{(F,R,-U)}(\mathcal{K} \times \mathbb{R}^+, \mathcal{K} \times \{0\}) := \bigcup_{T \geq 0} \text{Capt}^{(T)}_{(F,R,-U)}(\mathcal{K} \times \mathbb{R}^+, \mathcal{K} \times \{0\}) \tag{3.11}$$

by:

$$\overset{\infty\sup}{V}(x) = \sup_{(x,y) \in \text{Capt}_{(F,R,-U)}(\mathcal{K} \times I\!R^+, \mathcal{K} \times \{0\})} y \tag{3.12}$$

Proof: in Bonneuil (2012).

On this basis, we now innovate by finding all the Nash equilibria of {2.1, 2.2, 2.3}. We do this by solving the two optimization programs (3.9) jointly, with value functions y_1 and y_2. The idea is first to identify viable states in $\mathcal{K} \times [0, T] \times \mathbb{R}^2$ with target $\mathcal{K} \times \{T, 0, 0\}$ under the dynamic {2.2, 2.3} augmented with the three differential equations:

$$\left\{ \begin{array}{rcl} t' & = & 1 \\ y_j'(t) & := & -U(C_j(t), P(t)) e^{-\rho t}, \ j = 1, 2, \end{array} \right. \tag{3.13}$$

with initial conditions

$$x(0) = x^0, \ t(0) = 0; \ y_1(0) = y_1^0, y_2(0) = y_2^0. \tag{3.14}$$

This is achieved by Bonneuil (2006)'s viability algorithm applied to identify viable states belonging to $\text{Capt}_{\{2.2,2.3,3.13\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$ of the augmented dynamic. This is performed in the ten-dimensional state space (K_1, K_2, P, A_1, A_2, α_1, α_2, t, y_1, y_2) with the four controls t_1, t_2, C_1, and C_2. However, the initial conditions of t, A_j, and α_j are fixed, so there remain five dimensions in which to search for viable states.

The boundary in the direction of high y_1 of $\text{Capt}_{\{2.2,2.3,3.13\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$. gathers all initial conditions from which there exists at least one trajectory satisfying all state constraints and such that $y_1(0)$ is maximal, considering the variables taken by country 2 as given. The same is true for the boundary in the direction of high y_2, such that $y_2(0)$ is maximal, considering the values of the variables taken by country 1 as given. The set of Nash viable equilibria, where each country maximizes its value function taking into account that the other country also maximizes its value function, is then exactly the intersection of these two boundaries (necessary and sufficient conditions, by construction).

4. Numerical set-up

4.1. **Viability Algorithm.** The set of constraints $\mathcal{K}$ and the target Ω are defined by inequalities:

$$\mathcal{K} \cup \Omega =: \{\xi(z) \leq 0\}. \tag{4.1}$$

A viable state satisfies:

$$\text{Inf}_{z(.) \in S(z)} \text{Sup}_{t \in [0,T]} \xi(z(t)) \leq 0, \tag{4.2}$$

which expresses the fact that there exists at least one trajectory $z(.)$ belonging to the set of solutions $S(z)$ starting from z such that the constraint $\xi(z)$ is non positive for all t in $[0, T]$, and consequently, that $z(.)$ remains in the set $\mathcal{K}$.

Bonneuil (2006) uses stochastic optimization to solve (4.2). The state z is viable if and only if at least one solution $z(.)$ with $z(0) = z$ exists. Instead of trying z at random, a search for a viable z is done by traveling in the state space with a probability of decreasing the cost $\mathrm{Inf}_{z(.)\in S(z)}\mathrm{Sup}_{t\in[0,T]}\xi(z(t))$ over z, until this cost ever becomes non positive. This probability increases with the total number of trials. The search for z is proceeded by stochastic optimization, too.

With the addition of the auxiliary variables y_1, y_2, and time t now considered as a state variable in the augmented dynamic, the search for a viable optimum for population 1 consists in first finding a viable state (x, y_1^0, y_2^0), then increasing y_1^0 step by step at x and y_2^0 fixed until obtaining a non viable state. Refinement yields y_1^0 on the boundary of the capture-viability kernel in the direction of high y_1 of the viability kernel associated with the augmented dynamic (Bonneuil, 2012). There is no need to compute the whole capture-viability kernel, which is very time-consuming. The same applies when seeking the viable optimum for population 2.

4.2. **Three representative cases.** We consider three important cases, so as to understand the effect of the variation of productivity and parameters:

- case 1: the introduction of cleaner energy decreases the marginal contribution to pollution and the productivity of capital equally in both countries: $A_1^{(1)} = A_2^{(1)} = 1.5$, $A_1^{(2)} = A_2^{(2)} = 1.1$, $\alpha_1^{(1)} = \alpha_2^{(1)} = 0.05$, $\alpha_1^{(2)} = \alpha_2^{(2)} = 0.03$;
- case 2: the introduction of cleaner energy decreases the productivity of capital in both countries equally; it also decreases the marginal contribution to pollution in both countries, but in country 2 more than in country 1: $A_1^{(1)} = A_2^{(1)} = 1.5$, $A_1^{(2)} = 1.3$, $A_2^{(2)} = 1.0$, $\alpha_1^{(1)} = \alpha_2^{(1)} = 0.05$, $\alpha_1^{(2)} = \alpha_2^{(2)} = 0.03$;
- case 3: the introduction of cleaner energy decreases the productivity of capital in both countries, but in country 2 more than in country 1; it also decreases the marginal contribution to pollution equally in both countries: $A_1^{(1)} = A_2^{(1)} = 1.5$, $A_1^{(2)} = A_2^{(2)} = 1.1$, $\alpha_1^{(1)} = \alpha_2^{(1)} = 0.05$, $\alpha_1^{(2)} = 0.03$, $\alpha_2^{(2)} = 0.01$.

For all these cases, we take classical values for the parameters: discount rate $\rho = 0.05$, common capital share $\nu = 0.5$, $\delta_K = 0.1$, and $\delta_P = 0.1$.

We draw initial states $(K_1^0, K_2^0, P^0, y_1^0, y_2^0)$ at random, test their viability status by the viability algorithm, reject those tested not viable, until obtaining 300 viable states. For each of them, a one-dimensional optimization yields the highest value of y_1^0 (at y_2^0 fixed). These points belong to the upper boundary in the direction of high y_1 of the capture viability kernel $\mathrm{Capt}_{\{2.2,2.3,3.13\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$. We approximate this manifold by penalized least squares on the basis of the regular grid $\{0, 0.1, \cdots, 1.0\}^3$ of the cube (K_1^0, K_2^0, P^0) scaled to $[0, 1]^3$.

Similarly, we obtain 300 points $(K_1^0, K_2^0, P^0, y_1^0, y_2^0)$ viable in $\mathcal{K} \times \mathbb{R}^2$ and for which the value function y_2^0 is maximal. These points likewise delineate a manifold numerically approximated by penalized least squares.

The smoothed manifold built from the 300 scattered points obtained on each upper boundary must be truncated so as to retain only the states for which (K_1^0, K_2^0, P^0) is viable for {2.2, 2.3}. To do this, we smooth the viability kernel of problem {2.2, 2.3} by penalized least squares, successively P^0 as a function of K_1^0 and K_2^0, then K_1^0 as a function of K_2^0 and P^0, and finally K_2^0 as a function of K_1^0 and P^0. The computation of one viable state x with its two optima y_1^0 and y_2^0 requires 11 computing hours on a Dell Precision M6600, or 137 computing days for the 2×300 points. We have indeed demanding requirements for stochastic optimization, which already calls for a long computing time when run in a single procedure: to recall, combining viability with optimization —moreover in 6 state dimensions and with sufficient precision— involves a first double stochastic optimization to find a viable point (one optimization on the choice of the initial value, and one for testing the viability status by trying to satisfy (4.2) of each of these initial values). Once a viable state is found, we search for the optimum y_1 at (x, y_2) fixed by stochastic optimization on y_1, embedding stochastic optimization procedures to assess the viability status of each value (x, y_1, y_2). We repeat the double optimization for y_2 at (x, y_1) fixed. There is no free lunch: our ability to work in a large state dimension without any grid, which would exceed the capacity of most computers, is counterbalanced out by a long computing time, which is fortunately within tractable limits.

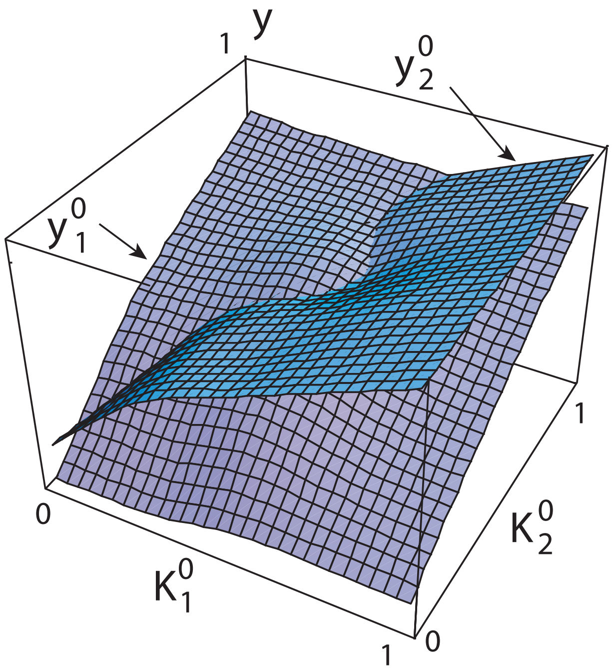

FIGURE 1. Section at $P^0 = 0.1$ (for P^0 normalized between 0 and 1) of the Nash equilibria as intersection of the boundaries in the direction of high y_j, $j = 1, 2$, of the two capture-viability kernels. Coordinates scaled between 0 (minimal computed value) and 1 (maximal computed value). Parameter values: $\rho = 0.05$, $\nu = 0.5$, $\delta_K = 0.1$, $\delta_P = 0.1$.

4.3. **Nash equilibria.** From the two truncated smoothed manifolds representing the two upper boundaries of $\mathrm{Capt}_{\{2.2,2.3,3.13\}}(\mathcal{K} \times \mathbb{R}^3, \mathcal{K} \times \{T, 0, 0\})$ in the direction of either y_1 or y_2, we compute the intersection of the two surfaces of viable optima. This intersection is the locus of viable Nash equilibria. There, by construction, each country maximizes its value function assuming that the other country does the same. This geometrical identification has the merit to be transparent

Figure 1 shows an example of a section at given P^0. The switching times t_1 and t_2, as well as the mean values of consumption $\overline{C_j}^{(t<t_j)}$ and $\overline{C_j}^{(t\geq t_j)}$, $j = 1, 2$, associated with these Nash equilibria likewise are obtained by penalized least squares from the values associated with the computed viable optima.

Figure 2 presents an example of a trajectory associated with a Nash equilibrium. The procedure uses smoothing procedures to produce viable states optimal with respect either to y_1 or to y_2. The intersection enables the determination of Nash equlibria. To obtain a trajectory associated with a viable Nash equlibrium, a further stochastic optimization is necessary, with now fixed starting and terminal states. Figure 2 shows the regular decline of the value functions to 0, and the rather irregular variation of control variables C_1 and C_2.

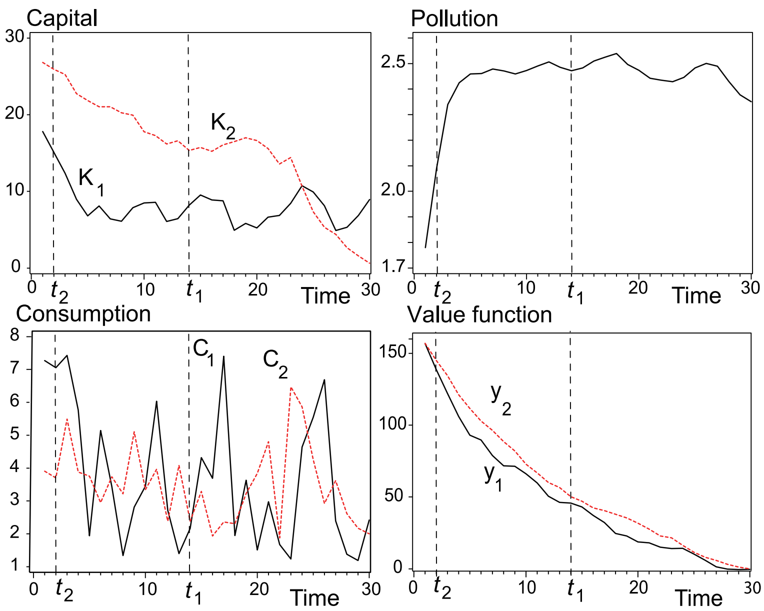

FIGURE 2. Example of a Nash equilibrium $(K_1(0), K_2(0), P(0), y_1(0), y_2(0))$ and its associated trajectory in state and control variables C_1, C_2. The switching times t_1 and t_2 are indicated by the vertical lines. Parameter values: $\rho = 0.05$, $\nu = 0.5$, $\delta_K = 0.1$, $\delta_P = 0.1$.

The value functions at *viable Nash equilibria* and their associated switching times and mean consumption values all depend only on the initial conditions K_1^0, K_2^0, and P^0 through the maximization (3.9). Therefore, in order to characterize Nash equilibria, we cannot take switching times or consumption levels as explanatory variables in regressions. These control variables are endogenous, and are entirely determined by the initial conditions K_1^0, K_2^0, and P^0. That is why we characterize the viable Nash equilibria in regressing the value function and its controls in a system of seemingly unrelated regressions having only these initial conditions as explanatory variables. The three systems, one for each case, are embedded in a single model, to allow testable comparisons (with τ denoting transposition):

$$Y = \sum_{i=1}^{3} B^{\tau} X 1_{\text{case } i} \tag{4.3}$$

where $Y^{\tau} = (y, t_1,\ t_2,\ \overline{C_1}^{(t<t_1)},\ \overline{C_2}^{(t<t_2)},\ \overline{C_1}^{(t\geq t_1)},\ \overline{C_2}^{(t\geq t_2)})$ is the vector made of the value function at the Nash equilibrium, the associated switching times, and the mean consumption values. The explanatory variables are $X^{\tau} = (1, K_1^0, K_2^0, P^0)$, B is a matrix of coefficients, and $1_{\text{case } i}$ is the indicator of case $i = 1, 2, 3$. The value function at *viable Nash equilibria* y, capital values K_1^0 and K_2^0, pollution level P^0, and consumption levels are re-scaled between 0 and 1 within their ranges of variation. Significant coefficients at 5% are indicated by a star; standard deviations appear in parentheses below the coefficients. We present the three cases successively, although they are estimated jointly in (4.3).

- Case 1 (after technological switch, both countries are of equal productivity and contribute equally to pollution)

$$\left\{\begin{array}{rcl}
E(y) & = & \underset{(0.01)}{0.67^*} + \underset{(0.02)}{0.20^*}\, K_1^0 + \underset{(0.02)}{0.20^*}\, K_2^0 - \underset{(0.01)}{0.10^*}\, P^0 \\
E(t_1) & = & \underset{(0.21)}{13.23^*} + \underset{(0.45)}{5.70^*}\, K_1^0 + \underset{(0.45)}{0.95^*}\, K_2^0 + \underset{(0.29)}{1.23^*}\, P^0 \\
E(t_2) & = & \underset{(0.17)}{12.87^*} + \underset{(0.36)}{1.01^*}\, K_1^0 + \underset{(0.36)}{5.81^*}\, K_2^0 + \underset{(0.23)}{1.71^*}\, P^0 \\
E(\overline{C_1}^{(t<t_1)}) & = & \underset{(0.019)}{0.16^*} + \underset{(0.041)}{0.12^*}\, K_1^0 + \underset{(0.04)}{0.38^*}\, K_2^0 + \underset{(0.026)}{0.013}\, P^0 \\
E(\overline{C_2}^{(t<t_2)}) & = & \underset{(0.02)}{0.11^*} + \underset{(0.04)}{0.41^*}\, K_1^0 + \underset{(0.04)}{0.14^*}\, K_2^0 - \underset{(0.022)}{0.010}\, P^0 \\
E(\overline{C_1}^{(t\geq t_1)}) & = & \underset{(0.02)}{0.30^*} - \underset{(0.04)}{0.22^*}\, K_1^0 + \underset{(0.04)}{0.34^*}\, K_2^0 - \underset{(0.025)}{0.001}\, P^0 \\
E(\overline{C_2}^{(t\geq t_2)}) & = & \underset{(0.02)}{0.25^*} + \underset{(0.04)}{0.35^*}\, K_1^0 - \underset{(0.04)}{0.19^*}\, K_2^0 - \underset{(0.024)}{0.008}\, P^0,
\end{array}\right. \tag{4.4}$$

where E denotes expectancy. As expected, because the two countries have the same parameters, the coefficients of K_1^0 and K_2^0 are not significantly different from each other. The higher the pollution level at the start, the lower the maximal value function. The higher the initial capital value, the higher the consumption level in the same country before the switching and the lower the consumption level thereafter: people consume more when their production capacity is higher. Expecting that the other country's inhabitants are doing the same, they increase consumption, while the higher the other country's initial capital, the more they delay their transition to the less polluting technology.

The higher the initial capital value, the more each country consumes before switching and the longer it delays switching, while reducing its consumption after switching. Combining the equations for consumption and timing, the mean consumption over the entire period $[0, T]$ increases with the initial capital values of both countries.

The Nash equilibria then reflect a race to consume more when the other participant is also expected to consume more. Logically, a longer time to switching is associated with more pollution, that is, with a higher initial pollution level.

- Case 2 (after technological switch, Country 2's productivity is less than Country 1's)

$$
(4.5)\qquad \begin{cases}
E(y) &= \underset{(0.01)}{0.70} + \underset{(0.02)}{0.23^*}\, K_1^0 + \underset{(0.005)}{0.030^*}\, K_2^0 - \underset{(0.01)}{0.10^*}\, P^0 \\
E(t_1) &= \underset{(0.98)}{23.71^*} - \underset{(2.18)}{23.89^*}\, K_1^0 - \underset{(0.47)}{1.97^*}\, K_2^0 - \underset{(0.45)}{5.23^*}\, P^0 \\
E(t_2) &= \underset{(0.33)}{16.51^*} + \underset{(0.73)}{12.51^*}\, K_1^0 - \underset{(0.16)}{9.96^*}\, K_2^0 - \underset{(0.15)}{7.08^*}\, P^0 \\
E(\overline{C_1}^{(t<t_1)}) &= \underset{(0.01)}{0.18^*} + \underset{(0.01)}{0.56^*}\, K_1^0 - \underset{(0.001)}{0.031^*}\, K_2^0 - \underset{(0.001)}{0.030^*}\, P^0 \\
E(\overline{C_2}^{(t<t_2)}) &= \underset{(0.01)}{0.24^*} - \underset{(0.01)}{0.05^*}\, K_1^0 + \underset{(0.01)}{0.46^*}\, K_2^0 - \underset{(0.01)}{0.04^*}\, P^0 \\
E(\overline{C_1}^{(t\geq t_1)}) &= \underset{(0.01)}{0.28^*} + \underset{(0.01)}{0.21^*}\, K_1^0 + \underset{(0.01)}{0.07^*}\, K_2^0 - \underset{(0.001)}{0.012^*}\, P^0 \\
E(\overline{C_2}^{(t\geq t_2)}) &= \underset{(0.01)}{0.32^*} - \underset{(0.01)}{0.04^*}\, K_1^0 + \underset{(0.01)}{0.11^*}\, K_2^0 - \underset{(0.01)}{0.08^*}\, P^0.
\end{cases}
$$

The asymmetry of productivity after the transition has modified the directions of association in switching times and initial capital values. The lower productivity of Country 2 makes Country 1 leader of the game: the value function depends mainly on consumption in Country 1, then on its initial capital endowment K_1^0 (coefficient 0.23 in the expression of $E(y)$), because Country 2's capital grows more slowly. Pollution then comes mainly from Country 1, whose capital grows faster after switching.

To compensate, the higher K_1^0 (coefficient -23.89 in the expression of $E(t_1)$), the earlier Country 1 switches. Meanwhile, Country 2 consumes with respect to its capital capacities allowed by its initial capital K_2^0 (coefficients 0.46 in $E(\overline{C_2}^{(t<t_2)})$ and 0.11 in $E(\overline{C_2}^{(t\geq t_2)})$), but, to keep up with Country 1, delays switching all the more as K_1^0 is high (coefficient 12.51 in $E(t_2)$), and continues using the more polluting technology as long as possible. However, delaying because of country 1's higher productivity is possible comes at the price of reduced consumption (coefficients -0.05 and -0.04 in $E(\overline{C_2}^{(t\geq t_2)})$). *The differential in productivity encourages the more productive country to switch earlier and the less productive country to switch later than they would do in the absence of a differential.*

More pollution in any case triggers earlier technological switching (coefficients -5.23 in $E(t_1)$ and -7.08 in $E(t_2)$), and limits consumption (coefficients -0.030 in $E(\overline{C_1}^{(t<t_1)})$, -0.04 in $E(\overline{C_2}^{(t<t_2)})$, -0.012 in $E(\overline{C_1}^{(t\geq t_1)})$, -0.08 in $E(\overline{C_2}^{(t\geq t_2)})$).

The higher the initial capital value K_1^0, the more Country 1 consumes before and after switching (coefficients 0.56 in $E(\overline{C_1}^{(t<t_1)})$ and 0.21 in $E(\overline{C_1}^{(t\geq t_1)})$), but also the sooner the switch (coefficient -23.89 in $E(t_1)$), and the later for Country 2 (coefficient 12.51 in $E(t_2)$). Country 1, expecting Country 2 to

switch to a less productive and less polluting technology, is better off consuming when Country 2 pollutes less. Conversely, Country 2, anticipating that Country 1 will use its advantage to hasten its switch, is better off postponing its own switching as long as K_1^0 is high.

As a result, combining the equations of consumption and timing, the mean consumption for each country over the entire period $[0, T]$ increases with initial capital then decreases: *"poor" countries consume less but for longer with the more polluting technology; while "rich" countries consume more (and pollute more) but for a shorter period of time.*

- Case 3 (after technological switch, Country 2 contributes less to total pollution than Country 1)

$$
(4.6) \quad \begin{cases}
E(y) &= \underset{(0.01)}{0.60^*} + \underset{(0.04)}{0.58^*}\, K_1^0 + \underset{(0.03)}{0.04}\, K_2^0 - \underset{(0.01)}{0.05^*}\, P^0 \\
E(t_1) &= \underset{(0.06)}{11.21^*} + \underset{(0.52)}{7.13^*}\, K_1^0 + \underset{(0.37)}{9.25^*}\, K_2^0 - \underset{(0.06)}{0.78^*}\, P^0 \\
E(t_2) &= \underset{(0.02)}{5.73^*} - \underset{(0.17)}{1.16^*}\, K_1^0 - \underset{(0.12)}{3.99^*}\, K_2^0 - \underset{(0.02)}{0.67^*}\, P^0 \\
E(\overline{C_1}^{(t<t_1)}) &= \underset{(0.01)}{0.15^*} + \underset{(0.01)}{0.67^*}\, K_1^0 + \underset{(0.01)}{0.02^*}\, K_2^0 - \underset{(0.01)}{0.03^*}\, P^0 \\
E(\overline{C_2}^{(t<t_2)}) &= -\underset{(0.01)}{0.03^*} + \underset{(0.07)}{0.58^*}\, K_1^0 + \underset{(0.05)}{0.32^*}\, K_2^0 - \underset{(0.01)}{0.13^*}\, P^0 \\
E(\overline{C_1}^{(t\geq t_1)}) &= \underset{(0.01)}{0.32^*} + \underset{(0.04)}{0.09^*}\, K_1^0 - \underset{(0.03)}{0.14^*}\, K_2^0 + \underset{(0.01)}{0.09^*}\, P^0 \\
E(\overline{C_2}^{(t\geq t_2)}) &= \underset{(0.01)}{0.14^*} + \underset{(0.01)}{0.10^*}\, K_1^0 + \underset{(0.01)}{0.21^*}\, K_2^0 - \underset{(0.001)}{0.013^*}\, P^0.
\end{cases}
$$

The value again depends mainly (coefficient 0.58 in $E(\overline{C_2}^{(t<t_2)})$) on the country contributing most to pollution, which is also the country that waits longer to switch, the higher the initial capital values (coefficients 7.13 and 9.25 in $E(t_1)$). This is because Country 1 has to produce longer under the more polluting regime to attain the same cumulative discounted utility as Country 2, whose utility is less penalized by pollution. At the Nash equilibrium, to obtain the same utility, the country with the higher pollution penalty after the transition remains as long as possible in the more productive regime. This strategy is more polluting, and consequently detrimental to Country 2. This continues to the point that in Country 1, higher consumption means more pollution (coefficients 0.03 in $E(\overline{C_1}^{(t<t_1)})$ and 0.09 in $E(\overline{C_1}^{(t\geq t_1)})$).

Meanwhile, expecting this behavior, Country 2's best strategy is to switch early so as to limit the pollution level, a behavior which is favorable to Country 1. More pollution entails less consumption (coefficients -0.13 in $E(\overline{C_2}^{(t<t_2)})$ and -0.013 in $E(\overline{C_2}^{(t\geq t_2)})$). The Nash equilibrium is cynical in that the polluter is encouraged to pollute by the other player, who is cleaner. Anticipating this, Country 2 consumes (and pollutes) all the more, the higher the initial capital values Country 1 starts from. *Non cooperation has the effect that reducing one's contribution results in more total consumption and more total pollution,.* Country 2, which gains utility by polluting less after the switch, is better off hastening this transition when both initial capital values are high.

The mean consumption over the entire period $[0, T]$ increases for both countries, as in the symmetric case.

Numerically, consumption levels C_1 and C_2 of countries 1 and 2, which are controls of the problem $\{2.1, 2.2, 2.3\}$, do not look bang bang. To confirm, we ran the optimization procedure from identical 100 initial values of the state variables, by admitting only extreme values $\min C_j$ and $\max C_j$ for the controls C_j, $j = 1, 2$. For each of these 100 initial points, we find smaller optimal values y_1 and y_2 with bang-bang controls than with control variables C_j admissible in $[\min C_j, \max C_j]$.

5. Conclusion

Taking Nash equilibria as located at the intersection of the two boundaries of the capture viability kernels in the directions of high y_1 and high y_2 allows us to compute Nash equilibria without resorting to simplifying endogenous assumptions on the controls, to take state constraints into account, to deal with non linearity, and to avoid the round-about method based on solving differential equations. This is an opportunity opened conjointly by the concept of capture-viability, by Bonneuil's (2012) theorem on the location of viable optima, and by Bonneuil's (2006) viability algorithm in large state dimension.

By examining three games: the symmetric case, the case where one country is less productive after switching to less polluting technology, and the case where one country contributes less to pollution after switching, we highlighted the interdependence of the two countries at the Nash equilibrium, again without having postulated any endogenous mechanism, as is often done. The model and the examined cases allow us to predict the conditions for technological switching: we showed that the absence of cooperation leads to undesirable consequences, where efforts to pollute less motivate the other country to pollute more, or encourage the country that will be cleaner or less productive after switching, to delay its transition.

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