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OPTIMALITY CONDITIONS FOR DISCRETE-TIME OPTIMAL CONTROL ON INFINITE HORIZON

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ABSTRACT. The paper presents first order necessary optimality conditions of Pontrygin's type for a general class of discrete-time optimal control problems on infinite horizon. The main novelty is that the adjoint function, for which the (local) maximum condition in the Pontryagin principle holds, is explicitly defined for any given optimal state-control process. This is done based on ideas from previous papers of the first and the last authors concerning continuous-time problems. In addition, the obtained (local) maximum principle is in a normal form, and the optimality has the general meaning of weakly overtaking optimality (hence unbounded processes are allowed), which is important for many economic applications. Two examples are given, which demonstrate the applicability of the obtained results in cases where the known necessary optimality conditions fail to identify the optimal solutions.

1. Introduction

Optimal control theory provides a relevant and widely used instrument for economic analysis. Many aspects in the development of control theory were motivated by economic applications, in particular such on infinite horizon. A large number of investigations have been devoted to that kind of problems, most of them for continuous-time (ODE) models. However, discrete-time economic models are often even more relevant due to the truly discrete nature of the economic decisions. Even more, the applications of such models are not limited to economics. For example, Model Predictive Control, which is a main mathematical tool for engineering process control, has intrinsic relations to optimal control of discrete-time infinite-horizon systems (see e.g. the book [9] by L. Gruene and J. Pannek). On the other hand, the optimal control theory for discrete-time problems on infinite horizon is far from being complete. A comprehensive account of the state of the art in the area, what concerns optimality conditions, is given in the recent book [6] by J. Blot and N. Hayek. For other aspects of this theory we refer to A. Zaslavski [13] and the bibliography therein.

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In the present paper we consider a general discrete time optimal problem of the form

$$\sum_{k=0}^{\infty} g(k, x_k, u_k) \to \max,$$

subject to the dynamics

$$x_{k+1} = f(k, x_k, u_k), \quad k = 0, 1, \dots,$$

with a given initial state x_0 and control constraints $u_k \in U_k$. The states x_k and the control vectors u_k belong to finite-dimensional spaces (an exact formulation is given in the next section).

One specific feature of discrete-time optimal control problems (in contrast to continuous-time ones) is that the Pontryagin-type maximum principles on a finite horizon (developed first in the book [7] by V.G. Boltyanskij) are known to have a local form. That is, if no additional concavity-type conditions are posed, the maximum condition for the associated Hamiltonian represents, in fact, only a necessary condition for a local maximum. This also applies to problems on infinite horizons, in particular to the results in the present paper.

The (local) maximum condition for the Hamiltonian, which for the problem indicated above has the form[1] $H(k, x, u, \psi, \lambda_0) := \lambda_0 g(k, x, u) + \psi f(k, x, u)$, holds along a specific solution $\{\psi_k\}_{k=1}^{\infty}$ of the so-called *adjoint equation*. In order to identify the "right" solution of the adjoint equation one needs additional conditions, usually in the form of *transversality conditions* for ψ_k when $k \to \infty$. In contrast to the continuous-time case, such conditions are present only in a few contributions that we discuss in the next lines.

In the very enlightening and general paper [12], P. Michel, introduced several types of transversality conditions (some of them known from earlier publications, see the account in [12]), among which we mention the following ones:

$$(1.1) \qquad (i)\ \lim_{k\to\infty} \psi_k = 0; \quad (ii)\ \lim_{k\to\infty} \psi_k\, x_k^* = 0; \quad (iii)\ \lim_{k\to\infty} \psi_k\,(x_k - x_k^*) \geq 0,$$

where $\{x_k^*\}_{k=1}^{\infty}$ is an optimal state sequence, and in (iii) $\{x_k\}_{k=1}^{\infty}$ is an arbitrary state sequence. This is done under concavity-type conditions on the problem.

In J. Blot and N. Hayek [6, Theorem 3.2, Section 3.2.3] it was proved that a local maximum principle holds with a solution $\{\psi_k\}_{k=1}^{\infty}$ of the adjoint equation, which satisfies the transversality condition (1.1)-(i), provided that (among several additional assumptions which, however, do not concavity) there are no control constraints and the optimal solution (control and trajectory) of the problem is bounded. It is remarkable that the maximum principle holds in a normal form, that is, with $\lambda_0 = 1$. We mention that for problems on infinite horizon the maximal principle does not necessarily hold in the normal form, even if no state constraints are involved (in contrast to the finite-horizon case).

[1] In order to avoid multiple use of the transposition sign, everywhere in the paper we consider the adjont vectors ψ as row-vectors, while x, u and f are column-vectors.

In the present paper we obtain that for any optimal solution $(\mathbf{x}^*, \mathbf{u}^*) = \{(x_k^*, u_k^*)\}_{k=1}^{\infty}$, the (local) maximum condition

$$\frac{\partial}{\partial u} H(k, x_k^*, u_k^*, \psi_{k+1}, \lambda_0)\, v \le 0 \quad \text{for every } v \in T_{U_k}(u_k^*),\ k = 0, 1, \dots, \tag{1.2}$$

holds with $\lambda_0 = 1$ (normal form) and with an adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$ which is defined by an explicit formula (see the next section). Here $T_{U_k}(u_k^*)$ is the contingent cone to the set U_k at the point $u_k^* \in U_k$. The optimality is understood in the sense of *weak overtaking optimality* (see e.g. [8] and the definition in the next section), so that problems with infinite vales of the objective function are included. In fact, the explicit definition of the adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$ selects the "right" solution of the usual adjoint equation, for which the maximum condition (1.2) holds. The result is based on an assumption, which guarantees that the definition of ψ_k produces a finite vector. Of course, this assumption rules out abnormal optimal processes for which the maximum principle cannot have a normal form. On the other hand, it is not restrictive for a large class of problems, including rather challenging ones, where the known optimality conditions do not hold or are not informative. In particular, the transversality conditions (1.1)-(i),(ii) are not necessarily satisfied for the explicitly defined adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$, while (1.1)-(iii) is too weak to identify a useful adjoint sequence. Two such examples are given in the last section: a complete solution of the discrete-time version of the well-known Halkin example [11] is obtained, and an economic model of optimal utilization of a non-renewable resource is solved.

The paper is structured as follows. The next section contains the exact formulation of the problem and the main results, including discussions. Section 3 presents two examples illustrating the advantages of the obtained results.

2. The discrete-time control problem

Denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{N}$ – the set of all non-negative integers. Let G be an open subset of $\mathbb{R}^n$, U_k, $k \in \mathbb{N}$, be nonempty subsets of $\mathbb{R}^m$, and let for every $k \in \mathbb{N}$, a function $f(k, \cdot, \cdot) : G \times \tilde{U}_k \to \mathbb{R}^n$, be given. Here $\tilde{U}_k$ denotes an open set containing U_k (the case $\tilde{U}_k = U_k$ is not excluded). We consider the discrete-time control system

$$x_{k+1} = f(k, x_k, u_k),\ u_k \in U_k,\ k \in \mathbb{N}, \tag{2.1}$$

with a given initial state $x_0 \in G$, where x_k and u_k, $k = 0, 1, \dots$, are regarded as state and control variables, respectively. Any sequence $\mathbf{u} = \{u_k\}_{k=0}^{\infty}$ with $u_k \in U_k$, $k \in \mathbb{N}$, is called admissible control. For a given an admissible control $\mathbf{u}$, equation (2.1) generates a trajectory $x_0, x_1, \dots$ extendible either to the minimal number k such that $f(k, x_k, u_k) \notin G$ (if such exists) or to infinity. In the latter case we call the pair $(\mathbf{u}, \mathbf{x} := \{x_k\}_{k=0}^{\infty})$ *admissible process.*

Given an admissible process $(\mathbf{u}, \mathbf{x})$, the trajectory $\{x_k\}_{k=0}^{\infty}$ can be represented from (2.1) as

$$x_{k+1} := f_{u_k}^{k} \circ f_{u_{k-1}}^{k-1} \circ \dots \circ f_{u_0}^{0}(x_0), \quad k \in \mathbb{N},$$

where $f_u^k(x) := f(k, x, u)$ for $x \in G$ and $u \in \tilde{U}_k$, and $\circ$ denotes the composition of the corresponding maps.

For a sequence of functions $g(k, \cdot, \cdot) : G \times \tilde{U}_k \to \mathbb{R}$, $k \in \mathbb{N}$, we consider the optimal control problem

$$\sum_{k=0}^{\infty} g(k, x_k, u_k) \to \max, \tag{2.2}$$

where $\mathbf{u} = \{u_k\}_{k=0}^{\infty}$ together with $\mathbf{x} = \{x_k\}_{k=0}^{\infty}$ is an admissible process. In order to define the meaning of this problem, for any $\omega \in \mathbb{N}$ we denote

$$J^{\omega}(\mathbf{u}, \mathbf{x}) = \sum_{k=0}^{\omega} g(k, x_k, u_k).$$

Definition 2.1. An admissible process $(\mathbf{u}^*, \mathbf{x}^*)$ is called weakly overtaking optimal if for each $\varepsilon > 0$, for each positive number ω and for each admissible process $(\mathbf{u}, \mathbf{x})$ there exists a positive integer $\tilde{\omega} > \omega$ such that $J^{\tilde{\omega}}(\mathbf{u}^*, \mathbf{x}^*) \geq J^{\tilde{\omega}}(\mathbf{u}, \mathbf{x}) - \varepsilon$.

Let $(\mathbf{u}^*, \mathbf{x}^*)$ be a weakly overtaking optimal process. For any $k \in \mathbb{N}$ and for each vector ξ we denote by $x^{k,\xi} = (x_k, x_{k+1}, \ldots)$ the trajectory of (2.1) starting from $x_k = \xi$ at the instant k, that is,

$$x_{s+1}^{k,\xi} := f_{u_s^*}^s \circ \cdots \circ f_{u_k^*}^k(\xi), \quad s = k, k+1, \ldots. \tag{2.3}$$

Clearly, $x^{k,\xi}$ may happen to be an infinite sequence or may terminate at the minimal $s > k$ such that $f_s(s, x_s^{k,\xi}, u_s^*) \notin G$.

We assume that for every $k \in \mathbb{N}$ the functions $f(k, \cdot, \cdot)$ and $g(k, \cdot, \cdot)$ are continuously differentiable on $G \times \tilde{U}_k$.

In the next assumption and further we use the notation $\mathbf{B}(x; \alpha)$ for the closed ball with radius α centered at x.

Assumption A. For each $k \in \mathbb{N}$ there exists $\alpha_k > 0$ and a sequence $\{\beta_s^k\}_{s=k}^{\infty}$ with $\sum_{s=k}^{\infty} \beta_s^k < \infty$ such that $\mathbf{B}(x_k^*; \alpha_k) \subset G$, for every $\xi \in \mathbf{B}(x_k^*; \alpha_k)$ the sequence $x^{k,\xi}$ is infinite, and

$$\sup_{\xi \in \mathbf{B}(x_k^*; \alpha_k)} \left\| \frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*) \right\| \leq \beta_s^k.$$

Assumption A implies that for every $k \in \mathbb{N}$ the series

$$\sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*)$$

is absolutely convergent, uniformly with respect to $\xi \in \mathbf{B}(x_k^*; \alpha_k)$. Due to the identity

$$g(s, x_s^{k,\xi}, u_s^*) = g\left(s, f_{u_{s-1}^*}^{s-1} \circ \cdots \circ f_{u_k^*}^k(\xi), u_s^*\right),$$

we have by the chain rule that

$$\frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*) = g_x'(s, x_s^{k,\xi}, u_s^*) \prod_{i=s-1}^{k} f_x'(i, x_i^{k,\xi}, u_i^*) \tag{2.4}$$

for $s \geq k$, where by definition for $s > k$

$$\prod_{i=s-1}^{k} A_i := A_{s-1}\ A_{s-2}\ \dots\ A_k \text{ with } A_i = f'_x(i, x_i^{k,\xi}, u_i^*)$$

$$\text{and } \prod_{i=s-1}^{k} A_i = I \text{ (the identity map) for } s \leq k.$$

We use the symbol $\prod_{i=s}^{k}$ instead of the usual symbol $\prod_{i=s}^{k}$ for products in order to indicate that the "increment" of the running index i is -1 (since $s \geq k$).

Next we define the *adjoint sequence* $\psi := \{\psi_k\}_{k=1}^{\infty}$ as follows

$$\psi_k = \sum_{s=k}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*)_{|\xi = x_k^*}, \quad k = 1, 2, \dots. \tag{2.5}$$

According to assumption A, we have that $\|\psi_k\| < \infty$. Also, taking into account (2.4) and the equality $x_s^{k,x_k^*} = x_s^*$, we obtain that

$$\psi_k = \sum_{s=k}^{\infty} g'_x(s, x_s^*, u_s^*) \prod_{i=s-1}^{k} f'_x(i, x_i^*, u_i^*) \tag{2.6}$$

and the above sum is absolutely convergent.

To formulate the main result of the paper, we denote by $T_{U_k}(u)$ the Bouligand tangent cone to the set U_k at the point $u \in U_k$. We remind that $T_{U_k}(u)$ consists of all $v \in \mathbb{R}^m$ such that there exist a sequence of positive real numbers $\{t_\mu\}_{\mu=1}^{\infty}$ convergent to 0 and a sequence $\{v_\mu\}_{\mu=1}^{\infty} \subset \mathbb{R}^m$ convergent to v such that $u + t_\mu v_\mu \in U_k$ for each $\mu = 1, 2, \dots$ (see, for example [1, Chapter 4.1]).

Theorem 2.2. *Let Assumption A be fulfilled and let* $(\mathbf{u}^*, \mathbf{x}^*) = (\{u_k^*\}_{k=0}^{\infty}, \{x_k^*\}_{k=0}^{\infty})$ *be a weakly overtaking optimal solution. Let the adjoint sequence* $\psi = \{\psi_i\}_{i=1}^{\infty}$ *be defined by (2.5) (or equivalently by (2.6)). Then for every* $k \in \mathbb{N}$ *the following* local maximum condition *holds:*

$$\left(g'_u(k, x_k^*, u_k^*) + \psi_{k+1} f'_u(k, x_k^*, u_k^*)\right) v \leq 0 \quad \text{for every } v \in T_{U_k}(u_k^*). \tag{2.7}$$

Proof. Let us fix arbitrarily $k \in \mathbb{N}$ and $v \in T_{U_k}(u_k^*)$. Without any restriction we may assume that $\|v\| = 1$. Then there exist an $\tilde{\alpha}_k > 0$, a sequence of positive reals $\{t_\mu\}_{\mu=1}^{\infty} \to 0$ and a sequence of elements $\{v_\mu\}_{\mu=1}^{\infty} \to v$ as $\mu \to \infty$ such that $\mathbf{B}(u_k^*; \tilde{\alpha}_k) \subset \tilde{U}_k$ and $u_k^* + t_\mu v_\mu \in U_k$ for each $\mu = 1, 2, \dots$. Without a restriction we may assume that $\|v_\mu\| = 1$ and $t_\mu \leq \tilde{\alpha}_k$. Define the number

$$c := \max_{u \in \mathbf{B}(u_k^*; \tilde{\alpha}_k)} \|f'_u(k, x_k^*, u)\|.$$

Let $\mu_0 > 0$ be such that $ct_\mu < \alpha_{k+1}$ for every $\mu \geq \mu_0$, where α_{k+1} is introduced in assumption A.

For a parameter $\mu \geq \mu_0$ we define the control sequence $\mathbf{u}^\mu$ as follows:

$$u_s^\mu := \begin{cases} u_s^* & \text{for } s \neq k, \\ u_k^* + t_\mu v_\mu, & \text{for } s = k. \end{cases}$$

Clearly, $\mathbf{u}^\mu(t_\mu)$ is an admissible control. The corresponding trajectory $\mathbf{x}^\mu$ can be represented as

$$x_s^\mu := \begin{cases} x_s^*, & \text{for } s = 0, 1, \ldots, k, \\ f(k, x_k^*, u_k^* + t_\mu v_\mu), & \text{for } s = k+1, \\ f(s-1, x_{s-1}^\mu, u_{s-1}^*), & \text{for } s > k+1, \end{cases}$$

i.e. for each $s \geq k+1$ we have that

$$x_s^\mu := f_{u_{s-1}^*}^{s-1} \circ \cdots \circ f_{u_{k+1}^*}^{k+1} \circ f_{u_k^* + t_\mu v_\mu}^{k}(x_k^*).$$

Notice that for $\mu \geq \mu_0$

$$|x_{k+1}^\mu - x_{k+1}^*| = |f(k, x_k^*, u_k^* + t_\mu v_k) - f(k, x_k^*, u_k^*)| \leq c t_\mu \leq \alpha_{k+1},$$

since $\operatorname{co}\{u_k^*, u_k^* + t_\mu v_k\} \subset \mathbf{B}(u_k^*; \tilde{\alpha}_k)$. Then according to assumption A, the trajectory $\{x_s^\mu\}_s$ is defined for all $s \in \mathbb{N}$.

Let us choose an arbitrary positive number ε and an arbitrary $\omega \geq k$ such that

$$c \sum_{s=\omega+1}^{\infty} \beta_s^{k+1} \leq \varepsilon.$$

For any $\mu > \mu_0$ we apply the definition of weak overtaking optimality for ω and for $t_\mu \varepsilon$ instead of ε: there exists a positive integer $\omega_\mu > \omega$ such that

$$\varepsilon t_\mu \geq J^{\omega_\mu}(\mathbf{u}^\mu, \mathbf{x}^\mu) - J^{\omega_\mu}(\mathbf{u}^*, \mathbf{x}^*).$$

Using the mean value theorem we obtain that

$$\begin{aligned} \varepsilon t_\mu \; &\geq \; \sum_{s=0}^{\omega_\mu} (g(s, x_s^\mu, u_s^\mu) - g(s, x_s^*, u_s^*)) \\ &= \; g(k, x_k^*, u_k^* + t_\mu v_\mu) - g(k, x_k^*, u_k^*) + \sum_{s=k+1}^{\omega_\mu} (g(s, x_s^\mu, u_s^*) - g(s, x_s^*, u_s^*)) \\ &= \; t_\mu g_u'(k, x_k^*, \tilde{u}_k^\mu) v_\mu \\ &\qquad + \sum_{s=k+1}^{\omega_\mu} \Big(g(s, x_s^{k+1, f(k, x_k^*, u_k^* + t_\mu v_\mu)}, u_s^*) - g(s, x_s^{k+1, f(k, x_k^*, u_k^*)}, u_s^*) \Big) \\ &= \; t_\mu g_u'(k, x_k^*, \tilde{u}_k^\mu) v_\mu \\ &\qquad + \; t_\mu \sum_{s=k+1}^{\omega_\mu} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s)_{|\xi = \tilde{\xi}_{k+1}^\mu} \, f_u'(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) v_\mu, \end{aligned} \tag{2.8}$$

where $\tilde{u}_k^\mu \in \operatorname{co}\{u_k^*, u_k^* + t_\mu v_\mu\}$, $\tilde{t}_\mu \in [0, t_\mu]$ and $\tilde{\xi}_{k+1}^\mu = f(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu)$.

Notice that

$$\|\tilde{\xi}_{k+1}^\mu - x_{k+1}^*\| = \|f(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) - f(k, x_k^*, u_k^*)\| \leq c t_\mu \leq \alpha_{k+1}.$$

Then due to assumption A we have

$$\begin{aligned}
&\left\| \sum_{s=\omega+1}^{\omega_\mu} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=\tilde{\xi}_{k+1}^\mu}\, f_u'(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) v_\mu \right\| \\
\leq \quad & \sum_{s=\omega+1}^{\infty} \left\| \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=\tilde{\xi}_{k+1}^\mu} \right\| \left\| f_u'(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) v_\mu \right\| \\
\leq \quad & c \sum_{s=\omega+1}^{\infty} \beta_s^{k+1} \leq \varepsilon.
\end{aligned}$$

Dividing (2.8) by t_μ and using this inequality we obtain that

$$\begin{aligned}
2\varepsilon \quad \geq \quad & g_u'(k, x_k^*, \tilde{u}_k^\mu) v_\mu + \sum_{s=k+1}^{\omega} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=\tilde{\xi}_{k+1}^\mu}\, f_u'(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) v_\mu \\
= \quad & g_u'(k, x_k^*, u_k^*) v_\mu + \sum_{s=k+1}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=x_{k+1}^*}\, f_u'(k, x_k^*, u_k^*) v_\mu \\
& - \sum_{s=\omega+1}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=x_{k+1}^*}\, f_u'(k, x_k^*, u_k^*) v_\mu + R(t_\mu) v_\mu, \qquad (2.9)
\end{aligned}$$

where

$$\begin{aligned}
R(t_\mu) \quad = \quad & (g_u'(k, x_k^*, \tilde{u}_k^\mu) - g_u'(k, x_k^*, u_k^*)) v_\mu \\
& + \sum_{s=k+1}^{\omega} \left[\frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=\tilde{\xi}_{k+1}^\mu}\, f_u'(k, x_k^*, u_k^* + \tilde{t}_\mu v_\mu) \right. \\
& \left. - \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=x_{k+1}^*}\, f_u'(k, x_k^*, u_k^*) \right] v_\mu.
\end{aligned}$$

From the continuity of f_u' and g_u' with respect to u and from the continuity of $\frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*)$ with respect to ξ (see e.g. (2.4)), we obtain that $R(t_\mu)$ converges to zero with t_μ. Then $\|R(t_\mu)\| \leq \varepsilon$ for all sufficiently large μ. Moreover, for the second last term in (2.9) we have from assumption A

$$\left\| \sum_{s=\omega+1}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k+1,\xi}, u_s^*)_{|\xi=x_{k+1}^*}\, f_u'(k, x_k^*, u_k^*) v_\mu \right\| \leq c \sum_{s=\omega+1}^{\infty} \beta_s^{k+1} \leq \varepsilon.$$

Thus, regarding (2.6), we obtain from (2.9) the inequality

$$\begin{aligned}
4\varepsilon \quad \geq \quad & g_u'(k, x_k^*, u_k^*) v_\mu + \sum_{s=k+1}^{\infty} \frac{\partial}{\partial \xi} g(s, x_s^{k,\xi}, u_s^*)_{|\xi=x_{k+1}}\, f_u'(k, x_k^*, u_k^*) v_\mu \\
= \quad & \left(g_u'(k, x_k^*, u_k^*) + \psi_{k+1}\, f_u'(k, x_k^*, u_k^*) \right) v_\mu
\end{aligned}$$

for all sufficiently large μ. Since ε was arbitrarily chosen and $v_\mu \to v$, we obtain the inequality (2.7). □

Notice that in the above theorem the "adjoint" sequence $\{\psi_k\}_{k=1}^{\infty}$, for which the local maximum condition (2.7) holds, is explicitly defined by (2.5) or equivalently by (2.6). The next corollary makes a link between this result and the familiar Pontryagin-type maximum principle. In the formulation we use the notation

$$Z_k := \prod_{s=k-1}^{1} f'_x(s, x^*_s, u^*_s), \quad k \geq 1. \tag{2.10}$$

Corollary 2.3. *Under the conditions of Theorem 2.2, there exists a sequence* $\{\psi_k\}_{k=1}^{\infty}$ *satisfying the* adjoint equation

$$\psi_k = \psi_{k+1} f'_x(k, x^*_k, u^*_k) + g'_x(k, x^*_k, u^*_k), \quad k = 1, 2, \dots,$$

and the transversality condition

$$\lim_{k\to+\infty} \psi_k Z_k = 0, \tag{2.11}$$

such that the local maximum condition (2.7) is fulfilled for every $k \in \mathbb{N}$.

As seen in the proof, the above corollary holds with the same adjoint sequence as defined explicitly in Theorem 2.2.

Proof. The first claim is fulfilled with $\{\psi_k\}_{k=1}^{\infty}$ defined by (2.6). To obtain the second claim we multiply (2.6) by Z_k. The right-hand side becomes

$$\psi_k Z_k = \sum_{s=k}^{\infty} g'_x(s, x^*_s, u^*_s) \prod_{i=s-1}^{1} f'_x(i, x^*_i, u^*_i). \tag{2.12}$$

The sum in the right-hand side is absolutely convergent since it is the same sum that appears in the definition of ψ_1. Hence, $\lim_{k\to 0} \psi_k Z_k = 0$. □

In the next paragraphs we give some explanations about assumption A. Clearly, its meaning is that the marginal effect of a disturbance of the optimal trajectory at time k on the future running objective values $g(s, \cdot, u^*_s)$ is summable. This assumption takes a simpler form (similar to that in assumption A2 in [4] in the continuous-time case) if the equation (2.1) is invertible. The latter means that for every $k \geq 0$, $x \in G$ and $u \in U_k$ the matrix $f'_x(k, x, u)$ is invertible (see [5], where this property is introduced). A somewhat weaker form of this condition (formulated along the reference process $(\{u^*_k\}, \{x^*_k\})$ is that for every k there is some $\alpha > 0$ such that $\mathbf{B}(x^*_k; \alpha) \subset \tilde{U}$ and the mapping $f^k_{u^*_k} : \mathbf{B}(x^*_k; \alpha) \to \mathbb{R}^n$ is a diffeomorphism (see [10]). In the lemma below the latter definition of invertibility suffices.

The invertibility condition is satisfied, for example, if the discrete-time equation (2.1) results from the Euler (or another Runge-Kutta discretization) of a controlled differential equation, provided that the discretization step is sufficiently small.

Lemma 2.4. *If equation (2.1) is invertible then assumption A is implied by the following one:*
there exist positive numbers α_0 *and* β_s, $s \in \mathbb{N}$, *such that* $\sum_{s=0}^{\infty} \beta_s < \infty$ *and*

$$\max_{\xi \in B(x_0;\alpha_0)} \left\| \frac{\partial}{\partial \xi} g(s, x^{0,\xi}_s, u^*_s) \right\| \leq \beta_s.$$

Proof. Let us fix an arbitrary positive integer k. The invertibility assumption implies that there exists $\alpha_0 > 0$ such that the map

$$F_k := f^{k-1}_{u^*_{k-1}} \circ f^{k-2}_{u^*_{k-2}} \circ \cdots \circ f^0_{u^*_0} : \mathbf{B}(x_0; \alpha_0) \to \mathbb{R}^n$$

is a diffeomorphism. Because of $x^*_k = F_k(x_0)$, the set $\Omega_k := \{y = F_k(\xi) : \ \xi \in \mathbf{B}(x_0; \alpha_0)\}$ contains an open neighborhood of the point x^*_k, and hence there exists $\alpha_k > 0$ such that $\mathbf{B}(x^*_k; \alpha_k) \subset \Omega_k \cap G$. Let ξ be an arbitrary point of $\mathbf{B}(x^*_k; \alpha_k)$. Then there exists $\xi_0 \in \mathbf{B}(x_0; \alpha_0)$ such that $\xi = F_k(\xi_0)$, and hence for each integer $s \geq k$ we have that

$$g(s, x^{k,\xi}_s, u^*_s) = g(s, x^{0,\xi_0}_s, u^*_s) = g\left(s, x^{0,F^{-1}_k(\xi)}_s, u^*_s\right),$$

and hence

$$\frac{\partial}{\partial \xi} g(s, x^{k,\xi}_s, u^*_s) = \frac{\partial}{\partial \xi} g\left(s, x^{0,F^{-1}_k(\xi)}_s, u^*_s\right) \circ \frac{\partial}{\partial \xi} F^{-1}_k(\xi).$$

Therefore, for $s \geq k$ we have

$$\begin{aligned} \left\| \frac{\partial}{\partial \xi} g(s, x^{k,\xi}_s, u^*_s) \right\| &\leq \left\| \frac{\partial}{\partial \xi} g\left(s, x^{0,F^{-1}_k(\xi)}_s, u^*_s\right) \right\| \left\| \frac{\partial}{\partial \xi} F^{-1}_k(\xi) \right\| \\ &\leq \beta_s \left\| \frac{\partial}{\partial \xi} F^{-1}_k(\xi) \right\| \leq \sup_{\xi \in B(x^*_k; \alpha_k)} \left\| \frac{\partial}{\partial \xi} F^{-1}_k(\xi) \right\| \beta_s =: \beta^k_s. \end{aligned}$$

This completes the proof, since $\sum_{s=k}^{\infty} \beta^k_s$ is convergent. ☐

3. Examples

We give two examples that illustrate the applicability of Theorem 2.2 in cases where the known to the authors previous results do not give comprehensive solutions.

Example 3.1.

Let us consider the following discrete version of the well known Halkins's example (see [11], also [12, Remark 4] for the discrete-time case):

$$\begin{aligned} & J(\mathbf{u}, \mathbf{x}) = \sum_{k=0}^{\infty} (1 - x_k)\, u_k \to \max, \\ & x_{k+1} = x_k + (1 - x_k)\, u_k, \quad x_0 = 0, \\ & u_k \in [0, 1], \qquad k \in \mathbb{N}. \end{aligned} \tag{3.1}$$

It is straightforward that for any admissible control $\mathbf{u} = \{u_k\}_{k=0}^{\infty}$ the corresponding trajectory $\mathbf{x} = \{x_k\}_{k=0}^{\infty}$ satisfies the inclusions $x_k \in [0, 1]$ for each $k \in \mathbb{N}$.

For each positive integer ω and admissible process $(\mathbf{u}, \mathbf{x})$ the truncated objective functional takes the form

$$J^{\omega}(\mathbf{u}, \mathbf{x}) = \textstyle\sum_{k=0}^{\omega} (1 - x_k)\, u_k = \sum_{k=0}^{\omega} (x_{k+1} - x_k) = x_{\omega}. \tag{3.2}$$

On the other hand, we can prove by induction (see the next paragraph for a similar proof) that the solution $\mathbf{x}$ of (3.1) for an admissible control $\mathbf{u}$ can be represented for $k > 0$ as

$$x_k = 1 - \prod_{s=0}^{k-1}(1 - u_s) \le 1. \tag{3.3}$$

(In this example we use the standard notation $\prod_{i=p}^{q}$ instead of $\prod_{i=q}^{p}$ since in the scalar case the order of multipliers does not matter; again $\prod_{i=p}^{q}$ is defined as 1 if $p > q$.) Since for the admissible control $\bar{u}_k \equiv 1$ we have $\bar{x}_k = 1$ for all $k \ge 1$, we obtain that $J(\bar{\mathbf{u}}, \bar{\mathbf{x}}) = 1$, thus $(\bar{\mathbf{u}}, \bar{\mathbf{x}})$ is optimal (in any reasonable sense). Taking into account also the monotonicity of the sequence $\{x_k\}_{k=1}^{\infty}$, we see from (3.3) that an admissible process $(\mathbf{u}^*, \mathbf{x}^*)$ is weakly overtaking optimal if and only if

$$\prod_{s=0}^{\infty}(1 - u_s^*) = 0. \tag{3.4}$$

Now, we analyze what Theorem 2.2 provides for this example. Let $(\mathbf{u}^*, \mathbf{x}^*) = \{(u_k^*, x_k^*)\}_{k=1}^{\infty}$ be an weakly overtaking optimal process. Again by induction we verify that

$$x_s^{k,\xi} = 1 - (1 - \xi)(1 - u_{s-1}^*) \dots (1 - u_k^*), \quad s = k + 1, \dots .$$

Indeed, having in mind the particular form of f, we have for $s = k + 1$ that

$$x_{k+1}^{k,\xi} = \xi + (1 - \xi)u_k^* = 1 - (1 - \xi)(1 - u_k^*)$$

and inductively,

$$x_{s+1}^{k,\xi} = x_s^{k,\xi} + (1 - x_s^{k,\xi})u_s^* = 1 - (1 - x_s^{k,\xi})(1 - u_s^*) = 1 - (1 - \xi)(1 - u_s^*) \dots (1 - u_k^*).$$

We have $g_x'(x, u) = -u$ (in this example the functions f and g do not depend on k), thus

$$\frac{\partial}{\partial \xi} g(x_s^{k,\xi}, u_s^*) = g_x'(x_s^{k,\xi}, u_s^*)\, \frac{\partial x_s^{k,\xi}}{\partial \xi} = -u_s^* \prod_{i=k}^{s-1}(1 - u_i^*). \tag{3.5}$$

In order to prove that assumption A is satisfied for the considered optimal process $(\mathbf{u}^*, \mathbf{x}^*)$ we define

$$\bar{\beta}_s^k := \left\| \frac{\partial}{\partial \xi} g(x_s^{k,\xi}, u_s^*) \right\| = u_s^* \prod_{i=k}^{s-1}(1 - u_i^*).$$

Let us prove that the series $\sum_{s=k}^{\infty} \bar{\beta}_s^k$ is convergent. In fact we will prove by induction that for every positive integer $\mu \ge k$

$$\sum_{s=k}^{\mu} \bar{\beta}_s^k = 1 - \prod_{i=k}^{\mu}(1 - u_i^*). \tag{3.6}$$

For $\mu = k$ we have that

$$\bar{\beta}_k^k = u_k^* = 1 - (1 - u_k^*),$$

Let us assume that (3.6) holds true for some positive integer $\mu \geq k$. Then

$$\sum_{s=k}^{\mu+1} \bar{\beta}_s^k = \bar{\beta}_k^{\mu+1} + \sum_{s=k}^{\mu} \bar{\beta}_s^k = u_{\mu+1}^* \prod_{i=k}^{\mu}(1-u_i^*) + \left(1 - \prod_{i=k}^{\mu}(1-u_i^*)\right) = 1 - \prod_{i=k}^{\mu+1}(1-u_i^*),$$

which proves (3.6).

Since the process $(\mathbf{u}^*, \mathbf{x}^*)$ is optimal, condition (3.4) holds. Then (3.6) implies that $\sum_{s=k}^{\infty} \bar{\beta}_s^k \leq 1$ Thus assumption A is fulfilled. Then Theorem 2.2 claims that the process $(\mathbf{u}^*, \mathbf{x}^*)$ satisfies the maximum condition (2.7) with $\{\psi_k\}_{k=1}^{\infty}$ defined by (2.5) (or equivalently by (2.6)). That is, the claim of Theorem 2.2 holds for every weakly overtaking optimal process in the considered example.

We note that the adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$ defined by (2.5) has the explicit form (see (3.5), the definition of the numbers $\bar{\beta}_s^k$ and (3.6))

$$\psi_k = -\sum_{s=k}^{\infty} u_s^* \prod_{i=k}^{s-1}(1-u_i^*) = -\sum_{s=k}^{\infty} \bar{\beta}_s^k = -1 + \prod_{i=k}^{\infty}(1-u_i^*). \tag{3.7}$$

Conversely, let the admissible control process $(\mathbf{u}^*, \mathbf{x}^*)$ satisfy assumption A and the maximum condition (2.7) in Theorem 2.2 with the adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$ defined by (2.5), that is by (3.7). Observe that, as argued after (2.5), ψ_k are all finite due to assumption A.

For the particular functions f and g the maximum condition takes the form

$$(1-x_k^*)(1+\psi_{k+1})v \leq 0 \quad \text{for every} \quad v \in T_{[0,1]}(u_k^*).$$

If for some k it happens that $u_k^* = 1$, then $\prod_{s=0}^{\infty}(1-u_s^*) = 0$, thus the process $(\mathbf{u}^*, \mathbf{x}^*)$ is optimal. Alternatively, if for every k it holds that $u_k^* < 1$, then $[0, \infty) \subset T_{[0,1]}(u_k^*)$. The maximum condition implies that $(1-x_k^*)(1+\psi_{k+1}) \leq 0$ for every k. Since $x_k \in [0, 1]$, this means that either $x_k = 1$ or $\psi_{k+1} \leq -1$. In the first case the process is optimal because $J^{\omega}(\mathbf{u}^*, \mathbf{x}^*) = x_{\omega}^* = 1$ for all $\omega \geq k$. In the second case (3.7) implies that $\prod_{i=k}^{\infty}(1-u_i^*) \leq 0$. Hence, $\prod_{i=0}^{\infty}(1-u_i^*) = 0$ and the considered process is again optimal.

Summarizing, we proved that Theorem 2.2 provides a complete characterization of the optimal controls in this discrete-time version of Halkin's example.

Notice that if an optimal control sequence $\mathbf{u}^* = \{u_k^*\}_{k=0}^{\infty}$ is such that $u_k^* \neq 1$ for all $k = 0, 1, 2\ldots$ then in view of (3.7), (3.3) and (3.4) both "natural" transversality conditions

$$\lim_{k\to\infty} \psi_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \psi_k x_k^* = 0$$

fail in this example. This is also noted in [12, Remark 4], where P. Michel argued that his transversality condition (1.1)-(iii), which for the example reads as $\lim_{k\to\infty} \psi_k \leq 0$, is satisfied. The trouble is, that for any admissible process there exists an adjoint sequence $\{\psi_k\}_{k=1}^{\infty}$ satisfying this transversality condition so that the local maximum condition is also satisfied. That is, the maximum principle with this transversality condition does not give any useful information for this example. According to Corollary 2.3, the "correct" transversality condition is the following:

$$\lim_{k\to\infty} \psi_k \prod_{s=1}^{k-1} (1-u_s^*) = 0.$$

Indeed (3.4) and (3.7) imply the last equality immediately.

Example 3.2.

Let us consider an application of Theorem 2.2 to a discrete version of the continuous-time model of optimal exploitation of a nonrenewable resource (see [2, Section 2.2, Example 1]):

$$J(\mathbf{u},\mathbf{x}) = \sum_{k=0}^{\infty} \theta^k \left[\ln(x_k - a) + \ln u_k\right] \to \max, \tag{3.8}$$

$$x_{k+1} = x_k - u_k(x_k - a), \quad x_0 = x_0 > a, \tag{3.9}$$
$$u_k \in U_k \equiv \tilde{U} = (0,1), \qquad k \in \mathbb{N}.$$

Here $0 < \theta < 1$ and $a \geq 0$ are given values.

In this example the state variable x_k represents the stock of the nonrenewable resource and the control variable u_k represents the extracted share of the available for exploitation stock $x_k - a$ of the resource at stage $k \in \mathbb{N}$ respectively. The discount factor θ^k characterizes subjective time preference, and the value $\ln(x_k - a) + \ln u_k$ represents the instantaneous utility of the extracted amount $u_k(x_k - a)$ of the resource at stage $k \in \mathbb{N}$. If $a = 0$ then the initial resource stock $x_0 > 0$ can be fully exhausted. If $a > 0$ then the initial resource stock $x_0 > a$ can be depleted only till the minimal possible level a.

Here we have $G = (a,\infty)$, $f(x,u) = x - u(x-a)$ and $g(k,x,u) = \theta^k\left[\ln(x-a) + \ln u\right]$ for $x \in G$, $u \in \tilde{U}$ and $k \in \mathbb{N}$ (in this example the function f does not depend on k).

It is easy to see that for any admissible control sequence $\mathbf{u} = \{u_k\}_{k=0}^{\infty}$, $u_k \in (0,1)$, $k \in \mathbb{N}$, the corresponding trajectory $\mathbf{x} = \{x_k\}_{k=0}^{\infty}$ (see (3.9)) is admissible, and for any integers $0 \leq \omega_1 < \omega_2$ we have

$$\sum_{k=\omega_1}^{\omega_2} \theta^k \left[\ln(x_k - a) + \ln u_k\right] \leq \sum_{k=\omega_1}^{\omega_2} \theta^k \ln(x_0 - a) \leq \frac{\theta^{\omega_1}}{1-\theta} \ln(x_0 - a).$$

This implies that for any admissible process $(\mathbf{u},\mathbf{x})$ the corresponding series (3.8) either converges to a finite number or diverges to $-\infty$. Thus, for any weakly overtaking optimal admissible process $(\mathbf{u}^*,\mathbf{x}^*)$ (if such exists) the corresponding value $J(\mathbf{u},\mathbf{x})$ of the functional (3.8) is finite.

For any admissible process $(\mathbf{u},\mathbf{x})$ we have $f'_x(x_k,u_k) = 1 - u_k$, and $g'_x(k,x_k,u_k) = \theta^k/(x_k - a)$, $k \in \mathbb{N}$. Since $f(x,u) = x(1-u) + au$ and $u \in (0,1)$, the mapping $x \mapsto f(\cdot,u)$ is a diffeomorphism. Then the system (3.9) is invertible (which is not the case in the previous example!). Further, it can be easily shown that the conditions of Lemma 2.4 are satisfied for any admissible process $(\mathbf{u},\mathbf{x})$ with $\alpha_0 = (x_0 - a)/2$ and $\beta_s = 2\theta^s/(x_0 - a)$, $s \in \mathbb{N}$. Thus, assumption A is fulfilled for any admissible process $(\mathbf{u},\mathbf{x})$.

Let $(\mathbf{u}^*, \mathbf{x}^*)$ be an optimal admissible process. Then due to Theorem 2.2 the adjoint sequence $\psi = \{\psi_k\}_{k=1}^{\infty}$, specified by (2.5) (or equivalently by (2.6)) for every $k \in \mathbb{N}$, satisfies the local maximum condition (2.7). Since the process $(\mathbf{u}^*, \mathbf{x}^*)$ is interior and

$$g'_u(k, x, u) = \frac{\theta^k}{u}, \quad f'_u(x, u) = -(x - a), \qquad x \in G, \quad u \in \tilde{U}, \quad k \in \mathbb{N},$$

the local maximum condition (2.7) reads as

$$\frac{\theta^k}{u_k^*} = \psi_{k+1}(x_k^* - a), \qquad k \in \mathbb{N}. \tag{3.10}$$

Due to (2.6) we have[2]

$$\psi_{k+1} = \sum_{s=k+1}^{\infty} \frac{\theta^s}{x_s^* - a} \prod_{i=k+1}^{s-1} (1 - u_i^*), \qquad k \in \mathbb{N},$$

and due to (3.9) we have

$$x_k^* - a = \prod_{i=0}^{k-1} (1 - u_i^*)(x_0 - a), \qquad k \in \mathbb{N}.$$

These equalities imply

$$\begin{aligned} \psi_{k+1}(x_k^* - a) &= \sum_{s=k+1}^{\infty} \frac{\theta^s}{x_s^* - a} \prod_{i=k+1}^{s-1} (1 - u_i^*) \prod_{i=0}^{k-1} (1 - u_i^*)(x_0 - a) \\ &= \sum_{s=k+1}^{\infty} \frac{\theta^s}{(x_s^* - a)(1 - u_k^*)} \prod_{i=0}^{s-1} (1 - u_i^*)(x_0 - a) \\ &= \frac{1}{1 - u_k^*} \sum_{s=k+1}^{\infty} \theta^s = \frac{\theta^{k+1}}{(1 - u_k^*)(1 - \theta)}, \qquad k \in \mathbb{N}. \end{aligned} \tag{3.11}$$

Due to the maximum condition (3.10) this implies

$$\frac{1 - u_k^*}{u_k^*} = \frac{\theta}{1 - \theta}, \qquad k \in \mathbb{N}.$$

Solving the last equality we finally get

$$u_k^* \equiv 1 - \theta, \quad x_k^* = a + \theta^k (x_0 - a), \qquad k \in \mathbb{N}. \tag{3.12}$$

Thus we have proved that if an optimal admissible process $(\mathbf{u}^*, \mathbf{x}^*)$ exists then it is uniquely defined by (3.12), and due to (3.11) the corresponding adjoint sequence $\psi = \{\psi_k\}_{k=0}^{\infty}$ is defined as follows:

$$\psi_{k+1} \equiv \frac{1}{(1 - \theta)(x_0 - a)}, \qquad k \in \mathbb{N}. \tag{3.13}$$

Now let us prove that the process $(\mathbf{u}^*, \mathbf{x}^*)$ defined by (3.12) is optimal.

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[2]Here as in the previous example we use the standard notation $\prod_{i=p}^{q}$ instead of $\prod_{i=q}^{p}$ since in the scalar case the order of multipliers does not matter; again $\prod_{i=p}^{q}$ is defined as 1 if $p > q$.

For this notice first that for any $k = 0, 1, 2, \dots$ the unique absolute maximum of the function $h(k, \cdot, \psi_{k+1}) : (0, \infty) \mapsto \mathbb{R}^1$, defined by

$$h(k, \zeta, \psi_{k+1}) = \theta^k \ln \zeta - \psi_{k+1}\zeta, \qquad \zeta > 0,$$

is reached at the point $\zeta_k = \theta^k/\psi_{k+1} = \theta^k(1-\theta)(x_0 - a) = u_k^*(x_k^* - a)$. Here the sequence $\psi = \{\psi_k\}_{k=1}^\infty$ is defined by (3.13) and the process $(\mathbf{u}^*, \mathbf{x}^*) = \{(u_k^*, x_k^*)\}_{k=0}^\infty$ is defined by (3.12).

Now let $\{(v_k, y_k)\}_{k=0}^\infty$ be an arbitrary admissible process. Then for any $k \in \mathbb{N}$ we have

$$h(k, u_k^*(x_k^* - a), \psi_{k+1}) \geq h(k, v_k(y_k - a), \psi_{k+1})$$

or, equivalently,

$$\begin{aligned} &\theta^k \left[\ln u_k^* + \ln(x_k^* - a)\right] - \psi_{k+1}u_k^*(x_k^* - a) \\ &\qquad \geq \theta^k \left[\ln v_k + \ln(y_k - a)\right] - \psi_{k+1}v_k(y_k - a). \end{aligned} \tag{3.14}$$

For an arbitrary integer $\omega \geq 1$ summing (3.14) we get

$$\begin{aligned} &\sum_{k=0}^{\omega} \theta^k \left[\ln u_k^* + \ln(x_k^* - a)\right] - \sum_{k=0}^{\omega} \theta^k \left[\ln v_k + \ln(y_k - a)\right] \\ &\qquad \geq \sum_{k=0}^{\omega} \psi_{k+1}u_k^*(x_k^* - a) - \sum_{k=0}^{\omega} \psi_{k+1}v_k(y_k - a) \\ &= \frac{1}{(1-\theta)(x_0 - a)} \sum_{k=0}^{\omega} \left[(x_k^* - x_{k+1}^*) - (y_k - y_{k+1})\right] = \frac{1}{(1-\theta)(x_0 - a)}(y_\omega - x_\omega^*). \end{aligned} \tag{3.15}$$

Since

$$\lim_{\omega\to\infty} y_\omega \geq a \quad \text{and} \quad \lim_{\omega\to\infty} x_\omega^* = a,$$

taking the limit in inequality (3.15) as $\omega \to \infty$ we get

$$\sum_{k=0}^{\infty} \theta^k \left[\ln u_k^* + \ln(x_k^* - a)\right] \geq \limsup_{\omega\to\infty} \sum_{k=0}^{\omega} \theta^k \left[\ln v_k + \ln(y_k - a)\right].$$

Thus, the process $(\mathbf{u}^*, \mathbf{x}^*)$ defined by (3.12) is optimal. As already proved above, if an optimal process exists then it is unique. Hence, the process $(\mathbf{u}^*, \mathbf{x}^*)$ defined by (3.12) is a unique optimal admissible process.

Notice that if $a > 0$ then in view of (3.12) and (3.13) both "natural" transversality conditions

$$\lim_{k\to\infty} \psi_k = 0 \quad \text{and} \quad \lim_{k\to\infty} \psi_k x_k^* = 0$$

fail in this example. Due to Corollary 2.3 the "correct" transversality condition is the following:

$$\lim_{k\to\infty} \psi_k \prod_{s=1}^{k-1}(1 - u_s^*) = 0.$$

Indeed, (3.12) and (3.13) imply

$$\lim_{k\to\infty} \psi_k \prod_{s=1}^{k-1}(1 - u_s^*) = \lim_{k\to\infty} \frac{\theta^{k-1}}{(1-\theta)(x_0 - a)} = 0.$$

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