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A NEW SPLIT INVERSE PROBLEM AND AN APPLICATION TO LEAST INTENSITY FEASIBLE SOLUTIONS

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Abstract. In this paper we propose a new prototypical split inverse problem (SIP) which we call the general split common fixed points problem. We introduce iterative projection methods for solving this problem in real Hilbert space. Two strong convergence theorems are presented in the spirit of Măruşter and Popirlan and of Baillon, Bruck and Reich.

As application we show how the new SIP modeling can be used to solve the best approximation problem of the split convex feasibility problem and obtain a minimum norm solution or least intensity feasible solution in the field of intensity-modulated radiation therapy. We compare the performance of our scheme on Dang and Xue example.

1. Introduction

The main purpose of this manuscript is to present a new instance of the *Split Inverse Problem* (SIP) (formulated in [19, Section 2]). In order to present the new problem formulation a survey of SIPs is given. Afterwards we present iterative projection methods for solving the problem and prove two strong convergence theorems of the generated sequences in real Hilbert space.

The SIP concerns a model in which there are given two vector spaces X and Y and a bounded linear operator $A : X \to Y$. In addition, two inverse problems are involved. The first one, denoted by IP_1, is formulated in the space X and the second one, denoted by IP_2, is formulated in the space Y. Given these data, the Split Inverse Problem (SIP) is formulated as follows:

$$\text{find a point } x^* \in X \text{ that solves } \mathrm{IP}_1 \tag{1.1}$$

and such that

$$\text{the point } y^* = Ax^* \in Y \text{ solves } \mathrm{IP}_2. \tag{1.2}$$

In order to illustrate the generality of this split inverse problem modeling we wish to present several special cases which are studied intensively in the literature. We first start with the formulation of the *Convex Feasibility Problem* (CFP) which stands at the core of the modeling of many inverse problems in various areas of mathematics

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and the physical sciences; for example in sensor networks, in radiation therapy treatment planning, in color imaging and in adaptive filtering, see e.g., [14, 7, 9, 16] and references therein.

Problem 1.1. The Convex *Feasibility Problem* (CFP).
Let $\mathcal{H}$ be real Hilbert space and for $i = 1, \ldots, p$ let $C_i \subseteq \mathcal{H}$ be closed and convex sets. The CFP is:

$$\text{find a point } x^* \in \boldsymbol{C} := \cap_{i=1}^{p} C_i. \tag{1.3}$$

The first instance of the split inverse problems is due to Censor and Elfving in [17] and called the *Split Convex Feasibility Problem* (SCFP) which is a SIP with IP_1 and IP_2 as CFPs. This reformulation was employed for solving an inverse problem in intensity-modulated radiation therapy (IMRT) treatment planning, see [15]. Other real-world application for the SCFP include the Multi-Domain Adaptive Filtering (MDAF) [41] and navigation on the Pareto frontier in Multi-Criteria Optimization, see [24].

The problem formulates as follows.

Problem 1.2. The Split Convex *Feasibility Problem* (SCFP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Let $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$ be two non-empty, closed and convex sets, in addition given a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, the SCFP is:

$$\text{find a point } x^* \in C \text{ such that } y^* = Ax^* \in Q. \tag{1.4}$$

The next natural development is a SCFP which allows a finite number of non-empty, closed and convex sets in the spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively, [18].

Problem 1.3. The Multiple Set Split Convex Feasibility Problem (MSS-CFP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let r and p be two natural numbers. Given C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively and a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$. The MSSCFP is to find a point x^* such that

$$x^* \in \boldsymbol{C} := \cap_{i=1}^{p} C_i \text{ such that } Ax^* \in \boldsymbol{Q} := \cap_{j=1}^{r} Q_j. \tag{1.5}$$

Masad and Reich [29] generalized the MSSCFP in which several bounded and linear operators $A_j : \mathcal{H}_1 \to \mathcal{H}_2$ are involved.

Problem 1.4. The Constrained Multiple Set Split Convex Feasibility Problem (CMSSCFP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let r and p be two natural numbers. Given C_i, $1 \leq i \leq p$ and Q_j, $1 \leq j \leq r$, closed and convex subsets of $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively; further, for $1 \leq j \leq r$ let $A_j : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators. In addition let Ω be another convex subsets of $\mathcal{H}_1$. The CMSSCFP is to find a point $x^* \in \Omega$ such that

$$x^* \in \cap_{i=1}^{p} C_i \text{ such that } A_j x^* \in Q_j. \tag{1.6}$$

Censor and Segal [21] replaced the CFPs in the above split inverse problem with fixed points problems.

Problem 1.5. The Split Common Fixed Points Problem (SCFPP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let r and p be two natural numbers. Given operators $U_i : \mathcal{H}_1 \to \mathcal{H}_1$, $1 \leq i \leq p$, and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$, $1 \leq j \leq r$, with non-empty fixed points sets $\operatorname{Fix}(U_i) = C_i$, $1 \leq i \leq p$ and $\operatorname{Fix}(T_j) = Q_j$, $1 \leq j \leq r$, respectively and bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$. The SCFPP is to find a point x^* such that

$$x^* \in \cap_{i=1}^{p} C_i \text{ such that } Ax^* \in \boldsymbol{Q} = \cap_{j=1}^{r} Q_j. \tag{1.7}$$

Censor, Gibali and Reich [19] introduced the following *Split Variational Inequality Problem* (SVIP).

Problem 1.6. The Split Variational Inequality Problem (SVIP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, and non-empty, closed and convex sets $C \subseteq \mathcal{H}_1$ and $Q \subseteq \mathcal{H}_2$. The SVIP is formulated as follows:

$$\text{find a point } x^* \in C \text{ such that } \langle f(x^*), x - x^* \rangle \geq 0 \text{ for all } x \in C \tag{1.8}$$

and such that

$$\text{the point } y^* = Ax^* \in Q \text{ solves } \langle g(y^*), y - y^* \rangle \geq 0 \text{ for all } y \in Q. \tag{1.9}$$

Moudafi [31] generalized the SVIP and introduced the *Split Monotone Variational Inclusion* (SMVI).

Problem 1.7. The S*plit Monotone Variational Inclusion (SMVI)*.
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given two operators $f : \mathcal{H}_1 \to \mathcal{H}_1$ and $g : \mathcal{H}_2 \to \mathcal{H}_2$, a bounded linear operator $A : \mathcal{H}_1 \to \mathcal{H}_2$, and two set-valued mappings $B_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ and $B_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, the SMVI is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in f(x^*) + B_1(x^*) \tag{1.10}$$

and such that the point

$$y^* = Ax^* \in \mathcal{H}_2 \text{ solves } 0 \in g(y^*) + B_2(y^*). \tag{1.11}$$

Byrne, Censor, Gibali and Reich [10] generalized and introduced the following *Split Common Null Point Problem* (SCNPP).

Problem 1.8. The Split Common Null Point Problem (SCNPP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces. Given set-valued mappings $B_i : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$, $1 \leq i \leq p$, and $F_j : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$, $1 \leq j \leq r$, respectively, and bounded linear operators $A_j : \mathcal{H}_1 \to \mathcal{H}_2$, $1 \leq j \leq r$, the SCNPP is formulated as follows:

$$\text{find a point } x^* \in \mathcal{H}_1 \text{ such that } 0 \in \cap_{i=1}^{p} B_i(x^*) \tag{1.12}$$

and such that the points

$$y_j^* = A_j x^* \in \mathcal{H}_2 \text{ solve } 0 \in \cap_{j=1}^{r} F_j(y_j^*). \tag{1.13}$$

Following the above SIPs we wish to introduce a new SIP which is a generalization of the split common fixed points problem.

Problem 1.9. The General Split Common Fixed Points Problem (GSCFPP).
Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let r and p be two natural numbers. Given operators $U_i : \mathcal{H}_1 \to \mathcal{H}_1$, $1 \leq i \leq p$, and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$, $1 \leq j \leq r$, with

non-empty fixed points sets $\text{Fix}(U_i) = C_i$, $1 \leq i \leq p$ and $\text{Fix}(T_j) = Q_j$, $1 \leq j \leq r$, respectively; further, for $1 \leq j \leq r$ let $A_j : \mathcal{H}_1 \to \mathcal{H}_2$ be bounded linear operators. The problem is

$$\text{find a point } x^* \in \boldsymbol{C} := \cap_{i=1}^{p} C_i \text{ such that } A_j x^* \in Q_j. \tag{1.14}$$

Our main contribution in this manuscript is not only the introductory of the new SIP instance but also the kind of operators involved, which are demicontractive, and the strong convergence theorems of the proposed algorithms. The analysis combines arguments from Moudafi [30] and Senter and Dotson [37]. We also present a special case of the problem with firmly nonexpansive operators.

The paper is organized as follows. In Section 2, we collect some definitions of operator classes that will be needed later on. In Section 3, we present and analyze our main result which consists of strong convergence theorems for demicontractive operators as well as for the special case of firmly nonexpansive operators. Finally in Section 4 we illustrate the performance of our proposed scheme to find a minimum solution of a SCFP which is motivated from the field of intensity-modulated radiation therapy (IMRT) treatment planning and called there least-intensity feasible solution.

2. Preliminaries

Let $\mathcal{H}$ be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$, and let D be a non-empty, closed and convex subset of $\mathcal{H}$. We write $x^k \rightharpoonup x$ and $x^k \to x$ to indicate that the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges weakly and strongly to x, respectively. Next we present some properties of operators, which will be useful later on.

Definition 2.1. Let $h : \mathcal{H} \to \mathcal{H}$ be an operator and $D \subseteq \mathcal{H}$.

- The operator h is called `ν-inverse strongly monotone` (ν-ism) on D if there exists $\nu \geq 0$ such that

$$\langle h(x) - h(y), x - y \rangle \geq \nu \|h(x) - h(y)\|^2 \text{ for all } x, y \in D. \tag{2.1}$$

- The operator h is called `firmly nonexpansive`, if

$$\langle h(x) - h(y), x - y \rangle \geq \|h(x) - h(y)\|^2 \text{ for all } x, y \in D, \tag{2.2}$$

  i. e., 1-ism.
- The operator h is called `Lipschitz continuous` with constant $\kappa > 0$ on D, if

$$\|h(x) - h(y)\| \leq \kappa \|x - y\| \text{ for all } x, y \in D. \tag{2.3}$$

- The operator h is called `nonexpansive,` if

$$\|h(x) - h(y)\| \leq \|x - y\| \text{ for all } x, y \in D. \tag{2.4}$$

- The operator h is called `quasi-nonexpansive,` if

$$\|h(x) - q\| \leq \|x - q\| \text{ for all } (x, q) \in D \times \text{Fix}(h) \tag{2.5}$$

  here $\text{Fix}(h)$ denotes the fixed points set of h, that is

$$\text{Fix}(h) := \{x \in \mathcal{H} \mid h(x) = x\}. \tag{2.6}$$

- The operator h is called `firmly quasi-nonexpansive` [39, 40, Section B] and [27], namely

$$\|h(x)-q\|^2 \leq \|x-q\|^2 - \|x-h(x)\|^2 \text{ for all } (x,q) \in D \times \operatorname{Fix}(h). \tag{2.7}$$

In the literature one can find different names for firmly quasi nonexpansive operators. The term `cutter` was introduced by Cegielski and Censor in [13], Bauschke and Combettes call it the $\mathcal{T}$`-class` [2]. Zaknoon [42], and Segal and Censor [36, 21] called it `directed operator` and in [11] these operators were called `separating operators`. Another equavalent definition for this class of operators is, $\operatorname{dom} h = \mathcal{H}$ and

$$\langle h(x)-x, h(x)-q\rangle \leq 0 \text{ for all } (x,q) \in D \times \operatorname{Fix}(h). \tag{2.8}$$

- The operator h is called `averaged` if there exists a nonexpansive operator $N : \mathcal{H} \to \mathcal{H}$ and a number $c \in (0,1)$ such that

$$h = (1-c)I + cN, \tag{2.9}$$

we say that h is c-av.

(i) It can be easily verified that if h is ν-ism, then it is Lipschitz continuous with constant $1/\nu$. (ii) It is known that an operator h is averaged if and only if its complement $G := I - h$ is ν-ism for some $\nu > 1/2$, see, e.g., [9, Lemma 2.1]. (iii) A well known identity relates an operator h to its complement G

$$\|x-y\|^2 - \|h(x)-h(y)\|^2 = 2\langle G(x)-G(y), x-y\rangle - \|G(x)-G(y)\|^2. \tag{2.10}$$

It follows immediately that the operator h is nonexpansive if and only if G is 1/2-ism.

- The operator h is called a `strongly nonexpansive` [6] if it is nonexpansive and whenever $\{x^k - y^k\}_{k=1}^{\infty}$ is bounded and $\|x^k - y^k\| - \|h(x^k) - h(y^k)\| \to 0$, it follows that $(x^k - y^k) - (h(x^k) - h(y^k)) \to 0$.
- The operator h is called `demicontractive operator` [28] (see also [5, 25, 32]), if the exists $\beta \in [0,1)$ such that

$$\|h(x)-q\|^2 \leq \|x-q\|^2 + \beta\|x-h(x)\|^2 \text{ for all } (x,q) \in D \times \operatorname{Fix}(h), \tag{2.11}$$

which is equivalent to

$$\langle x-h(x), x-q\rangle \geq \frac{1-\beta}{2}\|x-h(x)\|^2 \text{ for all } (x,q) \in D \times \operatorname{Fix}(h). \tag{2.12}$$

- The operator h is called `demiclosed` [4, Definition 2] at $y \in \mathcal{H}$, if for any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ such that $x^k \rightharpoonup \overline{x}$ and $h(x^k) \to y$, we have $h(\overline{x}) = y$.
- The operator h is called a `asymptotically regular` [6] if

$$\lim_{k\to\infty}(h^k(x) - h^{k+1}(y)) = 0 \text{ for all } x \in \mathcal{H}. \tag{2.13}$$

where h^k denotes the k iterate of h.
- The operator h is called `odd` if

$$h(-x) = -h(x) \text{ for all } x \in \mathcal{H}. \tag{2.14}$$

The Demiclosedness Principle. Let $\mathcal{H}$ be a real Hilbert space, D a closed and convex subset of $\mathcal{H}$, and $N : D \to \mathcal{H}$ a nonexpansive operator. Then $I - N$ (I is the identity operator of $\mathcal{H}$) is demiclosed at $y \in \mathcal{H}$.

If $I - h$ (I is the identity operator) is demiclosed at 0 we get $x^k \rightharpoonup \overline{x}$ and $(I - h)x^k \to y$, implies $\overline{x} \in \operatorname{Fix}(h)$.

Definition 2.2. Let $\mathcal{H}$ be a real Hilbert space and $\{x^k\}_{k=0}^{\infty} \subset \mathcal{H}$. Given a closed subset $M \subseteq \mathcal{H}$, we say that $\{x^k\}_{k=0}^{\infty}$ is `regular with respect to` M if

$$\lim_{k\to\infty} \operatorname{dist}(x^k, M) = 0. \tag{2.15}$$

Next we recall Opial's Theorem [33], also known in the literature as the Krasnosel'skiĭ-Mann Theorem.

Theorem 2.3. *Let $\mathcal{H}$ be a real Hilbert space and $D \subset \mathcal{H}$ be closed and convex. Assume that $h : D \to D$ is an averaged operator with $\operatorname{Fix}(h) \neq \emptyset$. Then, for an arbitrary $x^0 \in D$, the sequence*

$$x^{k+1} = h(x^k) \tag{2.16}$$

converges weakly to $x^* \in \operatorname{Fix}(h)$.

The convergence obtained in Theorem 2.3 is not strong in general [23, 3]!

3. Main result

In order to present our algorithm for solving Problem 1.9 and its strong convergence theorem we first discuss the case of two operators split common fixed point problem, and then by using an appropriate product space reformulation we show how the general case can be presented as a two operators split common fixed point problem.

3.1. **The two operators split common fixed points problem.** Now we are focus in Problem 1.9 where $p, r = 1$. In this case there is one linear bounded operator $A : \mathcal{H}_1 \to \mathcal{H}_2$ and $U : \mathcal{H}_1 \to \mathcal{H}_1$, $T : \mathcal{H}_2 \to \mathcal{H}_2$ demicontractive operators (with constants β and μ, respectively). We assume the non-emptness of $\operatorname{Fix}(U) = C$ and $\operatorname{Fix}(T) = Q$. The two operators split common fixed points problem is to find a point x^* such that

$$x^* \in C \text{ and } Ax^* \in Q. \tag{3.1}$$

Let us denote the solution set of the two operators SCFPP by

$$\Gamma \equiv \Gamma(U, T) := \{y \in C \mid Ay \in Q\}. \tag{3.2}$$

If (3.1) is restricted to Euclidean spaces and both U and T are cutters, then Censor and Segal [21] presented the following algorithm.

Algorithm 3.1.
Initialization: Let $x^0 \in \mathcal{H}_1$ be arbitrary.
Iterative step: For $k \geq 0$ let

$$x^{k+1} = U\left(x^k + \gamma A^*(T - I)(Ax^k)\right), \tag{3.3}$$

where $\gamma \in (0, 2/L)$, L is the spectral radius of the operator A^*A (A^* is the adjoint of A).

In case where U and T are demicontractive, Moudafi presented the following algorithm.

Algorithm 3.2.
Initialization: Let $x^0 \in \mathcal{H}_1$ be arbitrary.
Iterative step: For $k \geq 0$ set $u^k = x^k + \gamma A^*(T - I)Ax^k$ and let

$$x^{k+1} = (1 - \alpha_k)u^k + \alpha_k U\left(u^k\right), \tag{3.4}$$

where $\gamma \in (0, (1 - \mu)/L)$, L is the spectral radius of the operator A^*A, μ is the demicontractivity constant of U and $\{\alpha_k\}_{k=0}^{\infty} \subset (0, 1)$.

Moudafi's weak convergence theorem [30, Theorem 2.1] of Algorithm 3.2 is next.

Theorem 3.3. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear bounded operator. Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be demicontractive operators (with constants β and μ) with non-empty* $\mathrm{Fix}\,(U) = C$ *and* $\mathrm{Fix}\,(T) = Q$. *Assume that $U - I$ and $T - I$ are demiclosed at* 0. *If $\Gamma \neq \emptyset$ then any sequence $\left\{x^k\right\}_{k=0}^{\infty}$, generated by Algorithm 3.2 converges weakly to $x^* \in \Gamma$, provided that $\gamma \in (0, (1 - \mu)/L)$ and $\{\alpha_k\}_{k=0}^{\infty} \subset (\delta, 1 - \beta - \delta)$ for small enough $\delta > 0$.*

Since our goal is to obtain strong convergence theorem of Algorithm 3.2, we can discuss the following Senter and Dotson [37] condition.

Condition 3.4. There exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r > 0$, such that

$$f(\mathrm{dist}(x, \mathrm{Fix}(T)) \leq \|x - T(x)\|. \tag{3.5}$$

Petryshyn and Williamson [34], point out the significant role of the behavior of a sequence $\left\{x^k\right\}_{k=0}^{\infty}$ with respect to the set of fixed points in the following theorem.

Theorem 3.5. ***Petryshyn and Williamson*** [34]. *Let $\mathcal{H}$ be a real Hilbert space, $T : D \subset \mathcal{H} \to \mathcal{H}$ a quasi-nonexpansive operator such that* $\mathrm{Fix}(T)$ *is non-empty and closed set. Let $x^0 \in D$ such that $x^k = T^k(x^0)$. Then the sequence $\left\{x^k\right\}_{k=0}^{\infty}$ converges strongly to a fixed point of T if and only if $\left\{x^k\right\}_{k=0}^{\infty}$ is regular with respect to* $\mathrm{Fix}(T)$.

Following this theorem, Măruşter and Popirlan [28] proved that any sequence $\left\{x^k\right\}_{k=0}^{\infty}$ generated by the Mann iteration with demicontractive operator converges strongly to $\mathrm{Fix}(T)$ if and only if it is regular with respect to $\mathrm{Fix}(T)$.

Inspired by the above works we are able to present strong convergence theorem for Algorithm 3.2 using either Condition 3.4 or regularity assumption. The proof uses a similar arguments as in [26, Theorem 2].

Theorem 3.6. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear bounded operator. Let $U : \mathcal{H}_1 \to \mathcal{H}_1$ and $T : \mathcal{H}_2 \to \mathcal{H}_2$ be a demicontractive operators (with constants β and μ, respectively) with non-empty* $\mathrm{Fix}\,(U) = C$ *and*

Fix $(T) = Q$. *Assume that* $U - I$ *and* $T - I$ *are demiclosed at* 0 *and there exists* $0 \neq \sigma \in \mathcal{H}_1$ *such that*

$$\left\{ \begin{array}{c} \langle U(y) - y, \sigma \rangle \leq 0 \text{ for all } y \in \mathcal{H}_1, \\ \langle A^*(T - I)Ay, \sigma \rangle \leq 0 \text{ for all } y \in \mathcal{H}_1. \end{array} \right. \tag{3.6}$$

If $\Gamma \neq \emptyset$ *then for a suitable* $x^0 \in \mathcal{H}_1$ *any sequence* $\{x^k\}_{k=0}^{\infty}$, *generated by Algorithm 3.2 converges strongly to* $x^* \in \Gamma$, *provided that* $\gamma \in (0, (1 - \mu)/L)$ *and* $\{\alpha_k\}_{k=0}^{\infty} \subset (\delta, 1 - \beta - \delta)$ *for small enough* $\delta > 0$.

Proof. Let $x^* \in \Gamma$ and choose $x^0 \in \mathcal{H}_1$ such that

$$\langle x^0 - x^*, \sigma \rangle > 0, \tag{3.7}$$

then there exists $\epsilon > 0$ such that

$$\langle x^0 - x^*, \sigma \rangle \geq \epsilon \|x^0 - x^*\|^2. \tag{3.8}$$

We now prove by induction that

$$\langle x^k - x^*, \sigma \rangle \geq \epsilon \|x^k - x^*\|^2 \text{ for all } k \geq 0. \tag{3.9}$$

Assume it holds for $k > 0$,

$$\begin{aligned} \langle x^{k+1} - x^*, \sigma \rangle &= \langle x^{k+1} - x^k + x^k - x^*, \sigma \rangle \\ &= \langle x^{k+1} - x^k, \sigma \rangle + \langle x^k - x^*, \sigma \rangle \\ &= \gamma \langle A^*(T - I)Ax^k + \alpha_k(U(u^k) - u^k), \sigma \rangle + \langle x^k - x^*, \sigma \rangle. \end{aligned} \tag{3.10}$$

Since $\gamma > 0$, $\alpha_k > 0$ and by (3.6) we get

$$\langle x^{k+1} - x^*, \sigma \rangle \geq \langle x^k - x^*, \sigma \rangle \tag{3.11}$$

by the induction assumption, we get that

$$\langle x^{k+1} - x^*, \sigma \rangle \geq \epsilon \|x^k - x^*\|^2, \tag{3.12}$$

by [30, Lemma 2.1] the sequence $\{x^k\}_{k=0}^{\infty}$, generated by Algorithm 3.2 is Féjer-monotone with respect to Γ, i.e., for all $k \geq 0$, $\|x^{k+1} - x^*\| \leq \|x^k - x^*\|$. So we get

$$\langle x^{k+1} - x^*, \sigma \rangle \geq \epsilon \|x^{k+1} - x^*\|^2, \tag{3.13}$$

therefore (3.9) holds for all $k \geq 0$. Finally, by Theorem 3.3 $x^k \rightharpoonup x^*$, we get $\|x^k - x^*\| \to 0$, which completes the proof. □

3.2. **The general split common fixed points problem.** Now we employ a product space formulation similar to [21], originally due to Pierra [35], to derive and analyze a simultaneous algorithm for Problem 1.9. Let Γ be the solution set of Problem 1.9. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, we introduce the spaces $\mathbf{V} = \mathcal{H}_1$ and $\mathbf{W} = \mathcal{H}_1^p \times \mathcal{H}_2^r$, where p and r are as in Problem 1.9. Define the following sets in the product spaces

$$\tilde{\boldsymbol{C}} := \mathcal{H}_1 \text{ and} \tag{3.14}$$

$$\tilde{\boldsymbol{Q}} := \left(\prod_{i=1}^{p} \sqrt{\lambda_i} C_i \right) \times \left(\prod_{j=1}^{r} \sqrt{\beta_j} Q_j \right), \tag{3.15}$$

and the operator

$$\boldsymbol{A}: = \left(\sqrt{\lambda_1}I, \ldots, \sqrt{\lambda_p}I, \sqrt{\beta_1}A_1^*, \ldots, \sqrt{\beta_r}A_r^*\right)^*, \tag{3.16}$$

where $\lambda_i > 0$, for $i = 1, \ldots, p$, and $\beta_j > 0$, for $j = 1, \ldots, r$. Let $\mathbf{y} = (y_1, \ldots, y_p, \ldots, y_{p+r}) \in \mathbf{W}$, where $y_1, \ldots, y_p \in \mathcal{H}_1$ and $y_{p+1}, \ldots, y_{p+r} \in \mathcal{H}_2$. Define the operator $\boldsymbol{T} : \boldsymbol{W} \to \boldsymbol{W}$ by

$$\begin{aligned} \boldsymbol{T}(\boldsymbol{y}) = \boldsymbol{T}\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_{p+r} \end{pmatrix} &= \left(\left(U_1\left(y_1\right)\right), \left(U_2\left(y_2\right)\right), \ldots, \left(U_p\left(y_p\right)\right),\right. \\ &\left.\left(T_1\left(y_{p+1}\right)\right), \left(T_2\left(y_{p+2}\right)\right), \ldots, \left(T_r\left(y_{p+r}\right)\right)\right). \end{aligned} \tag{3.17}$$

We have obtained a two operators split fixed points problem in the product space, with sets $\tilde{\boldsymbol{C}} = \mathcal{H}_1$, $\tilde{\boldsymbol{Q}} \subseteq \boldsymbol{W}$, the operator $\boldsymbol{A}$, the identity operator $\boldsymbol{I} : \tilde{\boldsymbol{C}} \to \tilde{\boldsymbol{C}}$ and the operator $\boldsymbol{T} : \boldsymbol{W} \to \boldsymbol{W}$**.** This problem can be solved using Algorithm 3.2. It is also easy to verify that the following equivalence holds

$$x \in \Gamma \text{ if and only if } \boldsymbol{A}x \in \tilde{\boldsymbol{Q}}. \tag{3.18}$$

Therefore, we may apply Algorithm 3.2

$$\begin{aligned} x^{k+1} &= (1-\alpha_k)(x^k + \gamma\boldsymbol{A}^*(\boldsymbol{T}-\boldsymbol{I})\boldsymbol{A}x^k) + \alpha_k\boldsymbol{I}(x^k + \gamma\boldsymbol{A}^*(\boldsymbol{T}-\boldsymbol{I})\boldsymbol{A}x^k) \\ &= x^k + \gamma\boldsymbol{A}^*(\boldsymbol{T}-\boldsymbol{I})\boldsymbol{A}x^k \end{aligned} \tag{3.19}$$

to the problem (3.14)–(3.17) in order to obtain a solution of the original SCFPP. Observe that in this case Algorithm 3.2 and Algorithm 3.1 coincides. The translation of the iterative step (3.19) to the original spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ is obvious by using the relation

$$\boldsymbol{T}(\boldsymbol{A}x) = \left(\sqrt{\lambda_1}U_1(x), \ldots, \sqrt{\lambda_p}U_p(x), \sqrt{\beta_1}A_1T_1(x), \ldots, \sqrt{\beta_t}A_rT_r(x)\right)^*, \tag{3.20}$$

and obtain the following algorithm,

Algorithm 3.7.
Initialization: Select an arbitrary starting point $x^0 \in \mathcal{H}_1$.
Iterative step: Given the current iterate x^k, compute

$$x^{k+1} = x^k + \gamma\left(\sum_{i=1}^{p}\lambda_i\left(U_i(x^k) - x^k\right) + \sum_{j=1}^{r}\beta_jA_j^*(T_j - I)A_jx^k\right). \tag{3.21}$$

Here $\gamma \in (0, (1-\mu)/L)$, $L = \sum_{i=1}^{p}\lambda_i + \sum_{j=1}^{r}\beta_j\|T_j\|^2$ and μ is the maximum demicontractivity constant of $\{U_i\}_{i=1}^{p}$.

The following convergence result follows from Theorem 3.6.

Theorem 3.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces and let $A : \mathcal{H}_1 \to \mathcal{H}_2$ be a linear bounded operator. Let $U_i : \mathcal{H}_1 \to \mathcal{H}_1$, $i = 1, \ldots, p$, and $T_j : \mathcal{H}_2 \to \mathcal{H}_2$, $j = 1, \ldots, r$ be a demicontractive operators (with constants β and μ) with non-empty*

$\text{Fix}(U) = C$ *and* $\text{Fix}(T) = Q$. *Assume that* $(U_i - I)$, $i = 1, \dots, p$ *and* $(T_j - I)$, $j = 1, \dots, r$, *are demiclosed at* 0 *and there exists* $0 \neq \sigma \in \mathcal{H}_1$ *such that*

$$\begin{cases} \langle U_i(y) - y, \sigma \rangle \leq 0 \text{ for all } y \in \mathcal{H}_1, \ i = 1, \dots, p \\ \langle A_j^*(T_j - I)A_j y, \sigma \rangle \leq 0 \text{ for all } y \in \mathcal{H}_1, \ j = 1, \dots, r. \end{cases} \tag{3.22}$$

If $\Gamma \neq \emptyset$ *then any sequence* $\left\{x^k\right\}_{k=0}^{\infty}$, *generated by Algorithm 3.7 converges strongly to* $x^* \in \Gamma$, *provided that* $\gamma \in (0, (1 - \mu)/L)$, $L = \sum_{i=1}^{p} \lambda_i + \sum_{j=1}^{r} \beta_j \|T_j\|^2$ *and* μ *is the maximum demicontractivity constant of* $\{U_i\}_{i=1}^{p}$.

Proof. Applying Theorems 3.3 and 3.6 to the two operators split fixed points problem in the product space setting with $U = \boldsymbol{I} : \mathcal{H}_1 \to \mathcal{H}_1$, $\text{Fix}(U) = \tilde{\boldsymbol{C}}$ and $T = \boldsymbol{T} : \boldsymbol{W} \to \boldsymbol{W}$, $\text{Fix}(T) = \tilde{\boldsymbol{Q}}$ the proof follows. □

3.3. **Firmly nonexpansive operators.** In this subsection we wish to present an interesting special case of demicontractive operators, which is firmly nonexpansive operators and establish strong convergence theorem of algorithm Algorithm 3.2. Our analysis follows similar arguments from the proof of [1, Theorem 1.1], see also [29, Lemma 7].

Lemma 3.9. *Let* $\mathcal{X}$ *be a Banach space which is uniformly convex. If the operator* $S : \mathcal{X} \to \mathcal{X}$ *is nonexpansive, odd and asymptotically regular, then for any* $x \in \mathcal{X}$, *the sequence* $\left\{S^k(x)\right\}_{k=1}^{\infty}$ *converges strongly to a fixed point of* S.

The next theorem is focus in a special case of Algorithm 3.2, in which $\alpha_k \equiv 1$, that is a Landweber-type operator, see [12]. The theorem's proof follows the same lines as in [12] and is given for the convenient of the reader.

Theorem 3.10. *Let* $\mathcal{H}_1$ *and* $\mathcal{H}_2$ *be two real Hilbert spaces and let* $A : \mathcal{H}_1 \to \mathcal{H}_2$ *be a linear bounded operator. Let* $U : \mathcal{H}_1 \to \mathcal{H}_1$ *and* $T : \mathcal{H}_2 \to \mathcal{H}_2$ *be odd and firmly nonexpansive operators with non-empty* $\text{Fix}(U) = C$ *and* $\text{Fix}(T) = Q$. *If* $\Gamma \neq \emptyset$ *and* $\gamma \in (0, 2/L)$, *where* L *is the spectral radius of the operator* A^*A, *then for any* $x^0 \in \mathcal{H}_1$ *any sequence* $\left\{x^k\right\}_{k=0}^{\infty}$, *generated by Algorithm 3.2 with* $\alpha_k \equiv 1$, *i.e.,*

$$x^{k+1} = U\left(x^k - \gamma A^*(I - T)Ax^k\right), \tag{3.23}$$

converges strongly to $x^* \in \Gamma$.

Proof. First we prove that the operator $A^*(T - I)A$ is $1/L$-inverse strongly monotone. So

$$\begin{aligned} &\|A^*(I - T)Ax - A^*(I - T)Ay\|^2 \\ &= \langle A^*((I - T)Ax - (I - T)Ay), A^*((I - T)Ax - (I - T)Ay)\rangle \\ &= \langle (I - T)Ax - (I - T)Ay, AA^*((I - T)Ax - (I - T)Ay)\rangle \\ &\leq L\|(I - T)Ax - (I - T)Ay\|^2. \end{aligned} \tag{3.24}$$

In addition

$$\begin{aligned} \|(I - T)Ax - (I - T)Ay\|^2 &= \|(Ax - T(Ax) - (Ay - T(Ay))\|^2 \\ &= \|Ax - Ay\|^2 + \|T(Ay) - T(Ax)\|^2 \\ &\quad - 2\langle T(Ax) - T(Ay), Ax - Ay\rangle, \end{aligned} \tag{3.25}$$

since T is firmly nonexpansive

$$\langle T(Ax) - T(Ay), Ax - Ay\rangle \geq \|T\,(Ax) - T\,(Ay)\,\|^2. \tag{3.26}$$

Combining the above inequalities yields,

$$\begin{aligned}\|A^*(I-T)Ax - A^*(I-T)Ay\|^2 &\leq L(\|Ax - Ay\|^2 - \|T(Ax) - T(Ay)\|^2)\\ &\leq L\|Ax - Ay\|^2 = L\langle x - y, A^*A(x-y)\rangle\\ &= L\langle A(x-y), A(x-y)\rangle = L\langle x - y, A^*A(x-y)\rangle\\ &\leq L^2\|x - y\|^2. \qquad (3.27)\end{aligned}$$

So we get that the operator is L Lipschitz, i.e.,

$$\|A^*(I-T)Ax - A^*(I-T)Ay\|^2 \leq L^2\|x-y\|^2, \tag{3.28}$$

and it is easy to verify that it is $1/L$-inverse strongly monotone. From now on we abbreviate $G = A^*(I-T)A$. Now we get that for $\gamma \in (0, 2/L)$ the operator $I - \gamma G$ is averaged (see e.g., [29, Lemma 3]). Since U is firmly nonexpansive, it is averaged and therefore the composition $U\,(I - \gamma G)$. By [29, Lemma 5], $U\,(I - \gamma G)$ is strongly nonexpansive and as a result of [6, Corollary 1.1] it is asymptotically regular. Since T and U are odd then $I - \gamma G$ and also the composition $U\,(I - \gamma G)$. Finally the strong convergence is obtained by [29, Lemma 7] and [21, Lemma 3.4]. □

4. Applications

In this section we compare our new SIP instance with a standard split convex feasibility modeling. While the standard SCFP modeling allow to obtain some solution, ours enable to find the projection of a given point onto the solution set. In order to motivate to this advantage we briefly recall the Intensity-Modulated Radiation Therapy (IMRT) in which, for example, the minimum solution of a SCFP is called least-intensity feasible solution, see Xiao et al. [38].

Intensity-modulated radiation therapy (IMRT) is an advanced mode of high-precision radiotherapy that uses computer-controlled linear accelerators to deliver precise radiation doses to a malignant tumor or specific areas within the tumor. The idea is that beamlets of radiation with different intensities are transmitted into the body of the patient. Each voxel within the patient will then absorb a certain dose of radiation from each beamlet. The goal is to allows higher radiation doses to be focused to regions within the tumor while minimizing the dose to surrounding normal critical structures.

Censor et al. see e.g., [15], showed how the IMRT treatment planning can be formulated as a split convex feasibility problem. The idea is to consider (by discretization) J beamlets and divide also the region of interest of the patient into I voxels. So, the problem consists of two spaces, $\mathbb{R}^J$ - the radiation intensity space and $\mathbb{R}^I$ - the dose space. Denote by $x := (x_j)_{j=1}^J \in \mathbb{R}^J$ the vector of intensities and let $d_{ij} \geq 0$ denotes the dose absorbed in voxel i due to radiation of unit intensity from the j-th beamlet. In order to present the linear transformation between the two spaces $\mathbb{R}^J$ and $\mathbb{R}^I$ we denote by $h := (h_i)_{i=1}^I \in \mathbb{R}^I$ the dose vector, whose entries, h_i represent the total dose absorbed in voxel i, therefore

$$h_i = \Sigma_{j=1}^J d_{ij} x_j. \tag{4.1}$$

In each space there are several convex constraints sets, for example non-negativity of the intensities, that is

$$\mathbb{R}^J_+ = \{x \in \mathbb{R}^J \mid x_j \geq 0 \text{ for all } j \in J\}. \tag{4.2}$$

In the dose space typical constraints can be of the following nature. Let S_t be some volume of interest. In case that it represent an organ at risk, then it is natural to require that the dose should not exceed an upper bound U_t. This corresponds to the constraint set:

$$Q_{max,t} = \{h \in \mathbb{R}^I \mid h_i \leq U_t \text{ for all } i \in S_t\}. \tag{4.3}$$

Similarly, if S_t is a tumor volume, the dose should not fall below a lower bound l_t and we can write the set

$$Q_{\min,t} = \{h \in \mathbb{R}^I \mid h_i \geq l_t \text{ for all } i \in S_t\}. \tag{4.4}$$

There exists other kind of constraints sets but we our focus here is only in the above. Now it looks natural to search for the least-intensity feasible (LIF) solution of the above SCFP as in Xiao et al. paper [38].

4.1. **Numerical example.** In order to illustrate the performance and advantage of our proposed scheme, algorithm we compare the algorithm runs for a linear split feasibility problem taken from Dang and Xue [22]. It is worth mentioning again that our scheme allow to find the projection of a given point onto the solution set of the SCFP. All computations were performed using MATLAB R2015a on an Intel Core i5-4200U 2.3GHz running 64-bit Windows. The cpu time is measured in seconds using the intrinsic MATLAB function cputime. The exact solution of the problem $(0.2645, -0.6568, 0.4890, -0.7548, -0.3836)$ is obtained by using Matlab built-in function `fmincon`. The first 89 iterations of our scheme are presented in Table 1, where the iteration's number 89 is taken from Dang and Xue [22] when the stopping criteria is achieved. We obtain and approximate solution after 89 iterations after 0.0781 seconds. In Figure 1 we present the quintiles (in blue) and median (in red) of the iteration's trajectories of Algorithm 3.7 for different choices of parameters γ, λ_i, for $1 \leq i \leq 2$, and β_j for $1 \leq j \leq 3$.

Let the Euclidean spaces $\mathbb{R}^5$ and $\mathbb{R}^4$ and consider the constraints sets

$$\begin{aligned} C_1 &= \{x \in \mathbb{R}^5 \mid x_1 + 2x_2 + x_3 + x_4 \leq 5\} \\ C_2 &= \{x \in \mathbb{R}^5 \mid x_2 + 4x_4 + 4x_5 \leq 1\} \end{aligned} \tag{4.5}$$

and

$$\begin{aligned} Q_1 &= \{y \in \mathbb{R}^4 \mid y_1 + y_4 \leq 1\} \\ Q_2 &= \{y \in \mathbb{R}^4 \mid 2y_2 + 3y_3 \leq 6\} \\ Q_3 &= \{y \in \mathbb{R}^4 \mid y_3 + 2y_4 \leq 10\}. \end{aligned} \tag{4.6}$$

In addition

$$A = \begin{pmatrix} 2 & -1 & 3 & 2 & 3 \\ 1 & 2 & 5 & 2 & 1 \\ 2 & 0 & 2 & 1 & -2 \\ 2 & -1 & 0 & -3 & 5 \end{pmatrix}. \tag{4.7}$$

Given this data Dang and Xue considered the following SCFP

$$\text{find a point } x^* \in \cap_{i=1}^{2} C_i \tag{4.8}$$
$$\text{and such that}$$
$$\text{the point } y^* = Ax^* \cap_{j=1}^{3} Q_j. \tag{4.9}$$

They compared their scheme with Censor et al. method [18] which is Algorithm 3.7). Our SIP problem enable us to solve a more general problem, for example best approximation problem is obtained if IP_1 is variational inequality and IP_2 is feasibility problem. So, for a given point $p \in \mathbb{R}^5$ we wish to solve

$$\begin{array}{l} \min f(x) := \frac{1}{2}\|x-p\|^2 \\ \text{such that } x \text{ solves (4.8)–(4.9).} \end{array} \tag{4.10}$$

We take $p = (1, -1, 1, -1, 1)$ as x^0 in [22]. We also choose $U_i = P_{C_i}(I - \lambda(I - p))$ for positive λ and $i = 1, 2$ (this choice is due to the fact that $\nabla\left(\frac{1}{2}\|x-p\|^2\right) = I - p$). Moreover $T_j = P_{Q_j}$ for positive $j = 1, 2, 3$ and $A_j = A$. For simplicity we take $\lambda_i = 1$, $\beta_j = 1$, $\gamma = 0.1$. So the iterative step of our scheme translates to the following

(4.11)
$$x^{k+1} = x^k + 0.1\left(\sum_{i=1}^{2}\left(P_{C_i}\left(\frac{x^k+p}{2}\right) - x^k\right) + \sum_{j=1}^{3} A^t\left(P_{Q_j}\left(Ax^k\right) - Ax^k\right)\right).$$

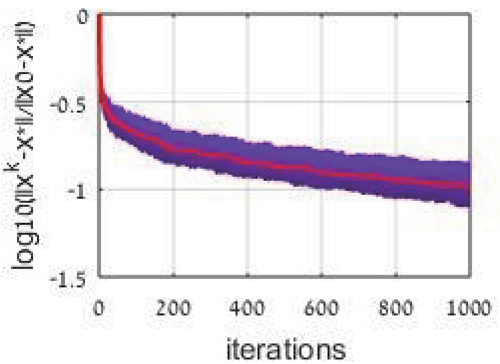

FIGURE 1. The quintiles (in blue) and median (in red) of the iteration's trajectories calculated via (4.11) for different choices of γ, λ_i, for $1 \le i \le 2$, and β_j for $1 \le j \le 3$.

TABLE 1. The first 89 coordinate wise iterations generated by the iterative step (4.11).

Iterations	x_1	x_2	x_3	x_4	x_5
1	1	-1	1	-1	1
2	0.9489	-0.9788	0.9658	-0.98	0.9117
3	0.90177426	-0.95898452	0.93444442	-0.9614787	0.82964858
4	0.85832628	-0.940456015	0.905712211	-0.944333172	0.753390518
5	0.818281537	-0.923123976	0.879399031	-0.928467843	0.682503979
6	0.781385862	-0.906904421	0.855315995	-0.913793957	0.616598063
⋮	⋮	⋮	⋮	⋮	⋮
40	0.300783969	-0.675872355	0.517753296	-0.761538378	-0.307677191
⋮	⋮	⋮	⋮	⋮	⋮
89	0.264826996	-0.656992447	0.48929388	-0.754801155	-0.382801781

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