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A NOTE ON VARIATIONAL PROBLEMS VIA THEORY OF YOUNG MEASURES

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Abstract. A triple of simple variational problems are compared. We give a brief exposition of the existence problem of solutions for them. And then, some relations between the optimized values of the problems are discussed. The problem is reformulated in terms of Young measures. Topological and geometric properties of Young measures as well as the basic results from set-valued analysis are effectively made use of.

1. Introduction

Let $x_1, x_2, \dots, x_p$ be any finite elements of $\mathbb{R}^l$. We denote by X the convex hull of $\{x_1, x_2, \dots, x_p\}$. X is, of course, a convex compact set. X is endowed with the Borel σ-field $\mathcal{B}(X)$. $(\Omega, \mathcal{E}, \mu)$ is a probability space and a function $u : \Omega \times X \to \mathbb{R}$ with certain nice properties is given.

The set of all measurable functions of Ω into $\{x_1, x_2, \dots, x_p\}$ (resp. X) is denoted by $\mathcal{P}$ (resp. $\mathcal{M}$). We also denote by $\mathfrak{Y}(\Omega, \mu; X)$ the set of all the Young measures on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$. The definition and basic properties of Young measures are discussed in the succeeding two sections.

Compare a triple of minimization problems:

$$\text{(I)} \quad \underset{x(\cdot)\in\mathcal{P}}{Minimize} \int_\Omega u(\omega, x(\omega))d\mu,$$

$$\text{(II)} \quad \underset{x(\cdot)\in\mathcal{M}}{Minimize} \int_\Omega u(\omega, x(\omega))d\mu,$$

$$\text{(III)} \quad \underset{\gamma\in\mathfrak{Y}(\Omega,\mu;X)}{Minimize} \int_{\Omega\times X} u(\omega, x)d\gamma.$$

Our targets are two fold. The first question is the existence of solutions for each problem of the three. The second puzzle concerns their relations to each other. Assume that there exists a solution $x^*(\cdot) \in \mathcal{M}$ for Problem(II). Is there any equivalent solution $y^*(\cdot) \in \mathcal{P}$ for Problem(I) in the sense that

$$\int_\Omega u(\omega, x^*(\omega))d\mu = \int_\Omega u(\omega, y^*(\omega))d\mu?$$

A similar question concerning the relation between Problems(II) and (III) should also be answered.

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My approach heavily depends upon the theory of Young measures.

2. Young measures and disintegrations

In this section, the basic concepts and properties of Young measures are briefly summarized in a simple framework without detailed proofs.[1]

Let $(\Omega, \mathcal{E}, \mu)$ be a finite complete measure space and X the convex hull of $\{x_1, x_2, \ldots, x_p\}$ in $\mathbb{R}^l$; i.e. $X = \mathrm{co}\{x_1, x_2, \ldots, x_p\}$. $\mathcal{B}(X)$ denotes the Borel σ-field on X. The projection of the product space $\Omega \times X$ into Ω(resp.X) is denoted by π_Ω(resp. π_X).

Definition 2.1. A (positive) measure γ on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ is called a **Young measure** if it satisfies.

$$\gamma \circ \pi_\Omega^{-1} = \mu. \tag{2.1}$$

The set of all the Young measures is denoted by $\mathfrak{Y}(\Omega, \mu; X)$.

Definition 2.2. A family $\{\nu_\omega \mid \omega \in \Omega\}$ of measures on $(X, \mathcal{B}(X))$ is called a **measurable family** if the mapping

$$\omega \mapsto \nu_\omega(B)$$

is measurable for any $B \in \mathcal{B}(X)$.[2]

Given a measurable family $\{\nu_\omega \mid \omega \in \Omega\}$ of finite measures on $(X, \mathcal{B}(X))$, the function

$$\omega \mapsto \int_X \chi_A(\omega, x) d\nu_\omega$$

is measurable for any $A \in \mathcal{E} \otimes \mathcal{B}(X)$. χ_A is the characteristic function of A. A set function γ defined by

$$\gamma(A) = \int_\Omega \{\int_X \chi_A(\omega, x) d\nu_\omega\} d\mu, \quad A \in \mathcal{E} \otimes \mathcal{B}(X) \tag{2.2}$$

is a measure on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$. In case an additional condition

$$\sup_{\omega \in \Omega} \nu_\omega(X) < \infty$$

is satisfied, γ defined by (2.2) is a finite measure. In particular, if every $\nu_\omega(\omega \in \Omega)$ is a probablity measure, then γ is a Young measure since

$$(\gamma \circ \pi_\Omega^{-1})(E) = \mu(E), \quad E \in \mathcal{E}.$$

The set of all the measurable families consisting of probability measures is denoted by $\mathfrak{P}(\Omega, \mu; X)$.

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[1]See Bourbaki [6] and Valadier [14], [15] for details.

[2]A measurable family can be characterized in several ways. For instance the following statements are equivalent to each other. Proof is not so hard.

(a) $\{\nu_\omega\}$ is a measurable family.

(b) The mapping $A : \omega \mapsto \nu_\omega(\Omega \to \mathfrak{M}(X))$ is $\mathcal{E} \otimes \mathcal{B}(\mathfrak{M}(X)_*)$-measurable, where $\mathcal{B}(\mathfrak{M}(X)_*)$ is the Borel σ-field generated by the w^*-topology on $\mathfrak{M}(X)$.

(c) The mapping $B : \omega \mapsto \int_X f(\omega, x) d\nu_\omega$ is measurable for any $f \in \mathfrak{G}_c(\Omega, \mu; X)$. ($\mathfrak{G}_c(\Omega, \mu; X)$ is defined in section 3.)

Conversely, is it possible to represent any Young measure on $\Omega \times X$ in the form (2.2) for some measurable family $\{\nu_\omega \mid \omega \in \Omega\}$? The next proposition gives a positive answer to this question.

Proposition 2.3 (Valadier). *For any Young measure γ on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$, there exists a measurable familly $\{\nu_\omega \mid \omega \in \Omega\}$ of probability measures such that (2.2) is satisfied.*[3]

If a Young measure γ is representable in the form(2.2), its right-hand side is called the **disintegration** of γ. In this case, we symbolically express γ as

$$\gamma = \int_\Omega \delta_\omega \otimes \nu_\omega d\mu. \tag{2.3}$$

3. Topology on $\mathfrak{Y}(\Omega, \mu; X)$

We now turn to defining some topology on the space $\mathfrak{Y}(\Omega, \mu; X)$ of Young measures.

A function $f : \Omega \times X \to \mathbb{R}$ is called a **Carathéodory function** if (i) $\omega \mapsto f(\omega, x)$ is measurable for each $x \in X$ and (ii) $x \mapsto f(\omega, x)$ is continuous for each $\omega \in \Omega$. It can be proved that f is $\mathcal{E} \otimes \mathcal{B}(X)$- measurable. We denote by $\mathfrak{G}_c(\Omega, \mu; X)$ the set of all the Carathéodory functions which satisfy

$$\int_\Omega \| f(\omega, \cdot) \|_\infty d\mu < \infty,$$

where $\| f(\omega, \cdot) \|_\infty = \sup_{x \in X} | f(\omega, x) |$.

We denote by $\mathfrak{M}(\Omega, \mu; X)$ the of all measurable families consisting of signed measures with $\sup_\omega \| \nu_\omega \| < \infty$ ($\| \nu_\omega \|$ is the total variation of ν_ω). Then $\mathfrak{M}(\Omega, \mu; X)$ can be regarded as a subspace of $\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$ which is the dual space of $\mathfrak{G}_c(\Omega, \mu; X) \cong \mathfrak{L}^1(\Omega, \mathfrak{C}(X, \mathbb{R}))$.[4]

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[3] Valadier [13] proved a much more general result.

> *Let $(\Omega, \mathcal{E}, \mu)$ be a finite complete measure space and X a Hausdorff topological space. If γ is a Young measure (similarly defined as in the text) on $(\Omega \times X, \mathcal{E} \otimes \mathcal{B}(X))$ and $\gamma \circ \pi_\Omega^{-1}$ is a Radon measure on X, then there exits a measurable family $\{\nu_\omega \mid \omega \in \Omega\}$ of Radon probability measures such that (2.2) is satisfied.*

In our text, X is assumed to be a compact set in $\mathbb{R}^l$. So $\gamma \circ \pi_X^{-1}$ is automatically a Radon measure. Proposition 2.3 follows immediately from Valadier's theorem as a simple corollary.

[4] $\mathfrak{M}(X)$ is the set of all the Radon measures on $\mathbb{R}$. $\mathfrak{M}_+^1(X)$ is the set of all Radon probability measures on X. See Billingsley [5] and Maruyama [11] Chap.8 for the topological properties of $\mathfrak{M}_+^1(X)$.

The following general theorem is well-known.

> *Let $(\Omega, \mathcal{E}, \mu)$ be a finite measure space, and $\mathfrak{X}$ a Banach space. Then the duality relation $\mathfrak{L}^p(\Omega, \mathfrak{X})' \cong \mathfrak{L}^q(\Omega, \mathfrak{X}')(1 \leqq p < \infty, 1/p + 1/q = 1)$ holds true if and only if $\mathfrak{X}'$ has the Radon-Nikodým property with respect to μ.*

The duality pair of $f \in \mathfrak{L}^p$ and $g \in \mathfrak{L}^q$ is given by

$$\int_\Omega \langle g(\omega), f(\omega) \rangle d\mu.$$

cf. Diestel and Uhl [9] pp.98-100.

Keeping the inclusions $\mathfrak{P}(\Omega,\mu;X) \subset \mathfrak{M}(\Omega,\mu;X) \subset \mathfrak{L}^\infty(\Omega,\mathfrak{M}(X))$ in mind, we give $\mathfrak{P}(\Omega,\mu;X)$ the relative topology induced from the w^*-topology on $\mathfrak{L}^\infty(\Omega,\mathfrak{M}(X))$; i.e. $\sigma(\mathfrak{L}^\infty(\Omega,\mathfrak{M}(X)),\ \mathfrak{L}^1(\Omega,\mathfrak{C}(X,\mathbb{R}))$. If we express $\mathfrak{L}^\infty(\Omega,\mathfrak{M}(X))$ endowed with the w^*-topology by $\mathfrak{L}^\infty_*(\Omega,\mathfrak{M}(X))$, it is a locally convex Hausdorff topological linear space.

Definition 3.1. The topology on $\mathfrak{Y}(\Omega,\mu;X)$ which is generated by the family of functions

$$\gamma \mapsto \int_{\Omega\times X} f(\omega,x)d\gamma, \quad f \in \mathfrak{G}_c(\Omega,\mu;X) \tag{3.1}$$

is called the **narrow topology**.

A measurable family $\{\nu_\omega \mid \Omega \in \Omega\} \in \mathfrak{P}(\Omega,\mu;X)$ of probability measures defines a Young measure by (2.2), and vice versa (Proposition 2.3). The next Proposition assures that $\mathfrak{P}(\Omega,\mu;X)$ and $\mathfrak{Y}(\Omega,\mu;X)$ are identified topologically.

Proposition 3.2. *Define a mapping $\Phi : \mathfrak{P}(\Omega,\mu;X) \to \mathfrak{Y}(\Omega,\mu;X)$ by*

$$\Phi : \{\nu_\omega \mid \omega \in \Omega\} \mapsto \int_\Omega \Big\{ \int_X \chi_A(\omega,x)d\nu_\omega \Big\} d\mu, \quad A \in \mathcal{E} \otimes \mathcal{B}(X).$$

Then Φ is a homeomorphism between $\mathfrak{P}(\Omega,\mu;X)$ with w^*-topology and $\mathfrak{Y}(\Omega,\mu;$ $X)$ with the narrow topology.

It can be proved that $\mathfrak{P}(\Omega,\mu;X)$ is a w^*-closed set contained in the unit ball of $\mathfrak{L}^\infty_*(\Omega,\mathfrak{M}(X))$, and so w^*-compact. Therefore, according to Proposition 3.2, $\mathfrak{Y}(\Omega,\mu;X)$ is also compact in the narrow topology.

Proposition 3.3. *$\mathfrak{P}(\Omega,\mu;X)$ is w^*-compact in $\mathfrak{L}^\infty_*(\Omega,\mathfrak{M}(X))$. $\mathfrak{Y}(\Omega,\mu;X)$ is compact in the narrow topology.*

We sometimes use expressions like "narrowly compact", "narrowly converge" and so on instead of "compact in the narrow topology", "converge in the narrow topology" ... for the sake of brevity.

The mapping Φ in Proposition 3.2 admits an extention to $\mathfrak{L}^\infty(\Omega,\mathfrak{M}(X))$ by the same formula. However each value of Φ is not necessarily a Young measure in this case.

4. Continuity of integral functionals

We are going to keep our notations used in the preceding sections : X is the convex hull of $\{x_1,x_2,\dots,x_p\}$ in $\mathbb{R}^l$, and $(\Omega,\mathcal{E},\mu)$ is a finite complete measure space. In this section, we investigate the continuity of the integral functional $J : \mathfrak{Y}(\Omega,\mu;X) \to \mathbb{R}$ defined by

$$J : \gamma \mapsto \int_{\Omega\times X} u(\omega,x)d\gamma, \tag{4.1}$$

where $u(\omega,x)$ is a given integrand.

A function $f : \Omega \times X \to \mathbb{R}$ is called a **normal integrand** if (i) f is $(\mathcal{E}\otimes\mathcal{B}(X))$-measurable, and (ii) $x \mapsto f(\omega,x)$ is lower semi-continuous ($l.s.c.$) for all $\omega \in \Omega$. We denote by $\mathfrak{G}(\Omega,\mu;X)$ the set of all the normal integrands. If, in addition to (i) and (ii), f satisfies the condition (iii) $x \mapsto f(\omega, x)$ is convex for all $\omega \in \Omega$, f is called a **convex normal integrand**.

It is well-known [5] that for any positive normal integrand $f : \Omega \times X \to \mathbb{R}$, there exists an increasing sequence $\psi_n : \Omega \times X \to \mathbb{R}$ of positive Carathéodory functions such that

$$\int_{\Omega} \| \psi_n(\omega, \cdot) \|_{\infty} \, d\mu < \infty \tag{4.2}$$

and

$$f(\omega, x) = \lim_{n \to \infty} \psi_n(\omega, x) \quad \text{for all} \quad \omega. \tag{4.3}$$

Hence

$$\int_{\Omega \times X} f(\omega, x) d\gamma = \sup_n \int_{\Omega \times X} \psi_n(\omega, x) d\gamma \tag{4.4}$$

for any $\gamma \in \mathfrak{Y}(\Omega, \mu; X)$.

By definition of the narrow topology, the functional

$$\gamma \mapsto \int_{\Omega \times X} \psi_n(\omega, x) d\gamma \;\; ; \;\; n = 1, 2, \ldots$$

is narrowly continuous. Combining these observations with (4.4), we obtain the following result.

Proposition 4.1. *For any positive normal integrand $f : \Omega \times X \to \mathbb{R}$, the functional*

$$J : \gamma \mapsto \int_{\Omega \times X} f(\omega, x) d\gamma$$

is narrowly l.s.c. on $\mathfrak{Y}(\Omega, \mu; X)$.

5. Existence of solutions (1)

We start with examining Problems(II) and (III).

$(\Omega, \mathcal{E}, \mu)$ is a finite complete measure space and $X = \mathrm{co}\{x_1, x_2, \ldots, x_p\}$ as in the preceding sections. However it is required to impose some additional assumpitons concerning $(\Omega, \mathcal{E}, \mu)$ and $u(\omega, x)$.

Assumption 1 $(\Omega, \mathcal{E}, \mu)$ is non-atomic.
Assumption 2 A function $u : \Omega \times X \to \mathbb{R}$ is a positive convex normal integrand.

It is obvious that Problem(III) has a solution

$$\gamma^* = \int_{\Omega} \delta_{\omega} \otimes \nu^*_{\omega} d\mu$$

in $\mathfrak{Y}(\Omega, \mu; X)$ in view of Proposition 3.3 and Proposition 4.1.

Theorem 5.1. *Under Assumptions 2, there exists a solution for Problem(III).*

[5] See Berliocchi-Lasry [4] and Valadier [14], [15] for a characterization of positive normal integrands by using Carathéodory functions. It is not so easy because we need the "projection theorem". cf. Maruyama [11](pp.411-426).

How about the existence of a solution for Problem(II)? The answer is positive again. However we need a little bit sophisticated reasonings for its proof.

Definition 5.2. Let $\mathfrak{X}$ be a real linear space and C a nonempty convex set of $\mathfrak{X}$. Suppose that x is any point of C. Then $v \in \mathfrak{X}$ is called a **facial direction** of C at x if $x \pm tv \in C$ for sufficiently small $t > 0$. The set of all the facial directions of C at x is called the **facial space** of C at x, and is denoted by $L(x\,|\,C)$.[6]

$L(x\,|\,C)$ is a linear subspace of $\mathfrak{X}$. It is easy to see that $x \in C$ is an extreme point of C if and only if $\dim L(x\,|\,L) = 0$.

Theorem 5.3. *Under Assumptions 1 and 2, there exists a solution for Problem (II).*

Proof. By Theorem 5.1, we know that Problem(III) has a solution γ^* of the form

$$\gamma^* = \int_\Omega \delta_\omega \otimes \nu^*_\omega d\mu. \tag{5.1}$$

By Proposition 3.3, $\mathfrak{Y}(\Omega, \mu; X)$ can be regarded as a convex and narrowly compact set in $\Phi(\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X))$. If we denote by K the set of Young measures which are equivalent to γ^* in the following sense:

$$\int_{\Omega\times X} u(\omega, x) d\gamma - \int_{\Omega\times X} u(\omega, x) d\gamma^* = 0. \tag{5.2}$$

Since K is nonempty, convex and narrowly compact, K has an extreme point, say γ(i.e. $\gamma \in \ddot{K}$: the set of extreme points of K). By the Karlin-Castaing Theorem,[7] γ can be expressed in the form

$$\gamma = \int_\Omega \delta_\omega \otimes \delta_{y(\omega)} d\mu$$

for some measurable mapping $y : \Omega \to X$.

If we define a linear operator $T : \Phi(\mathfrak{L}^\infty(\Omega, \mathfrak{M}(X)) \to \mathbb{R}$ by

$$T\gamma = \int_{\Omega\times X} u(\omega, x) d\gamma$$

and denote by U^* the optimized value of Problem(III); i.e.

$$U^* = \int_{\Omega\times X} u(\omega, x) d\gamma^*,$$

then the left-hand side of (5.2) can be written as

$$A(\gamma) \equiv T\gamma - U^*,\ \gamma \in \mathfrak{Y}(\Omega, \mu; X).$$

It is clear that $K = T^{-1}(U^*) \cap \mathfrak{Y}(\Omega, \mu; X)$. Since $\gamma \in \ddot{K}$, it is obvious that γ is an extreme point of $\mathfrak{Y}(\Omega, \mu; X)$.

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[6] For the concept of facial spaces, consult Arrow and Hahn [1] pp.389-390. See also Artstein [2].

[7] Let $\mathfrak{X}$ be a locally convex topological linear space. Assume that a set-valued mapping $\Gamma : \Omega \twoheadrightarrow \mathfrak{X}$ is compact, convex-valued, measurable and $\mathfrak{L}^1$-integrably bounded. $\ddot{\Gamma} : \omega \twoheadrightarrow \ddot{\Gamma}(\omega)$ (the set of extreme points of $\Gamma(\omega)$). $\mathcal{F}_\Gamma$(resp. $\mathcal{F}_{\ddot{\Gamma}}$) denotes the set of all measurable selections of Γ(resp. $\ddot{\Gamma}$). Then

$$\text{(i)}\ddot{\mathcal{F}}_\Gamma \neq \emptyset, \quad \text{(ii)}\mathcal{F}_{\ddot{\Gamma}} \neq \emptyset, \quad \text{(iii)}\ddot{\mathcal{F}}_\Gamma = \mathcal{F}_{\ddot{\Gamma}}.$$

See Castaing-Valadier [7] Theorem IV. 15(p.109), Maruyama [11] Chap.12, §2.

We observe that the restriction of T to $L(\gamma|\mathfrak{Y}(\Omega,\mu;X))$ is an injection.[8] Consequently,

$$\dim L(\gamma \mid \mathfrak{Y}(\Omega,\mu;X)) \leqq 1. \tag{5.3}$$

γ is an element of the compact convex set

$$H = [\gamma + L(\gamma \mid \mathfrak{Y}(\Omega,\mu;X))] \cap \mathfrak{Y}(\Omega,\mu;X).$$

By Carathéodory's theorem, γ can be expressed as a convex combination of at most two extreme points of H. Any extreme point of H is an extreme point of $\mathfrak{Y}(\Omega,\mu;X)$. So γ can be expressed as a convex combination of at most (or exactly) two extreme points of $\mathfrak{Y}(\Omega,\mu;X)$.

Consequently γ can be expressed as

$$\gamma = (1-t)\int_\Omega \delta_\omega \otimes \delta_{y(\omega)} d\mu + t\int_\Omega \delta_\omega \otimes \delta_{z(\omega)} d\mu, \tag{5.4}$$

for some measurable mappings $y(\cdot), z(\cdot) : \Omega \to X$ and $t \in [0,1]$.[9] Hence

$$\int_{\Omega\times X} u(\omega,x)d\gamma = (1-t)\int_\Omega u(\omega,y(\omega))d\mu + t\int_\Omega u(\omega,z(\omega))d\mu.$$

By Ljapunov's convexity theorem, there exists a decomposition $E_1, E_2 \in \mathcal{E}$ of Ω such that

$$\int_{\Omega\times X} u(\omega,x)d\gamma = \int_{E_1} u(\omega,y(\omega))d\mu + \int_{E_2} u(\omega,z(\omega))d\mu.$$ [10]

Defining

$$x^*(\omega) = \chi_{E_1}(\omega)y(\omega) + \chi_{E_2}(\omega)z(\omega)$$

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[8]Suppose that it is not. Then there must exist nonzero $\theta \in L(\gamma \mid K)$ such that $T\theta = 0$. By definition of facial space,

$$w = \gamma + t\theta \in K \quad \text{and} \quad w' = \gamma - t\theta \in K$$

for sufficiently small $t > 0$. So $Tw = Tw' = T\gamma$, which implies $Aw = Aw' = A\gamma = U^*$. Thus w and w' are distinct elements of $T^{-1}(U^*) \cap K$ and satisfy $\gamma = (w+w)/2$. This contradicts to the fact that γ is an extreme point of $T^{-1}(U^*) \cap K$.

[9]We had recourse to a geometric theory of the facial space due to Arrow-Hahn [1]. Berliocchi-Lasry [4](pp.145-146) established the following result in order to get (5.4), sharing a common idea.

> *Let $\mathfrak{X}$ be a locally convex Hausdorff real topological linear space. Assume that $K \subset \mathfrak{X}$ is a nonempty compact convex set and $\varphi_i : \mathfrak{X} \to (-\infty,\infty](i = 1,2,\ldots,n)$ are affine mappings Then any extreme point of the set $G = \{x \in K \mid \varphi_i(x) \leqq 0; i = 1,2,\ldots,n\}$ can be expressed as a convex combination of $(n+1)$ extreme points of K.*

[10]Let $(\Omega,\mathcal{E},\mu)$ be a finite complete non-atomic measure space. Assume that $f_1, f_2, \ldots, f_m$ are any elements of $\mathfrak{L}^1(\Omega,\mathbb{R}^l)$ and a mapping $\lambda : \Omega \to \Lambda_m$ is measurable, where Λ_m is the fundamental simplex in $\mathbb{R}^m$; i.e. $\Lambda_m = \{\lambda \in \mathbb{R}^m \mid \lambda_i \geqq 0 (i = 1,2,\ldots,m), \sum_{i=1}^m \lambda_i = 1\}$. Then there exists a decomposition $E_1, E_2, \ldots, E_m$ of $\Omega (E_i \in \mathcal{E}$ for all $i)$ such that

$$\int_\Omega \sum_{i=1}^m \lambda_i(\omega)f_i(\omega)d\mu = \sum_{i=1}^m \int_{E_i} f_i(\omega)d\mu.$$

cf. Castaing-Valadier [7] Theorem IV. 17(pp.112-117), Maruyama [11] Chap.12, §2.

(χ_{E_i} is the characteristic function of E_i), we obtain

$$\int_\Omega u(\omega, x^*(\omega))d\mu = U^*.$$

Thus $x^*(\cdot)$ is clearly a solution for Problem(II). □

Remark 5.4. It is Berliocchi-Lasry [4] which first reformulated so called Aumann-Perles' variational problem in terms of Young measures. Their basic ideas successfully transformed a nonlinear problem to a linear one. I am much indebted to their ideas here as well as in my previous works.

Remark 5.5. Theorem 5.3 can be proved without the theory of Young measures. First of all, the set $\mathcal{M}$ of all the measurable mappings of Ω into X is weakly compact in $\mathfrak{L}^1(\Omega, \mathbb{R}^l)$ by the Dunford-Pettis-Nagumo theorem. Furthermore we know that the integral functional $J' : \mathfrak{L}^1(\Omega, \mathbb{R}^l) \to \mathbb{R}$ defined by

$$J' : f \mapsto \int_\Omega u(\omega, f(\omega))d\mu$$

is sequentially *l.s.c.* with respect to the weak topology of $\mathfrak{L}^1(\Omega, \mathbb{R}^l)$. This comes from Ioffe's [9] fundamental theorem. Combining these observations, we can assure the existence of a solution for Problem(II).(cf. Maruyama [11] pp.286-295.)

Theorem 5.3 guaranttees the equivalence of Problems(II) and (III) in the sense that the optimized values of the two problems are equal.

6. Existence of a solution(2)

We turn next to Problem(I).

Theorem 6.1. *Under Assumption 1 and Assumption 2, Problem(I) has a solution.*

Proof. The finite set $K = \{x_1, x_2, \ldots, x_p\}$ is compact. It is easy to check that the set of all the measurable families $\mathfrak{P}(\Omega, \mu; K)$ consisting of probability measures on K is compact in $\mathfrak{L}^\infty_*(\Omega, \mathfrak{M}(X))$. Hence, by Proposition 3.2, the set $\mathfrak{Y}(\Omega, \mu; K)$ of Young measures on $(\Omega \times K, \mathcal{E} \otimes \mathcal{B}(K))$ is narrowly compact. Since the integral functional J defined by (4.1) is narrowly *l.s.c.* on $\mathfrak{Y}(\Omega, \mu; X)$, so is on $\mathfrak{Y}(\Omega, \mu; K)$. Thus there exists a solution $\gamma^* \in \mathfrak{Y}(\Omega, \mu; K)$ of the problem:

$$\underset{\gamma \in \mathfrak{Y}(\Omega,\mu;K)}{Minimize} \int_{\Omega \times K} u(\omega, x)d\gamma. \tag{6.1}$$

The measurable family ν^*_ω which determines γ^* is of the form:

$$\nu^*_\omega = \sum_{i=1}^p \lambda_i(\omega)\delta_{x_i}$$

where $\lambda(\omega) = (\lambda_1(\omega), \lambda_2(\omega), \ldots, \lambda_p(\omega)) : \Omega \to \Lambda_p$ (the fundamental simplex in $\mathbb{R}^p$) is measurable. Hence the optimized value W^* of Problem(I) can be calculated as

$$W^* = \int_{\Omega \times K} u(\omega, x)d\gamma^* = \int_\Omega \sum_{i=1}^p \lambda_i(\omega)u(\omega, x_i)d\mu.$$

Again by Ljapunov's convexity theorem, there exists a decomposition $E_1, E_2, \ldots, E_p$ of $\Omega(E_i \in \mathcal{E}$ for all $i)$ such that

$$W^* = \sum_{i=1}^{p} \int_{E_i} u(\omega, x_i) d\mu.$$

Defining

$$x^*(\omega) = \sum_{i=1}^{p} \chi_{E_i}(\omega) x_i,$$

we obtain

$$W^* = \int_{\Omega} u(\omega, x^*(w)) d\mu.$$

$x^*(\cdot)$ is clearly a solution for Problem(I). □

As we already saw in the previous section, Problems(II) and (III) are equivalent. Then it is natural to ask if Problems(I) and (II) are equivalent. Is there any solution for Problem(I) which attains the optimized value V^* of Problem(II)?

Game theorists interprete each element of the set $\mathcal{P}$(resp. $\mathcal{M}$) as a pure strategy (resp. mixed strategy). If the optimized value V^* of Problem(II) is attained by some pure strategy, game theorists say that Problem(II) can be **purified**. So the problem stated above is expressed as "Can Problem(II) be purified? "

The answer is negative as the following counter-example illuminates.

Counter-example Let Ω be the unit interval [0,1] with Lebesgue measure m. X is also specified as $X = [0,1] = \text{co}\{0,1\}$. We define an integrand $u : [0,1] \times [0,1] \to \mathbb{R}$ by

$$u(\omega, x) = \begin{cases} -2x + 1 & \text{on } [0, 1/2], \\ 2x - 1 & \text{on } [1/2, 1] \end{cases}$$

for any $\omega \in [0,1]$. Then the only solution $x^*(\omega)$ for Problem(II)

$$\underset{x(\cdot):[0,1]\to[0,1]}{Minimize} \int_0^1 u(\omega, x(\omega)) dm(\omega)$$

is given by $x^*(\omega) = 1/2$ a.e. The optimized value $V^* = 0$. However it is impossible to find a measurable function $y^*(\omega)$ of [0, 1] into $\{0,1\}$ which is equivalent to $x^*(\omega)$; i.e.

$$\int_0^1 u(\omega, y^*(\omega)) dm = \int_0^1 u(\omega, x^*(\omega)) dm = 0.$$

Thus $x^*(\omega)$ can not be purified.

However the purification is proved to be possible if we impose an additional condition on $u(\omega, x)$.

Assumption 3 $u(\omega, (1-\lambda)x_1 + \lambda x_2) = (1-\lambda)u(\omega, x_1) + \lambda u(\omega, x_2)$ for any $\omega \in \Omega, x_1, x_2 \in X$ and $\lambda \in [0,1]$.

Assumption 3 requires the graph of the function $x \mapsto u(\omega, x)$ to be flat for each fixed $\omega \in \Omega$.

Let $x^*(\cdot)$ be a solution for Problem(II). The measurable implicit function theorem assures the existence of measurable function$\lambda^* : \Omega \to \Lambda_p$ such that

$$x^*(\omega) = \sum_{i=1}^{p} \lambda_i^*(\omega) x_i.$$

Since the optimized value of Problem(II) is attained by $x^*(\cdot)$, it follows that

$$V^* = \int_\Omega u(\omega, x^*(\omega)) d\mu = \int_\Omega u\big(\omega, \sum_{i=1}^{p} \lambda_i^*(\omega) x_i\big) d\mu = \int_\Omega \sum_{i=1}^{p} \lambda_i^*(\omega) u(\omega, x_i) d\mu$$

by Assumption 3. we now apply Ljapunov's convexity theorem again to get a function $y^* : \Omega \to K$ which satisfies

$$V^* = \int_\Omega u(\omega, y^*(\omega)) d\mu.$$

We conclude the possibility of the purification of Problem(II).

Theorem 6.2. *Under Assumptions 1-3, Problem(II) can be purified.*

Since Problem(II) and Problem(III) are equivalent, Problem(III) can also be purified, that is, there exists a solution $z^*(\cdot)$ for Problem(I) which realizes the optimized value U^* of Problem(III).

Although the full purification is difficult without very strict conditions, what can we say about an approximate purification? Let $x^*(\cdot)$ be a solution for Problem(II). If there exists a solution $y^*(\cdot)$ for Problem(I) such that

$$\Big|\int_\Omega u(\omega, x^*(\omega)) d\mu - \int_\Omega u(\omega, y^*(\omega)) d\mu\Big| < \varepsilon$$

for some $\varepsilon > 0$, we say that Problem(II) can be **ε-purified**. Aumann et al. [3] examined a similar problem in the context of game theory. However we leave it to another occasion.

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