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# DEPENDENCE ON PARAMETER OF THE SOLUTION TO AN INFINITE HORIZON LINEAR-QUADRATIC OPTIMAL CONTROL PROBLEM FOR SYSTEMS WITH STATE DELAYS

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Abstract. An infinite horizon linear-quadratic optimal control problem with point-wise and distributed state delays in the dynamics is considered. The coefficients of the problem, as well as the initial conditions, depend on a parameter. Subject to the assumption on a proper smoothness of this data of the optimal control problem in the parameter, a smooth dependence of its solution on this parameter is established.

## 1. Introduction

Dependence on a parameter of solutions to optimal control problems is an important topic of the control theory. Various problems of this topic have been studied in the literature. Different optimal control problems for ordinary differential equations (deterministic and stochastic) without delays, perturbed by a small parameter (or by a parameter with a small deviation from a nominal value), were extensively considered in the literature (see e.g. [1, 6, 7, 8, 10, 12, 13, 26, 30, 37] and references therein). Stability and sensitivity of solutions to these problems, as well as an asymptotic solution with respect to the parameter, for regular and singular types of the perturbation were analyzed. Dependence on a parameter of solutions to optimal control problems for various types of partial differential equations without delays also was studied extensively in the literature (see e.g. [2, 28, 29, 31, 33, 36] and references therein). Dependence on a parameter of solutions to optimal control problems for ordinary differential equations with delays was studied in the literature mainly in three cases: (i) dependence on a small positive parameter of singular perturbation in dynamics; (ii) dependence on a constant time delay; (iii) dependence on a small positive weight of control cost in the cost functional. The first case was analyzed in e.g. [11, 14, 16, 20, 22, 25, 38] (see also references therein). For the second case, one can see [3, 32] and references therein. The third case can be found in [15, 19, 21, 23, 24] and references therein. Sensitivity analysis of optimal control problems for some classes of partial differential equations with delays and a singular perturbation of geometrical domain of integration was carried out in [9, 27].

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In the present paper, we consider an infinite horizon linear-quadratic optimal control problem with point-wise and distributed state delays in the dynamics. The coefficients of the equation of dynamics in this problem and the corresponding initial conditions depend on a parameter. Using the control optimality conditions, the solution of this problem is reduced to solution of a set of three Riccati-type matrix equations, one algebraic and two differential equations (ordinary and partial) with deviating arguments. Based on the assumption of a proper smoothness of the data of the optimal control problem with respect to the parameter varying in a given interval, a smooth dependence of the solution to this set of Riccati-type equations on the parameter is established. Using this result, a smoothness with respect to the parameter of the state-feedback optimal control, the optimal trajectory and the optimal value of the cost functional in the considered optimal control problem is obtained. To the best of our knowledge, such a problem has not yet been studied in the literature.

Although the results of the present paper are rather theoretical, we would like to present here one their important application. Namely, these results are extremely useful in qualitative analysis and quadratic optimization of singularly perturbed linear time-dependent systems with small delays (see e.g. [14, 17, 18]) by application of the Boundary Function Method [34] and the Separation of Time-Scales Method [26]. Both methods are based on an asymptotic decomposition of the original singularly perturbed system into two simpler unperturbed subsystems (slow and fast ones). The slow subsystem is undelayed, and its independent variable is the original one (time). The fast subsystem is a time delay system. Its independent variable is a new one (stretched time), while the original independent variable becomes a parameter. Thus, the fast subsystem depends on the stretched time (as an independent variable) and on the original time (as a parameter). The stretched time varies from zero to infinity, while the original time varies in a given bounded and closed interval. One of important assumptions in the analysis and optimization of the original singularly perturbed system is the requirement on the first-order smoothness with respect to the parameter of the solution to a proper infinite-horizon linear-quadratic optimal control problem associated with the parameter dependent fast subsystem. In the papers [14, 17, 18] such a smoothness either was proven for the simplest particular case of a single point-wise delay in the fast subsystem [14], or was assumed to be valid in the general case of multiple point-wise delays and a distributed delay in the fast subsystem [17, 18]. In the present paper this smoothness is rigorously proven in the general case.

The paper is organized as follows. In Section 2, the optimal control problem is formulated rigorously. Its reduction to the set of three Riccati-type matrix equations is presented. Main assumptions are made. Objectives of the paper are stated. In Section 3, three important auxiliary lemmas and their corollaries are formulated and proven. Two main lemmas on continuity and first-order smoothness with respect to the parameter of the solution to the set of Riccati-type matrix equations are formulated and proven in Section 4. In Section 5, the smoothness with respect to the parameter of the state-feedback optimal control, the optimal trajectory and the optimal value of the cost functional of the considered optimal control problem is proven.

The following notations are applied in this paper:

(1) $E^n$ is the $n$-dimensional real Euclidean space.
(2) The Euclidean norm of either a matrix or a vector is denoted by $\|\cdot\|$.
(3) The superscript "$T$" denotes the transposition either of a vector $x$ ($x^T$) or of a matrix $A$ ($A^T$).
(4) $I_n$ denotes the identity matrix of dimension $n$.
(5) $L^2[\eta_1, \eta_2; E^n]$ denotes the Hilbert space of all functions $f(\eta) : [\eta_1, \eta_2) \to E^n$ square integrable on the interval $[\eta_1, \eta_2)$. The inner product of the elements $f(\eta)$ and $g(\eta)$ in this space is $\langle f(\eta), g(\eta)\rangle_{L^2} = \int_{\eta_1}^{\eta_2} f^T(\eta)g(\eta)d\eta$.
(6) $\mathcal{M}[\eta_1, \eta_2; n]$ denotes the Hilbert space of all pairs $f = \big(f_E, f_L(\eta)\big)$, $f_E \in E^n$, $f_L(\eta) \in L^2[\eta_1, \eta_2; E^n]$. The inner product of the elements $f = \big(f_E, f_L(\eta)\big)$ and $g = \big(g_E, g_L(\eta)\big)$ in this space is $\langle f, g\rangle_{\mathcal{M}} = f_E^T g_E + \langle f_L(\eta), g_L(\eta)\rangle_{L^2}$.
(7) Re$\lambda$ denotes the real part of a complex number $\lambda$.

## 2. Problem Statement

2.1. **Control problem formulation.** Consider the controlled system

$$\frac{dx(t)}{dt} = \sum_{j=0}^{N} A_j(\omega)x(t-h_j) + \int_{-h}^{0} G(\eta,\omega)x(t+\eta)d\eta + B(\omega)u(t), \quad t \geq 0, \tag{2.1}$$

where $x(t) \in E^n$; $u(t) \in E^r$ ($u$ is a control); $N \geq 0$ is an integer; $0 = h_0 < h_1 < h_2 < ... < h_N = h$ are some given constants; $\omega$ is a parameter; $A_j(\omega)$, $(j = 0, ..., N)$, $G(\eta, \omega)$ and $B(\omega)$ are matrix-valued functions of corresponding dimensions, given for $\eta \in [-h, 0]$ and $\omega \in [\omega_1, \omega_2]$; $A_j(\omega)$, $(j = 0, ..., N)$ and $B(\omega)$ are continuously differentiable with respect to $\omega \in [\omega_1, \omega_2]$; the function $G(\eta, \omega)$ is piecewise continuous with respect to $\eta \in [-h, 0]$ for each $\omega \in [\omega_1, \omega_2]$, its partial derivative $\partial G(\eta, \omega)/\partial\omega$ is continuous with respect to $\omega \in [\omega_1, \omega_2]$ uniformly in $\eta \in [-h, 0]$.

The initial conditions for the system (2.1) have the form

$$x(\eta) = \varphi(\eta, \omega), \quad \eta \in [-h, 0); \qquad x(0) = x_0(\omega), \tag{2.2}$$

where the vector-valued function $\varphi(\eta, \omega) \in L^2[-h, 0; E^n]$ for any $\omega \in [\omega_1, \omega_2]$, its partial derivative $\partial\varphi(\eta, \omega)/\partial\omega$ is continuous with respect to $\omega \in [\omega_1, \omega_2]$ uniformly in $\eta \in [-h, 0]$, and $\partial\varphi(\eta, \omega)/\partial\omega \in L^2[-h, 0; E^n]$ for any $\omega \in [\omega_1, \omega_2]$; the vector-valued function $x_0(\omega)$ is continuously differentiable for $\omega \in [\omega_1, \omega_2]$.

The cost functional, to be minimized by a proper choice of the control $u(t)$, is

$$J(u) = \int_{0}^{+\infty} \Big(x^T(t)x(t) + u^T(t)u(t)\Big)dt. \tag{2.3}$$

2.2. **Control optimality conditions in the problem (2.1)-(2.3).** Let us denote

$$S(\omega) \stackrel{\triangle}{=} B(\omega)B^T(\omega), \quad \omega \in [\omega_1, \omega_2]. \tag{2.4}$$

Using the matrix $S(\omega)$, we consider for any $\omega \in [\omega_1, \omega_2]$ the following set, consisting of one algebraic and two differential equations (ordinary and partial) for matrices $P$, $Q$, and $R$:

$$PA_0(\omega) + A_0^T(\omega)P - PS(\omega)P + Q(0) + Q^T(0) + I_n = 0, \tag{2.5}$$

$$\begin{aligned} \frac{dQ(\eta)}{d\eta} &= \Big(A_0^T(\omega) - PS(\omega)\Big)Q(\eta) + PG(\eta,\omega) \\ &\quad + \sum_{j=1}^{N-1} PA_j(\omega)\delta(\eta + h_j) + R(0,\eta), \end{aligned} \tag{2.6}$$

$$\begin{aligned} \Big(\frac{\partial}{\partial\eta} + \frac{\partial}{\partial\chi}\Big)R(\eta,\chi) &= G^T(\eta,\omega)Q(\chi) \\ &\quad + Q^T(\eta)G(\chi,\omega) + \sum_{j=1}^{N-1} A_j^T(\omega)Q(\chi)\delta(\eta + h_j) \\ &\quad + \sum_{j=1}^{N-1} Q^T(\eta)A_j(\omega)\delta(\chi + h_j) - Q^T(\eta)S(\omega)Q(\chi), \end{aligned} \tag{2.7}$$

where $\eta \in [-h, 0]$ and $\chi \in [-h, 0]$ are independent variables; $\delta(\cdot)$ is the Dirac delta-function.

The set of equations (2.5)-(2.7) is subject to the boundary conditions

$$\begin{gathered} Q(-h) = PA_N(\omega), \\ R(-h,\eta) = A_N^T(\omega)Q(\eta), \qquad R(\eta,-h) = Q^T(\eta)A_N(\omega). \end{gathered} \tag{2.8}$$

For a given $\omega \in [\omega_1, \omega_2]$, consider the state-feedback control in the system (2.1)

$$\tilde{u}(x_t) = \widetilde{K}_1(\omega)x(t) + \int_{-h}^{0} \widetilde{K}_2(\eta,\omega)x(t+\eta)d\eta, \tag{2.9}$$

where $\widetilde{K}_1(\omega)$ and $\widetilde{K}_2(\eta,\omega)$ are $r \times n$-matrices; $\widetilde{K}_2(\eta,\omega)$ is piece-wise continuous in $\eta \in [-h, 0]$; $x_t \overset{\triangle}{=} x(t+\eta)$, $\eta \in [-h, 0]$.

**Definition 2.1.** For a given $\omega \in [\omega_1, \omega_2]$, the system (2.1) is called $L^2$-stabilizable if there exists the state-feedback control (2.9) such that for any $x_0(\omega) \in E^n$, $\varphi(\eta,\omega) \in L^2[-h, 0; E^n]$, the solution $\tilde{x}(t)$ of (2.1) with $u(t) = \tilde{u}(x_t)$ and subject to the initial conditions (2.2) satisfies the inclusion $\tilde{x}(t) \in L^2[0, +\infty; E^n]$.

In what follows, we assume
**(A)** For all $\omega \in [\omega_1, \omega_2]$ and any complex number $\lambda$ with $\mathrm{Re}\lambda \geq 0$, the following equality is valid:

$$\mathrm{rank}\left[\sum_{j=0}^{N} A_j(\omega)\exp(-\lambda h_j) + \int_{-h}^{0} G(\eta,\omega)\exp(\lambda\eta)d\eta - \lambda I_n\ ,\ B(\omega)\right] = n. \tag{2.10}$$

The following assertion is a direct consequence of the results of [35].

**Proposition 2.2.** *Let the assumption (A) be valid. Then, for each $\omega \in [\omega_1, \omega_2]$, the system (2.1) is $L^2$-stabilizable.*

**Lemma 2.3.** *Let the assumption (A) be valid. Then, for each $\omega \in [\omega_1, \omega_2]$, the set of equations (2.5)-(2.7) subject to the boundary conditions (2.8) has the unique solution $\{P(\omega),\ Q(\eta,\omega),\ R(\eta,\chi,\omega),\ (\eta,\chi) \in [-h,0]\times[-h,0]\}$ such that:*

(i) *the matrix*

$$\begin{pmatrix} P(\omega) & Q(\chi,\omega) \\ \big(Q(\eta,\omega)\big)^T & R(\eta,\chi,\omega) \end{pmatrix} \tag{2.11}$$

*defines a linear bounded self-adjoint nonnegative operator mapping the space $\mathcal{M}[-h,0;n]$ into itself;*

(ii) *the matrix $P(\omega)$ is positive definite;*

(iii) *the matrix-valued function $Q(\eta,\omega)$ is piece-wise absolutely continuous in $\eta \in [-h,0]$ with the bounded jumps at $\eta = -h_j$, $(j = 1,...,N-1)$;*

(iv) *the matrix-valued function $R(\eta,\chi,\omega)$ is piece-wise absolutely continuous in $\eta \in [-h,0]$ and in $\chi \in [-h,0]$ with the bounded jumps at $\eta = -h_{j_1}$ and $\chi = -h_{j_2}$, $(j_1 = 1,...,N-1;\ j_2 = 1,...,N-1)$;*

(v) *the unique state-feedback optimal control in the problem (2.1)-(2.3) has the form*

$$u = u^*_\omega(x_t) = -B^T(\omega)\left[P(\omega)x(t) + \int_{-h}^{0} Q(\eta,\omega)x(t+\eta)d\eta\right]; \tag{2.12}$$

(vi) *the optimal value $J^*(\omega)$ of the cost functional (2.3) has the form*

$$\begin{aligned} J^*(\omega) &= x_0^T P(\omega)x_0 + 2x_0^T \int_{-h}^{0} Q(\eta,\omega)\varphi(\eta)d\eta \\ &+ \int_{-h}^{0}\int_{-h}^{0} \varphi^T(\eta)R(\eta,\rho,\omega)\varphi(\rho)d\eta d\rho; \end{aligned} \tag{2.13}$$

(vii) *the closed-loop system (2.1),(2.12) is $L^2$-stable, implying that all roots $\lambda(\omega)$ of the equation*

$$\begin{aligned} \det\Big[\lambda I_n - \Big(A_0(\omega) - S(\omega)P(\omega)\Big) - \sum_{j=1}^{N} A_j(\omega)\exp(-\lambda h_j) \\ - \int_{-h}^{0}\Big(G(\eta,\omega) - S(\omega)Q(\eta,\omega)\Big)\exp(\lambda\eta)d\eta\Big] = 0 \end{aligned} \tag{2.14}$$

*satisfy the inequality*

$$\mathrm{Re}\lambda(\omega) < -2\gamma(\omega), \qquad \omega \in [\omega_1,\omega_2], \tag{2.15}$$

*where $\gamma(\omega) > 0$ is some function of $\omega$.*

*Proof.* The statements of the lemma immediately follow from Proposition 2.2 and the results of [4]. □

2.3. **Objectives of the paper.** The objectives of the paper are the following:

(I) to establish a continuity with respect to $\omega$ of the solution to the set (2.5)-(2.7),(2.8);

(II) to establish a first-order smoothness with respect to $\omega$ of the solution to the set (2.5)-(2.7),(2.8);

(III) to establish a first-order smoothness with respect to $\omega$ of the state-feedback optimal control in the problem (2.1)-(2.3);

(IV) to establish a first-order smoothness with respect to $\omega$ of the optimal trajectory in the problem (2.1)-(2.3);

(V) to establish a first-order smoothness with respect to $\omega$ of the optimal value of the cost functional in the problem (2.1)-(2.3).

## 3. Auxiliary Lemmas

Consider two quasi-polynomial equations with respect to $\lambda$

$$\det\left[\lambda I_n - \sum_{j=0}^{N}\mathcal{A}_j\exp(-\lambda h_j) - \int_{-h}^{0}\mathcal{G}(\eta)\exp(\lambda\eta)d\eta\right] = 0, \tag{3.1}$$

$$\det\left[\lambda I_n - \sum_{j=0}^{N}\big(\mathcal{A}_j + \Delta\mathcal{A}_j\big)\exp(-\lambda h_j) - \int_{-h}^{0}\big(\mathcal{G}(\eta)+\Delta\mathcal{G}(\eta)\big)\exp(\lambda\eta)d\eta\right] = 0, \tag{3.2}$$

where $\mathcal{A}_j$, $\Delta\mathcal{A}_j$, $(j = 0, ..., N)$, $\mathcal{G}(\eta)$, $\Delta\mathcal{G}(\eta)$ are matrices of the dimension $n\times n$; the matrix-valued functions $\mathcal{G}(\eta)$ and $\Delta\mathcal{G}(\eta)$ are piece-wise continuous in the interval $[-h, 0]$.

**Lemma 3.1.** *Let all roots $\lambda$ of the equation (3.1) satisfy the inequality* $\mathrm{Re}\lambda < -2\beta$, *where $\beta > 0$ is some constant. Then, there exists a positive number $\nu$ such that for all $\Delta\mathcal{A}_j$, $(j = 0, ..., N)$ and $\Delta\mathcal{G}(\eta)$, $\eta \in [-h, 0]$, satisfying the inequalities*

$$\big\|\Delta\mathcal{A}_j\big\| \le \nu, \quad j = 0, ..., N; \qquad \big\|\Delta\mathcal{G}(\eta)\big\| \le \nu, \quad \eta \in [-h, 0], \tag{3.3}$$

*all roots $\lambda_\Delta$ of the equation (3.2) satisfy the inequality* $\mathrm{Re}\lambda_\Delta < -2\beta$.

*Proof.* We prove the lemma by contradiction, i.e., we assume that the statement of the lemma is wrong. This means the existence of the sequences $\{\nu_k\}$, $\{\Delta\mathcal{A}_{j,k}\}$, $(j = 0, 1, ..., N)$, $\{\Delta\mathcal{G}_k(\eta)\}$ and $\{\lambda_{\Delta,k}\}$ with the following properties: (a) $\nu_k > 0$, $(k = 1, 2, ...)$, and $\lim_{k\to+\infty}\nu_k = 0$; (b) the $n\times n$-matrices $\Delta\mathcal{A}_{j,k}$ satisfy the inequalities $\big\|\Delta\mathcal{A}_{j,k}\big\| \le \nu_k$, $(j = 0, 1, ..., N;\ k = 1, 2, ...)$; (c) the $n\times n$-matrix-valued functions $\Delta\mathcal{G}_k(\eta)$ are piece-wise continuous in the interval $[-h, 0]$ and satisfy the inequalities $\big\|\Delta\mathcal{G}_k(\eta)\big\| \le \nu_k$, $\eta \in [-h, 0]$, $(k = 1, 2, ...)$; (d) $\mathrm{Re}\lambda_{\Delta,k} \ge -2\beta$, $(k = 1, 2, ...)$; (e) $\lambda_{\Delta,k}$ is a root of the equation (3.2) with $\Delta\mathcal{A}_j = \Delta\mathcal{A}_{j,k}$, $\Delta\mathcal{G}(\eta) = \Delta\mathcal{G}_k(\eta)$, $(k = 1, 2, ...)$.

The following two cases can be distinguished with respect to the sequence $\{\lambda_{\Delta,k}\}$: (i) $\{\lambda_{\Delta,k}\}$ is bounded; (ii) $\{\lambda_{\Delta,k}\}$ is unbounded. We start with the case (i). In this case, there exists a convergent subsequence of $\{\lambda_{\Delta,k}\}$. For the sake of simplicity (but without a loss of generality), we assume that the sequence $\{\lambda_{\Delta,k}\}$ itself is convergent. Let $\bar\lambda_\Delta = \lim_{k\to+\infty}\lambda_{\Delta,k}$. Due to the above mentioned property (d), Re$\bar{\lambda}_\Delta \geq -2\beta$. Substitution of $\Delta\mathcal{A}_j = \Delta\mathcal{A}_{j,k}$, $\Delta\mathcal{G}(\eta) = \Delta\mathcal{G}_k(\eta)$, $\lambda = \lambda_{\Delta,k}$ into (3.2), followed by calculation of the limit of the resulting equality for $k \to +\infty$, yields:

$$\det\left[\bar{\lambda}_\Delta I_n - \sum_{j=0}^{N} \mathcal{A}_j \exp(-\bar{\lambda}_\Delta h_j) - \int_{-h}^{0} \mathcal{G}(\eta)\exp(\bar{\lambda}_\Delta \eta)d\eta\right] = 0. \tag{3.4}$$

The latter means that $\bar{\lambda}_\Delta$ is a root of the equation (3.1). Thus, due to the assumption of the lemma on the roots of this equation, we obtain that Re$\bar{\lambda}_\Delta < -2\beta$. This contradicts the above obtained inequality Re$\bar{\lambda}_\Delta \geq -2\beta$.

Proceed to the case (ii) where the sequence $\{\lambda_{\Delta,k}\}$ is unbounded. In this case, there exists a subsequence of $\{\lambda_{\Delta,k}\}$, modules of elements of which tend to infinity. Similarly to the case (i), we assume that $\{\lambda_{\Delta,k}\}$ itself is such a subsequence, i.e., $\lim_{k\to+\infty} |\lambda_{\Delta,k}| = +\infty$. By substituting $\Delta\mathcal{A}_j = \Delta\mathcal{A}_{j,k}$, $\Delta\mathcal{G}(\eta) = \Delta\mathcal{G}_k(\eta)$, $\lambda = \lambda_{\Delta,k}$ into (3.2), dividing the resulting equality by $\left(\lambda_{\Delta,k}\right)^n$ and, then, calculating the limit of the last equality as $k \to +\infty$, one obtains the contradiction $(-1)^n = 0$.

The contradictions, obtained in the cases (i) and (ii), prove the lemma. □

Consider the initial-value problem with respect to the $n \times n$-matrix-valued function $\Phi(t)$

$$\frac{d\Phi(t)}{dt} = \sum_{j=0}^{N} \mathcal{H}_j(\omega)\Phi(t-h_j) + \int_{-h}^{0} \mathcal{K}(\eta,\omega)\Phi(t+\eta)d\eta, \quad t \geq 0, \tag{3.5}$$

$$\Phi(\eta) = 0, \quad \eta \in [-h,0); \qquad \Phi(0) = I_n, \tag{3.6}$$

where $\mathcal{H}_j(\omega)$, $(j = 0,...,N)$ and $\mathcal{K}(\eta,\omega)$ are $n \times n$-matrix-valued functions, given for $\eta \in [-h,0]$ and $\omega \in [\omega_1,\omega_2]$; $\mathcal{H}_j(\omega)$, $(j = 0,...,N)$ are continuously differentiable with respect to $\omega \in [\omega_1,\omega_2]$; the function $\mathcal{K}(\eta,\omega)$ is piecewise continuous with respect to $\eta \in [-h,0]$ for each $\omega \in [\omega_1,\omega_2]$, and its partial derivative $\partial\mathcal{K}(\eta,\omega)/\partial\omega$ is continuous with respect to $\omega \in [\omega_1,\omega_2]$ uniformly in $\eta \in [-h,0]$.

Due to the results of [5], for any $\omega \in [\omega_1,\omega_2]$, the problem (3.5)-(3.6) has the unique locally absolutely continuous solution $\Phi(t) = \Phi(t,\omega)$, $t \geq 0$.

Consider the quasi-polynomial equation with respect to $\lambda$

$$\det\left[\lambda I_n - \sum_{j=0}^{N} \mathcal{H}_j(\omega)\exp(-\lambda h_j) - \int_{-h}^{0} \mathcal{K}(\eta,\omega)\exp(\lambda\eta)d\eta\right] = 0, \tag{3.7}$$

**Lemma 3.2.** *Let for any $\omega \in [\omega_1,\omega_2]$, all roots $\lambda(\omega)$ of the equation (3.7) satisfy the inequality*

$$\lambda(\omega) < -2\kappa(\omega), \tag{3.8}$$

*where $\kappa(\omega) > 0$ is some function of $\omega \in [\omega_1,\omega_2]$. Then, there exists a positive number $\bar{\kappa}$ such that all these roots satisfy the inequality*

$$\lambda(\omega) < -2\bar{\kappa} \quad \forall\omega \in [\omega_1,\omega_2]. \tag{3.9}$$

*Proof.* Using the above mentioned smoothness of $\mathcal{H}_j(\omega)$ and $\mathcal{K}(\eta,\omega)$ with respect to $\omega \in [\omega_1,\omega_2]$, the lemma is proven by contradiction similarly to the proof of Lemma 3.1. □

Let $\omega_0$ be an arbitrary but fixed point in the interval $[\omega_1, \omega_2]$, and $\Delta\omega \neq 0$ be an arbitrary number such that $\omega_0 + \Delta\omega \in [\omega_1, \omega_2]$. Let us denote

$$\Delta\Phi(t) = \Phi(t, \omega_0 + \Delta\omega) - \Phi(t, \omega_0), \quad t \geq 0. \tag{3.10}$$

**Lemma 3.3.** *Let the condition of Lemma 3.2 be valid. Then, for all sufficiently small* $|\Delta\omega|$*, the matrix-valued function* $\Delta\Phi(t)$ *satisfies the inequality*

$$\big\|\Delta\Phi(t)\big\| \leq a \exp(-\bar{\kappa}_1 t)|\Delta\omega|, \qquad t \geq 0, \tag{3.11}$$

*where* $a > 0$ *and* $0 < \bar{\kappa}_1 < \bar{\kappa}$ *are some constants independent of* $\Delta\omega$*.*

*Proof.* First of all, let us note that, due to Lemma 3.2 and the results of [4], the matrix-valued function $\Phi(t, \omega_0)$ satisfies the inequality

$$\big\|\Phi(t, \omega_0)\big\| \leq a_1 \exp(-\bar{\kappa} t), \qquad t \geq 0, \tag{3.12}$$

where $a_1 > 0$ is some constant.

Denote

$$\Delta\mathcal{H}_j = \mathcal{H}_j(\omega_0 + \Delta\omega) - \mathcal{H}_j(\omega_0), \tag{3.13}$$

$$\Delta\mathcal{K}(\eta) = \mathcal{K}(\eta, \omega_0 + \Delta\omega) - \mathcal{K}(\eta, \omega_0), \quad \eta \in [-h, 0]. \tag{3.14}$$

Due to the smoothness of of $\mathcal{H}_j(\omega)$ and $\mathcal{K}(\eta, \omega)$ with respect to $\omega \in [\omega_1, \omega_2]$, we obtain the inequalities for all sufficiently small $|\Delta\omega|$:

$$\big\|\Delta\mathcal{H}_j\big\| \leq a_2|\Delta\omega|, \quad j = 0, 1, ..., N; \qquad \big\|\Delta\mathcal{K}(\eta)\big\| \leq a_2|\Delta\omega|, \quad \eta \in [-h, 0], \tag{3.15}$$

where $a_2 > 0$ is some constant independent of $\Delta\omega$.

Using the problem (3.5)-(3.6) at $\omega = \omega_0$ and $\omega = \omega_0 + \Delta\omega$, we obtain the following initial-value problem for $\Delta\Phi(t)$:

$$\begin{aligned} \frac{d\Delta\Phi(t)}{dt} &= \sum_{j=0}^{N} \mathcal{H}_j(\omega_0 + \Delta\omega)\Delta\Phi(t - h_j) \\ &+ \int_{-h}^{0} \mathcal{K}(\eta, \omega_0 + \Delta\omega)\Delta\Phi(t + \eta)d\eta + \mathcal{D}(t), \quad t \geq 0, \end{aligned} \tag{3.16}$$

$$\Delta\Phi(\eta) = 0, \qquad \eta \in [-h, 0], \tag{3.17}$$

where

$$\mathcal{D}(t) = \sum_{j=0}^{N} \Delta\mathcal{H}_j \Phi(t - h_j, \omega_0) + \int_{-h}^{0} \Delta\mathcal{K}(\eta)\Phi(t + \eta, \omega_0)d\eta, \quad t \geq 0. \tag{3.18}$$

Using inequalities (3.12),(3.15) and the initial conditions (3.6), we immediately have for all sufficiently small $|\Delta\omega|$

$$\big\|\mathcal{D}(t)\big\| \leq a_3 \exp(-\bar{\kappa} t)|\Delta\omega|, \qquad t \geq 0, \tag{3.19}$$

where $a_3 > 0$ is some constant independent of $\Delta\omega$.

Now, by virtue of the variation-of-constant formula (see e.g. [5]), we obtain the unique solution of the problem (3.16)-(3.17)

$$\Delta\Phi(t) = \int_0^t \Theta(t - s, \Delta\omega)\mathcal{D}(s)ds, \qquad t \geq 0, \tag{3.20}$$

where the $n\times n$-matrix-valued function $\Theta(t,\Delta\omega)$ is the unique solution of the initial-value problem

$$\frac{d\Theta(t,\Delta\omega)}{dt} = \sum_{j=0}^{N}\mathcal{H}_j(\omega_0+\Delta\omega)\Theta(t-h_j,\Delta\omega) + \int_{-h}^{0}\mathcal{K}(\eta,\omega_0+\Delta\omega)\Theta(t+\eta,\Delta\omega)d\eta,\quad t\geq 0, \tag{3.21}$$

$$\Theta(\eta,\Delta\omega)=0,\quad \eta\in[-h,0);\qquad \Theta(0,\Delta\omega)=I_n. \tag{3.22}$$

Consider the following quasi-polynomial equation with respect to $\lambda$, which is the characteristic equation of the delay differential equation (3.21):

$$\det\left[\lambda I_n-\sum_{j=0}^{N}\mathcal{H}_j(\omega_0+\Delta\omega)\exp(-\lambda h_j)-\int_{-h}^{0}\mathcal{K}(\eta,\omega_0+\Delta\omega)\exp(\lambda\eta)d\eta\right]=0. \tag{3.23}$$

Due to the inequalities (3.15), and Lemmas 3.1 and 3.2, we obtain that all roots $\lambda(\Delta\omega)$ of (3.23) satisfy the inequality $\mathrm{Re}\lambda(\Delta\omega)\leq -2\bar{\kappa}$ for all sufficiently small $\Delta\omega$. The latter, along with the results of [4], yields for such $\Delta\omega$ the inequality

$$\left\|\Theta(t,\Delta\omega)\right\|\leq a_4\exp(-\bar{\kappa}t),\qquad t\geq 0, \tag{3.24}$$

where $a_4>0$ is some constant independent of $\Delta\omega$.

Now, the statement of the lemma (the inequality (3.11)) follows immediately from the equation (3.20) and the inequalities (3.19),(3.24). □

**Corollary 3.4.** *Let the condition of Lemma 3.2 be valid. Then, the solution $\Phi(t,\omega)$ of the problem (3.5)-(3.6) is continuous with respect to $\omega\in[\omega_1,\omega_2]$ uniformly in $t\in[0,+\infty)$.*

*Proof.* The corollary is a direct consequence of Lemmas 3.2 and 3.3. □

**Corollary 3.5.** *Let the condition of Lemma 3.2 be valid. Then, the derivative $\partial\Phi(t,\omega)/\partial\omega$ exists and is continuous with respect to $\omega\in[\omega_1,\omega_2]$ uniformly in $t\in[0,+\infty)$.*

*Proof.* In the proof of this corollary, we use the notations introduced in the proof of Lemma 3.3.

Applying Corollary 3.4 to the problem (3.21)-(3.22), we obtain that $\lim_{\Delta\omega\to 0}\Theta(t,\Delta\omega)$ exists and satisfies the initial-value problem, obtained from (3.21)-(3.22) by setting there $\Delta\omega=0$. Thus,

$$\lim_{\Delta\omega\to 0}\Theta(t,\Delta\omega)=\Phi(t,\omega_0). \tag{3.25}$$

Now, dividing the equation (3.20) by $\Delta\omega$, calculating the limit of the resulting equality for $\Delta\omega\to 0$ and using the equations (3.18),(3.25), we obtain

$$\frac{\partial\Phi(t,\omega_0)}{\partial\omega} = \lim_{\Delta\omega\to 0}\frac{\Delta\Phi(t)}{\Delta\omega}=\int_0^t\Phi(t-s,\omega_0)\left[\sum_{j=0}^{N}\frac{d\mathcal{H}_j(\omega_0)}{d\omega}\Phi(s-h_j,\omega_0) + \int_{-h}^{0}\frac{\partial\mathcal{K}(\eta,\omega_0)}{\partial\omega}\Phi(s+\eta,\omega_0)d\eta\right]ds,\qquad t\geq 0. \tag{3.26}$$

Since $\omega_0$ is an arbitrary point of the interval $[\omega_1, \omega_2]$, the equation (3.26) is valid for all $\omega \in [\omega_1, \omega_2]$. This observation, along with Corollary 3.4, directly yields the statement of the corollary. □

## 4. Main Lemmas

### 4.1. **Continuity with respect to the parameter of the solution to the set (2.5)-(2.7),(2.8).**

**Lemma 4.1.** *Let the assumption (A) be valid. Then, the matrices $P(\omega)$, $Q(\eta,\omega)$, $R(\eta,\chi,\omega)$, constituting the solution of the set (2.5)-(2.7),(2.8), are continuous functions of $\omega \in [\omega_1, \omega_2]$ uniformly in $(\eta,\chi) \in [-h,0]\times[-h,0]$.*

*Proof.* Let $\omega_0$ be an arbitrary but fixed point in the interval $[\omega_1, \omega_2]$, and $\Delta\omega \neq 0$ be an arbitrary number such that $\omega_0 + \Delta\omega \in [\omega_1, \omega_2]$. Let us denote

$$\Delta A_j = A_j(\omega_0+\Delta\omega) - A_j(\omega_0), \quad j = 0,1,...,N,$$

$$\Delta S = S(\omega_0+\Delta\omega) - S(\omega_0), \quad \Delta G(\eta) = G(\eta,\omega_0+\Delta\omega) - G(\eta,\omega_0), \tag{4.1}$$

$$\Delta P = P(\omega_0+\Delta\omega) - P(\omega_0),$$

$$\Delta Q(\eta) = Q(\eta,\omega_0+\Delta\omega) - Q(\eta,\omega_0) - P(\omega_0)\Delta A_N,$$

$$\begin{aligned}\Delta R(\eta,\chi) = {} & R(\eta,\chi,\omega_0+\Delta\omega) - R(\eta,\chi,\omega_0) - Q^T(\eta,\omega_0)\Delta A_N \\ & -\big(\Delta A_N\big)^T Q(\chi,\omega_0) - \big(\Delta A_N\big)^T P(\omega_0)\Delta A_N.\end{aligned} \tag{4.2}$$

Using the set of equations (2.5)-(2.7) at $\omega = \omega_0$ and $\omega = \omega_0 + \Delta\omega$, we obtain the following problem for $\Delta P$, $\Delta Q(\eta)$, $\Delta R(\eta,\chi)$:

$$\Delta P\alpha(\omega_0) + \alpha^T(\omega_0)\Delta P + \Delta Q(0) + \big(\Delta Q(0)\big)^T + \Upsilon_P(\Delta P) = 0, \tag{4.3}$$

$$\begin{aligned}\frac{d\Delta Q(\eta)}{d\eta} = {} & \alpha^T(\omega_0)\Delta Q(\eta) + \Delta P\theta(\eta,\omega_0) \\ & + \sum_{j=1}^{N-1}\Delta P A_j(\omega_0)\delta(\eta+h_j) + \Delta R(0,\eta) + \Upsilon_Q\big(\Delta P, \Delta Q(\eta)\big),\end{aligned} \tag{4.4}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\right)\Delta R(\eta,\chi) = {} & \big(\Delta Q(\eta)\big)^T\theta(\chi,\omega_0) \\ & +\theta^T(\eta,\omega_0)\Delta Q(\chi) + \sum_{j=1}^{N-1} A_j^T(\omega_0)\Delta Q(\chi)\delta(\eta+h_j) \\ & + \sum_{j=1}^{N-1}\big(\Delta Q(\eta)\big)^T A_j(\omega_0)\delta(\chi+h_j) + \Upsilon_R\big(\Delta Q(\eta),\Delta Q(\chi)\big),\end{aligned} \tag{4.5}$$

$$\begin{aligned}\Delta Q(-h) &= \Delta P\big(A_N(\omega_0)+\Delta A_N\big), \\ \Delta R(-h,\eta) &= \big(A_N(\omega_0)+\Delta A_N\big)^T\Delta Q(\eta), \\ \Delta R(\eta,-h) &= \big(\Delta Q(\eta)\big)^T\big(A_N(\omega_0)+\Delta A_N\big),\end{aligned} \tag{4.6}$$

where

$$\alpha(\omega) = A_0(\omega) - S(\omega)P(\omega), \qquad \theta(\eta,\omega) = G(\eta,\omega) - S(\omega)Q(\eta,\omega), \tag{4.7}$$

$$\begin{aligned} \Upsilon_P(\Delta P) &= \big(P(\omega_0)+\Delta P\big)\Delta A_0 + \big(\Delta A_0\big)^T\big(P(\omega_0)+\Delta P\big) \\ &\quad -\big(P(\omega_0)+\Delta P\big)\Delta S\big(P(\omega_0)+\Delta P\big) - \Delta P S(\omega_0)\Delta P \\ &\quad -P(\omega_0)\Delta A_N - \big(\Delta A_N\big)^T P(\omega_0), \end{aligned} \tag{4.8}$$

$$\begin{aligned} \Upsilon_Q\big(\Delta P, \Delta Q(\eta)\big) &= \big[\Delta A_0 - \Delta S\big(P(\omega_0)+\Delta P\big)\big]^T\big(Q(\eta,\omega_0)+\Delta Q(\eta)\big) \\ &\quad -\Delta P S(\omega_0)\Delta Q(\eta) + \big(P(\omega_0)+\Delta P\big)\Delta G(\eta) \\ &\quad + \sum_{j=1}^{N-1}\big(P(\omega_0)+\Delta P\big)\Delta A_j\delta(\eta+h_j) + \alpha^T(\omega_0)P(\omega_0)\Delta A_N, \end{aligned} \tag{4.9}$$

$$\begin{aligned} \Upsilon_R\big(\Delta Q(\eta), \Delta Q(\chi)\big) &= \big(\Delta G(\eta)\big)^T\big(Q(\chi,\omega_0)+\Delta Q(\chi)\big) \\ &\quad +\big(Q(\eta,\omega_0)+\Delta Q(\eta)\big)^T\Delta G(\chi) \\ &\quad + \sum_{j=1}^{N-1}\big(\Delta A_j\big)^T\big(Q(\chi,\omega_0)+\Delta Q(\chi)\big)\delta(\eta+h_j) \\ &\quad + \sum_{j=1}^{N-1}\big(Q(\eta,\omega_0)+\Delta Q(\eta)\big)^T\Delta A_j\delta(\chi+h_j) \\ &\quad -\big(\Delta Q(\eta)\big)^T S(\omega_0)\Delta Q(\chi) \\ &\quad -\big(Q(\eta,\omega_0)+\Delta Q(\eta)\big)^T\Delta S\big(Q(\chi,\omega_0)+\Delta Q(\chi)\big) \\ &\quad +\left(\frac{dQ(\eta,\omega_0)}{d\eta}\right)^T\Delta A_N + \big(\Delta A_N\big)^T\frac{dQ(\chi,\omega_0)}{d\chi} \\ &\quad +\big(\Delta A_N\big)^T P(\omega_0)\theta(\chi,\omega_0) + \theta^T(\eta,\omega_0)P(\omega_0)\Delta A_N \\ &\quad + \sum_{j=1}^{N-1} A_j^T(\omega_0)P(\omega_0)\Delta A_N\delta(\eta+h_j) \\ &\quad + \sum_{j=1}^{N-1}\big(\Delta A_N\big)^T P(\omega_0)A_j(\omega_0)\delta(\chi+h_j). \end{aligned} \tag{4.10}$$

By virtue of the results of [4], we can rewrite the problem (4.3)-(4.6) in the equivalent integral form

$$\begin{aligned} \Delta P &= \int_0^{+\infty}\Big[\mathcal{L}^T(\sigma,\omega_0,\Delta\omega)\Upsilon_P(\Delta P)\mathcal{L}(\sigma,\omega_0,\Delta\omega) \\ &\quad + \int_{-h}^0 \mathcal{L}^T(\sigma,\omega_0,\Delta\omega)\Upsilon_Q\big(\Delta P,\Delta Q(\eta)\big)\mathcal{L}(\sigma+\eta,\omega_0,\Delta\omega)d\eta \\ &\quad + \int_{-h}^0 \mathcal{L}^T(\sigma+\eta,\omega_0,\Delta\omega)\Upsilon_Q^T\big(\Delta P,\Delta Q(\eta)\big)\mathcal{L}(\sigma,\omega_0,\Delta\omega)d\eta \end{aligned}$$

$$+\int_{-h}^{0}\int_{-h}^{0}\mathcal{L}^T(\sigma+\eta,\omega_0,\Delta\omega)\Upsilon_R\big(\Delta Q(\eta),\Delta Q(\chi)\big)$$

$$\times\mathcal{L}(\sigma+\chi,\omega_0,\Delta\omega)d\eta d\chi\bigg]d\sigma, \tag{4.11}$$

$$\begin{aligned}\Delta Q(\eta) &= \int_0^{+\infty}\bigg[\mathcal{L}^T(\sigma,\omega_0,\Delta\omega)\Upsilon_P(\Delta P)\widetilde{\mathcal{L}}(\sigma,\eta,\omega_0,\Delta\omega)\\ &\quad+\int_{-h}^{0}\mathcal{L}^T(\sigma,\omega_0,\Delta\omega)\Upsilon_Q\big(\Delta P,\Delta Q(\chi)\big)\widetilde{\mathcal{L}}(\sigma+\chi,\eta,\omega_0,\Delta\omega)d\chi\\ &\quad+\int_{-h}^{0}\mathcal{L}^T(\sigma+\chi,\omega_0,\Delta\omega)\Upsilon_Q^T\big(\Delta P,\Delta Q(\chi)\big)\widetilde{\mathcal{L}}(\sigma,\eta,\omega_0,\Delta\omega)d\chi\\ &\quad+\int_{-h}^{0}\int_{-h}^{0}\mathcal{L}^T(\sigma+\chi,\omega_0,\Delta\omega)\Upsilon_R\big(\Delta Q(\chi),\Delta Q(\chi_1)\big)\\ &\quad\times\widetilde{\mathcal{L}}(\sigma+\chi_1,\eta,\omega_0,\Delta\omega)d\chi d\chi_1\bigg]d\sigma\\ &+\int_0^{\eta+h}\bigg[\mathcal{L}^T(\sigma,\omega_0,\Delta\omega)\Upsilon_Q\big(\Delta P,\Delta Q(\eta-\sigma)\big)\\ &\quad+\int_{-h}^{0}\mathcal{L}^T(\sigma+\chi,\omega_0,\Delta\omega)\Upsilon_R\big(\Delta Q(\chi),\Delta Q(\eta-\sigma)\big)d\chi\bigg]d\sigma,\end{aligned} \tag{4.12}$$

$$\begin{aligned}\Delta R(\eta,\chi) &= \int_0^{+\infty}\bigg[\widetilde{\mathcal{L}}^{\,T}(\sigma,\eta,\omega_0,\Delta\omega)\Upsilon_P(\Delta P)\widetilde{\mathcal{L}}(\sigma,\chi,\omega_0,\Delta\omega)\\ &\quad+\int_{-h}^{0}\widetilde{\mathcal{L}}^{\,T}(\sigma,\eta,\omega_0,\Delta\omega)\Upsilon_Q\big(\Delta P,\Delta Q(\chi_1)\big)\widetilde{\mathcal{L}}(\sigma+\chi_1,\chi,\omega_0,\Delta\omega)d\chi_1\\ &\quad+\int_{-h}^{0}\widetilde{\mathcal{L}}^{\,T}(\sigma+\chi_1,\eta,\omega_0,\Delta\omega)\Upsilon_Q^T\big(\Delta P,\Delta Q(\chi_1)\big)\widetilde{\mathcal{L}}(\sigma,\chi,\omega_0,\Delta\omega)d\chi_1\\ &\quad+\int_{-h}^{0}\int_{-h}^{0}\widetilde{\mathcal{L}}^{\,T}(\sigma+\chi_1,\eta,\omega_0,\Delta\omega)\Upsilon_R\big(\Delta Q(\chi_1),\Delta Q(\chi_2)\big)\\ &\quad\times\widetilde{\mathcal{L}}(\sigma+\chi_2,\chi,\omega_0,\Delta\omega)d\chi_1 d\chi_2\bigg]d\sigma\\ &+\int_0^{\eta+h}\bigg[\Upsilon_Q^T\big(\Delta P,\Delta Q(\eta-\sigma)\big)\widetilde{\mathcal{L}}(\sigma,\chi,\omega_0,\Delta\omega)\\ &\quad+\int_{-h}^{0}\Upsilon_R\big(\Delta Q(\chi_1),\Delta Q(\eta-\sigma)\big)\widetilde{\mathcal{L}}(\sigma+\chi_1,\chi,\omega_0,\Delta\omega)d\chi_1\bigg]d\sigma\\ &+\int_0^{\chi+h}\bigg[\widetilde{\mathcal{L}}^{\,T}(\sigma,\eta,\omega_0,\Delta\omega)\Upsilon_Q\big(\Delta P,\Delta Q(\chi-\sigma)\big)\\ &\quad+\int_{-h}^{0}\widetilde{\mathcal{L}}^{\,T}(\sigma+\chi_1,\eta,\omega_0,\Delta\omega)\Upsilon_R\big(\Delta Q(\chi_1),\Delta Q(\chi-\sigma)\big)d\chi_1\bigg]d\sigma\end{aligned}$$

$$+\int_0^{\min(\eta+h,\chi+h)} \Upsilon_R\big(\Delta Q(\eta-\sigma),\Delta Q(\chi-\sigma)\big)d\sigma, \tag{4.13}$$

where $\mathcal{L}(\sigma,\omega_0,\Delta\omega)$ is the solution of the problem

$$\begin{aligned}\frac{d\mathcal{L}(\sigma,\omega_0,\Delta\omega)}{d\sigma} &= \alpha(\omega_0)\mathcal{L}(\sigma,\omega_0,\Delta\omega)+\sum_{j=1}^{N-1}A_j(\omega_0)\mathcal{L}(\sigma-h_j,\omega_0,\Delta\omega)\\ &\quad +A_N(\omega_0+\Delta\omega)\mathcal{L}(\sigma-h_N,\omega_0,\Delta\omega)\\ &\quad +\int_{-h}^{0}\theta(\eta,\omega_0)\mathcal{L}(\sigma+\eta,\omega_0,\Delta\omega)d\eta,\quad \sigma>0,\end{aligned}$$

$$\mathcal{L}(0,\omega_0,\Delta\omega)=I_n,\qquad \mathcal{L}(\sigma,\omega_0,\Delta\omega)=0\ \ \forall\sigma<0, \tag{4.14}$$

and $\widetilde{\mathcal{L}}(\sigma,\eta,\omega_0,\Delta\omega)$ is defined as follows:

$$\begin{aligned}\widetilde{\mathcal{L}}(\sigma,\eta,\omega_0,\Delta\omega) &\triangleq \sum_{j=1}^{N-1}\left\{\begin{array}{ll}\mathcal{L}(\sigma-\eta-h_j,\omega_0,\Delta\omega)A_j(\omega_0), & \eta-\sigma<-h_j\le\eta\\ 0, & otherwise\end{array}\right\}\\ &\quad +\left\{\begin{array}{ll}\mathcal{L}(\sigma-\eta-h,\omega_0,\Delta\omega)A_N(\omega_0+\Delta\omega), & \eta-\sigma<-h\le\eta\\ 0, & otherwise\end{array}\right\}\\ &\quad +\int_{-\eta}^{h}\mathcal{L}(\sigma-\eta-\chi,\omega_0,\Delta\omega)\theta(-\chi,\omega_0)d\chi,\quad \sigma\ge0,\quad \eta\in[-h,0].\end{aligned} \tag{4.15}$$

Using Lemmas 2.3 and 3.1, the equation (4.7) and the continuity of $A_N(\omega)$, we obtain that, for all sufficiently small $|\Delta\omega|$, all roots $\lambda$ of the equation

$$\begin{aligned}\det\Big[\lambda I_n-\alpha(\omega_0)-\sum_{j=1}^{N-1}A_j(\omega_0)\exp(-\lambda h_j)\\ -A_N(\omega_0+\Delta\omega)\exp(-\lambda h)-\int_{-h}^{0}\theta(\eta,\omega_0)\exp(\lambda\eta)d\eta\Big]=0\end{aligned} \tag{4.16}$$

satisfy the inequality

$$\mathrm{Re}\lambda<-2\gamma(\omega_0). \tag{4.17}$$

The latter, along with the results of [4], and the equations (4.14) and (4.15), yields the following inequalities for all sufficiently small $|\Delta\omega|$, and all $\sigma\ge0$, $\eta\in[-h,0]$:

$$\begin{aligned}\big\|\mathcal{L}(\sigma,\omega_0,\Delta\omega)\big\| &\le a\exp\big(-\gamma(\omega_0)\sigma\big),\\ \big\|\widetilde{\mathcal{L}}(\sigma,\eta,\omega_0,\Delta\omega)\big\| &\le a\exp\big(-\gamma(\omega_0)\sigma\big),\end{aligned} \tag{4.18}$$

where $a>0$ is some constant independent of $\Delta\omega$.

Due to the smooth dependence of $A_j(\omega)$, $(j=0,1,...,N)$, $G(\eta,\omega)$ and $S(\omega)$ on the parameter $\omega$, we have the following inequalities for all sufficiently small $|\Delta\omega|$:

$$\begin{aligned}&\big\|\Delta A_j\big\|\le a|\Delta\omega|,\qquad j=0,1,...,N,\\ &\big\|\Delta S\big\|\le a|\Delta\omega|,\qquad \big\|\Delta G(\eta)\big\|\le a|\Delta\omega|,\quad \eta\in[-h,0],\end{aligned} \tag{4.19}$$

where $a>0$ is some constant independent of $\Delta\omega$.

Now, applying the procedure of successive approximations with zero initial approximation to the set (4.11)-(4.13), and using the equations (4.7)-(4.10) and the inequalities (4.18)-(4.19), one can show after a routine algebra that for all sufficiently small $|\Delta\omega|$ there exists the unique solution $\big\{\Delta P,\ \Delta Q(\eta),\ \Delta R(\eta,\chi)\big\}$ of this set, such that the matrix

$$\begin{pmatrix} \Delta P & \Delta Q(\chi) \\ \big(\Delta Q(\eta)\big)^T & \Delta R(\eta,\chi) \end{pmatrix} \tag{4.20}$$

defines a linear bounded self-adjoint operator mapping the space $\mathcal{M}[-h,0;n]$ into itself. Moreover, the following inequality is satisfied:

$$\max\Big[\big\|\Delta P\big\|,\ \big\|\Delta Q(\eta)\big\|,\ \big\|\Delta R(\eta,\chi)\big\|\Big] \le a\Delta\omega,$$
$$(\eta,\chi)\in[-h,0]\times[-h,0], \tag{4.21}$$

where $a>0$ is some constant independent of $\Delta\omega$.

Since (4.11)-(4.13) is a set of nonlinear equations and may have multiple solutions, we must show that its solution, satisfying (4.21), indeed satisfies the equation (4.2), where $\big\{P(\omega_0+\Delta\omega),\ Q(\eta,\omega_0+\Delta\omega),\ R(\eta,\chi,\omega_0+\Delta\omega)\big\}$ and $\big\{P(\omega_0),\ Q(\eta,\omega_0),\ R(\eta,\chi,\omega_0)\big\}$ are the solutions of the set (2.5)-(2.8) at the parameter values $\omega=\omega_0+\Delta\omega$ and $\omega=\omega_0$, satisfying Lemma 2.3.

Consider the matrices

$$\begin{aligned} \widetilde{P} &\triangleq P(\omega_0)+\Delta P, \qquad \widetilde{Q}(\eta)\triangleq Q(\eta,\omega_0)+\Delta Q(\eta)+P(\omega_0)\Delta A_N,\\ \widetilde{R}(\eta,\chi) &\triangleq R(\eta,\chi,\omega_0)+\Delta R(\eta,\chi)+Q^T(\eta,\omega_0)\Delta A_N\\ &\quad +\big(\Delta A_N\big)^TQ(\chi,\omega_0)+\big(\Delta A_N\big)^TP(\omega_0)\Delta A_N. \end{aligned} \tag{4.22}$$

It is clear that the triplet $\big\{\widetilde{P},\ \widetilde{Q}(\eta),\ \widetilde{R}(\eta,\chi)\big\}$ satisfies the set (2.5)-(2.7),(2.8) for $\omega=\omega_0+\Delta\omega$, i.e.,

$$\widetilde{P}A_0(\omega_0+\Delta\omega)+A_0^T(\omega_0+\Delta\omega)\widetilde{P}-\widetilde{P}S(\omega_0+\Delta\omega)\widetilde{P}+\widetilde{Q}(0)+\widetilde{Q}^T(0)+I_n=0, \tag{4.23}$$

$$\begin{aligned} \frac{d\widetilde{Q}(\eta)}{d\eta} &= \Big(A_0^T(\omega_0+\Delta\omega)-\widetilde{P}S(\omega_0+\Delta\omega)\Big)\widetilde{Q}(\eta)+\widetilde{P}G(\eta,\omega_0+\Delta\omega)\\ &\quad+\sum_{j=1}^{N-1}\widetilde{P}A_j(\omega_0+\Delta\omega)\delta(\eta+h_j)+\widetilde{R}(0,\eta), \end{aligned} \tag{4.24}$$

$$\begin{aligned} \Big(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\Big)\widetilde{R}(\eta,\chi) &= G^T(\eta,\omega_0+\Delta\omega)\widetilde{Q}(\chi)\\ &\quad+\widetilde{Q}^{\,T}(\eta)G(\chi,\omega_0+\Delta\omega)+\sum_{j=1}^{N-1}A_j^T(\omega_0+\Delta\omega)\widetilde{Q}(\chi)\delta(\eta+h_j)\\ &\quad+\sum_{j=1}^{N-1}\widetilde{Q}^{\,T}(\eta)A_j(\omega_0+\Delta\omega)\delta(\chi+h_j)\\ &\quad-\widetilde{Q}^{\,T}(\eta)S(\omega_0+\Delta\omega)\widetilde{Q}(\chi), \end{aligned} \tag{4.25}$$

$$\begin{aligned}\widetilde{Q}(-h) &= \widetilde{P}A_N(\omega_0+\Delta\omega),\\ \widetilde{R}(-h,\eta) &= A_N^T(\omega_0+\Delta\omega)\widetilde{Q}(\eta),\ \ \widetilde{R}(\eta,-h)=\widetilde{Q}^{\ T}(\eta)A_N(\omega_0+\Delta\omega),\end{aligned} \tag{4.26}$$

where $\eta\in[-h,0]$, $\chi\in[-h,0]$.

The set (4.23)-(4.25) is transformed equivalently as:

$$\widetilde{P}\tilde{\alpha}+\tilde{\alpha}^T\widetilde{P}+I_n+\widetilde{\Upsilon}_P=0, \tag{4.27}$$

$$\begin{aligned}\frac{d\widetilde{Q}(\eta)}{d\eta} &= \tilde{\alpha}^T\widetilde{Q}(\eta)+\widetilde{P}\tilde{\theta}(\eta)\\ &+\sum_{j=1}^{N-1}\widetilde{P}A_j(\omega_0+\Delta\omega)\delta(\eta+h_j)+\widetilde{R}(0,\eta)+\widetilde{\Upsilon}_Q(\eta),\end{aligned} \tag{4.28}$$

$$\begin{aligned}\left(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\right)\widetilde{R}(\eta,\chi) &= \tilde{\theta}^T(\eta)\widetilde{Q}(\chi)\\ &+\widetilde{Q}^{\ T}(\eta)\tilde{\theta}(\chi)+\sum_{j=1}^{N-1}A_j^T(\omega_0+\Delta\omega,0)\widetilde{Q}(\chi)\delta(\eta+h_j)\\ &+\sum_{j=1}^{N-1}\widetilde{Q}^{\ T}(\eta)A_j(\omega_0+\Delta\omega,0)\delta(\chi+h_j)+\widetilde{\Upsilon}_R(\eta,\chi),\end{aligned} \tag{4.29}$$

where $\tilde{\alpha}=A_0(\omega_0+\Delta\omega)-S(\omega_0+\Delta\omega)\widetilde{P}$, $\tilde{\theta}(\eta)=G(\eta,\omega_0+\Delta\omega)-S(\omega_0+\Delta\omega)\widetilde{Q}(\eta)$, $\widetilde{\Upsilon}_P=\widetilde{P}S(\omega_0+\Delta\omega)\widetilde{P}$, $\widetilde{\Upsilon}_Q(\eta)=\widetilde{P}S(\omega_0+\Delta\omega)\widetilde{Q}(\eta)$, $\widetilde{\Upsilon}_R(\eta,\chi)=\widetilde{Q}^{\ T}(\eta))S(\omega_0+\Delta\omega)\widetilde{Q}(\chi)$.

Consider the following quasi-polynomial equation with respect to $\tilde{\lambda}$:

$$\det\left[\tilde{\lambda}I_n-\tilde{\alpha}-\sum_{j=1}^{N}A_j(\omega_0+\Delta\omega)\exp(-\tilde{\lambda}h_j)-\int_{-h}^{0}\tilde{\theta}(\eta)\exp(\tilde{\lambda}\eta)d\eta\right]=0. \tag{4.30}$$

Due to Lemma 2.3, all roots $\tilde{\lambda}$ of this equation satisfy the inequality

$$\mathrm{Re}\tilde{\lambda}<-2\gamma(\omega_0+\Delta\omega). \tag{4.31}$$

Also, consider the operator, defined by the matrix

$$\begin{pmatrix}\widetilde{\Upsilon}_P & \widetilde{\Upsilon}_Q(\chi)\\ \widetilde{\Upsilon}_Q^T(\eta) & \widetilde{\Upsilon}_R(\eta,\chi)\end{pmatrix}, \tag{4.32}$$

and mapping the space $\mathcal{M}[-h,0;n]$ into itself. Since $S(\omega_0+\Delta\omega)$ is a symmetric and positive semi-definite matrix, then this operator is self-adjoint and non-negative.

Now, based on the inequality (4.31) and the properties of the operator defined by (4.32), we obtain by virtue of the results of [4] that the set (4.27)-(4.29),(4.26) has the unique solution $\{\widetilde{P},\ \widetilde{Q}(\eta),\ \widetilde{R}(\eta,\chi)\}$, $(\eta,\chi)\in[-h,0]\times[-h,0]$. Moreover, the operator, defined by the matrix

$$\begin{pmatrix} \widetilde{P} & \widetilde{Q}(\chi) \\ \widetilde{Q}^{\,T}(\eta) & \widetilde{R}(\eta,\chi) \end{pmatrix}, \tag{4.33}$$

and mapping the space $\mathcal{M}[-h,0;n]$ into itself, is self-adjoint and non-negative.

Note that the set (4.27)-(4.29),(4.26) is equivalent to the set (4.23)-(4.25), (4.26). Moreover, the latter coincides with the set (2.5)-(2.7),(2.8) for $\omega = \omega_0 + \Delta\omega$. Therefore, by the above mentioned properties of the solution $\{\widetilde{P},\ \widetilde{Q}(\eta),\ \widetilde{R}(\eta,\chi)\}$ to the set (4.27)-(4.29),(4.26), and by virtue of Lemma 2.3, we immediately have that for all $(\eta,\chi) \in [-h,0]\times[-h,0]$:

$$\widetilde{P} = P(\omega_0+\Delta\omega),\quad \widetilde{Q}(\eta) = Q(\eta,\omega_0+\Delta\omega),\quad \widetilde{R}(\eta,\chi) = R(\eta,\chi,\omega_0+\Delta\omega). \tag{4.34}$$

These equalities, along with the equation (4.2) and the inequality (4.21), imply the continuity of the matrix-valued functions $P(\omega)$, $Q(\eta,\omega)$, $R(\eta,\chi,\omega)$ with respect to $\omega$ at $\omega = \omega_0$ uniformly in $(\eta,\chi) \in [-h,0]\times[-h,0]$. Since $\omega_0$ is any point of the interval $[\omega_1,\omega_2]$, then $P(\omega)$, $Q(\eta,\omega)$, $R(\eta,\chi,\omega)$ are continuous functions of $\omega \in [\omega_1,\omega_2]$ uniformly in $(\eta,\chi) \in [-h,0]\times[-h,0]$. This completes the proof of the lemma. □

## 4.2. **Smoothness with respect to the parameter of the solution to the set (2.5)-(2.7),(2.8).**

**Lemma 4.2.** *Let the assumption (A) be valid. Then, the derivatives $dP(\omega)/d\omega$, $\partial Q(\eta,\omega)/\partial\omega$, $\partial R(\eta,\chi,\omega)/\partial\omega$ exist and are continuous functions of $\omega \in [\omega_1,\omega_2]$ uniformly in $(\eta,\chi) \in [-h,0]\times[-h,0]$.*

*Proof.* In the proof of this lemma, we use the notations introduced in the proof of Lemma 4.1.

Let us divide the equalities (4.11)-(4.13) by $\Delta\omega$. Then, let us calculate the limits of the resulting equalities for $\Delta\omega \to 0$. Using the equations (4.1)-(4.2),(4.7)-(4.10),(4.14)-(4.15) and the inequalities (4.19),(4.21), we obtain that the derivatives $dP(\omega_0)/d\omega$, $\partial Q(\omega_0,\eta)/\partial\omega$, $\partial R(\omega_0,\eta,\chi)/\partial\omega$ exist for any pair $(\eta,\chi) \in [-h,0]\times[-h,0]$, and these derivatives have the form

$$\begin{aligned}
\frac{dP(\omega_0)}{d\omega} &= \lim_{\Delta\omega\to 0}\frac{\Delta P}{\Delta\omega} \\
&= \int_0^{+\infty}\Big[\mathcal{L}^T_{\lim}(\sigma,\omega_0)\Pi_P(\omega_0)\mathcal{L}_{\lim}(\sigma,\omega_0) \\
&\quad + \int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma,\omega_0)\Pi_Q(\eta,\omega_0)\mathcal{L}_{\lim}(\sigma+\eta,\omega_0)d\eta \\
&\quad + \int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma+\eta,\omega_0)\Pi_Q^T(\eta,\omega_0)\mathcal{L}_{\lim}(\sigma,\omega_0)d\eta \\
&\quad + \int_{-h}^0\int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma+\eta,\omega_0)\Pi_R(\eta,\chi,\omega_0)\mathcal{L}_{\lim}(\sigma+\chi,\omega_0)d\eta d\chi\Big]d\sigma,
\end{aligned} \tag{4.35}$$

$$
\begin{aligned}
\frac{\partial Q(\eta,\omega_0)}{\partial \omega} &= \lim_{\Delta\omega\to 0}\frac{\Delta Q(\eta)}{\Delta\omega} + P(\omega_0)\frac{dA_N(\omega_0)}{d\omega} \\
&= \int_0^{+\infty}\Big[\mathcal{L}^T_{\lim}(\sigma,\omega_0)\Pi_P(\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega_0) \\
&\quad + \int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma,\omega_0)\Pi_Q(\chi,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi,\eta,\omega_0)d\chi \\
&\quad + \int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma+\chi,\omega_0)\Pi^T_Q(\chi,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega_0)d\chi \\
&\quad + \int_{-h}^0\int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma+\chi,\omega_0)\Pi_R(\chi,\chi_1,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\eta,\omega_0)d\chi d\chi_1\Big]d\sigma \\
&\quad + \int_0^{\eta+h}\Big[\mathcal{L}^T_{\lim}(\sigma,\omega_0)\Pi_Q(\eta-\sigma,\omega_0) \\
&\quad + \int_{-h}^0 \mathcal{L}^T_{\lim}(\sigma+\chi,\omega_0)\Pi_R\big(\chi,\eta-\sigma,\omega_0)d\chi\Big]d\sigma + P(\omega_0)\frac{dA_N(\omega_0)}{d\omega},
\end{aligned}
\tag{4.36}
$$

$$
\begin{aligned}
\frac{\partial R(\eta,\chi,\omega_0)}{\partial \omega} &= \lim_{\Delta\omega\to 0}\frac{\Delta R(\eta,\chi)}{\Delta\omega} + Q^T(\eta,\omega_0)\frac{dA_N(\omega_0)}{d\omega} \\
&\quad + \Big(\frac{dA_N(\omega_0)}{d\omega}\Big)^T Q(\chi,\omega_0) \\
&= \int_0^{+\infty}\Big[\widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma,\eta,\omega_0)\Pi_P(\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega_0) \\
&\quad + \int_{-h}^0 \widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma,\eta,\omega_0)\Pi_Q(\chi_1,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\chi,\omega_0)d\chi_1 \\
&\quad + \int_{-h}^0 \widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma+\chi_1,\eta,\omega_0)\Pi^T_Q(\chi_1,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega_0)d\chi_1 \\
&\quad + \int_{-h}^0\int_{-h}^0 \widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma+\chi_1,\eta,\omega_0)\Pi_R\big(\chi_1,\chi_2,\omega_0) \\
&\quad \times\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_2,\chi,\omega_0)d\chi_1 d\chi_2\Big]d\sigma \\
&\quad + \int_0^{\eta+h}\Big[\Pi^T_Q(\eta-\sigma,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega_0) \\
&\quad + \int_{-h}^0 \Pi_R(\chi_1,\eta-\sigma,\omega_0)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\chi,\omega_0)d\chi_1\Big]d\sigma \\
&\quad + \int_0^{\chi+h}\Big[\widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma,\eta,\omega_0)\Pi_Q\big(\chi-\sigma,\omega_0) \\
&\quad + \int_{-h}^0 \widetilde{\mathcal{L}}^{\,T}_{\lim}(\sigma+\chi_1,\eta,\omega_0)\Pi_R(\chi_1,\chi-\sigma,\omega_0)d\chi_1\Big]d\sigma \\
&\quad + \int_0^{\min(\eta+h,\chi+h)} \Pi_R(\eta-\sigma,\chi-\sigma,\omega_0)d\sigma
\end{aligned}
$$

$$+Q^T(\eta,\omega_0)\frac{dA_N(\omega_0)}{d\omega}+\left(\frac{dA_N(\omega_0)}{d\omega}\right)^T Q(\chi,\omega_0), \tag{4.37}$$

where

$$\begin{aligned}\Pi_P(\omega) &= P(\omega)\frac{dA_0(\omega)}{d\omega}+\left(\frac{dA_0(\omega)}{d\omega}\right)^T P(\omega)-P(\omega)\frac{dS(\omega)}{d\omega}P(\omega)\\ &\quad -P(\omega)\frac{dA_N(\omega)}{d\omega}-\left(\frac{dA_N(\omega)}{d\omega}\right)^T P(\omega),\end{aligned} \tag{4.38}$$

$$\begin{aligned}\Pi_Q(\eta,\omega) &= \left[\frac{dA_0(\omega)}{d\omega}-\frac{dS(\omega)}{d\omega}P(\omega)\right]^T Q(\eta,\omega)+P(\omega)\frac{\partial G(\eta,\omega)}{\partial\omega}\\ &\quad +\sum_{j=1}^{N-1}P(\omega)\frac{dA_j(\omega)}{d\omega}\delta(\eta+h_j)+\alpha^T(\omega)P(\omega)\frac{dA_N(\omega)}{d\omega},\end{aligned} \tag{4.39}$$

$$\begin{aligned}\Pi_R(\eta,\chi,\omega) &= \left(\frac{\partial G(\eta,\omega)}{\partial\omega}\right)^T Q(\chi,\omega)+Q^T(\eta,\omega)\frac{\partial G(\chi,\omega)}{\partial\omega}\\ &\quad +\sum_{j=1}^{N-1}\left(\frac{dA_j(\omega)}{d\omega}\right)^T Q(\chi,\omega)\delta(\eta+h_j)\\ &\quad +\sum_{j=1}^{N-1}Q^T(\eta,\omega)\frac{dA_j(\omega)}{d\omega}\delta(\chi+h_j)\\ &\quad -Q^T(\eta,\omega)\frac{dS(\omega)}{d\omega}Q(\chi,\omega)\\ &\quad +\left(\frac{dQ(\eta,\omega)}{d\eta}\right)^T\frac{dA_N(\omega)}{d\omega}+\left(\frac{dA_N(\omega)}{d\omega}\right)^T\frac{dQ(\chi,\omega)}{d\chi}\\ &\quad +\left(\frac{dA_N(\omega)}{d\omega}\right)^T P(\omega)\theta(\chi,\omega)+\theta^T(\eta,\omega)P(\omega)\frac{dA_N(\omega)}{d\omega}\\ &\quad +\sum_{j=1}^{N-1}A_j^T(\omega)P(\omega)\frac{dA_N(\omega)}{d\omega}\delta(\eta+h_j)\\ &\quad +\sum_{j=1}^{N-1}\left(\frac{dA_N(\omega)}{d\omega}\right)^T P(\omega)A_j(\omega)\delta(\chi+h_j),\end{aligned} \tag{4.40}$$

$\mathcal{L}_{\lim}(\sigma,\omega)$ is the solution of the problem

$$\begin{aligned}\frac{d\mathcal{L}_{\lim}(\sigma,\omega)}{d\sigma} &= \alpha(\omega)\mathcal{L}_{\lim}(\sigma,\omega)+\sum_{j=1}^{N}A_j(\omega)\mathcal{L}_{\lim}(\sigma-h_j,\omega)\\ &\quad +\int_{-h}^{0}\theta(\eta,\omega)\mathcal{L}_{\lim}(\sigma+\eta,\omega)d\eta,\quad \sigma>0,\end{aligned}$$

$$\mathcal{L}_{\lim}(0,\omega)=I_n,\qquad \mathcal{L}_{\lim}(\sigma,\omega)=0\ \ \forall\sigma<0, \tag{4.41}$$

and $\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega)$ is defined as follows:

$$\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega) \triangleq \sum_{j=1}^{N}\left\{ \begin{array}{ll} \mathcal{L}_{\lim}(\sigma-\eta-h_j,\omega)A_j(\omega), & \eta-\sigma<-h_j\le\eta \\ 0, & otherwise \end{array}\right\}$$

$$+\int_{-\eta}^{h}\mathcal{L}_{\lim}(\sigma-\eta-\chi,\omega)\theta(-\chi,\omega)d\chi, \quad \sigma\ge 0, \quad \eta\in[-h,0]. \tag{4.42}$$

Since $\omega_0$ is any point of the interval $[\omega_1,\omega_2]$, then the derivatives $dP(\omega)/d\omega$, $\partial Q(\eta,\omega)/\partial\omega$, $\partial R(\eta,\chi,\omega)/\partial\omega$ exist for any $\omega\in[\omega_1,\omega_2]$ and any pair $(\eta,\chi)\in[-h,0]\times[-h,0]$. Now, let as show that these derivatives are continuous in $\omega\in[\omega_1,\omega_2]$ uniformly with respect to $(\eta,\chi)\in[-h,0]\times[-h,0]$.

Let us denote

$$\mathcal{P}(\omega)\triangleq\frac{dP(\omega)}{d\omega}, \qquad \mathcal{Q}(\eta,\omega)\triangleq\frac{\partial Q(\eta,\omega)}{\partial\omega}-P(\omega)\frac{dA_N(\omega)}{d\omega},$$

$$\mathcal{R}(\eta,\chi,\omega)\triangleq\frac{\partial R(\eta,\chi,\omega)}{\partial\omega}-Q^T(\eta,\omega)\frac{dA_N(\omega)}{d\omega}-\left(\frac{dA_N(\omega)}{d\omega}\right)^T Q(\chi,\omega). \tag{4.43}$$

Using these notations and the equations (4.35)-(4.37), we have for any $\omega\in[\omega_1,\omega_2]$

$$\begin{aligned}\mathcal{P}(\omega) = \int_0^{+\infty}\Big[&\mathcal{L}_{\lim}^T(\sigma,\omega)\Pi_P(\omega)\mathcal{L}_{\lim}(\sigma,\omega)\\ &+\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma,\omega)\Pi_Q(\eta,\omega)\mathcal{L}_{\lim}(\sigma+\eta,\omega)d\eta\\ &+\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma+\eta,\omega)\Pi_Q^T(\eta,\omega)\mathcal{L}_{\lim}(\sigma,\omega)d\eta\\ &+\int_{-h}^{0}\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma+\eta,\omega)\Pi_R(\eta,\chi,\omega)\mathcal{L}_{\lim}(\sigma+\chi,\omega)d\eta d\chi\Big]d\sigma,\end{aligned} \tag{4.44}$$

$$\begin{aligned}\mathcal{Q}(\eta,\omega) = \int_0^{+\infty}\Big[&\mathcal{L}_{\lim}^T(\sigma,\omega)\Pi_P(\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega)\\ &+\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma,\omega)\Pi_Q(\chi,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi,\eta,\omega)d\chi\\ &+\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma+\chi,\omega)\Pi_Q^T(\chi,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega)d\chi\\ &+\int_{-h}^{0}\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma+\chi,\omega)\Pi_R(\chi,\chi_1,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\eta,\omega)d\chi d\chi_1\Big]d\sigma\\ &+\int_0^{\eta+h}\Big[\mathcal{L}_{\lim}^T(\sigma,\omega)\Pi_Q(\eta-\sigma,\omega)\\ &+\int_{-h}^{0}\mathcal{L}_{\lim}^T(\sigma+\chi,\omega)\Pi_R\big(\chi,\eta-\sigma,\omega\big)d\chi\Big]d\sigma,\end{aligned} \tag{4.45}$$

$$\mathcal{R}(\eta,\chi,\omega) = \int_0^{+\infty}\Big[\widetilde{\mathcal{L}}_{\lim}^{\,T}(\sigma,\eta,\omega)\Pi_P(\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega)$$

$$+\int_{-h}^{0}\widetilde{\mathcal{L}}^{T}_{\lim}(\sigma,\eta,\omega)\Pi_Q(\chi_1,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\chi,\omega)d\chi_1$$

$$+\int_{-h}^{0}\widetilde{\mathcal{L}}^{T}_{\lim}(\sigma+\chi_1,\eta,\omega)\Pi_Q^T(\chi_1,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega)d\chi_1$$

$$+\int_{-h}^{0}\int_{-h}^{0}\widetilde{\mathcal{L}}^{T}_{\lim}(\sigma+\chi_1,\eta,\omega)\Pi_R\big(\chi_1,\chi_2,\omega)$$

$$\times\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_2,\chi,\omega)d\chi_1 d\chi_2\Big]d\sigma$$

$$+\int_{0}^{\eta+h}\Big[\Pi_Q^T(\eta-\sigma,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma,\chi,\omega)$$

$$+\int_{-h}^{0}\Pi_R(\chi_1,\eta-\sigma,\omega)\widetilde{\mathcal{L}}_{\lim}(\sigma+\chi_1,\chi,\omega)d\chi_1\Big]d\sigma$$

$$+\int_{0}^{\chi+h}\Big[\widetilde{\mathcal{L}}^{T}_{\lim}(\sigma,\eta,\omega)\Pi_Q\big(\chi-\sigma,\omega)$$

$$+\int_{-h}^{0}\widetilde{\mathcal{L}}^{T}_{\lim}(\sigma+\chi_1,\eta,\omega)\Pi_R(\chi_1,\chi-\sigma,\omega)d\chi_1\Big]d\sigma$$

$$+\int_{0}^{\min(\eta+h,\chi+h)}\Pi_R(\eta-\sigma,\chi-\sigma,\omega)d\sigma, \tag{4.46}$$

where $\mathcal{L}_{\lim}(\sigma,\omega)$ and $\widetilde{\mathcal{L}}_{\lim}(\sigma,\eta,\omega)$ are given by (4.41) and (4.42), respectively.

Remember that in the proof of Lemma 4.1, we transformed equivalently the set (4.3)-(4.6) to the set of integral equations (4.11)-(4.13). In the present proof, we apply the inverse transformation of the set (4.44)-(4.46). Due to this transformation, we obtain the following set of equations, equivalent to (4.44)-(4.46):

$$\mathcal{P}(\omega)\alpha(\omega)+\alpha^T(\omega)\mathcal{P}(\omega)+\mathcal{Q}(0,\omega)+\mathcal{Q}^T(0,\omega)+\Pi_P(\omega)=0, \tag{4.47}$$

$$\begin{aligned}\frac{d\mathcal{Q}(\eta,\omega)}{d\eta} &= \alpha^T(\omega)\mathcal{Q}(\eta,\omega)+\mathcal{P}(\omega)\theta(\eta,\omega)\\ &\quad+\sum_{j=1}^{N-1}\mathcal{P}(\omega)A_j(\omega)\delta(\eta+h_j)+\mathcal{R}(0,\eta,\omega)+\Pi_Q(\eta,\omega),\end{aligned} \tag{4.48}$$

$$\begin{aligned}\Big(\frac{\partial}{\partial\eta}+\frac{\partial}{\partial\chi}\Big)\mathcal{R}(\eta,\chi,\omega) &= \mathcal{Q}^T(\eta,\omega)\theta(\chi,\omega)\\ &\quad+\theta^T(\eta,\omega)\mathcal{Q}(\chi,\omega)+\sum_{j=1}^{N-1}A_j^T(\omega)\mathcal{Q}(\chi,\omega)\delta(\eta+h_j)\\ &\quad+\sum_{j=1}^{N-1}\mathcal{Q}^T(\eta,\omega)A_j(\omega)\delta(\chi+h_j)+\Pi_R(\eta,\chi,\omega),\end{aligned} \tag{4.49}$$

$$\mathcal{Q}(-h,\omega)=\mathcal{P}(\omega)A_N(\omega),$$

$$\mathcal{R}(-h,\eta,\omega)=A_N^T(\omega)\mathcal{Q}(\eta,\omega),\quad \mathcal{R}(\eta,-h,\omega)=\mathcal{Q}^T(\eta,\omega)A_N(\omega), \tag{4.50}$$

where $\omega\in[\omega_1,\omega_2]$ is the parameter.

Now, based on the set (4.47)-(4.50) and using the statement of Lemma 4.1, one can show (similarly to the proof of Lemma 4.1) the continuity of $\mathcal{P}(\omega)$, $\mathcal{Q}(\eta,\omega)$, $\mathcal{R}(\eta,\chi,\omega)$ in $\omega\in[\omega_1,\omega_2]$ uniformly with respect to $(\eta,\chi)\in[-h,0]\times[-h,0]$. The latter, along with the equation (4.43) implies the continuity of $dP(\omega)/d\omega$, $\partial Q(\eta,\omega)/\partial\omega$, $\partial R(\eta,\chi,\omega)/\partial\omega$ in $\omega\in[\omega_1,\omega_2]$ uniformly with respect to $(\eta,\chi)\in[-h,0]\times[-h,0]$. This completes the proof of the lemma. □

## 5. Main Theorems

5.1. **Dependence on the parameter of the state-feedback optimal control.** Based on the expression (2.12) for the state-feedback optimal control of the problem (2.1)-(2.3), let us consider for any $\omega\in[\omega_1,\omega_2]$ the operator $\mathcal{F}_\omega:\mathcal{M}[-h,0;n]\to E^n$, given as:

$$\mathcal{F}_\omega\big(f_E,f_L(\eta)\big)\overset{\triangle}{=}P_{\mathcal{F}}(\omega)f_E+\int_{-h}^{0}Q_{\mathcal{F}}(\eta,\omega)f_L(\eta)d\eta, \tag{5.1}$$

where

$$P_{\mathcal{F}}(\omega)=-B^T(\omega)P(\omega),\qquad Q_{\mathcal{F}}(\eta,\omega)=-B^T(\omega)Q(\eta,\omega), \tag{5.2}$$

$\big(f_E,f_L(\eta)\big)\in\mathcal{M}[-h,0;n]$, $f_E\in E^n$, $f_L(\eta)\in L^2[-h,0;E^n]$; $P(\omega)$ and $Q(\eta,\omega)$ are the components of the unique solution to the set (2.5)-(2.8) mentioned in Lemma 2.3.

**Theorem 5.1.** *Let the assumption (A) be valid. Then, for any given* $\big(f_E,f_L(\eta)\big)\in\mathcal{M}[-h,0;n]$*, the derivative* $d\mathcal{F}_\omega\big(f_E,f_L(\eta)\big)/d\omega$ *of the image* $\mathcal{F}_\omega\big(f_E,f_L(\eta)\big)$ *exists and is a continuous function of* $\omega$ *in the interval* $[\omega_1,\omega_2]$*. Moreover, for any given bounded set* $\mathcal{S}\subset\mathcal{M}[-h,0;n]$*, this derivative is continuous in the interval* $[\omega_1,\omega_2]$ *uniformly with respect to* $\big(f_E,f_L(\eta)\big)\in\mathcal{S}$*.*

*Proof.* Using the smoothness of the matrix $B(\omega)$ with respect to $\omega\in[\omega_1,\omega_2]$, as well as the equation (5.2) and Lemma 4.2, we directly obtain that the derivatives $dP_{\mathcal{F}}(\omega)/d\omega$ and $\partial Q_{\mathcal{F}}(\eta,\omega)/\partial\omega$ exist. The first of these derivatives is continuous in $\omega\in[\omega_1,\omega_2]$, and the second is continuous in $\omega\in[\omega_1,\omega_2]$ uniformly with respect to $\eta\in[-h,0]$. Therefore, for any given $\big(f_E,f_L(\eta)\big)\in\mathcal{M}[-h,0;n]$, the derivative $d\mathcal{F}_\omega\big(f_E,f_L(\eta)\big)/d\omega$ exists, has the form

$$\frac{d\mathcal{F}_\omega\big(f_E,f_L(\eta)\big)}{d\omega}=\frac{dP_{\mathcal{F}}(\omega)}{d\omega}f_E+\int_{-h}^{0}\frac{\partial Q_{\mathcal{F}}(\eta,\omega)}{\partial\omega}f_L(\eta)d\eta,\quad \omega\in[\omega_1,\omega_2], \tag{5.3}$$

and is continuous for $\omega\in[\omega_1,\omega_2]$.

Proceed to the proof of the second statement of the theorem. Since the set $\mathcal{S}\subset\mathcal{M}[-h,0;n]$ is bounded, there exists a positive number $K_{\mathcal{S}}$ such that any $\big(f_E,f_L(\eta)\big)\in\mathcal{S}$ satisfies the inequalities

$$\|f_E\|\le K_{\mathcal{S}},\qquad \big\|f_L(\eta)\big\|_{L^2}\le K_{\mathcal{S}}. \tag{5.4}$$

Let $\omega_0$ be an arbitrary but fixed point in the interval $[\omega_1,\omega_2]$, and $\Delta\omega\ne0$ be an arbitrary number such that $\omega_0+\Delta\omega\in[\omega_1,\omega_2]$. Using the equation (5.3), the inequalities (5.4) and the Cauchy-Bunyakovsky-Schwarz inequality, we obtain the following inequality for any $\big(f_E, f_L(\eta)\big) \in \mathcal{S}$:

$$
\begin{aligned}
&\left\| \frac{d\mathcal{F}_{\omega_0+\Delta\omega}\big(f_E, f_L(\eta)\big)}{d\omega} - \frac{d\mathcal{F}_{\omega_0}\big(f_E, f_L(\eta)\big)}{d\omega} \right\| \leq K_{\mathcal{S}} \Bigg[ \left\| \frac{dP_{\mathcal{F}}(\omega_0+\Delta\omega)}{d\omega} - \frac{dP_{\mathcal{F}}(\omega_0)}{d\omega} \right\| \\
&\qquad + \left( \int_{-h}^{0} \left\| \frac{\partial Q_{\mathcal{F}}(\eta,\omega_0+\Delta\omega)}{\partial\omega} - \frac{\partial Q_{\mathcal{F}}(\eta,\omega_0)}{d\omega} \right\|^2 d\eta \right)^{1/2} \Bigg].
\end{aligned} \tag{5.5}
$$

This inequality implies immediately the continuity of $d\mathcal{F}_{\omega}\big(f_E, f_L(\eta)\big)/d\omega$ at $\omega = \omega_0$ uniformly with respect to $\big(f_E, f_L(\eta)\big) \in \mathcal{S}$. The observation that $\omega_0$ is an arbitrary point in the interval $[\omega_1, \omega_2]$ completes the proof of the second statement of the theorem. □

5.2. **Dependence on the parameter of the optimal trajectory.** Substitution of the expression (2.12) for the state-feedback optimal control into the system (2.1) and using (2.4) yield the system

$$
\begin{aligned}
\frac{dx(t)}{dt} &= \alpha(\omega)x(t) + \sum_{j=1}^{N} A_j(\omega)x(t-h_j) \\
&\quad + \int_{-h}^{0} \theta(\eta,\omega)x(t+\eta)d\eta, \qquad t \geq 0,
\end{aligned} \tag{5.6}
$$

where $\alpha(\omega)$ and $\theta(\eta,\omega)$ are given by (4.7).

The system (5.6) and the initial conditions (2.2) constitute the initial-value problem for obtaining the optimal trajectory of the optimal control problem (2.1)-(2.3). Due to the results of [5], for any $\omega \in [\omega_1, \omega_2]$, the problem (5.6),(2.2) has the unique locally absolutely continuous solution $x(t) = x(t,\omega)$, $t \geq 0$.

**Theorem 5.2.** *Let the assumption (A) be valid. Then, the derivative $\partial x(t,\omega)/\partial\omega$ exists and is continuous with respect to $\omega \in [\omega_1, \omega_2]$ uniformly in $t \in [0, +\infty)$.*

*Proof.* Using the variation-of-constant formula (see e.g. [5]), for any $\omega \in [\omega_1, \omega_2]$, we obtain the unique solution of the problem (5.6),(2.2) in the form:

$$
x(t,\omega) = \mathcal{L}_{\lim}(t,\omega)x_0(\omega) + \int_{-h}^{0} \widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)\varphi(\eta,\omega)d\eta, \quad t \geq 0, \tag{5.7}
$$

$\mathcal{L}_{\lim}(t,\omega)$ is the solution of the problem (4.41) and $\widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)$ is given by (4.42), where $\sigma$ is replaced with $t$.

Due to the equation (4.7), Lemma 2.3 (item vii, equations (2.14)-(2.15)) and Corollaries 3.4-3.5, the functions $\mathcal{L}_{\lim}(t,\omega)$, $\partial\mathcal{L}_{\lim}(t,\omega)/\partial\omega$ are continuous with respect to $\omega \in [\omega_1, \omega_2]$ uniformly in $t \geq 0$. Therefore, the functions $\widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)$, $\partial\widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)/\partial\omega$ also are continuous with respect to $\omega \in [\omega_1, \omega_2]$ uniformly in $(t,\eta) \in [0, +\infty) \times [-h, 0]$.

Using (5.7) and the assumptions on the smoothness of $x_0(\omega)$ and $\varphi(\eta,\omega)$ with respect to $\omega \in [\omega_1,\omega_2]$, we obtain

$$\begin{aligned}\frac{\partial x(t,\omega)}{\partial \omega} &= \frac{\partial \mathcal{L}_{\lim}(t,\omega)}{\partial \omega} x_0(\omega) + \mathcal{L}_{\lim}(t,\omega)\frac{dx_0(\omega)}{d\omega} \\ &+ \int_{-h}^{0}\left[\frac{\partial \widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)}{\partial \omega}\varphi(\eta,\omega) + \widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega)\frac{\partial \varphi(\eta,\omega)}{\partial\omega}\right]d\eta, \\ & t\geq 0, \quad \omega\in[\omega_1,\omega_2]. \end{aligned} \tag{5.8}$$

By virtue of Lemma 2.3 (item vii, equations (2.14)-(2.15)), as well as the proofs of Lemma 3.3 (equation (3.12)) and Corollary 3.5 (equation (3.26)), we obtain the following inequalities for any prechosen $\omega = \omega_0 \in [\omega_1,\omega_2]$ and all $t \geq 0$:

$$\big\|\mathcal{L}_{\lim}(t,\omega_0)\big\| \leq c_1 \exp\big(-\gamma(\omega_0)t\big), \quad \left\|\frac{\partial \mathcal{L}_{\lim}(t,\omega_0)}{\partial\omega}\right\| \leq c_1 \exp\big(-\gamma(\omega_0)t\big), \tag{5.9}$$

where $c_1 > 0$ is some constant.

The equation (4.42) and the inequalities (5.9) yield the following inequalities for all $t \geq 0$ and $\eta \in [-h,0]$:

$$\big\|\widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega_0)\big\| \leq c_2 \exp\big(-\gamma(\omega_0)t\big), \quad \left\|\frac{\partial \widetilde{\mathcal{L}}_{\lim}(t,\eta,\omega_0)}{\partial\omega}\right\| \leq c_2 \exp\big(-\gamma(\omega_0)t\big), \tag{5.10}$$

where $c_2 > 0$ is some constant.

Now, using the equation (5.8), the inequalities (5.9)-(5.10), and the continuity of all the functions in the right-hand side of (5.8) with respect to $\omega$ at $\omega = \omega_0$ uniformly in $(t,\eta) \in [0,+\infty) \times [-h,0]$, we obtain that the derivative $\partial x(t,\omega)/\partial\omega$ is continuous with respect to $\omega$ at $\omega = \omega_0$ uniformly in $t \geq 0$. The observation that $\omega_0$ is any point of the interval $[\omega_1,\omega_2]$ implies the continuity of the derivative $\partial x(t,\omega)/\partial\omega$ with respect to $\omega \in [\omega_1,\omega_2]$ uniformly in $t \geq 0$. Thus, the theorem is proven. □

### 5.3. Dependence on the parameter of the optimal value of the cost functional.

**Theorem 5.3.** *Let the assumption (A) be valid. Then, the derivative $dJ^*(\omega)/d\omega$ exists and is continuous with respect to $\omega \in [\omega_1,\omega_2]$, where the optimal value of the cost functional $J^*(\omega)$ of the problem (2.1)-(2.3) is given by (2.13).*

*Proof.* The statement of the theorem directly follows from the equation (2.13) and Lemma 4.2. □

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