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STOCHASTIC PROCESSING TIME MODELLING IN MANUFACTURING

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ABSTRACT. This paper supports sustainable supply chain development by creation of an appropriate framework for a system efficiency evaluating process. The paper introduces and incorporates a concept of rework in an assembly production system under random conditions using a semi-regenerative stochastic process. The modelling has been done to obtain various busy period durations over finite-time duration for transient state analysis without any limitations on distribution of processing times. This helps to obtain desired busy, blocked or idle durations and to experiment on sensitivity of their changes for tuning the manufacturing rate. Such a transient state analysis will be highly beneficial at the design stage of transfer-line production systems under random conditions, and when it is needed, to monitor the system condition over a finite time horizon. Knowing the sensitivity of various durations with respect to changes in processing rates makes a production system more flexible and adaptable. This helps to avoid overproduction or products shortage that is reflected in financial performance of a supply chain. Our results create a framework to control rework process efficiency.

1. Introduction

Production system modelling and analysis is one of the oldest fields in Operational Research. Scientific investigations have being carried out for a long time, over years and decades, under various assumptions. At initial stages, research has been conducted under deterministic assumptions. Later, randomness was incorporated but confined to exponential distribution only. There is quite an amount of literature on modelling and analysis of production systems under the Markov assumption. That is, researchers assume that processing times involved in the analysis follow an exponential distribution. The unique memory less feature of the exponential distribution assumption enables us to model a system as a Markov process. Markov models can be developed using traditional difference-differential equation and solved using Laplace transforms and inverse transforms for a steady state solution and analysis. The present model makes no assumption about the distributions of any processing times involved and provides transient state solution and analysis. A general framework was developed, and for particular cases one can

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substitute desired distributions for various processing times involved. Nowadays, a sustainable manufacturing strategy is becoming very important as companies have to respond to continuous changes, environmental and social requirements. Some organizations have adopted sustainable thinking paradigm in order to optimise performance and competitive advantage [35]. Nevertheless, the majority of companies are still looking for scientific support allowing them to proceed with transformations into sustainable business effectively [34]. Some studies are already done on "sustainable" ("green") supply chains modelling [5, 18, 29, 30, 39, 43, 36]. One of the issues of green supply chain development is closing the "material loop" of products moving from resource extraction through production use to the end of life [29]. There are two approaches how to proceed with it: the first is to minimise amount of waste at all stages of product life from production till expiration through different ways of recycling and rework [5, 18, 29, 36]; the second is to decrease the amount of rework in production through reengineering technical and business processes [19, 30, 43]. These ways seem to be conflicting only at first sight. However, they supplement each other if we consider them as 2 stages: Stage I is waste minimisation through all possible ways; Stage II is production quality maximisation in order to avoid any other activities that cause additional resources spending. Each production tends to be flawless, but in reality there is still a huge amount of defects introduced [31] where rework is a better solution than scrap. That is why rework becomes vital in production for sustainable supply chains development. The current paper presents modelling of a production system with rework aiming to be a helpful tool to support waste reduction process. System can be defined as a set of interacting elements or processes that operate in a coordinated and combined structure to achieve a predefined common goal. Production system can be defined as a way, by which resource inputs are transformed to create goods and services. In manufacturing industries, inputs include various raw materials such as energy, labour, machine, etc. In service industries, inputs are likely to be dominated by labour [8]. This paper focuses on a discrete part of manufacturing system featuring a distinct processed item and non-Markovian processing times; this is common for industries that are producing parts for computers, cars, home appliances, etc. Apparently, production line is an essential part of manufacturing industry, but only few studies were made on interactions between stages in it [10]. Production systems analysis is a very old discipline of industrial engineering, but it still needs to be extended [22]. Analysis of systems with 2 phases prompts how to deal with n-stage systems, and this question has been in the spotlight for the last few years. This is a result of possibility to analyse any multi n-stage system as a series of two-stage systems [13]. If we take a closer look at relevant researches, we can notice that several authors such as Lau (1986) [24], Prasaka Rao (1975) [33], Avi-Itzhak (1965) [3], Avi-Itzhak and Yadin (1965) [4], Muth (1973) [27], Gupta and Sharma (1983) [17], Berman (1982) [6], Commault and Dallery (1990) [12] have analysed such production systems with special attention at processing times and times between states. But the blind sides of these researches are either absence of inspection or rework as all defected items were scrapped [38]. However, this is not applying for all production situations, for example, when the cost of an item is very high. In fact, as it has been acutely noticed by Gupta and Chakraborty (1984) [16], rework is essential for the big part of production systems and situations. Furthermore, now we live in a "cradle-to-grave" society, which considers only the environmental impact of disposal. But we need to minimise the natural resource consumption, waste generation, and environmental impact in the life cycle of products; we should aim at realisation of the "cradle-to-cradle" system that will enable full use of recycled materials and reusable parts [5, 39, 43, 36]. Nevertheless, not much research has been done on rework. Tayi and Ballou (1988) made a research on rework of items that were rejected, but their study is limited by deterministic models [41]. A big part of literature is dedicated to analysis of steady-state behaviour of production system. It is a rough abstraction of reality, because in the majority of cases systems will collapse before reaching the steady state. Few researchers like Prabhu (1966-1967) [32], Altiok (1982) [1], Kumar (1992) [23], Grassman (1977) [15] have made a research on the transient behaviour of production systems. In their works, a system was presented as a queuing complex, dealing only with Markovian distributions without including a possibility of rework. Processing times become important with implementation of lean manufacturing production shifted from batch manufacturing to a combination of batch and balanced mixed-model assembly system [40]. On the one hand, new practice helps to keep production load even and effectively manage both production and inventory. On the other hand, it requires extra efficient loading and processing times management. Many applications and information systems include an idea that an input queue has unlimited capacity. This concept is applied in many serial production systems featuring an infinite buffer [28]. Though an assumption of the initial buffer of infinite capacity sounds artificial in production lines, it only implies that input is always available, meaning that the initial stage never gets starved [37]. Jingshan Li (2004) has developed an aggregation procedure to calculate production rate of production systems with rework loops when machines have identical cycle times [25]. Recently, Buscher and Lindner (2007) have analysed a production system with rework, but this model is deterministic [9]. Assembly production systems have been analysed without taking into consideration the idea of rework in Gopalan and Kumar (1994) and Liu and Li (2009) [14, 26]. We are considering the below quality control problem in an assembly production system. At Stage I, components are made and processed. At Stage II, components are assembled. Assembled products coming out of Stage II are checked at the inspection point, and properly done items are moved out of the production line, while bad ones are further classified as products that can be reworked or scrapped due to major defects. We use semi-regenerative processes to model the system described above. It requires understanding and knowledge of renewal processes. Studies of Uematsu et al. (1984) and Birolini (1985) could be used as a reference [7, 42]. For state probabilities, we have developed an integral representation. It is done by identifying the system at suitable regeneration epochs as explained by Cinlar (1975) [11]. Created convolution equations have been solved using existing numerical methods [20]. This provides means to employ a variety of distributions, for example, Erlang (two-stage) distribution that is particularly useful in practical situations. It provides more possibilities than the traditional Laplace Transform technique, because it is not limited to exponential distribution only [2]. In our paper, we have reflected a numerical example for this particular case; it is presented considering Erlang distributions of processing times for Machines 1, 2 and

3. We made an assumption that distributions of processing times of rework and all other random variables used in modelling are arbitrary. Therefore, we obtained the following system [21]:

- Expected time of Machine 1 at Stage I being busy in [0, t].
- Expected time of Machine 2 at Stage I being busy in [0, t].
- Expected time of both Machines at Stage I being busy in [0, t]. Expected time of Machine 1 at Stage I being blocked in [0, t].
- Expected time of Machine 2 at Stage I being blocked in [0, t].
- Expected time of Stage II (i.e., Machine 3) being busy in [0, t].
- Expected time of Stage II being idle in [0, t].
- Expected time of Stage II being busy with rework in [0, t].
- Expected time of Stage II being busy with rework of type i in [0, t].
- Below is a list of assumptions made to model the system under consideration.

2. Assumptions

In this paper, we assume the following:

- Instantaneous type of items delivery from the initial buffer to Machines in Stage I and from Stage I to Inspection Station and from Inspection Station to Stage II.
- Inspection is instantaneous.
- Whenever an assembly operation should be reworked, the Stage II machine will start reworking it immediately.
- Duration of the both stages is random, arbitrarily distributed and independent.
- Products from Stage I will be inspected at the inspection station only when Stage II is free.
- Never blocked Stage II (i.e., machine 3).
- All items are reworked properly.
- Machine 1/2/3 is perfect (i.e., reliable).
- Instantaneous setup time.

3. Notations and System Modeling

Let us introduce the following symbols:

- pdf: Probability density function,
- cdf: Cumulative distribution function,
- sf: Survivor function (also known as reliability function or complementary cumulative distribution function),
- $f(\cdot)/g(\cdot)$: pdf for Machine 1/Machine 2 processing time at Stage I,
- $F(\cdot)/G(\cdot)$: cdf for Machine 1/Machine 2 processing time at Stage I,
- $\bar{F}(\cdot)/\bar{G}(\cdot)$: sf for Machine 1/Machine 2 processing time at Stage I,
- $h(\cdot)/r(\cdot)$: pdf of assembling and rework time at Stage II,
- $H(\cdot)/R(\cdot)$: cdf of assembling and rework time at Stage II,
- $\bar{H}(\cdot)/\bar{R}(\cdot)$: sf of assembling and rework time at Stage II,
- p_g: Probability of an assembly operation to be performed properly at Stage II,

State	Machine 1 in Stage 1	Machine 2 in Stage 1	Stage II (Machine 3)
1	Busy	Busy	Free
2	Blocked	Busy	Free
3	Busy	Blocked	Free
4	Busy	Busy	Busy
5	Blocked	Busy	Busy
6	Busy	Blocked	Busy
7	Blocked	Blocked	Busy
8	Busy	Busy	Busy with Rework
9	Blocked	Busy	Busy with Rework
10	Busy	Blocked	Busy with Rework
11	Blocked	Blocked	Busy with Rework

Table 1. State space

- p_r: Probability of an assembly operation to be performed improperly at Stage II but can be reworked,
- p_s: Probability of an assembly operation to be neither performed properly at Stage II nor reworked. Clearly, $p_g + p_r + p_s = 1$.
- $*$: Convolution:

$$(f * g)(t) = \int_a^b f(u)g(t-u)du; \quad a,\, b \in \mathbb{R}, \quad a < b.$$

According to commutative property of convolution [2]:

$$(f * g)(t) = (g * f)(t).$$

Conformably to the convolution theorem, two functions' convolution in the time domain has a simple effect of multiplying their Fourier transforms (T) in the frequency domain [2]:

$$T[f(t) * g(t)] = \int_a^b (\int_a^b f(u)g(t-u)dt)e^{-u}du.$$

The change of variables, $\xi = t - u$, and reversing the order of integration give

$$\begin{aligned} T[f(t) * g(t)] &= \int_a^b f(u)(\int_a^b (g(\xi)e^{-i(\xi+u)})d\xi du \\ &= (\int_a^b f(u)e^{-iu}du)(\int_a^b g(\xi)e^{-i\xi}d\xi) \\ &= T[f(t)]T[g(t)]. \end{aligned}$$

To model the system under consideration, we have identified the state of the system at any instant t. An exhaustive list of probable states of the system is given in Table 1.

Schematic diagram of the production system is presented in Figure 3.

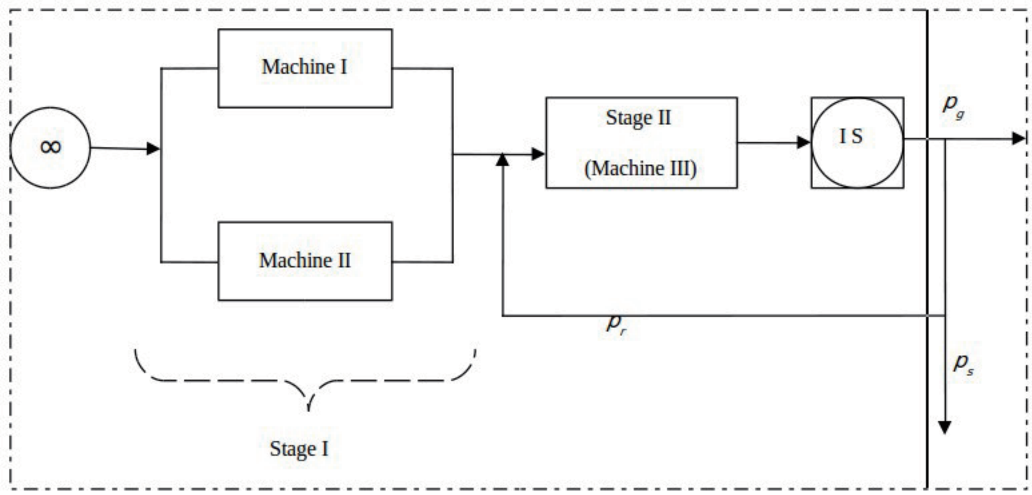

Figure 1. Schematic diagram of the production system.

4. Evaluation of System Characteristics

In this section, we have obtained analytical expressions for various measures of the system performance such as busy period durations.

4.1. Expected duration of Machine 1 or Machine 2 at Stage I being busy in $[0, t]$.

An expression for the expected duration when Machine 1 or Machine 2 at Stage I is busy in [0, t] is obtained as follows.

Let $Av_1^{1B}(t)$ for Machine 1 and $Av_1^{2B}(t)$ for Machine 2 denote the probability of a corresponding machine at Stage I to be busy at instant t, given that the system was in State 1 at time $t = 0$. Starting with State 1, the next regenerative transition is to State 4 (i.e., finished components from Machines 1 and 2 are transferred to Machine 3 for merging):

$$Av_1^{1B}(t) = [f(t)G(t) + g(t)F(t)] * Av_4^{1B}(t) + \bar{F}(t). \tag{4.1}$$

The term $\bar{F}(t)$ captures the busy period duration of Machine 1. Now, starting from State 4, assembling may be performed properly with probability p_g or with defect, which can be reworked with probability p_r, or it should be scrapped with probability p_s. In other words, starting from State 4, depending on whether the assembling has been done properly or not, the system makes a transition to State 4 with probability $p_g + p_s$ and to State 8 with probability p_r. This means:

$$\begin{aligned} Av_4^{1B}(t) = {} & \{f(t)[h(t)G(t) + g(t)H(t)] + g(t)[f(t)H(t) + h(t)F(t)] \\ & + h(t)[f(t)G(t) + g(t)F(t)]\} \\ & * \{p_g Av_4^{1B}(t) + p_r Av_8^{1B}(t) + p_s Av_4^{1B}(t)\} \\ & + \bar{F}(t). \end{aligned} \tag{4.2}$$

Similarly, for State 8, the transition equation is

$$\begin{aligned}(4.3)\qquad Av_8^{1B}(t) = \{&f(t)[r(t)G(t)+g(t)R(t)]+g(t)[f(t)R(t)+r(t)F(t)]\\ &+r(t)[f(t)G(t)+g(t)F(t)]\}\\ &* Av_4^{1B}(t)\\ &+\bar{F}(t).\end{aligned}$$

The above set of integral equations can be arranged in a matrix form with reference to [2, 20] as follows:

$$G(t) - \int_0^t W(u)G(t-u)du = L(t), \tag{4.4}$$

where W is a square matrix of order n (n = the number of equations) consisting of coefficients of the functions Av_i^{1B}; G and L are column matrices of format $n \times 1$ consisting of functions Av_i^I and terms independent of the Av_i^{1B}, respectively.

The above set of integral equations, being of convolution type, can be solved by the method suggested by Jones (1961) [2, 20].

Expected duration of Machine 1 being busy in $[0,\ t]$ is given by

$$\mu^{1B}(t) = \int_0^t Av_1^{1B}(u)du. \tag{4.5}$$

For Machine 2, the matrices G and W remain the same, while the matrix L is

$$L^T = [L_1, L_2, L_3], \tag{4.6}$$

where

$$L_1 = \bar{G}(t),\ L_2 = \bar{G}(t),\ L_3 = \bar{G}(t).$$

Expected time of Machine 2 being busy in $[0,\ t]$ is given by

$$\mu^{2B}(t) = \int_0^t Av_1^{2B}(u)du. \tag{4.7}$$

4.2. **Expected duration of Machine 1 or Machine 2 at Stage I being blocked in** $[0,t]$**.** We have obtained the following expression for the expected duration of Machine 1 or Machine 2 at Stage I being blocked in $[0,t]$.

Let $Av_1^{1BL}(t)$ for Machine 1 and $Av_1^{2BL}(t)$ for Machine 2 reflect the probability of each machine at Stage I to be blocked at instant t, given that the system was in State 1 at time $t=0$. A system of equations will be as follows:

$$Av_1^{1BL}(t) = [f(t)G(t)+g(t)F(t)] * Av_4^{1BL}(t) + F(t)\bar{G}(t), \tag{4.8}$$

$$\begin{aligned}(4.9)\qquad Av_4^{1BL}(t) = \{&f(t)[h(t)G(t)+g(t)H(t)]+g(t)[f(t)H(t)+h(t)F(t)]\\ &+h(t)[f(t)G(t)+g(t)F(t)]\}\\ &*\ \{p_g Av_4^{1BL}(t) + p_r Av_8^{1BL}(t) + p_s Av_4^{1BL}(t)\}\\ &+F(t)[\bar{G}(t)+G(t)\bar{H}(t)],\end{aligned}$$

$$\begin{aligned}(4.10)\qquad Av_8^{1BL}(t) = \{&f(t)[r(t)G(t)+g(t)R(t)]+g(t)[f(t)R(t)+r(t)F(t)]\\ &+r(t)[f(t)G(t)+g(t)F(t)]\}\end{aligned}$$

$$* \ Av_4^{1BL}(t) + F(t)[\bar{G}(t) + G(t)\bar{H}(t)].$$

The above set of integral equations can be reflected in matrix form [20] as in Equation (4.4), where W is a square matrix of type $n \times n$, and n is a number of equations consisting of coefficients of the functions Av_i^{1BL}. Furthermore, L is a column matrix of dimension $n \times 1$ consisting of the functions Av_i^{1BL} and terms independent of the Av_i^{1BL}, respectively. Clearly, the matrices G and W remain the same, while the matrix L changes.

This set of convolution integral equations can be solved as mentioned before [20]. The expected time of Machine 1 being busy in $[0, t]$ is given by

$$\mu^{1BL}(t) = \int_0^t Av_1^{1BL}(u)du. \tag{4.11}$$

For Machine 2, the matrices G and W stay the same, while elements of the matrix L will be the following:

$$L_1 = G(t)\bar{F}(t),\ L_2 = G(t)[\bar{F}(t) + F(t)\bar{H}(t)],\ L_3 = G(t)[\bar{F}(t) + F(t)\bar{R}(t)].$$

The expected time of Machine 2 at Stage I being blocked in $[0, t]$ is given by

$$\mu^{2BL}(t) = \int_0^t Av_1^{2BL}(u)du. \tag{4.12}$$

4.3. **Expected time of Stage II being busy, busy with rework or idle in $[0, t]$.** Let $Av_1^{S2B}(t)$, $Av_1^{S2R}(t)$ and $Av_1^{S2I}(t)$ denote the probability that Stage II (i.e., Machine 3) is busy, busy with rework or idle respectively at instant t, given that the system was in State 1 at time $t = 0$.

For Stage II being busy at instant t, the matrices G and W remain the same, while elements of the matrix L will be the following:

$$L_1 = 0,\ L_2 = \bar{H}(t),\ L_3 = \bar{R}(t).$$

The expected time of Stage II being busy in $[0, t]$ is given by [21]

$$\mu^{S2B}(t) = \int_0^t Av_1^{S2B}(u)du. \tag{4.13}$$

For Stage II being busy with rework at instant t, the matrices G and W stay the same, while the elements of the matrix L corresponding to this case are:

$$L_1 = 0,\ L_2 = 0,\ L_3 = \bar{R}(t).$$

The expected time of Stage II being busy with rework in $[0, t]$ is given by

$$\mu^{S2R}(t) = \int_0^t Av_1^{S2R}(u)du. \tag{4.14}$$

For Stage II being idle at instant t, the elements of the matrix L corresponding to this case are:

$$L_1 = \bar{F}(t) + F(t)\bar{G}(t),\ L_2 = H(t)[\bar{F}(t) + F(t)\bar{G}(t)],\ L_3 = R(t)[\bar{F}(t) + F(t)\bar{G}(t)].$$

The expected time of Stage II being idle in $[0, t]$ is given by

State	Machine 1 in Stage I	Machine 2 in Stage I	Stage II (Machine 3)
1	Busy	Busy	Free
2	Blocked	Busy	Free
3	Busy	Blocked	Free
4	Busy	Busy	Busy
5	Blocked	Busy	Busy
6	Busy	Blocked	Busy
7	Blocked	Blocked	Busy
8	Busy	Busy	Busy with Rework of Type 1
9	Busy	Busy	Busy with Rework of Type 2
10	Blocked	Busy	Busy with Rework of Type 1
11	Blocked	Busy	Busy with Rework of Type 2
12	Busy	Blocked	Busy with Rework of Type 1
13	Busy	Blocked	Busy with Rework of Type 2
14	Blocked	Blocked	Busy with Rework of Type 1
15	Blocked	Blocked	Busy with Rework of Type 2

TABLE 2. State space with two types of rework.

$$\mu^{S2I}(t) = \int_0^t Av_1^{S2I}(u)du. \tag{4.15}$$

4.4. **Expected duration of Machine 1 at Stage I being busy in $[0,t]$ when there are two types of rework.** Now, we may increase the number of possible types of defects from 1 to 2. Later, we shall generalise the number of defects to M ($M \geq 2$). Assume that an assembling operation can result in two types of defects: Type 1 defect with probability p_{r1} and Type 2 defect with probability p_{r1}. Clearly, $p_g+p_s+p_r = 1$. This means that after assembling at Stage II, products are checked at the inspection station, and the assembling operation clears inspection with probability p_g. Afterwards, products could have Type 1 defect with probability p_{r1} or Type 2 with probability p_{r2}. Therefore, the operation is neither performed properly nor can be reworked (scrap) with probability p_s. Additional notations pertaining to the two types of rework are:

- p_{r1}: Probability that an assembled product has Type 1 defect,
- p_{r2}: Probability that an assembled product has Type 2 defect,
- $f_1(\cdot)/f_2(\cdot)$: pdf of time to rework Type 1/Type 2 defect at Stage II,
- $F_1(\cdot)/F_2(\cdot)$: cdf of time to rework Type 1/Type 2 defect at Stage II,
- $\bar{F}_1(\cdot)/\bar{F}_2(\cdot)$: sf of time to rework Type 1/Type 2 defect at Stage II.

The total number of states in the state space increases to 15. The *state space* is presented in Table 2.

A schematic diagram, corresponding to two types of defects cases, is provided in Figure 4.4.

Under varied assumptions on types of rework, we shall obtain an expression for the expected time of Machine 1 being busy in $[0,t]$.

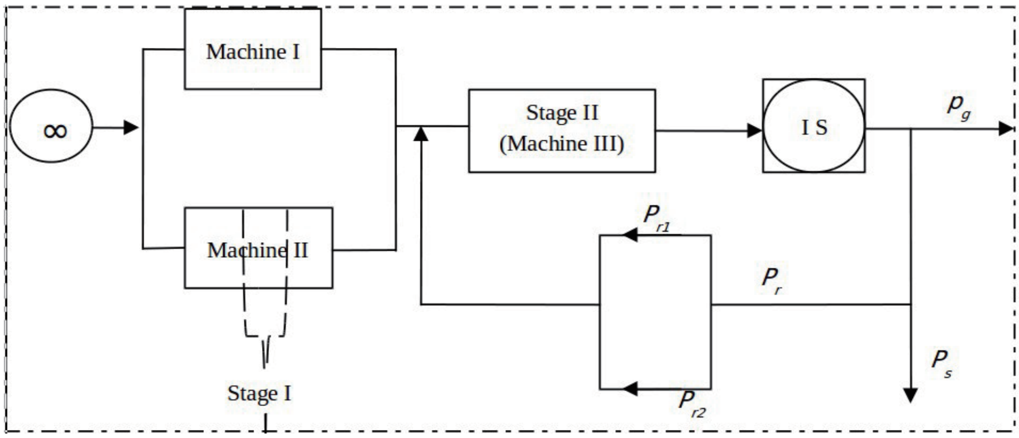

FIGURE 2. Schematic diagram of the production system.
Note: I.S: Inspection Station.

The expression for the expected time of Machine 1 at Stage I being busy in $[0, t]$ is modelled as follows.

Let $Av_1^{1B}(t)$ denote the probability that Machine 1 at Stage I is busy at instant t, given that the system was in State 1 at time $t = 0$.

$$Av_1^{1B}(t) = [f(t)G(t) + g(t)F(t)] * Av_4^{1B}(t) + \bar{F}(t), \tag{4.16}$$

$$\begin{aligned} Av_4^{1B}(t) = &\{f(t)[h(t)G(t) + g(t)H(t)] + g(t)[f(t)H(t) + h(t)F(t)] \\ &+ h(t)[f(t)G(t) + g(t)F(t)]\} \\ &* \{p_g Av_4^{1B}(t) + p_{r1} Av_8^{1B}(t) + p_{r2} Av_9^{1B}(t) + p_s Av_4^{1B}(t)\} + \bar{F}(t), \end{aligned} \tag{4.17}$$

$$\begin{aligned} Av_8^{1B}(t) = &\{f(t)[r_1(t)G(t) + g(t)R_1(t)] + g(t)[f(t)R_1(t) + r_1(t)F(t)] \\ &+ r_1(t)[f(t)G(t) + g(t)F(t)]\} \\ &* Av_4^{1B}(t) + \bar{F}(t), \end{aligned} \tag{4.18}$$

$$\begin{aligned} Av_9^{1B}(t) = &\{f(t)[r_2(t)G(t) + g(t)R_2(t)] + g(t)[f(t)R_2(t) + r_2(t)F(t)] \\ &+ r_2(t)[f(t)G(t) + g(t)F(t)]\} \\ &* Av_4^{1B}(t) + \bar{F}(t). \end{aligned} \tag{4.19}$$

The expected time of Machine 1 being busy in $[0, t]$ is given by

$$\mu^{1B}(t) = \int_0^t Av_1^{1B}(u)du. \tag{4.20}$$

One can compare the systems of equations obtained in Subsections 4.1–4.4 to understand the difference due to the assumption about two types of defects and consequently two types of rework.

4.5. **Expected time of Stage II being busy with rework of Type 1, Type 2 in** $[0, t]$**.** Let $Av_1^{R1}(t)$ and $Av_1^{R2}(t)$ denote the probability that Stage II is busy with rework of Type 1 and Type 2 defects at instant t respectively, given that the system was in State 1 at time $t = 0$.

For rework of Type 1 defect, the matrices G and W remain the same, while the matrix L will be the following:

$$L^T = [L_1, L_2, L_3, L_4], \tag{4.21}$$

where

$$L_1 = 0,\ L_2 = 0,\ L_3 = \bar{R}_1(t),\ L_4 = 0.$$

The expected time of Stage II being busy with Type 1 defect rework in $[0,\ t]$ is given by

$$\mu^{R1}(t) = \int_0^t Av_1^{R1}(u)du. \tag{4.22}$$

For rework of Type 2 defect, elements of the matrix L will be the following:

$$L_1 = 0,\ L_2 = 0,\ L_3 = 0,\ L_4 = \bar{R}_2(t).$$

Using the same logic as for Type 1 rework, we can define the expected time of Stage II being busy with rework of Type 2 in $[0, t]$ as given by

$$\mu^{R2}(t) = \int_0^t Av_1^{R2}(u)du. \tag{4.23}$$

4.6. **Expected time of Stage II being busy with rework of Type i defect in** $[0, t]$**.** Now, we may increase and generalise the number of defects from 2 to M ($M \geq 2$). We assume that an assembly operation can result in M types of defects. Completed assembling is defected with Type 1 with probability p_{r1}, with Type 2 with probability p_{r2}, and so on. Clearly, $p_g + p_s + p_{r1} + p_{r2} + \ldots + p_{rM} = 1$. This means that after assembling at Stage II products are inspected at the inspection station, and the assembly operation clears inspection with probability p_g. Afterwards, products could have Type 1 defect with probability p_{r1} or Type 2 with probability p_{r2},..., or Type M with probability p_{rM}. Therefore, the operation is neither performed properly nor can be reworked with probability p_s. The additional notation pertaining to M types of rework are:

(1) p_{ri}: probability that an assembled product is defective of type i ($i \in \{2, \ldots, M\}$),
(2) $f_i(\cdot)$: pdf of time to rework Type 1/Type 2 defect in Stage II,
(3) $F_i(\cdot)$: cdf of time to rework Type 1/Type 2 defect in Stage II,
(4) $\bar{F}_i(\cdot)$: sf of time to rework Type 1/Type 2 defect in Stage II.

When the number of defects types increases and gets generalised to M, the number of states in the state space increases to $4M + 7$. The state space is presented in Table 4.6.

A schematic diagram, corresponding to M types of defect cases, is provided in Figure 4.6.

States in the state space can be clustered together, and the abbreviated version of State Space (the clustered state space) is presented in Table 4.

The system of equations have been developed using the clustered state space of Table 4. For a specific value of M, the state space can be extended to get the system of equations. An expression for the expected time of Stage II (i.e., Machine 3) being busy with rework Type i defect in $[0, t]$ is obtained as follows.

State	Machine 1 in Stage I	Machine 2 in Stage I	Stage II (Machine 3)
1	Busy	Busy	Free
2	Blocked	Busy	Free
3	Busy	Blocked	Free
4	Busy	Busy	Busy
5	Blocked	Busy	Busy
6	Busy	Blocked	Busy
7	Blocked	Blocked	Busy
8	Busy	Busy	Busy with Rework of Type 1
9	Busy	Busy	Busy with Rework of Type 2
...	...	...	...
M+7	Busy	Busy	Busy with Rework of Type M
M+8	Blocked	Busy	Busy with Rework of Type 1
M+9	Blocked	Busy	Busy with Rework of Type 2
...	...	...	...
2M+7	Blocked	Busy	Busy with Rework of Type M
2M+8	Busy	Blocked	Busy with Rework of Type 1
2M+9	Busy	Blocked	Busy with Rework of Type 2
...	...	...	...
3M+7	Busy	Blocked	Busy with Rework of Type M
3M+8	Blocked	Blocked	Busy with Rework of Type 1
3M+9	Blocked	Blocked	Busy with Rework of Type 2
...	...	...	...
4M+7	Blocked	Blocked	Busy with Rework of Type M

TABLE 3. State Space with M Types of rework.

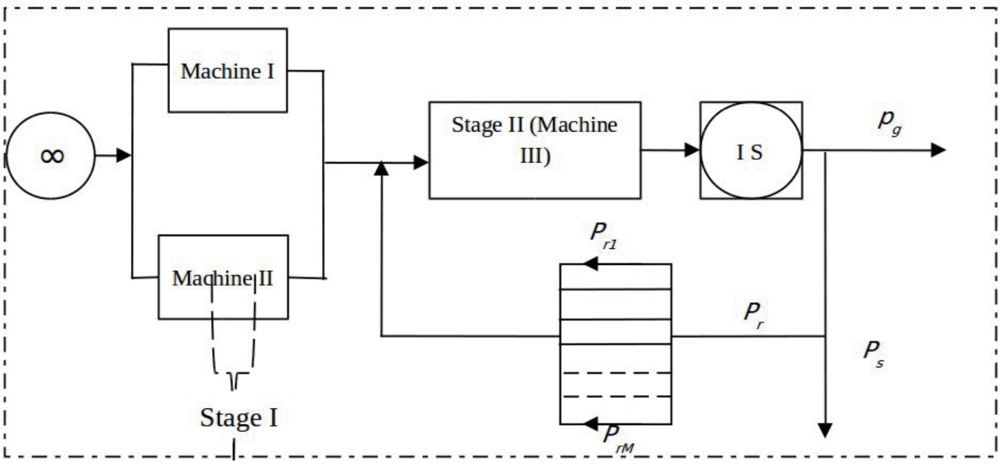

FIGURE 3. Schematic diagram of the production system.
Note: I.S: Inspection Station.

State	Machine 1 in Stage I	Machine II in Stage II	Stage II (Machine 3)
1	Busy	Busy	Free
2	Blocked	Busy	Free
3	Busy	Blocked	Free
4	Busy	Busy	Busy
5	Blocked	Busy	Busy
6	Busy	Blocked	Busy
7	Blocked	Blocked	Busy
8(i)	Busy	Busy	Busy with Rework of Type i
9(i)	Blocked	Busy	Busy with Rework of Type i
10(i)	Busy	Blocked	Busy with Rework of Type i
11(i)	Blocked	Blocked	Busy with Rework of Type i

TABLE 4. Clustered state space (with M types of rework).

Let $Av_1^{Ri}(t)$ denote probability that Stage II is busy with rework of type i at instant t, given that the system was in state 1 at time $t = 0$. Then, the system of equations can be written following similar logic:

$$Av_1^{Ri}(t) = [f(t)G(t) + g(t)F(t)] * Av_4^{Ri}(t), \tag{4.24}$$

$$\begin{aligned} Av_4^{Ri}(t) = \{&f(t)[h(t)G(t) + g(t)H(t)] + g(t)[f(t)H(t) + h(t)F(t)] \\ &+ h(t)[f(t)G(t) + g(t)F(t)]\} \\ * \ \{&p_g Av_4^{Ri}(t) + \sum_{i=1}^{M} p_{r1} Av_8^{Ri}(t) + p_s Av_4^{Ri}(t)\}, \end{aligned} \tag{4.25}$$

$$\begin{aligned} Av_8^{Ri}(t) = \{&f(t)[r_i(t)G(t) + g(t)R_i(t)] + g(t)[f(t)R_i(t) + r_i(t)F(t)] \\ &+ r_i(t)[f(t)G(t) + g(t)F(t)]\} \\ * \ &Av_4^{Ri}(t) + \bar{F}(t) \ (i \in \{1, \ldots, M\}). \end{aligned} \tag{4.26}$$

The expected time of Stage II being busy with rework Type 1 defect in $[0, t]$ is given by

$$\mu^{Ri}(t) = \int_0^t Av_1^{Ri}(u)du. \tag{4.27}$$

5. Numerical illustration

Analysis of production system (Figure 3, Figure 4.4, Figure 4.6) with rework performed in the article resulted in analytical expressions for expected time of Machine 1/2/3 being busy, blocked, idle, and busy with rework of any given type in a given time interval. It was used for the development of a tool that allows smart processing of our problem. We have written a program using VBA language in MS Excel application in order to obtain numerical values of blocked and busy time durations. Number of defects types is taken to be 2 (an assembly operation, when not correct, can result in an item that has Type 1 or Type 2 defect). With respect to changes in

$\lambda_2 = 3$; $\lambda_3 = 4$; $\lambda_4 = 1$							
λ_1	T	Expected Duration					
		M/c I of Stage is		M/c II of Stage I is		M/c III of Stage II is	
		Busy	Blocked	Busy	Blocked	Busy	Idle
	1.0	0.878	0.122	0.720	0.280	0.124	0.876
	2.0	1.700	0.300	1.270	0.729	0.507	1.493
2.0	3.0	2.510	0.489	1.808	1.191	0.934	2.066
	4.0	3.307	0.693	2.337	1.662	1.378	2.622
	5.0	4.096	0.904	2.862	2.138	1.832	3.168
	1.0	0.787	0.213	0.787	0.213	0.184	0.816
	2.0	1.465	0.535	1.465	0.535	0.669	1.331
3.0	3.0	2.120	0.880	2.121	0.879	1.198	1.802
	4.0	2.761	1.239	2.761	1.239	1.744	2.256
	5.0	3.396	1.604	3.396	1.604	2.295	2.705

TABLE 5. Effect of change in Machine 1 processing rate on system characteristics; Part 1.

$\lambda_1 = 2$; $\lambda_3 = 4$; $\lambda_4 = 1$; $\lambda_5 = 1$				
λ_1	T	Expected duration (Machine III at Stage II)		
		Busy with Rework	Busy with	
			Rework of Type 1	Rework of Type 2
	1.0	0.001	0.001	0.000
	2.0	0.027	0.040	0.001
2.0	3.0	0.110	0.161	0.002
	4.0	0.226	0.323	0.003
	5.0	0.352	0.496	0.006
	1.0	0.002	0.002	0.000
	2.0	0.051	0.075	0.001
3.0	3.0	0.171	0.246	0.002
	4.0	0.319	0.450	0.004
	5.0	0.474	0.660	0.007

TABLE 6. Effect of change in Machine 1 processing rate on system characteristics; Part 2.

processing rates of Machine 1, numerical values for the above mentioned measures of the system performance are presented in Table 5 and Table 6.

Using the same logic as for Machine 1, we have obtained results for Busy and Blocked durations for Machine 2, presented in Table 7 and Table 8.

$\lambda_2 = 3$; $\lambda_3 = 4$; $\lambda_4 = 1$							
λ_1	T	Expected Duration					
		M/c I of Stage is		M/c II of Stage I is		M/c III of Stage II is	
		Busy	Blocked	Busy	Blocked	Busy	Idle
2.0	1.0	0.912	0.088	0.623	0.377	0.151	0.849
	2.0	1.786	0.214	1.056	0.944	0.561	1.439
	3.0	2.640	0.360	1.480	1.520	1.014	1.986
	4.0	3.479	0.521	1.896	2.104	1.484	2.516
	5.0	4.310	0.690	2.310	2.690	1.962	3.038
3.0	1.0	0.934	0.066	0.543	0.457	0.168	0.832
	2.0	1.832	0.168	0.893	1.107	0.590	1.410
	3.0	2.707	0.293	1.239	1.761	1.055	1.946
	4.0	3.565	0.435	1.580	2.420	1.537	2.464
	5.0	4.415	0.585	1.919	3.081	2.026	2.974

TABLE 7. Effect of change in Machine 2 processing rate on system characteristics; Part 1.

$\lambda_1 = 2$; $\lambda_3 = 4$; $\lambda_4 = 1$; $\lambda_5 = 1$				
λ_2	T	Expected duration (Machine III at Stage II)		
		Busy with Rework	Busy with	
			Rework of Type 1	Rework of Type 2
2.0	1.0	0.001	0.002	0.000
	2.0	0.036	0.053	0.001
	3.0	0.130	0.188	0.002
	4.0	0.253	0.360	0.004
	5.0	0.387	0.542	0.006
3.0	1.0	0.002	0.002	0.000
	2.0	0.041	0.061	0.001
	3.0	0.140	0.202	0.002
	4.0	0.267	0.379	0.004
	5.0	0.404	0.565	0.006

TABLE 8. Effect of change in Machine 2 processing rate on system characteristics; Part 2.

This is done with respect to changes in the processing rates of Machine 2 and has been given in Appendix B, where $f(t) = \lambda_1^2 t\exp(-\lambda_1 t)$, $g(t) = \lambda_2^2 t\exp(-\lambda_2 t)$, $h(t) = \lambda_3^2 t\exp(-\lambda_3 t)$, $r_1(t) = \lambda_4\exp(-\lambda_4 t)$ and $r_2(t) = \lambda_5\exp(-\lambda_5 t)$.

Sensitivity of the numerical values of Busy and Blocked durations of Machine 1 with respect to changes in time (Figure 5): blocked duration increase with time passed, proportions that are measured in terms of % value of the time are actually decreasing are presented on Figure 5.

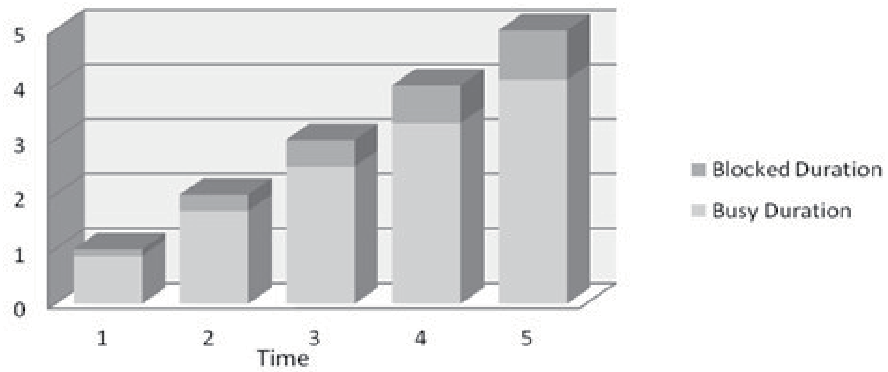

FIGURE 4. Busy and blocked durations of Machine 1 ($\lambda_1 = 2$; $\lambda_2 = 3$; $\lambda_3 = 4$; $\lambda_4 = 1$).

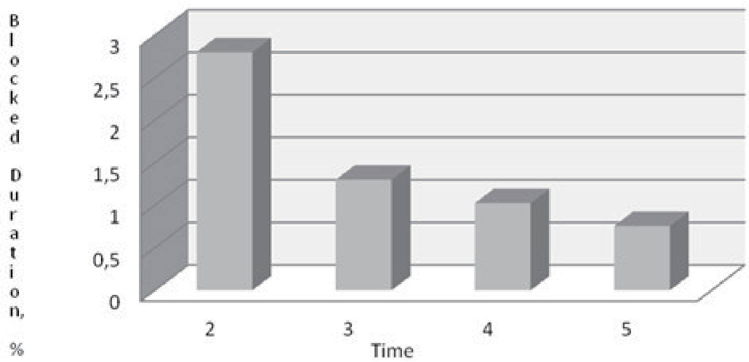

FIGURE 5. Rate of % Decrease in Blocked Durations ($\lambda_1 = 2$; $\lambda_2 = 3$; $\lambda_3 = 4$; $\lambda_4 = 1$).

Similarly, one can analyse busy or blocked/idle durations of Machines 2 and 3. Using such analyses, one can test sensitivity of various durations with respect to changes in processing rates of Machines 1, 2 and 3. Such a sensitivity analysis can be useful in two ways. The first one is to fix desired busy, blocked or idle durations (including that of rework) and to experiment on processing rates of various machines until desired durations are obtained. It is very helpful for tuning manufacturing rate when production needs to be speeded up or slowed down. The second way is to fix processing times of machines and to experiment on the sensitivity of changes in busy, blocked or idle durations. Such a transient state analysis will be highly useful at the design stage of transfer-line production systems under random conditions [6] and, likewise, when it is needed, to monitor the system condition over a finite time horizon. These situations are also common during testing working conditions when new machines are installed or regular inspection is done.

We are considering usage of regularization techniques to further stabilize the obtained numerical result. In general case of our problem dealing with production systems, we are facing multiple Machines, Products and types of Operations and Rework. Therefore, either dimension reduction techniques could be applied, or it is possible to subdivide the problem into smaller problems with the further parallelizing their solutions.

In order to make our model closer to real life situations, it could be embedded into a time-continuous model with a possibly infinite number of scenarios, and Monte-Carlo Markov-Chain methods and Regime-Switching models could be incorporated.

6. Conclusion and outlook

In this paper, the concept of rework and its generalisation to multi-type rework was investigated in the probabilistic modelling of two-stage assembly production systems with an inspection point. Analytical expressions for expected time of Machine 1/2/3 being busy, blocked, idle, and busy with rework of any given type in a given time interval have been obtained. Such an analysis of transient time is highly useful at the design stage of transfer-line production systems.

Results create a basis to model recycling process efficiency that could be used to develop a closed-loop supply chain as a part of reverse logistics process. In this case, types of defects could be replaced by recycling reasons (product expiration, power unit expiration, power unit defect, quality defect, etc.). It will give an opportunity to evaluate change of consumption of resources (time, cost, energy, etc.) caused by recycling process and manage its efficiency.

It is beneficial to have such a tool when we need to monitor a system over a finite time horizon. For example, to save energy, reduce waste, and efficiently rework a product without compromising production time or quality control processes, a greener solution is going to be implemented. Hence, we need to monitor a system in order to tune it up.

Also, it is helpful to learn when a machine could finish its task to enable a real-time adjustment. Knowing sensitivity of various durations with respect to changes in processing rates makes a production system more flexible and adaptable to changes. It is important to be able to respond to volatile customer demand by manipulation of busy, blocked and idle durations. This could be used for setting needed parameters for situations when demand is changing and manufacturing rate should be increased or decreased. That helps to solve the problem of controlling machines' usage within acceptable parameters of work. Furthermore, it facilitates adjusting to demand changes while avoiding product shortage or overstock. This allows to elude extra expenses (additional storage costs in case of overstock, penalties from clients in case of shortage, etc.) and to receive expected financial outcome.

Finally, our presented model creates a basis for a further algorithm and software development that will automate a system's efficiency evaluation.

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